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Center Manifolds for Non-instantaneous Impulsive Equations Under Nonuniform Hyperbolicity

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Abstract. In this paper, we establish the existence of smooth center manifolds for a class of nonautonomous differential equations with non-instantaneous impulses under sufficiently small perturbations of the linear homogeneous part which has a nonuniform exponential trichotomy. In addition, we show the C^1 smoothness of center manifolds outside the jumping times.

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1. Introduction

Instantaneous impulsive effects arise naturally in physics, biology and control theory [1, 2, 23]. Non-instantaneous impulsive effects start at an arbitrary fixed point and remain active on a finite time interval and this effect models certain dynamics of evolution processes in pharmaceutics. Noninstantaneous impulsive differential equations was introduced by Hernández and O'Regan [17] and is an extension of classical instantaneous impulsive differential equations [27, 30]; we refer the reader to [12, 18, 22, 24–26, 29] and the reference therein for results on qualitative and stability theory.

It is well known that the notion of (uniform) exponential dichotomy and exponential trichotomy play an important role in stability theory for differential equations and dynamical systems, both with continuous and discrete time. The theory of exponential dichotomies and exponential trichotomy and its applications are widely developed and we refer the reader to [8, 15, 16] for details and further references. Analogously, the more general notions of nonuniform exponential dichotomy and nonuniform exponential trichotomy play a similar role although under much weaker assumptions, and thus also for much larger classes of dynamics. In particular, the notion of nonuniform exponential dichotomy and nonuniform exponential trichotomy are essentially as weak as the (uniform) exponential dichotomy and exponential trichotomy.

A significant result in the theory of ordinary differential equations is the stable manifold theorem. The concept of the invariant manifold for rest points arises from the study of linear systems. Recall that if $\mathscr{A}$ is a linear operator on $\mathbb{R}^n$, then the spectrum of $\mathscr{A}$ splits naturally (from the point of view of the stability theorem) into three subsets: the eigenvalues with negative, zero, or positive real parts. With a linear change of coordinates that transforms $\mathscr{A}$ to its real Jordan normal form, we find that the differential equation $\mathfrak{u}' = \mathscr{A}\mathfrak{u}$ decouples into an equivalent system

$$x' = \mathscr{Q}x, \quad y' = \mathscr{W}y, \quad z' = \mathscr{R}z,$$

where $(x, y, z) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m$ with $k + l + m = n$, and $\mathscr{Q}$, $\mathscr{W}$ and $\mathscr{R}$ are linear operator whose eigenvalues have all negative, zero, and positive real parts, respectively. The subspace $\mathbb{R}^k$ is called the stable manifold of the rest point of the original system $\mathfrak{u}' = \mathscr{A}\mathfrak{u}$, the subspace $\mathbb{R}^l$ is called the center manifold, and the subspace $\mathbb{R}^m$ is called the unstable manifold; we refer the reader to [7, Chapter 4] for details and further references. If a center manifold has dimension less than the dimension of the phase space, then the most important dynamics can be studied by considering the restriction of the original system to a center manifold. We refer the reader to [3,5,6,9,10,19,20] for more details and further references.

It should also be noted that the study of invariant manifolds has a long history. Fenner and Pinto [13] introduced the notion of (h, k) manifolds and gave conditions under which the property of being a manifold with asymptotic phase holds. In [14], Fenner and Pinto studied discrete nonautonomous nonlinear systems possessing (h, k)-trichotomies and (h, k)-hyperbolic. Li et al. [21, 28] studied Lyapunov regularity and the existence of stable invariant manifolds and stable invariant manifolds of C^1 regularity for non-instantaneous impulsive equations.

In this paper we consider the following non-instantaneous linear impulsive differential equation:

$$\begin{cases} y'(t) = A(t)y(t), \ t \in (s_i, t_{i+1}] \cup (t_{-(i+1)}, s_{-i}], & i = 0, 1, 2, \ldots, \\ y(t_i^+) = B_i(t_i^+)y(t_i^-), & i = \pm 1, \pm 2, \ldots, \\ y(t) = B_{\pm i}(t)y(t_{\pm i}^-), \ t \in (t_i, s_i] \cup (s_{-i}, t_{-i}], & i = 1, 2, \ldots, \\ y(s_i^+) = y(s_i^-), & i = \pm 1, \pm 2, \ldots, \end{cases} \tag{1}$$

in $\mathbb{R}^n$, where we consider $n \times n$ matrices $A(t)$ and $B_{\pm i}(t)$ varying continuously for $t \in \mathbb{R}$ and impulsive point $t_{\pm i}$ and junction point $s_{\pm i}$ satisfying the relation $t_{-(i+1)} < s_{-i} < t_{-i}$ and $s_i < t_{i+1} < s_{i+1}$, $i \in \mathbb{N}$. The symbols $y(\varrho_{\pm i}^+)$ and $y(\varrho_{\pm i}^-)$ represent the right and left limits of $y(t)$ at $t = \varrho_{\pm i}$, respectively and set $y(\varrho_{\pm i}^-) = y(\varrho_{\pm i})$. We consider the perturbed equation:

$$\begin{cases} y'(t) = A(t)y(t) + f(t, y(t)), \ t \in (s_i, t_{i+1}] \cup (t_{-(i+1)}, s_{-i}], & i = 0, 1, 2, \ldots, \\ y(t_i^+) = B_i(t_i^+)y(t_i^-) + g_i(t_i^+, y(t_i^-)), & i = \pm 1, \pm 2, \ldots, \\ y(t) = B_{\pm i}(t)y(t_{\pm i}^-) + g_{\pm i}(t, y(t_{\pm i}^-)), \ t \in (t_i, s_i] \cup (s_{-i}, t_{-i}], & i = 1, 2, \ldots, \\ y(s_i^+) = y(s_i^-), & i = \pm 1, \pm 2, \ldots, \end{cases} \tag{2}$$

where $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ and $g_i : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ satisfy $f(t,0) = 0$ and $g_i(t,0) = 0$ for each $t \in \mathbb{R}, i \in \mathbb{Z}$. We assume f and g_i are piecewise continuous in t with at most discontinuities of the first kind at t_i.

The rest of the paper is organized as follows. In Section 2, we recall the notion of nonuniform exponential trichotomy and use Example 2 to present nonuniform exponential trichotomy for non- instantaneous impulsive differential equations, and some important lemmas are given. In Section 3, we establish the existence of center manifolds under sufficiently small perturbations of the linear homogeneous part which has a nonuniform exponential trichotomy. Existence of center manifolds are formulated and proved.

2. Preliminaries

Set $PC(\mathbb{R},\mathbb{R}^n) := \{x : \mathbb{R} \to \mathbb{R}^n : x \in C((t_i, t_{i+1}],\mathbb{R}^n), i \in \mathbb{Z}$ and there exist $x(t_i^-)$ and $x(t_i^+)$ with $x(t_i^-) = x(t_i)\}$ with the norm $\|x\|_{PC} := \sup_{t\in R} \|x(t)\|$, and $C(\mathbb{R},\mathbb{R}^n)$ denotes the Banach space of vector-valued continuous functions from $\mathbb{R} \to \mathbb{R}^n$ endowed with the norm $\|x\|_{C(\mathbb{R})} = \sup_{t\in\mathbb{R}} \|x(t)\|$ for a norm $\|\cdot\|$ on $\mathbb{R}^n$.

We assume that the impulsive points $t_{\pm i}$ and the junction points $s_{\pm i}$ satisfy the following relation

$$\cdots < t_{-(i+1)} < s_{-i} < t_{-i} \cdots < s_{-1} < t_{-1} < 0 = s_0 = t_0 < t_1 < s_1 < \cdots < t_i < s_i < t_{i+1} \ldots,$$

with $\lim_{i\to\pm\infty} t_i = \pm\infty$, $\lim_{i\to\pm\infty} s_i = \pm\infty$, and

$$\rho := \limsup_{t,s\in\mathbb{R}} \frac{r(t,s)}{t-s} < \infty, \quad \text{and} \quad \inf_{i\in Z, t\in\mathbb{R}} |\det(B_i(t))| > 0, \tag{3}$$

where $r(t,s)$ denotes the number of impulsive points which belong to (s,t).

In [28], the authors introduced a bounded linear operator $W(\cdot,\cdot)$ and any nontrivial solution of (1) can be formulated as $y(t) = W(t,s)y(s)$ for every $t,s \in \mathbb{R}$. In addition, we obtained the fact that any nontrivial solution of (1) has a finite Lyapunov exponent provided (3) holds. Note $W(t,s)W(s,\tau) = W(t,\tau)$ and $W(t,t) = \mathrm{Id}$ for every $t,s,\tau \in \mathbb{R}$, where Id denotes the identity operator.

Definition 1 (see [5]). *We say that* (1) *admits a nonuniform exponential trichotomy if there exist projections $P(t)$, $Q_1(t)$, $Q_2(t)$ varying continuously with $t \in \mathbb{R}$ such that $P(t)+Q_1(t)+Q_2(t) = \mathrm{Id}$ and*

$$W(t,s)P(s) = P(t)W(t,s), \quad W(t,s)Q_i(s) = Q_i(t)W(t,s), \quad i = 1,2 \tag{4}$$

for every $t \geq s$, and there exist constants $b > a \geq 0$, $d > c \geq 0$ and $\varepsilon, D > 0$ such that

$$\|W(t,s)P(s)\| \leq De^{a(t-s)+\varepsilon|s|}, \quad \|W(t,s)^{-1}Q_2(t)\| \leq De^{-b(t-s)+\varepsilon|t|}, \quad t,s \in \mathbb{R} \text{ with } t \geq s, \tag{5}$$

and

$$\|W(s,t)^{-1}P(s)\| \leq De^{c(s-t)+\varepsilon|s|}, \quad \|W(s,t)Q_1(t)\| \leq De^{-d(s-t)+\varepsilon|t|}, \quad t,s \in \mathbb{R} \text{ with } t \leq s. \tag{6}$$

Let $E(t) = P(t)(\mathbb{R}^n)$, $F_1(t) = Q_1(t)(\mathbb{R}^n)$ and $F_2(t) = Q_2(t)(\mathbb{R}^n)$ be the center, stable and unstable subspaces for each $t \in \mathbb{R}$, respectively. We now present an example of nonuniform exponential trichotomy.

Example 2. Let $\omega, \mu, \nu > 0$. We consider the non-instantaneous impulsive differential equation

$$\begin{cases} y'(t) = A(t)y(t), & t \in (s_i, t_{i+1}] \cup (t_{-(i+1)}, s_{-i}],\ i = 0,1,2,\ldots, \\ y(t_i^+) = B_i(t_i^+)y(t_i^-), & i = \pm1,\pm2,\ldots, \\ y(t) = B_{\pm i}(t)y(t_{\pm i}^-), & t \in (t_i, s_i] \cup (s_{-i}, t_{-i}],\ i = 1,2,\ldots, \\ y(s_i^+) = y(s_i^-), & i = \pm1,\pm2,\ldots, \\ y(s) = y_s^\top = (u(s), v_1(s), v_2(s)), & s \in (s_0, t_1), \end{cases} \tag{7}$$

where $y^\top(t) = (u, v_1, v_2)$ and

$$A(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\omega - \mu\sin t & 0 \\ 0 & 0 & \omega + \mu \sin t \end{pmatrix}, \quad B_i(t) = \begin{pmatrix} \nu & 0 & 0 \\ 0 & (1+\nu)e^{-\omega(t-t_i)} & 0 \\ 0 & 0 & \frac{1}{1+\nu}e^{\omega(t-t_i)} \end{pmatrix}$$

in $\mathbb{R}^3$. We assume that $\omega > \mu + \rho \ln(1+\nu)$ with ρ as in (3).

For $t_i < t \le s_i$, $i = 1, 2, \dots$, the solutions of (7) are given by

$$\begin{aligned} u(t) &= \nu^{r(t,s)} u(s), \\ v_1(t) &= U(t,s)(1+\nu)^{r(t,s)} v_1(s) \\ v_2(t) &= U(t,s)^{-1}(1+\nu)^{-r(t,s)} v_1(s), \end{aligned}$$

where

$$U(t,s) = e^{-\omega(t-s)+\mu(\sum_{j=2}^{r(t,s)}(\cos t_j - \cos s_{j-1}) + (\cos t_1 - \cos s))}.$$

For $s_i < t \le t_{i+1}$, $i = 0, 1, 2, \dots$, the solutions of (7) are given by

$$\begin{aligned} u(t) &= \nu^{r(t,s)} u(s), \\ v_1(t) &= U(t,s)(1+\nu)^{r(t,s)} v_1(s) \\ v_2(t) &= U(t,s)^{-1}(1+\nu)^{-r(t,s)} v_1(s), \end{aligned}$$

where

$$U(t,s) = e^{-\omega(t-s)+\mu((\cos t - \cos s_{r(t,s)}) + \sum_{j=2}^{r(t,s)}(\cos t_j - \cos s_{j-1}) + (\cos t_1 - \cos s))}.$$

Now we consider projections $P(\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$, $Q_1(\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ and $Q_2(\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$P(t)(u, v_1, v_2) = u, \quad Q_1(t)(u, v_1, v_2) = v_1, \quad Q_2(t)(u, v_1, v_2) = v_2.$$

Note (4) holds. For $t_i < t \le s_i$ and $s_i < t \le t_{i+1}$, there exists a constant $D > 0$ such that

$$\begin{aligned} U(t,s) &\le e^{-\omega(t-s)+\mu((\cos t - \cos s_{r(t,s)}) + \sum_{j=2}^{r(t,s)}(\cos t_j - \cos s_{j-1}) + (\cos t_1 - \cos s))} \\ &\le De^{(-\omega+\mu)(t-s)+2\mu s}. \end{aligned}$$

Therefore, using (3),

$$\begin{aligned} |v_1(t)| &\le De^{(-\omega+\mu)(t-s)+2\mu s} e^{\rho \ln(1+\nu)(t-s)} |v_1(s)| \\ &\le De^{(-\omega+\mu+\rho\ln(1+\nu))(t-s)+2\mu s} |v_1(s)|. \end{aligned}$$

The remaining cases (when $t < s$) can be treated in a similar manner. Similar inequalities hold for the component v_2. This shows that (7) admits a nonuniform exponential trichotomy.

To obtain the smoothness of the center manifolds, we consider the following result. Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces, and let $\mathscr{U} \subset \mathscr{X}$ be an open set. Given constants $\alpha \in (0,1]$ and $b > 0$, we consider the set $\mathscr{D}_b^\alpha(\mathscr{U}, \mathscr{Y}) = \{u \in C^{1,\alpha}(\mathscr{U}, \mathscr{Y}) : \|u\|_{1,\alpha} \le b\}$, where $\|u\|_{1,\alpha} = \max\{\|u\|_\infty, \|Du\|_\infty, H_\alpha(Du)\}$; here $\|\cdot\|_\infty$ denotes the supremum norm and $H_\alpha(Du) = \sup\{\|Du(x) - Du(y)\| / \|x-y\|^\alpha : x, y \in \mathscr{U} \text{ with } x \ne y\}$.

The following result shows that $\mathscr{D}_b^\alpha(\mathscr{U}, \mathscr{Y})$ is closed with respect to the supremum norm.

Lemma 3 (see [11, Lemma 2.2])**.** *Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces, and let $\mathscr{U} \subset \mathscr{X}$ be an open set. Then the following properties hold:*

(1) *If $u_n \in \mathscr{D}_b^\alpha(\mathscr{U}, \mathscr{Y})$ for each $n \in \mathbb{N}$ and the function $u : \mathscr{U} \to \mathscr{Y}$ is such that $\|u_n - u\|_\infty \to 0$ when $n \to \infty$, then $u \in \mathscr{D}_b^\alpha(\mathscr{U}, \mathscr{Y})$ and $Du_n(x) \to Du(x)$ when $n \to \infty$, for each $x \in \mathscr{U}$.*
(2) *If in addition $\mathscr{V} \subset \mathscr{U}$ is uniformly bounded away from the boundary of $\mathscr{U}$, then $Du_n \to Du$ uniformly on $\mathscr{V}$.*

Lemma 4 (see [23, p. 14, Lemma 3])**.** *Let $x : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a piecewise continuous function at most with discontinuities of the first kind at the points t_i. If*

$$x(t) \le \alpha + \int_s^t w(\tau)x(\tau)\mathrm{d}\tau + \sum_{s\le t_i<t} \gamma_i x(t_i),\ t \ge s$$

for some constants $\alpha, \gamma_i \ge 0$, and some function $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, then the following estimate holds

$$x(t) \le \alpha \prod_{s\le t_i<t} (1+\gamma_i)\exp\left(\int_s^t w(\tau)\mathrm{d}\tau\right).$$

Throughout the paper, we will always denote the norm $\|(x, y, z)\| = \|x\| + \|y\| + \|z\|$ for $(x, y, z) \in \mathbb{R}^n$. We assume that:

(1) Each restriction $A|(s_i, t_{i+1}] \cup (t_{-(i+1)}, s_{-i}]$ has an extension of class C^1 to some open interval containing $[s_i, t_{i+1}] \cup [t_{-(i+1)}, s_{-i}]$, and each restriction $f|((s_i, t_{i+1}] \cup (t_{-(i+1)}, s_{-i}]) \times \mathbb{R}^n$ has an extension of class C^1 to some open interval containing $([s_i, t_{i+1}] \cup [t_{-(i+1)}, s_{-i}]) \times \mathbb{R}^n$;
(2) Each restriction $B_i|(t_i, s_i] \cup (s_{-i}, t_{-i}]$ has an extension of class C^1 to some open interval containing $[t_i, s_i] \cup [s_{-i}, t_{-i}]$, and each restriction $g_i|((t_i, s_i] \cup (s_{-i}, t_{-i}]) \times \mathbb{R}^n$ has an extension of class C^1 to some open interval containing $([t_i, s_i] \cup [s_{-i}, t_{-i}]) \times \mathbb{R}^n$;
(3) $\frac{\partial f}{\partial x}(t, 0) = 0$ for every $t \in (s_i, t_{i+1}] \cup (t_{-(i+1)}, s_{-i}]$ and $\frac{\partial g_i}{\partial x}(t, 0) = 0$ for every $t \in (t_i, s_i] \cup (s_{-i}, t_{-i}]$;
(4) There exists sufficiently small $\delta > 0$ such that for each $t \in \mathbb{R}$, $i \in \mathbb{Z}$ and $x, y \in \mathbb{R}^n$ we have

$$\left\|\frac{\partial f}{\partial x}(t,x)\right\| \le \delta e^{-2\varepsilon|t|},\quad \left\|\frac{\partial^j f}{\partial x}(t,x) - \frac{\partial^j f}{\partial x}(t,y)\right\| \le \delta e^{-2\varepsilon|t|}\|x-y\|,\quad j = 0, 1, \tag{8}$$

and

$$\left\|\frac{\partial g_i}{\partial x}(t,x)\right\| \le \delta e^{-2\varepsilon|t|},\quad \left\|\frac{\partial^j g_i}{\partial x}(t,x) - \frac{\partial^j g_i}{\partial x}(t,y)\right\| \le \delta e^{-2\varepsilon|t|}\|x-y\| \quad j = 0, 1, \tag{9}$$

where ε is defined in Definition 1.

Note in (8) and (9) the $\delta > 0$ is sufficiently small so that some constants in the following Lemma's can be appropriately chosen.

Without loss of generality, we consider only the case when $t \ge s$, the case when $t \le s$ can be proved in a similar fashion, we assume that (1) admits a nonuniform exponential trichotomy. Concerning the position relationship between the impulsive point t_i and the junction point s_i, the unique solution $(P(t)y(t), Q_1(t)y(t), Q_2(t)y(t)) = (u(t), v_1(t), v_2(t)) \in E(t) \times F_1(t) \times F_2(t)$ of (2) with initial condition $(\xi, \eta_1, \eta_2) \in E(s) \times F_1(s) \times F_2(s)$ and fixed point s with $0 < s_j < s < t_{j+1} < +\infty$, $j \in \mathbb{N}$ satisfies the following conditions:

Let $s_{j+r(t,s)} < t \le t_{j+r(t,s)+1}$ and $r(t, s) \ge 1$, and we have

$$\begin{aligned}
u(t) = W(t,s)\xi &+ \int_s^{t_{j+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),v_1(\tau),v_2(\tau))\mathrm{d}\tau \\
&+ \int_{s_{j+r(t,s)}}^t W(t,\tau)P(\tau)f(\tau,u(\tau),v_1(\tau),v_2(\tau))\mathrm{d}\tau \\
&+ \sum_{k=1}^{r(t,s)-1} \int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),v_1(\tau),v_2(\tau))\mathrm{d}\tau \\
&+ \sum_{k=1}^{r(t,s)} W(t,s_{j+k})P(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),v_1(t_{j+k}),v_2(t_{j+k})),
\end{aligned} \tag{10}$$

and

$$
\begin{aligned}
v_i(t) = W(t,s)\eta_i &+ \int_s^{t_{j+1}} W(t,\tau)Q_i(\tau)f(\tau,u(\tau),v_1(\tau),v_2(\tau))\mathrm{d}\tau \\
&+ \int_{s_{j+r(t,s)}}^{t} W(t,\tau)Q_i(\tau)f(\tau,u(\tau),v_1(\tau),v_2(\tau))\mathrm{d}\tau \\
&+ \sum_{k=1}^{r(t,s)-1} \int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q_i(\tau)f(\tau,u(\tau),v_1(\tau),v_2(\tau))\mathrm{d}\tau \\
&+ \sum_{k=1}^{r(t,s)} W(t,s_{j+k})Q_i(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),v_1(t_{j+k}),v_2(t_{j+k})), \quad i=1,2.
\end{aligned} \tag{11}
$$

Let $t_{j+r(t,s)} < t \le s_{j+r(t,s)}$ and $r(t,s) \ge 1$, and we have

$$
\begin{aligned}
u(t) = W(t,s)\xi &+ \int_s^{t_{j+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),v_1(\tau),v_2(\tau))\mathrm{d}\tau \\
&+ P(t)g_{j+r(t,s)}(t,u(t_{j+r(t,s)}),v_1(t_{j+r(t,s)}),v_2(t_{j+r(t,s)})) \\
&+ \sum_{k=1}^{r(t,s)-1} \int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),v_1(\tau),v_2(\tau))\mathrm{d}\tau \\
&+ \sum_{k=1}^{r(t,s)-1} W(t,s_{j+k})P(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),v_1(t_{j+k}),v_2(t_{j+k})),
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
v_i(t) = W(t,s)\eta_i &+ \int_s^{t_{j+1}} W(t,\tau)Q_i(\tau)f(\tau,u(\tau),v_1(\tau),v_2(\tau))\mathrm{d}\tau \\
&+ Q_i(t)g_{j+r(t,s)}(t,u(t_{j+r(t,s)}),v_1(t_{j+r(t,s)}),v_2(t_{j+r(t,s)})) \\
&+ \sum_{k=1}^{r(t,s)-1} \int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q_i(\tau)f(\tau,u(\tau),v_1(\tau),v_2(\tau))\mathrm{d}\tau \\
&+ \sum_{k=1}^{r(t,s)-1} W(t,s_{j+k})Q_i(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),v_1(t_{j+k}),v_2(t_{j+k})), \quad i=1,2.
\end{aligned} \tag{13}
$$

From the properties of the projection operator, the function $u(\cdot)$ defined in (10) is a solution of the center subspace of the phase space of (1) and the function $v_i(\cdot)$ defined in (11) are the solutions of the stable subspace and the unstable subspace of the phase space of (1). Similar comments apply to the formula (12) and (13). Note, (10), (11), (12) and (13) are useful in studying the center manifold of the perturbed equation (2).

For each $(s,\xi,\eta_1,\eta_2) \in (s_j,t_{j+1}) \times E(s) \times F_1(s) \times F_2(s)$, $j \in \mathbb{N}$, we consider the semiflow

$$\Psi_t(s,\xi,\eta_1,\eta_2) = (s+t,u(s+t),v_1(s+t),v_2(s+t)).$$

3. Smooth center manifold results

In this section, using ideas from [5], we consider the existence of smooth center manifolds under sufficiently small perturbations of nonuniform exponential trichotomies. We first describe a certain class of functions (in fact each center manifold is a graph of one of these functions (see [5]).

Let $\mathscr{S}$ be the space of functions $\phi : \mathbb{R} \times E(\cdot) \to F_1(\cdot) \times F_2(\cdot)$ such that

(1) Each restriction $\phi|((s_i,t_{i+1}] \cup (t_{-(i+1)},s_{-i}]) \times \mathbb{R}^n$ has an extension of class C^1 to some open interval containing $([s_i,t_{i+1}] \cup [t_{-(i+1)},s_{-i}]) \times \mathbb{R}^n$, and each restriction $\phi|((t_i,s_i] \cup (s_{-i},t_{-i}]) \times \mathbb{R}^n$ has an extension of class C^1 to some open interval containing $([t_i,s_i] \cup [s_{-i},t_{-i}]) \times \mathbb{R}^n$;

(2) For each $s \neq t_k$ we have $\phi(s,0)=0$, $\frac{\partial \phi}{\partial x}(s,0)=0$ and $\phi(s,E(s)) \subset F_1(s)\times F_2(s)$;
(3) There exists a constant $L>0$ such that

$$\left\|\frac{\partial \phi}{\partial x}(s,x)\right\| \le L, \quad \left\|\frac{\partial^j \phi}{\partial x}(s,x)-\frac{\partial^j \phi}{\partial x}(s,y)\right\| \le L\|x-y\|, \quad s\in\mathbb{R},\ j=0,1. \tag{14}$$

We equip the space $\mathscr{S}$ with the distance

$$d(\phi,\varphi)=\sup\{\|\phi(s,x)-\varphi(s,x)\|/\|x\| : s\in\mathbb{R} \text{ and } x\in E(s)\backslash\{0\}\}. \tag{15}$$

Lemma 3 can be used to show that $\mathscr{S}$ is a complete metric space with the distance in (15) (see [4, Proposition 3]). Given a $\phi\in\mathscr{S}$ we consider the set

$$\mathcal{V}_\phi^c=\{(s,\xi,\phi(s,\xi)) : (s,\xi)\in\mathbb{R}\times E(s)\}.$$

Moreover, for arbitrarily fixed constant $\sigma>0$ we let

$$\begin{aligned} r_j^\sigma &= \sum_{k=1}^{+\infty} e^{-\sigma|s_{j+k}|}<+\infty, \qquad \bar{r}_j^\sigma = \sum_{k=1}^{+\infty} e^{-\sigma|t_{j+k}|}<+\infty,\\ r^\sigma &= \sum_{k=1}^{+\infty} e^{-\sigma|s-s_{-k}|}<+\infty, \quad \widetilde{r}^\sigma = \sum_{k=1}^{+\infty} e^{-\sigma|s_{j+k}-s|}<+\infty. \end{aligned} \tag{16}$$

Definition 5. *$\mathcal{V}_\phi^c$ is called the center manifold of* (2) *if the semiflow*

$$\Psi_t(s,\xi,\phi(s,\xi))\in\mathcal{V}_\phi^c, \quad \textit{for every } t\ge s,$$

where $\phi\in\mathscr{S}$ and $\xi\in E(s)$.

Using Definition 5, each solution in $\mathcal{V}_\phi^c$ must be of the form $(t,u(t),\phi(t,u(t)))$ for some $\phi\in\mathscr{S}$ and $t\in\mathbb{R}$. In particular, writing $\phi(t,u(t))=(\phi_1(t,u(t)),\phi_2(t,u(t)))\in F_1(t)\times F_2(t)$, the equations (10)–(11) for $s_{j+r(t,s)}<t\le t_{j+r(t,s)+1}$ and (12)–(13) for $t_{j+r(t,s)}<t\le s_{j+r(t,s)}$ can be replaced by

$$\begin{aligned} u(t) = {} & W(t,s)\xi+\int_s^{t_{j+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau\\ & + \sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau\\ & + \int_{s_{j+r(t,s)}}^{t} W(t,\tau)P(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau\\ & + \sum_{k=1}^{r(t,s)} W(t,s_{j+k})P(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\phi(t_{j+k},u(t_{j+k}))), \end{aligned} \tag{17}$$

$$\begin{aligned} \phi_l(t,u(t)) = {} & W(t,s)\phi_l(s,u(s))+\int_s^{t_{j+1}} W(t,\tau)Q_l(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau\\ & + \sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q_l(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau\\ & + \int_{s_{j+r(t,s)}}^{t} W(t,\tau)Q_l(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau\\ & + \sum_{k=1}^{r(t,s)} W(t,s_{j+k})Q_l(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\phi(t_{j+k},u(t_{j+k}))), \quad l=1,2, \end{aligned} \tag{18}$$

and

$$\begin{aligned}u(t) = W(t,s)\xi &+ \int_s^{t_{j+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau \\ &+ \sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau \\ &+ \sum_{k=1}^{r(t,s)-1} W(t,s_{j+k})P(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\phi(t_{j+k},u(t_{j+k}))) \\ &+ P(t)g_{j+r(t,s)}(t,u(t_{j+r(t,s)}),\phi(t_{j+r(t,s)},u(t_{j+r(t,s)}))),\end{aligned} \tag{19}$$

$$\begin{aligned}\phi_l(t,u(t)) = W(t,s)\phi_l(s,u(s)) &+ \int_s^{t_{j+1}} W(t,\tau)Q_l(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau \\ &+ \sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q_l(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau \\ &+ \sum_{k=1}^{r(t,s)-1} W(t,s_{j+k})Q_l(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\phi(t_{j+k},u(t_{j+k}))) \\ &+ Q_l(t)g_{j+r(t,s)}(t,u(t_{j+r(t,s)}),\phi(t_{j+r(t,s)},u(t_{j+r(t,s)}))), \quad l=1,2.\end{aligned} \tag{20}$$

For a fixed s such that $0 < s_j < s < t_{j+1} < +\infty,\ j \in \mathbb{N}$, set

$$\kappa(t) = \begin{cases}(a+\alpha)(t-s)+\varepsilon|s|, & t \ge s,\\ (c+\alpha)(s-t)+\varepsilon|s|, & t \le s,\end{cases} \qquad \alpha = 2\delta D.$$

Let $\mathbb{B}$ be the space of functions $x : \mathbb{R}\times E(s) \to \mathbb{R}^n$ such that:

(1) $x(s,\xi) = \xi$ and $x(t,\xi) \in E(t)$ for each $t \in \mathbb{R}$;
(2) Each restriction $x|(s_{j+r(t,s)}, t_{j+r(t,s)+1}] \times E(s)$ or $x|(t_{j+r(t,s)}, s_{j+r(t,s)}] \times E(s)$ has a C^1 extension to some open set containing $[s_{j+r(t,s)}, t_{j+r(t,s)+1}] \times E(s)$ or $[t_{j+r(t,s)}, s_{j+r(t,s)}] \times E(s)$;
(3)

$$\|x\|_0 := \sup\left\{\frac{\|x(t,\xi)\|}{\|\xi\|}e^{-\kappa(t)} : t \ge s, \xi \in E(s)\backslash\{0\}\right\} \le 2D, \tag{21}$$

$$\begin{aligned}\|x\|_1 &:= \sup\left\{\left\|\frac{\partial x}{\partial \xi}(t,\xi)\right\| e^{-\kappa(t)} : t \ge s, \xi \in E(s)\right\} \le 2D,\\ \|x\|_2 &:= \sup\left\{\frac{\|(\partial x/\partial\xi)(t,\xi) - (\partial x/\partial\xi)(t,\bar{\xi})\|}{\|\xi - \bar{\xi}\|}e^{-2\kappa(t)}\right\} \le 2D,\end{aligned} \tag{22}$$

with the last supremum taken over $t \ge s$ and $\xi, \bar{\xi} \in E(s)$ with $\xi \ne \bar{\xi}$.

Using Lemma 3, then $\mathbb{B}$ is a complete metric space with the norm in (21).

Lemma 6. *Assume that* (1) *admits a nonuniform exponential trichotomy. Given $\delta > 0$ sufficiently small and $(s,\xi) \in \mathbb{R}\times E(s)$, for each $\phi \in \mathscr{S}$ there exists a unique function $u_\phi \in \mathbb{B} : [s,+\infty) \to \mathbb{R}^n$ with $u_\phi(s) = \xi$ and $u_\phi(t) \in E(t)$ satisfying* (17) *and* (19) *with $t \ge s$.*

Proof. Given $(s,\xi) \in (s_j, t_{j+1}) \times E(s)$ with $\xi \ne 0$, $j \in \mathbb{N}$ and $\phi \in \mathscr{S}$, we consider the operator Λ (see below) defined in the two intervals $(s_{j+r(t,s)}, t_{j+r(t,s)+1}]$ and $(t_{j+r(t,s)}, s_{j+r(t,s)}]$.

Case 1. For $s_{j+r(t,s)} < t \le t_{j+r(t,s)+1}$, we consider

$$\begin{aligned}(\Lambda u)(t) = W(t,s)\xi &+ \int_s^{t_{j+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau \\ &+ \sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau \\ &+ \int_{s_{j+r(t,s)}}^{t} W(t,\tau)P(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau \\ &+ \sum_{k=1}^{r(t,s)} W(t,s_{j+k})P(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\phi(t_{j+k},u(t_{j+k}))),\end{aligned}$$

for each $u \in \mathbb{B}$. Let $u_1, u_2 \in \mathbb{B}$ and $\tau \ge s$, Note from (8) and (14), we obtain

$$\begin{aligned} q(\tau) &= \|f(\tau,u_1(\tau),\phi(\tau,u_1(\tau))) - f(\tau,u_2(\tau),\phi(\tau,u_2(\tau)))\| \\ &\le \delta(1+L)e^{\kappa(\tau)}e^{-2\varepsilon|\tau|}\|\xi\|\|u_1-u_2\|_0, \end{aligned} \tag{23}$$

and

$$\begin{aligned} b_i &= \|g_i(s_i,u_1(t_i),\phi(t_i,u_1(t_i))) - g_i(s_i,u_2(t_i),\phi(t_i,u_2(t_i)))\| \\ &\le \delta(1+L)e^{\kappa(t_i)}e^{-2\varepsilon|s_i|}\|\xi\|\|u_1-u_2\|_0,\ i = j+k, k = 1,2,\dots,r(t,s), \\ b_i(t) &= \|g_i(t,u_1(t_i),\phi(t_i,u_1(t_i))) - g_i(t,u_2(t_i),\phi(t_i,u_2(t_i)))\| \\ &\le \delta(1+L)e^{\kappa(t_i)}e^{-2\varepsilon|t|}\|\xi\|\|u_1-u_2\|_0,\ i = j+r(t,s). \end{aligned} \tag{24}$$

Using (23) and (24), we obtain

$$\begin{aligned} &\|(\Lambda u_1)(t) - (\Lambda u_2)(t)\| \\ &\le \int_s^{t_{j+1}} \|W(t,\tau)P(\tau)\|q(\tau)\mathrm{d}\tau + \sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}} \|W(t,\tau)P(\tau)\|q(\tau)\mathrm{d}\tau \\ &\quad + \int_{s_{j+r(t,s)}}^{t} \|W(t,\tau)P(\tau)\|q(\tau)\mathrm{d}\tau + \sum_{k=1}^{r(t,s)} \|W(t,s_{j+k})P(s_{j+k})\|b_{j+k} \\ &\le \int_s^t \|W(t,\tau)P(\tau)\|q(\tau)\mathrm{d}\tau + \sum_{k=1}^{r(t,s)} \|W(t,s_{j+k})P(s_{j+k})\|b_{j+k} \\ &\le D\delta(1+L)\|\xi\|\|u_1-u_2\|_0 e^{\kappa(t)}\left(\int_s^t e^{-\alpha(t-\tau)-\varepsilon|\tau|}\mathrm{d}\tau + \sum_{k=1}^{r(t,s)} e^{-\alpha(t-t_{j+k})-a(s_{j+k}-t_{j+k})-\varepsilon|s_{j+k}|}\right) \\ &\le D\delta(1+L)\|\xi\|\|u_1-u_2\|_0 e^{\kappa(t)}\left(\int_s^t e^{-\varepsilon|\tau|}\mathrm{d}\tau + \sum_{k=1}^{r(t,s)} e^{-\varepsilon|s_{j+k}|}\right) \\ &\le D\delta(1+L)\|\xi\|\|u_1-u_2\|_0 e^{\kappa(t)}\left(\frac{1}{\varepsilon} + r_j^{\varepsilon}\right), \end{aligned}$$

where ε is a given positive constant. This implies that

$$\|\Lambda u_1 - \Lambda u_2\|_0 \le \theta\|u_1-u_2\|_0,$$

where $\theta = D\delta(1+L)(\frac{1}{\varepsilon} + r_j^{\varepsilon})$. Take δ sufficiently small so that $\theta < \frac{1}{2}$. Therefore, the operator Λ becomes a contraction mapping. Moreover

$$\|\Lambda u\|_0 \le \|W(\cdot,s)\xi\|_0 + \theta\|u\|_0 \le D + \frac{1}{2}\|u\|_0 \le 2D.$$

Now we consider the partial derivative $\frac{\partial\Lambda(u)}{\partial\xi}$. For simplicity, we write

$$x_\phi(t) = (t,u(t,\xi),\phi(t,u(t,\xi))),\ \widetilde{x}_\phi(s_i,t_i) = (s_i,u(t_i,\xi),\phi(t_i,u(t_i,\xi))),\ z(t) = (t,u(t,\xi)).$$

Then, for $s_{j+r(t,s)} < t \le t_{j+r(t,s)+1}$, we have

$$\frac{\partial}{\partial\xi}f(x_\phi(t)) = \frac{\partial f}{\partial u}(x_\phi(t))\frac{\partial u}{\partial\xi}(t,\xi) + \frac{\partial f}{\partial\phi}(x_\phi(t))\frac{\partial\phi}{\partial u}(z(t))\frac{\partial u}{\partial\xi}(t,\xi), \tag{25}$$

where $(u,\phi) \in E(t)\times(F_1(t)\times F_2(t))$, and using (8), (14) and (22), we have

$$\begin{aligned}\left\|\frac{\partial}{\partial\xi}f(x_\phi(t))\right\| &\le \left\|\frac{\partial f}{\partial u}(x_\phi(t))\frac{\partial u}{\partial\xi}(t,\xi)\right\| + \left\|\frac{\partial f}{\partial\phi}(x_\phi(t))\frac{\partial\phi}{\partial u}(z(t))\frac{\partial u}{\partial\xi}(t,\xi)\right\|\\ &\le (1+L)\delta e^{-2\varepsilon|t|}\left\|\frac{\partial u}{\partial\xi}(t,\xi)\right\|\\ &\le 2D(1+L)\delta e^{-2\varepsilon|t|+\kappa(t)}.\end{aligned} \tag{26}$$

Similarly, for each $i = j+k, k = 1,2,\dots,r(t,s)$, we have

$$\frac{\partial}{\partial\xi}g_i(\widetilde{x}_\phi(s_i,t_i)) = \frac{\partial g_i}{\partial u}(\widetilde{x}_\phi(s_i,t_i))\frac{\partial u}{\partial\xi}(t_i,\xi) + \frac{\partial g_i}{\partial\phi}(\widetilde{x}_\phi(s_i,t_i))\frac{\partial\phi}{\partial u}(z(t_i))\frac{\partial u}{\partial\xi}(t_i,\xi). \tag{27}$$

and

$$\begin{aligned}\left\|\frac{\partial}{\partial\xi}g_i(\widetilde{x}_\phi(s_i,t_i))\right\| &\le \left\|\frac{\partial g_i}{\partial u}(\widetilde{x}_\phi(s_i,t_i))\frac{\partial u}{\partial\xi}(t_i,\xi)\right\| + \left\|\frac{\partial g_i}{\partial\phi}(\widetilde{x}_\phi(s_i,t_i))\frac{\partial\phi}{\partial u}(z(t_i))\frac{\partial u}{\partial\xi}(t_i,\xi)\right\|\\ &\le 2D(1+L)\delta e^{-2\varepsilon|s_i|+\kappa(t_i)}.\end{aligned} \tag{28}$$

Using the first inequality in (5), we obtain

$$\begin{aligned}\left\|\frac{\partial(\Lambda u)}{\partial\xi}(t,\xi)\right\| &\le \|W(t,s)\| + \int_s^{t_{j+1}}\|W(t,\tau)P(\tau)\|\left\|\frac{\partial}{\partial\xi}f(x_\phi(t))\right\|\mathrm{d}\tau\\ &\quad + \sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}}\|W(t,\tau)P(\tau)\|\left\|\frac{\partial}{\partial\xi}f(x_\phi(t))\right\|\mathrm{d}\tau\\ &\quad + \int_{s_{j+r(t,s)}}^{t}\|W(t,\tau)P(\tau)\|\left\|\frac{\partial}{\partial\xi}f(x_\phi(t))\right\|\mathrm{d}\tau\\ &\quad + \sum_{k=1}^{r(t,s)}\|W(t,s_{j+k})P(s_{j+k})\|\left\|\frac{\partial}{\partial\xi}g_{j+k}(\widetilde{x}_\phi(s_{j+k},t_{j+k}))\right\|\\ &\le De^{\kappa(t)} + 2D^2(1+L)\delta e^{\kappa(t)}\left(\int_s^t e^{-\alpha(t-\tau)-\varepsilon|\tau|}\mathrm{d}\tau + \sum_{k=1}^{r(t,s)}e^{-\varepsilon|s_{j+k}|}\right)\\ &\le De^{\kappa(t)}\left(1+\frac{2D(1+L)\delta}{\varepsilon} + 2D\delta(1+L)r_j^\varepsilon\right).\end{aligned}$$

Therefore, for δ sufficiently small, we have

$$\begin{aligned}\|\Lambda u\|_1 &\le D\left(1+\frac{2D(1+L)\delta}{\varepsilon} + 2D\delta(1+L)r_j^\varepsilon\right)\\ &\le 2D.\end{aligned}$$

Next, we also write

$$\bar{x}_\phi(t) = (t,u(t,\bar{\xi}),\phi(t,u(t,\bar{\xi}))),\quad \widehat{x}_\phi(s_i,t_i) = (s_i,u(t_i,\bar{\xi}),\phi(t_i,u(t_i,\bar{\xi}))),\quad \bar{z}(t) = (t,u(t,\bar{\xi})).$$

Using (25) and (27), we obtain

$$
\begin{aligned}
\overline{q}(t) &= \left\| \frac{\partial}{\partial \xi} f(x_\phi(t)) - \frac{\partial}{\partial \bar{\xi}} f(\bar{x}_\phi(t)) \right\| \\
&\le \left\| \frac{\partial f}{\partial u}(x_\phi(t)) - \frac{\partial f}{\partial u}(\bar{x}_\phi(t)) \right\| \left\| \frac{\partial u}{\partial \xi}(t,\xi) \right\| + \left\| \frac{\partial f}{\partial u}(\bar{x}_\phi(t)) \right\| \left\| \frac{\partial u}{\partial \xi}(t,\xi) - \frac{\partial u}{\partial \bar{\xi}}(t,\bar{\xi}) \right\| \\
&\quad + \left\| \frac{\partial f}{\partial \phi}(x_\phi(t)) - \frac{\partial f}{\partial \phi}(\bar{x}_\phi(t)) \right\| \left\| \frac{\partial \phi}{\partial u}(z(t)) \right\| \left\| \frac{\partial u}{\partial \xi}(t,\xi) \right\| \\
&\quad + \left\| \frac{\partial f}{\partial \phi}(\bar{x}_\phi(t)) \right\| \left(\left\| \frac{\partial \phi}{\partial u}(z(t)) - \frac{\partial \phi}{\partial u}(\bar{z}(t)) \right\| \left\| \frac{\partial u}{\partial \xi}(t,\xi) \right\| \right. \\
&\quad \left. + \left\| \frac{\partial \phi}{\partial u}(\bar{z}(t)) \right\| \left\| \frac{\partial u}{\partial \xi}(t,\xi) - \frac{\partial u}{\partial \bar{\xi}}(t,\bar{\xi}) \right\| \right),
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{b}_i &= \left\| \frac{\partial}{\partial \xi} g_i(\widetilde{x}_\phi(s_i,t_i)) - \frac{\partial}{\partial \bar{\xi}} g_i(\widehat{x}_\phi(s_i,t_i)) \right\| \\
&\le \left\| \frac{\partial g_i}{\partial u}(\widetilde{x}_\phi(s_i,t_i)) - \frac{\partial g_i}{\partial u}(\widehat{x}_\phi(s_i,t_i)) \right\| \left\| \frac{\partial u}{\partial \xi}(t_i,\xi) \right\| \\
&\quad + \left\| \frac{\partial g_i}{\partial u}(\widehat{x}_\phi(s_i,t_i)) \right\| \left\| \frac{\partial u}{\partial \xi}(t_i,\xi) - \frac{\partial u}{\partial \bar{\xi}}(t_i,\bar{\xi}) \right\| \\
&\quad + \left\| \frac{\partial g_i}{\partial \phi}(\widetilde{x}_\phi(s_i,t_i)) - \frac{\partial g_i}{\partial \phi}(\widehat{x}_\phi(s_i,t_i)) \right\| \left\| \frac{\partial \phi}{\partial u}(z(t_i)) \right\| \left\| \frac{\partial u}{\partial \xi}(t_i,\xi) \right\| \\
&\quad + \left\| \frac{\partial g_i}{\partial \phi}(\widehat{x}_\phi(s_i,t_i)) \right\| \left(\left\| \frac{\partial \phi}{\partial u}(z(t_i)) - \frac{\partial \phi}{\partial u}(\bar{z}(t_i)) \right\| \left\| \frac{\partial u}{\partial \xi}(t_i,\xi) \right\| \right. \\
&\qquad \left. + \left\| \frac{\partial \phi}{\partial u}(\bar{z}(t_i)) \right\| \left\| \frac{\partial u}{\partial \xi}(t_i,\xi) - \frac{\partial u}{\partial \bar{\xi}}(t_i,\bar{\xi}) \right\| \right), \quad i = j+k, \quad k = 1,2,\dots,r(t,s).
\end{aligned}
$$

Note that

$$
\|u(t,\xi) - u(t,\bar{\xi})\| \le \sup_{\vartheta \in [0,1]} \left\| \frac{\partial u}{\partial \xi}(t, \xi + \vartheta(\bar{\xi} - \xi)) \right\| \|\xi - \bar{\xi}\| \le 2De^{\kappa(t)} \|\xi - \bar{\xi}\|. \tag{29}
$$

Let $\beta = 2D(L + (1+L)^2) + (1+L) > 0$. It follows from (22) and (29) that

$$
\begin{aligned}
\overline{q}(t) &\le 2D\delta(L + (1+L)^2) e^{-2\varepsilon|t| + \kappa(t)} \|u(t,\xi) - u(t,\bar{\xi})\| \\
&\quad + 2D\delta(1+L) e^{-2\varepsilon|t| + 2\kappa(t)} \|\xi - \bar{\xi}\| \\
&\le 2D\delta\beta e^{-2\varepsilon|t| + 2\kappa(t)} \|\xi - \bar{\xi}\|,
\end{aligned} \tag{30}
$$

and for $i = j+k, k = 1,2,\dots,r(t,s)$, we obtain

$$
\begin{aligned}
\overline{b}_i &\le 2D\delta(L + (1+L)^2) e^{-2\varepsilon|s_i| + \kappa(t_i)} \|u(t_i,\xi) - u(t_i,\bar{\xi})\| \\
&\quad + 2D\delta(1+L) e^{-2\varepsilon|s_i| + 2\kappa(t_i)} \|\xi - \bar{\xi}\| \\
&\le 2D\delta\beta e^{-2\varepsilon|s_i| + 2\kappa(t_i)} \|\xi - \bar{\xi}\|.
\end{aligned} \tag{31}
$$

Therefore, for $\xi,\bar{\xi}\in E(s)$ with $\xi\neq\bar{\xi}$ and $t\geq s$, we have

$$\begin{aligned}
&\left\|\frac{\partial(\Lambda u)}{\partial\xi}(t,\xi)-\frac{\partial(\Lambda u)}{\partial\bar{\xi}}(t,\bar{\xi})\right\| \\
&\quad\leq\int_s^t\|W(t,\tau)P(\tau)\|\overline{q}(\tau)\mathrm{d}\tau+\sum_{k=1}^{r(t,s)}\|W(t,s_{j+k})P(s_{j+k})\|\overline{b}_{j+k} \\
&\quad\leq 2D^2\delta\beta\|\xi-\bar{\xi}\|\left(\int_s^t e^{a(t-\tau)-\varepsilon|\tau|+2\kappa(\tau)}\mathrm{d}\tau+\sum_{k=1}^{r(t,s)}e^{a(t-s_{j+k})-\varepsilon|s_{j+k}|+2\kappa(t_{j+k})}\right) \\
&\quad\leq 2D^2\delta\beta\|\xi-\bar{\xi}\|e^{2\kappa(t)}\left(\int_s^t e^{(a+2\alpha)(\tau-t)-\varepsilon|\tau|}\mathrm{d}\tau+\sum_{k=1}^{r(t,s)}e^{-(a+2\alpha)(t-t_{j+k})-a(s_{j+k}-t_{j+k})-\varepsilon|s_{j+k}|}\right) \\
&\quad\leq 2D^2\delta\beta\|\xi-\bar{\xi}\|e^{2\kappa(t)}\left(\frac{1}{\varepsilon}+r_j^{\varepsilon}\right).
\end{aligned}$$

For δ sufficiently small, we obtain

$$\|\Lambda u\|_2\leq 2D^2\delta\beta\left(\frac{1}{\varepsilon}+r_j^{\varepsilon}\right)\leq 2D.$$

This shows that $\Lambda(\mathbb{B})\subset\mathbb{B}$. Therefore, Λ has a unique fixed point $u_\phi\in\mathbb{B}$ such that $u_\phi=\Lambda u_\phi$.

Case 2. For $t_{j+r(t,s)}<t\leq s_{j+r(t,s)}$, we have

$$\begin{aligned}
(\Lambda u)(t)=W(t,s)\xi&+\int_s^{t_{j+1}}W(t,\tau)P(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau \\
&+\sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}}W(t,\tau)P(\tau)f(\tau,u(\tau),\phi(\tau,u(\tau)))\mathrm{d}\tau \\
&+\sum_{k=1}^{r(t,s)-1}W(t,s_{j+k})P(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\phi(t_{j+k},u(t_{j+k}))) \\
&+P(t)g_{j+r(t,s)}(t,u(t_{j+r(t,s)}),\phi(t_{j+r(t,s)},u(t_{j+r(t,s)}))).
\end{aligned}$$

From (5) and (6) with $s=t\in\mathbb{R}$, we obtain

$$\|P(t)\|\leq De^{\varepsilon|t|},\quad\|Q_1(t)\|\leq De^{\varepsilon|t|}\quad\text{and}\quad\|Q_2(t)\|\leq De^{\varepsilon|t|}.$$

Using (23) and (24), we have

$$\begin{aligned}
&\|(\Lambda u_1)(t)-(\Lambda u_2)(t)\| \\
&\quad\leq\int_s^{t_{j+1}}\|W(t,\tau)P(\tau)\|q(\tau)\mathrm{d}\tau+\sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}}\|W(t,\tau)P(\tau)\|q(\tau)\mathrm{d}\tau \\
&\qquad+\sum_{k=1}^{r(t,s)-1}\|W(t,s_{j+k})P(s_{j+k})\|b_{j+k}+\|P(t)b_{j+r(t,s)}(t)\| \\
&\quad\leq\int_s^t\|W(t,\tau)P(\tau)\|q(\tau)\mathrm{d}\tau+\sum_{k=1}^{r(t,s)-1}\|W(t,s_{j+k})P(s_{j+k})\|b_{j+k}+\|P(t)\|b_{j+r(t,s)}(t) \\
&\quad\leq D\delta(1+L)\|\xi\|\|u_1-u_2\|_0e^{\kappa(t)}\left(\int_s^t e^{-\varepsilon|\tau|}\mathrm{d}\tau+\sum_{k=1}^{r(t,s)-1}e^{-\varepsilon|s_{j+k}|}+e^{-\varepsilon|t|-(a+\alpha)(t-t_{j+r(t,s)})}\right) \\
&\quad\leq D\delta(1+L)\|\xi\|\|u_1-u_2\|_0e^{\kappa(t)}\left(\int_s^t e^{-\varepsilon|\tau|}\mathrm{d}\tau+\sum_{k=1}^{r(t,s)}e^{-\varepsilon|t_{j+k}|}\right) \\
&\quad\leq D\delta(1+L)\|\xi\|\|u_1-u_2\|_0e^{\kappa(t)}\left(\frac{1}{\varepsilon}+\bar{r}_j^{\varepsilon}\right),
\end{aligned}$$

which implies that

$$\|\Lambda u_1-\Lambda u_2\|_0\leq\theta\|u_1-u_2\|_0,$$

where $\theta = D\delta(1+L)(\frac{1}{\varepsilon}+\bar{r}_j^{\varepsilon})$. Take δ sufficiently small so that $\theta < \frac{1}{2}$. Therefore, the operator Λ becomes a contraction. Moreover

$$\|\Lambda u\|_0 \le \|W(\cdot,s)\xi\|_0 + \theta\|u\|_0 \le D + \theta\|u\|_0 \le 2D.$$

Now we consider the partial derivative $\frac{\partial\Lambda(u)}{\partial\xi}$. For simplicity of the notation, we always write $\widetilde{x}(t,t_i) = (t, u(t_i,\xi), \phi(t, u(t_i,\xi)))$ and $\widehat{x}_\phi(t,t_i) = (t, u(t_i,\bar{\xi}), \phi(t_i, u(t_i,\bar{\xi})))$. For each $i = j + r(t,s)$, we have

$$\begin{aligned}
\frac{\partial}{\partial\xi} g_i(\widetilde{x}_\phi(t,t_i)) &= \frac{\partial g_i}{\partial u}(\widetilde{x}_\phi(t,t_i))\frac{\partial u}{\partial\xi}(t_i,\xi) + \frac{\partial g_i}{\partial\phi}(\widetilde{x}_\phi(t,t_i))\frac{\partial\phi}{\partial u}(z(t_i))\frac{\partial u}{\partial\xi}(t_i,\xi),\\
\left\|\frac{\partial}{\partial\xi} g_i(\widetilde{x}_\phi(t,t_i))\right\| &\le \left\|\frac{\partial g_i}{\partial u}(\widetilde{x}_\phi(t,t_i))\frac{\partial u}{\partial\xi}(t_i,\xi)\right\| + \left\|\frac{\partial g_i}{\partial\phi}(\widetilde{x}_\phi(t,t_i))\frac{\partial\phi}{\partial u}(z(t_i))\frac{\partial u}{\partial\xi}(t_i,\xi)\right\|\\
&\le 2D(1+L)\delta e^{-2\varepsilon|t|+\kappa(t_i)}
\end{aligned}\tag{32}$$

and

$$\begin{aligned}
\overline{b}_i(t) &= \left\|\frac{\partial}{\partial\xi} g_i(\widetilde{x}_\phi(t,t_i)) - \frac{\partial}{\partial\bar{\xi}} g_i(\widehat{x}_\phi(t,t_i))\right\|\\
&\le \left\|\frac{\partial g_i}{\partial u}(\widetilde{x}_\phi(t,t_i)) - \frac{\partial g_i}{\partial u}(\widehat{x}_\phi(t,t_i))\right\|\left\|\frac{\partial u}{\partial\xi}(t_i,\xi)\right\|\\
&\quad + \left\|\frac{\partial g_i}{\partial u}(\widehat{x}_\phi(t,t_i))\right\|\left\|\frac{\partial u}{\partial\xi}(t_i,\xi) - \frac{\partial u}{\partial\bar{\xi}}(t_i,\bar{\xi})\right\|\\
&\quad + \left\|\frac{\partial g_i}{\partial\phi}(\widetilde{x}_\phi(t,t_i)) - \frac{\partial g_i}{\partial\phi}(\widehat{x}_\phi(t,t_i))\right\|\left\|\frac{\partial\phi}{\partial u}(z(t_i))\right\|\left\|\frac{\partial u}{\partial\xi}(t_i,\xi)\right\|\\
&\quad + \left\|\frac{\partial g_i}{\partial\phi}(\widehat{x}_\phi(t,t_i))\right\|\left(\left\|\frac{\partial\phi}{\partial u}(z(t_i)) - \frac{\partial\phi}{\partial u}(\bar{z}(t_i))\right\|\left\|\frac{\partial u}{\partial\xi}(t_i,\xi)\right\|\right.\\
&\qquad\left. + \left\|\frac{\partial\phi}{\partial u}(\bar{z}(t_i))\right\|\left\|\frac{\partial u}{\partial\xi}(t_i,\xi) - \frac{\partial u}{\partial\bar{\xi}}(t_i,\bar{\xi})\right\|\right),\\
\overline{b}_i(t) &\le 2D\delta(L+(1+L)^2)e^{-2\varepsilon|t|+\kappa(t_i)}\|u(t_i,\xi)-u(t_i,\bar{\xi})\|\\
&\quad + 2D\delta(1+L)e^{-2\varepsilon|t|+2\kappa(t_i)}\|\xi-\bar{\xi}\|\\
&\le 2D\delta\beta e^{-2\varepsilon|t|+2\kappa(t_i)}\|\xi-\bar{\xi}\|.
\end{aligned}\tag{33}$$

Using (26), (28) and (32) we obtain

$$\begin{aligned}
\left\|\frac{\partial\Lambda(u)}{\partial\xi}(t,\xi)\right\| &= \|W(t,s)\| + \int_s^{t_{j+1}}\|W(t,\tau)P(\tau)\|\left\|\frac{\partial f}{\partial\xi}(x_\phi(t))\right\|\mathrm{d}\tau\\
&\quad + \sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}}\|W(t,\tau)P(\tau)\|\left\|\frac{\partial f}{\partial\xi}(x_\phi(t))\right\|\mathrm{d}\tau\\
&\quad + \sum_{k=1}^{r(t,s)-1}\|W(t,s_{j+k})P(s_{j+k})\|\left\|\frac{\partial}{\partial\xi}g_{j+k}(\widetilde{x}_\phi(s_{j+k},t_{j+k}))\right\|\\
&\quad + \|P(t)\|\left\|\frac{\partial}{\partial\xi}g_{j+r(t,s)}(\widetilde{x}_\phi(t,t_{j+r(t,s)}))\right\|\\
&\le De^{\kappa(t)} + 2D^2(1+L)\delta e^{\kappa(t)}\left(\int_s^t e^{-\alpha(t-\tau)-\varepsilon|\tau|}\mathrm{d}\tau + \sum_{k=1}^{r(t,s)} e^{-\varepsilon|t_{j+k}|}\right)\\
&\le De^{\kappa(t)}\left(1 + \frac{2D(1+L)\delta}{\varepsilon} + 2D\delta(1+L)\bar{r}_j^{\varepsilon}\right)
\end{aligned}$$

Therefore, for δ sufficiently small, we have

$$\|\Lambda u\|_1 \le D\left(1 + \frac{2D(1+L)\delta}{\varepsilon} + 2D\delta(1+L)\bar{r}_j^{\varepsilon}\right) \le 2D.$$

Using (30), (31) and (33) we obtain

$$\begin{aligned}\left\|\frac{\partial(\Lambda u)}{\partial\xi}(t,\xi)-\frac{\partial(\Lambda u)}{\partial\bar{\xi}}(t,\bar{\xi})\right\| &\le \int_s^t \|W(t,\tau)P(\tau)\|\overline{q}(\tau)\mathrm{d}\tau+\sum_{k=1}^{r(t,s)-1}\|W(t,s_{j+k})P(s_{j+k})\|\overline{b}_{j+k}+\|P(t)\|\overline{b}_{j+r(t,s)}(t)\\ &\le 2D^2\delta\beta\|\xi-\bar{\xi}\|e^{2\kappa(t)}\left(\frac{1}{\varepsilon}+\bar{r}_j^{\varepsilon}\right).\end{aligned}$$

For δ sufficiently small, we obtain

$$\|\Lambda u\|_2\le 2D^2\delta\beta\left(\frac{1}{\varepsilon}+\bar{r}_j^{\varepsilon}\right)\le 2D.$$

This shows that $\Lambda(\mathbb{B})\subset\mathbb{B}$. Therefore, Λ has a unique fixed point $u_\phi\in\mathbb{B}$ such that $u_\phi=\Lambda u_\phi$. The proof is complete. □

Next, we present some auxiliary results for the function $u_\phi\in\mathbb{B}$. Let

$$\kappa_\rho(t)=\begin{cases}(a+\alpha+\rho\ln(1+D\delta(1+L)))(t-s)+\varepsilon|s|, & t\ge s,\\ (c+\alpha+\rho\ln(1+D\delta(1+L)))(s-t)+\varepsilon|s|, & t\le s,\end{cases}\qquad \alpha=2\delta D.$$

Lemma 7. *Assume that* (1) *admits a nonuniform exponential trichotomy. Given* $\delta>0$ *sufficiently small and* $\phi,\varphi\in\mathscr{S}$ *and* $(s,\xi)\in(s_j,t_{j+1})\times E(s)$*,* $j\in\mathbb{N}$*, there exists a* $K>0$ *such that*

$$\|u_\phi(t,\xi)-u_\varphi(t,\xi)\|\le Ke^{\kappa_\rho(t)}\|\xi\|d(\phi,\varphi),$$

for every $t\ge s$*.*

Proof. For each $\tau\ge s$, we have

$$\begin{aligned}&\|f(\tau,u_\phi(\tau),\phi(\tau,u_\phi(\tau)))-f(\tau,u_\varphi(\tau),\varphi(\tau,u_\varphi(\tau)))\|\\ &\quad\le\delta e^{-2\varepsilon|\tau|}\|(u_\phi(\tau)-u_\varphi(\tau),\phi(\tau,u_\phi(\tau))-\varphi(\tau,u_\varphi(\tau)))\|\\ &\quad\le\delta e^{-2\varepsilon|\tau|}(\|u_\phi(\tau)-u_\varphi(\tau)\|+\|\phi(\tau,u_\phi(\tau))-\varphi(\tau,u_\phi(\tau))+\varphi(\tau,u_\phi(\tau))-\varphi(\tau,u_\phi(\tau))\|)\\ &\quad\le\delta e^{-2\varepsilon|\tau|}(\|u_\phi(\tau)\|d(\phi,\varphi)+(1+L)\|u_\phi(\tau)-u_\varphi(\tau)\|),\end{aligned}$$

and

$$\begin{aligned}&\|g_i(s_i,u_\phi(t_i),\phi(t_i,u_\phi(t_i)))-g_i(s_i,u_\varphi(t_i),\varphi(t_i,u_\varphi(t_i)))\|\\ &\quad\le\delta e^{-2\varepsilon|s_i|}(\|u_\phi(t_i)\|d(\phi,\varphi)+(1+L)\|u_\phi(t_i)-u_\varphi(t_i)\|),\quad i=j+k,\quad k=1,2,\dots,r(t,s),\\ &\|g_i(t,u_\phi(t_i),\phi(t_i,u_\phi(t_i)))-g_i(t,u_\varphi(t_i),\varphi(t_i,u_\varphi(t_i)))\|\\ &\quad\le\delta e^{-2\varepsilon|t|}(\|u_\phi(t_i)\|d(\phi,\varphi)+(1+L)\|u_\phi(t_i)-u_\varphi(t_i)\|),\quad i=j+r(t,s).\end{aligned}$$

Set $\psi(t)=\|u_\phi(t)-u_\varphi(t)\|$. We have two cases to consider:

Case 1. For $s_{j+r(t,s)} < t \le t_{j+r(t,s)+1}$, we have

$$\begin{aligned}\psi(t) &\le \delta\int_s^t \|W(t,\tau)P(\tau)\|e^{-2\varepsilon|\tau|}\|u_\phi(\tau)\|d(\phi,\varphi)\mathrm{d}\tau\\ &\quad+\delta(1+L)\int_s^t \|W(t,\tau)P(\tau)\|e^{-2\varepsilon|\tau|}\psi(\tau)\mathrm{d}\tau\\ &\quad+\delta\sum_{k=1}^{r(t,s)}\|W(t,s_{j+k})P(s_{j+k})\|e^{-2\varepsilon|s_{j+k}|}\|u_\phi(t_{j+k})\|d(\phi,\varphi)\\ &\quad+\delta(1+L)\sum_{k=1}^{r(t,s)}\|W(t,s_{j+k})P(s_{j+k})\|e^{-2\varepsilon|s_{j+k}|}\psi(t_{j+k})\\ &\le 2D^2\delta\|\xi\|d(\phi,\varphi)e^{\kappa(t)}\int_s^t e^{-\alpha(t-\tau)-\varepsilon|\tau|}\mathrm{d}\tau + D\delta(1+L)\int_s^t e^{a(t-\tau)-\varepsilon|\tau|}\psi(\tau)\mathrm{d}\tau\\ &\quad+2D^2\delta\|\xi\|d(\phi,\varphi)e^{\kappa(t)}\sum_{k=1}^{r(t,s)}e^{-\varepsilon|s_{j+k}|}+D\delta(1+L)\sum_{k=1}^{r(t,s)}e^{a(t-s_{j+k})-\varepsilon|s_{j+k}|}\psi(t_{j+k})\\ &\le 2D^2\delta\|\xi\|d(\phi,\varphi)e^{\kappa(t)}\left(\frac{1}{\varepsilon}+r_j^\varepsilon\right)+D\delta(1+L)\int_s^t e^{a(t-\tau)-\varepsilon|\tau|}\psi(\tau)\mathrm{d}\tau\\ &\quad+D\delta(1+L)\sum_{k=1}^{r(t,s)}e^{a(t-s_{j+k})-\varepsilon|s_{j+k}|}\psi(t_{j+k}).\end{aligned}$$

Setting $\Gamma(t)=e^{-\kappa(t)}\psi(t)$, we obtain

$$\begin{aligned}\Gamma(t) &\le 2D^2\delta\|\xi\|d(\phi,\varphi)\left(\frac{1}{\varepsilon}+r_j^\varepsilon\right)+D\delta(1+L)\int_s^t e^{-\alpha(t-\tau)-\varepsilon|\tau|}\Gamma(\tau)\mathrm{d}\tau\\ &\quad+D\delta(1+L)\sum_{k=1}^{r(t,s)}e^{-\alpha(t-t_{j+k})-a(s_{j+k}-t_{j+k})-\varepsilon|s_{j+k}|}\Gamma(t_{j+k})\\ &\le 2D^2\delta\|\xi\|d(\phi,\varphi)\left(\frac{1}{\varepsilon}+r_j^\varepsilon\right)+D\delta(1+L)\left(\int_s^t e^{-\varepsilon|\tau|}\Gamma(\tau)\mathrm{d}\tau+\sum_{k=1}^{r(t,s)}\Gamma(t_{j+k})\right).\end{aligned}$$

Using Lemma 4, we have

$$\begin{aligned}\Gamma(t) &\le 2D^2\delta\|\xi\|d(\phi,\varphi)\left(\frac{1}{\varepsilon}+r_j^\varepsilon\right)\prod_{k=1}^{r(t,s)}(1+D\delta(1+L))\exp\left(\int_s^t D\delta(1+L)e^{-\varepsilon|\tau|}\mathrm{d}\tau\right)\\ &\le 2D^2\delta\|\xi\|e^{\frac{D\delta(1+L)}{\varepsilon}}d(\phi,\varphi)\left(\frac{1}{\varepsilon}+r_j^\varepsilon\right)(1+D\delta(1+L))^{r(t,s)}.\end{aligned}$$

Using (3) and taking δ sufficiently small so that $e^{\frac{D\delta(1+L)}{\varepsilon}}\le 2$, we obtain

$$\psi(t)\le 4D^2\delta\left(\frac{1}{\varepsilon}+r_j^\varepsilon\right)\|\xi\|d(\phi,\varphi)e^{\rho\ln(1+D\delta(1+L))(t-s)+\kappa(t)},$$

Case 2. For $t_{j+r(t,s)} < t \le s_{j+r(t,s)}$, we have

$$\begin{aligned}\psi(t) &\le \delta\int_s^t \|W(t,\tau)P(\tau)\|e^{-2\varepsilon|\tau|}(\|u_\phi(\tau)\|d(\phi,\varphi)+(1+L)\psi(\tau))\mathrm{d}\tau\\ &\quad+\delta\sum_{k=1}^{r(t,s)-1}\|W(t,s_{j+k})P(s_{j+k})\|e^{-2\varepsilon|s_{j+k}|}\Big(\|u_\phi(t_{j+k})\|d(\phi,\varphi)+(1+L)\psi(t_{j+k})\Big)\\ &\quad+\delta De^{-2\varepsilon|t|}(\|u_\phi(t_{j+r(t,s)})\|d(\phi,\varphi)+(1+L)\psi(t_{j+r(t,s)})\\ &\le 2D^2\delta\|\xi\|d(\phi,\varphi)e^{\kappa(t)}\left(\frac{1}{\varepsilon}+\bar{r}_j^\varepsilon\right)+D\delta(1+L)\int_s^t e^{a(t-\tau)-\varepsilon|\tau|}\psi(\tau)\mathrm{d}\tau\\ &\quad+D\delta(1+L)\left(\sum_{k=1}^{r(t,s)-1}e^{a(t-s_{j+k})-\varepsilon|s_{j+k}|}\psi(t_{j+k})+e^{-\varepsilon|t|}\psi(t_{j+r(t,s)})\right),\end{aligned}$$

where we use $\|P(t)\| \le De^{\varepsilon|t|}$.

Setting $\Gamma(t) = e^{-\kappa(t)}\psi(t)$, we obtain

$$\Gamma(t) \le 2D^2\delta\|\xi\| d(\phi,\varphi)\left(\frac{1}{\varepsilon} + \bar{r}_j^\varepsilon\right) + D\delta(1+L)\left(\int_s^t e^{-\varepsilon|\tau|}\Gamma(\tau)\mathrm{d}\tau + \sum_{k=1}^{r(t,s)} \Gamma(t_{j+k})\right).$$

Using Lemma 4, we have

$$\begin{aligned}\Gamma(t) &\le 2D^2\delta\|\xi\| d(\phi,\varphi)\left(\frac{1}{\varepsilon} + \bar{r}_j^\varepsilon\right)\prod_{k=1}^{r(t,s)}(1+D\delta(1+L))\exp\left(\int_s^t D\delta(1+L)e^{-\varepsilon|\tau|}\mathrm{d}\tau\right)\\ &\le 2D^2\delta\|\xi\| e^{\frac{D\delta(1+L)}{\varepsilon}} d(\phi,\varphi)\left(\frac{1}{\varepsilon} + \bar{r}_j^\varepsilon\right)(1+D\delta(1+L))^{r(t,s)}.\end{aligned}$$

Using (3) and taking δ sufficiently small so that $e^{\frac{D\delta(1+L)}{\varepsilon}} \le 2$, we obtain

$$\psi(t) \le 4D^2\delta\left(\frac{1}{\varepsilon} + \bar{r}_j^\varepsilon\right)\|\xi\| d(\phi,\varphi)e^{\rho\ln(1+D\delta(1+L))(t-s)+\kappa(t)}.$$

Therefore, we can choose the constant $K = \max\{4D^2\delta\left(\frac{1}{\varepsilon} + r_j^\varepsilon\right), 4D^2\delta\left(\frac{1}{\varepsilon} + \bar{r}_j^\varepsilon\right)\}$ such that

$$\psi(t) \le K\|\xi\| d(\phi,\varphi)e^{\rho\ln(1+D\delta(1+L))(t-s)+\kappa(t)}, \quad t \ge s.$$

The proof is complete. □

Now we transform (18), (20) into an equivalent problem. We use the notation $\phi(t,u(t)) = (\phi_1(t,u(t)), \phi_2(t,u(t))) \in F_1(t) \times F_2(t)$ with $t \in \mathbb{R}$ and $\phi \in \mathscr{S}$.

Lemma 8. *Assume that* (1) *admits a nonuniform exponential trichotomy. Given* $\delta > 0$ *sufficiently small, and*

$$2a - b + \varepsilon < 0, \quad 2c - d + \varepsilon < 0, \tag{34}$$

then the following properties hold:

(A) *For each* $(s,\xi) \in (s_j, t_{j+1}) \times E(s)$, *with* $t \le s$ *for* $l = 1$ *and with* $t \ge s$ *for* $l = 2$, *if*

$$\begin{aligned}\phi_l(t,u_\phi(t,\xi)) = W(t,s)\phi_l(s,\xi) &+ \int_s^{t_{j+1}} W(t,\tau)Q_l(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\ &+ \sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q_l(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\ &+ \int_{s_{j+r(t,s)}}^{t} W(t,\tau)Q_l(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\ &+ \sum_{k=1}^{r(t,s)} W(t,s_{j+k})Q_l(s_{j+k})g_{j+k}(s_{j+k},u_\phi(t_{j+k},\xi),\phi(t_{j+k},u_\phi(t_{j+k},\xi))),\end{aligned} \tag{35}$$

or

$$\begin{aligned}\phi_l(t,u_\phi(t,\xi)) = W(t,s)\phi_l(s,\xi) &+ \int_s^{t_{j+1}} W(t,\tau)Q_l(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\ &+ \sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q_l(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\ &+ \sum_{k=1}^{r(t,s)-1} W(t,s_{j+k})Q_l(s_{j+k})g_{j+k}(s_{j+k},u_\phi(t_{j+k},\xi),\phi(t_{j+k},u_\phi(t_{j+k},\xi)))\\ &+ Q_l(t)g_{j+r(t,s)}(t,u_\phi(t_{j+r(t,s)},\xi),\phi(t_{j+r(t,s)},u_\phi(t_{j+r(t,s)},\xi))).\end{aligned} \tag{36}$$

then

$$
\begin{aligned}
\phi_1(s,\xi) = &\int_{s_{r(s,0)}}^{s} W(s,\tau)Q_1(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau \\
&+ \sum_{k=1}^{r(s,0)} \int_{s_{k-1}}^{t_k} W(s,\tau)Q_1(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau \\
&+ \sum_{k=1}^{+\infty} \int_{t_{-k}}^{s_{-k+1}} W(s,\tau)Q_1(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau \\
&+ \sum_{k=1}^{r(s,0)} W(s,s_k)Q_1(s_k)g_k(s_k,u_\phi(t_k,\xi),\phi(t_k,u_\phi(t_k,\xi))) \\
&+ \sum_{k=1}^{+\infty} W(s,s_{-k})Q_1(s_{-k})g_{-k}(s_{-k},u_\phi(t_{-k},\xi),\phi(t_{-k},u_\phi(t_{-k},\xi))), \\
\phi_2(s,\xi) = &-\int_{s}^{t_{j+1}} W(\tau,s)^{-1}Q_2(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau \\
&- \sum_{k=1}^{+\infty} \int_{s_{j+k}}^{t_{j+k+1}} W(\tau,s)^{-1}Q_2(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau \\
&- \sum_{k=1}^{+\infty} W(s_{j+k},s)^{-1}Q_2(s_{j+k})g_{j+k}(s_{j+k},u_\phi(t_{j+k},\xi),\phi(t_{j+k},u_\phi(t_{j+k},\xi))).
\end{aligned}
\tag{37}
$$

(B) *If* (37) *holds for each* $(s,\xi)\in(s_j,t_{j+1})\times E(s)$, *then* (35) *and* (36) *hold for each* $(s,\xi)\in(s_j,t_{j+1})\times E(s)$ *with* $t\le s$ *for* $l=1$ *and with* $t\ge s$ *for* $l=2$.

Proof. We first show that the integral and the series in (37) are well-defined for each $(s,\xi)\in(s_j,t_{j+1})\times E(s)$. From Lemma 6, we have

$$
\|f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\| \le (1+L)\delta e^{-2\varepsilon|\tau|}\|u_\phi(\tau,\xi)\| \le 2D\delta(1+L)e^{-2\varepsilon|\tau|}e^{\kappa(\tau)}\|\xi\|, \tag{38}
$$

and

$$
\begin{cases}
\|g_i(s_i,u_\phi(t_i,\xi),\phi(t_i,u_\phi(t_i,\xi)))\| \le 2D\delta(1+L)e^{\kappa(t_i)-2\varepsilon|s_i|}\|\xi\|, & i=j+k, k=1,\dots,r(t,s), \\
\|g_i(t,u_\phi(t_i,\xi),\phi(t_i,u_\phi(t_i,\xi)))\| \le 2D\delta(1+L)e^{\kappa(t_i)-2\varepsilon|t|}\|\xi\|, & i=j+r(t,s).
\end{cases}
\tag{39}
$$

Using (6) and the inequality $|s|\le|s-\tau|+|\tau|$, it follows from (38) and (39) that

$$
\begin{aligned}
&\int_{t_{j+1}}^{s} \|W(s,\tau)Q_1(\tau)\|\|f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\|\mathrm{d}\tau \\
&\qquad + \sum_{-\infty}^{k=1} \int_{s_{j+k}}^{t_{j+k+1}} \|W(s,\tau)Q_1(\tau)\|\|f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\|\mathrm{d}\tau \\
&\qquad\qquad \le 2D^2\delta(1+L)\|\xi\| \int_{-\infty}^{s} e^{(c-d+\alpha+\varepsilon)(s-\tau)}\mathrm{d}\tau.
\end{aligned}
$$

From (34), we obtain $c-d+\varepsilon\le 2c-d+\varepsilon<0$, and taking δ sufficiently small, we have (recall $\alpha=2\delta D$) $c-d+\alpha+\varepsilon<0$. This shows that this integral is well defined. Similarly,

$$
\begin{aligned}
&\int_{s}^{t_{j+1}} \|W(\tau,s)^{-1}Q_2(\tau)\|\|f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\|\mathrm{d}\tau \\
&\qquad + \sum_{k=1}^{+\infty} \int_{s_{j+k}}^{t_{j+k+1}} \|W(\tau,s)^{-1}Q_2(\tau)\|\|f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\|\mathrm{d}\tau \\
&\qquad\qquad \le 2D^2\delta(1+L)\|\xi\| \int_{s}^{+\infty} e^{(a-b+\alpha+\varepsilon)(\tau-s)}\mathrm{d}\tau < \infty,
\end{aligned}
$$

since $a-b+\alpha+\varepsilon<0$, and

$$\begin{aligned}
\sum_{k=1}^{r(s,0)} &\|W(s,s_k)Q_1(s_k)\|\|g_k(s_k,u_\phi(t_k,\xi),\phi(t_k,u_\phi(t_k,\xi)))\| \\
&\qquad + \sum_{k=1}^{+\infty}\|W(s,s_{-k})Q_1(s_{-k})\|\|g_{-k}(s_{-k},u_\phi(t_{-k},\xi),\phi(t_{-k},u_\phi(t_{-k},\xi)))\| \\
&\le 2D^2\delta(1+L)\|\xi\|\Big(\sum_{k=1}^{+\infty} e^{(c-d+\alpha+\varepsilon)(s-s_{-k})}+\sum_{k=1}^{r(s,0)} e^{(c+\alpha)(s_k-t_k)}\Big) \\
&\le 2D^2\delta(1+L)\|\xi\|\Big(r^{|c-d+\alpha+\varepsilon|}+\sum_{k=1}^{r(s,0)} e^{(c+\alpha)(s_k-t_k)}\Big)<\infty,
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{+\infty} &\|W(s_{j+k},s)^{-1}Q_2(s_{j+k})\|\|g_{j+k}(s_{j+k},u_\phi(t_{j+k},\xi),\phi(t_{j+k},u_\phi(t_{j+k},\xi)))\| \\
&\le 2D^2\delta(1+L)\|\xi\|\sum_{k=1}^{+\infty} e^{(a-b+\alpha+\varepsilon)(s_{j+k}-s)} \\
&\le 2D^2\delta(1+L)\|\xi\|\widetilde{r}^{|a-b+\alpha+\varepsilon|}<\infty.
\end{aligned}$$

This implies that the right-hand side of (37) is well-defined.

(A). Assume that (35) holds for each $(s,\xi)\in(s_j,t_{j+1})\times E(s)$ and $j\in\mathbb{N}$. Therefore, we will consider the following two cases:

Case i. Let $s_{j+r(t,s)}<t\le t_{j+r(t,s)+1}$. Then (35) can be written in the form

$$\begin{aligned}
\phi_l(s,\xi) &= W(t,s)^{-1}Q_l(t)\phi(t,u_\phi(t,\xi))-\int_s^{t_{j+1}} W(\tau,s)^{-1}Q_l(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau \\
&\quad -\sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}} W(\tau,s)^{-1}Q_l(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau \\
&\quad -\int_{s_{j+r(t,s)}}^{t} W(\tau,s)^{-1}Q_l(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau \\
&\quad -\sum_{k=1}^{r(t,s)} W(s_{j+k},s)^{-1}Q_l(s_{j+k})g_{j+k}(s_{j+k},u_\phi(t_{j+k},\xi),\phi(t_{j+k},u_\phi(t_{j+k},\xi))),
\end{aligned}\tag{40}$$

Case ii. Let $t_{j+r(t,s)}<t\le s_{j+r(t,s)}$. Then (35) can be written in the form

$$\begin{aligned}
\phi_l(s,\xi) &= W(t,s)^{-1}Q_l(t)\phi(t,u_\phi(t,\xi))-\int_s^{t_{j+1}} W(t,s)^{-1}Q_l(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau \\
&\quad -\sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}} W(t,s)^{-1}Q_l(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau \\
&\quad -\sum_{k=1}^{r(t,s)-1} W(s_{j+k},s)^{-1}Q_l(s_{j+k})g_{j+k}(s_{j+k},u_\phi(t_{j+k},\xi),\phi(t_{j+k},u_\phi(t_{j+k},\xi))) \\
&\quad -W(t,s)^{-1}Q_l(t)g_{j+r(t,s)}(t,u_\phi(t_{j+r(t,s)},\xi),\phi(t_{j+r(t,s)},u_\phi(t_{j+r(t,s)},\xi))).
\end{aligned}\tag{41}$$

From Lemma 6, for $t\le s$ we have

$$\begin{aligned}
\|W(t,s)^{-1}Q_1(t)\phi(t,u_\phi(t,\xi))\| &\le LDe^{-d(s-t)+\varepsilon|t|}\|u_\phi(t)\| \\
&\le 2LD^2\|\xi\|e^{(c-d+a+\varepsilon)(s-t)+2\varepsilon|s|}.
\end{aligned}\tag{42}$$

We note that since $c-d+a+\varepsilon<0$, the right-hand side of (42) tends to zero as $t\to-\infty$. Thus, (40) yields the first identity (37) by letting $t\to-\infty$. Similarly, for $t\geq s$ we have

$$\begin{aligned}\|W(t,s)^{-1}Q_2(t)\phi(t,u_\phi(t,\xi))\| &\leq LDe^{-d(s-t)+\varepsilon|t|}\|u_\phi(t)\|\\ &\leq 2LD^2\|\xi\|e^{(a-b+a+\varepsilon)(t-s)+2\varepsilon|s|}.\end{aligned}$$

Thus, letting $t\to+\infty$ in (40) yields the second identity (37).

A similar argument works if we assume (36).

(B). Assume that (37) holds. Let $s_{j+r(t,s)}<t\leq t_{j+r(t,s)+1}$, and we have

$$\begin{aligned}&W(t,s)\phi_1(s,\xi)+\int_s^{t_{j+1}}W(t,\tau)Q_1(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\
&\quad+\sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}}W(t,\tau)Q_1(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\
&\quad+\int_{s_{j+r(t,s)}}^{t}W(t,\tau)Q_1(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\
&\quad+\sum_{k=1}^{r(t,s)}W(t,s_{j+k})Q_1(s_{j+k})g_{j+k}(s_{j+k},u_\phi(t_{j+k},\xi),\phi(t_{j+k},u_\phi(t_{j+k},\xi)))\\
&=\int_{s_{r(t,0)}}^{t}W(t,\tau)Q_1(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\
&\quad+\sum_{k=1}^{r(t,0)}\int_{s_{k-1}}^{t_k}W(t,\tau)Q_1(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\
&\quad+\sum_{k=1}^{+\infty}\int_{t_{-k}}^{s_{-k+1}}W(t,\tau)Q_1(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\
&\quad+\sum_{k=1}^{r(t,0)}W(t,s_k)Q_1(s_k)g_k(s_k,u_\phi(t_k,\xi),\phi(t_k,u_\phi(t_k,\xi)))\\
&\quad+\sum_{k=1}^{+\infty}W(t,s_{-k})Q_1(s_{-k})g_{-k}(s_{-k},u_\phi(t_{-k},\xi),\phi(t_{-k},u_\phi(t_{-k},\xi))).\end{aligned}$$

By the first identity of (37), we have

$$\begin{aligned}&W(t,s)\phi_1(s,\xi)+\int_s^{t_{j+1}}W(t,\tau)Q_1(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\
&\quad+\sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}}W(t,\tau)Q_1(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\
&\quad+\int_{s_{j+r(t,s)}}^{t}W(t,\tau)Q_1(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\
&\quad+\sum_{k=1}^{r(t,s)}W(t,s_{j+k})Q_1(s_{j+k})g_{j+k}(s_{j+k},u_\phi(t_{j+k},\xi),\phi(t_{j+k},u_\phi(t_{j+k},\xi)))\\
&=\phi_1(t,u_\phi(t,\xi)).\end{aligned}$$

which implies (35) holds.

Using the second identity of (37), we have

$$\begin{aligned}
&W(t,s)\phi_2(s,\xi)+\int_s^{t_{j+1}} W(t,\tau)Q_2(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\
&\qquad+\sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q_2(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\\
&\qquad+\int_{s_{j+r(t,s)}}^{t} W(t,\tau)Q_2(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\\
&\qquad+\sum_{k=1}^{r(t,s)} W(t,s_{j+k})Q_2(s_{j+k})g_{j+k}(s_{j+k},u_\phi(t_{j+k},\xi),\phi(t_{j+k},u_\phi(t_{j+k},\phi)))\\
&=-\int_t^{t_{j+r(t,s)+1}} W(t,\tau)Q_2(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\
&\qquad-\sum_{r(t,s)+1}^{\infty}\int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q_2(\tau)f(\tau,u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))\mathrm{d}\tau\\
&\qquad-\sum_{r(t,s)+1}^{\infty} W(t,s_{j+k})Q_2(s_{j+k})g_{j+k}(s_{j+k},u_\phi(t_{j+k},\xi),\phi(t_{j+k},u_\phi(t_{j+k},\phi))).
\end{aligned}$$

which implies (35) holds.

A similar argument shows (36). The proof is complete. □

Lemma 9. *Assume that* (1) *admits a nonuniform exponential trichotomy. If* (34) *holds, then for* $\delta>0$ *is sufficiently small, there exists a unique function* $\phi\in\mathscr{S}$ *such that* (37) *holds for every* $(s,\xi)\in(s_j,t_{j+1})\times E(s)$.

Proof. Let $u(t)=u_\phi(t)$ be the unique function given in Lemma 6 such that $u_\phi(s)=\xi$. When $\xi=0$ we have $u_\phi(t)=0$ for every $\phi\in\mathscr{S}$ and $t\geq s$. For each $\phi\in\mathscr{S}$ and $(s,\xi)\in(s_j,t_{j+1})\times E(s)$, we define the operator $\mathscr{J}$ by

$$(\mathscr{J}\phi)(s,\xi)=(\phi_1(s,\xi),\phi_2(s,\xi)). \tag{43}$$

Since $f(t,0)=g_i(t,0)=0$, we have $(\mathscr{J}\phi)(s,0)=0$ for every $s>0$. By the chain rule, the function $\mathscr{J}(\phi)$ satisfies condition 1 in the definition of the space $\mathscr{S}$, and for each $s\neq t_i$, we have

$$\frac{\partial(\mathscr{J}\phi)}{\partial\xi}(s,0)=\left(\frac{\partial\phi_1}{\partial\xi}(s,0),\frac{\partial\phi_2}{\partial\xi}(s,0)\right).$$

Let $\omega(\tau,\xi)=(u_\phi(\tau,\xi),\phi(\tau,u_\phi(\tau,\xi)))$. We have

$$\begin{aligned}
\frac{\partial\phi_1}{\partial\xi}(s,0)&=\int_{s_{r(s,0)}}^{s} W(s,\tau)Q_1(\tau)\frac{\partial f}{\partial\omega}(\tau,0)\frac{\partial\omega}{\partial\xi}(\tau,0)\mathrm{d}\tau\\
&\quad+\sum_{k=1}^{r(s,0)}\int_{s_{k-1}}^{t_k} W(s,\tau)Q_1(\tau)\frac{\partial f}{\partial\omega}(\tau,0)\frac{\partial\omega}{\partial\xi}(\tau,0)\mathrm{d}\tau\\
&\quad+\sum_{k=1}^{+\infty}\int_{t_{-k}}^{s_{-k+1}} W(s,\tau)Q_1(\tau)\frac{\partial f}{\partial\omega}(\tau,0)\frac{\partial\omega}{\partial\xi}(\tau,0)\mathrm{d}\tau\\
&\quad+\sum_{k=1}^{r(s,0)} W(s,s_k)Q_1(s_k)\frac{\partial g_k}{\partial\omega}(s_k,0)\frac{\partial\omega}{\partial\xi}(t_k,0)\\
&\quad+\sum_{k=1}^{+\infty} W(s,s_{-k})Q_1(s_{-k})\frac{\partial g_{-k}}{\partial\omega}(s_{-k},0)\frac{\partial\omega}{\partial\xi}(t_{-k},0),
\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \phi_2}{\partial \xi}(s,0) = &-\int_s^{t_{j+1}} W(\tau,s)^{-1} Q_2(\tau)\frac{\partial f}{\partial \omega}(\tau,0)\frac{\partial \omega}{\partial \xi}(\tau,0)\mathrm{d}\tau \\ &-\sum_{k=1}^{+\infty}\int_{s_{j+k}}^{t_{j+k+1}} W(\tau,s)^{-1} Q_2(\tau)\frac{\partial f}{\partial \omega}(\tau,0)\frac{\partial \omega}{\partial \xi}(\tau,0)\mathrm{d}\tau \\ &-\sum_{k=1}^{+\infty} W(s_{j+k},s)^{-1} Q_2(s_{j+k})\frac{\partial g_{j+k}}{\partial \omega}(s_{j+k},0)\frac{\partial \omega}{\partial \xi}(t_{j+k},0).\end{aligned}$$

Since $\frac{\partial f}{\partial \omega}(\tau,0) = \frac{\partial g_k}{\partial \omega}(s_i,0) = 0$, we have $\frac{\partial(\mathscr{J}\phi)}{\partial \xi}(s,0) = 0$ for $s \in (s_j, t_{j+1})$.

Now let $\mathfrak{b}(\tau,\xi) = \frac{\partial}{\partial \xi} f(x_\phi(t))$ and $\mathfrak{b}_i(s_i,\xi) = \frac{\partial}{\partial \xi} g_i(\widetilde{x}_\phi(s_i,t_i))$, and it follows from (26) and (28) that

$$\begin{aligned}\left\|\frac{\partial(\mathscr{J}\phi)}{\partial \xi}(s,\xi)\right\| &= \left\|\left(\frac{\partial \phi_1}{\partial \xi}(s,\xi), \frac{\partial \phi_2}{\partial \xi}(s,\xi)\right)\right\| \le \left\|\frac{\partial \phi_1}{\partial \xi}(s,\xi)\right\| + \left\|\frac{\partial \phi_2}{\partial \xi}(s,\xi)\right\| \\ &:= J_1 + J_2, \end{aligned} \tag{44}$$

where

$$\begin{aligned}J_1 \le& \int_{-\infty}^{s} \|W(s,\tau)Q_1(\tau)\|\|\mathfrak{b}(\tau,\xi)\|\mathrm{d}\tau + \sum_{k=1}^{r(s,0)} \|W(s,s_k)Q_1(s_k)\|\|\mathfrak{b}_k(s_k,\xi)\| \\ &+ \sum_{k=1}^{+\infty} \|W(s,s_{-k})Q_1(s_{-k})\|\|\mathfrak{b}_{-k}(s_{-k},\xi)\| \\ \le& 2D^2(1+L)\delta\left(\int_{-\infty}^{s} e^{(c-d+\alpha+\varepsilon)(s-\tau)}\mathrm{d}\tau + \sum_{k=1}^{r(s,0)} e^{(c+\alpha)(s_k-t_k)} + \sum_{k=1}^{+\infty} e^{(c-d+\alpha+\varepsilon)(s-s_{-k})}\right) \\ \le& 2D^2(1+L)\delta\left(\frac{1}{|c-d+\alpha+\varepsilon|} + \sum_{k=1}^{r(s,0)} e^{(c+\alpha)(s_k-t_k)} + r^{|c-d+\alpha+\varepsilon|}\right),\end{aligned} \tag{45}$$

and

$$\begin{aligned}J_2 \le& \int_{s}^{+\infty} \|W(\tau,s)^{-1}Q_2(\tau)\|\|\mathfrak{b}(\tau,\xi)\|\mathrm{d}\tau + \sum_{k=1}^{+\infty} \|W(s_{j+k},s)^{-1}Q_2(s_{j+k})\|\|\mathfrak{b}_{j+k}(s_{j+k},\xi)\| \\ \le& 2D^2(1+L)\delta\left(\int_{s}^{+\infty} e^{(a-b+\alpha+\varepsilon)(\tau-s)}\mathrm{d}\tau + \sum_{k=1}^{+\infty} e^{(a-b+\alpha+\varepsilon)(s_{j+k}-s)}\right) \\ \le& 2D^2(1+L)\delta\left(\frac{1}{|a-b+\alpha+\varepsilon|} + \widetilde{r}^{|a-b+\alpha+\varepsilon|}\right).\end{aligned} \tag{46}$$

Put (45) and (46) into (44) and provided that δ is sufficiently small, we have

$$\begin{aligned}\left\|\frac{\partial(\mathscr{J}\phi)}{\partial \xi}(s,\xi)\right\| \le 2D^2(1+L)\delta\Bigg(&\frac{1}{|c-d+\alpha+\varepsilon|} + \frac{1}{|a-b+\alpha+\varepsilon|} \\ &+ \sum_{k=1}^{r(s,0)} e^{(c+\alpha)(s_k-t_k)} + r^{|c-d+\alpha+\varepsilon|} + \widetilde{r}^{|a-b+\alpha+\varepsilon|}\Bigg) \le L.\end{aligned}$$

Let $u = u_\phi$ and $\bar{u} = \bar{u}_\phi$ be the unique functions given by Lemma 6 such that $u(s) = \xi$ and $\bar{u}(s) = \bar{\xi}$, and we obtain

$$\|(\mathscr{J}\phi)(s,\xi) - (\mathscr{J}\phi)(s,\bar{\xi})\| \le \sup_{\widetilde{\vartheta}\in[0,1]} \left\|\frac{\partial u}{\partial \xi}(s,\xi+\widetilde{\vartheta}(\bar{\xi}-\xi))\right\|\|\xi-\bar{\xi}\| \le L\|\xi-\bar{\xi}\|.$$

Moreover, set $\beta = 2D(L+(1+L)^2)+(1+L) > 0$ and

$$\begin{aligned}\widetilde{\mathfrak{b}}(\tau,\xi) &= \frac{\partial}{\partial \xi} f(x_\phi(t)) - \frac{\partial}{\partial \bar{\xi}} f(\bar{x}_\phi(t)) \\ \widetilde{\mathfrak{b}}_i(s_i,\xi) &= \frac{\partial}{\partial \xi} g_i(\widetilde{x}_\phi(s_i,t_i)) - \frac{\partial}{\partial \bar{\xi}} g_i(\widehat{x}_\phi(s_i,t_i)).\end{aligned}$$

It follows from (30) and (31) that

$$\left\|\frac{\partial(\mathscr{J}\phi)}{\partial\xi}(s,\xi)-\frac{\partial(\mathscr{J}\phi)}{\partial\bar{\xi}}(s,\bar{\xi})\right\| \le \left\|\frac{\partial\phi_1}{\partial\xi}(s,\xi)-\frac{\partial\phi_1}{\partial\bar{\xi}}(s,\bar{\xi})\right\|+\left\|\frac{\partial\phi_2}{\partial\xi}(s,\xi)-\frac{\partial\phi_2}{\partial\bar{\xi}}(s,\bar{\xi})\right\| := J_3+J_4, \tag{47}$$

where

$$\begin{aligned}J_3 &\le \int_{-\infty}^{s}\|W(s,\tau)Q_1(\tau)\|\|\widetilde{\mathfrak{b}}(\tau,\xi)\|\mathrm{d}\tau+\sum_{k=1}^{r(s,0)}\|W(s,s_k)Q_1(s_k)\|\|\widetilde{\mathfrak{b}}_k(s_k,\xi)\|\\ &\quad+\sum_{k=1}^{+\infty}\|W(s,s_{-k})Q_1(s_{-k})\|\|\widetilde{\mathfrak{b}}_{-k}(s_{-k},\xi)\|\\ &\le 2D^2\beta\delta\|\xi-\bar{\xi}\|e^{\varepsilon s}\left(\int_{-\infty}^{s}e^{(2c-d+2\alpha+\varepsilon)(s-\tau)}\mathrm{d}\tau+\sum_{k=1}^{r(s,0)}e^{2(c+\alpha)(s_k-t_k)}+\sum_{k=1}^{+\infty}e^{(2c-d+2\alpha+\varepsilon)(s-s_{-k})}\right)\\ &\le 2D^2\beta\delta\|\xi-\bar{\xi}\|e^{\varepsilon s}\left(\frac{1}{|2c-d+2\alpha+\varepsilon|}+\sum_{k=1}^{r(s,0)}e^{2(c+\alpha)(s_k-t_k)}+r^{|2c-d+2\alpha+\varepsilon|}\right),\end{aligned} \tag{48}$$

and

$$\begin{aligned}J_4 &\le \int_{s}^{+\infty}\|W(\tau,s)^{-1}Q_2(\tau)\|\|\widetilde{\mathfrak{b}}(\tau,\xi)\|\mathrm{d}\tau\\ &\quad+\sum_{k=1}^{+\infty}\|W(s_{j+k},s)^{-1}Q_2(s_{j+k})\|\|\widetilde{\mathfrak{b}}_{j+k}(s_{j+k},\xi)\|\\ &\le 2D^2\beta\delta\|\xi-\bar{\xi}\|e^{\varepsilon s}\left(\int_{s}^{+\infty}e^{(2a-b+2\alpha+\varepsilon)(\tau-s)}\mathrm{d}\tau+\sum_{k=1}^{+\infty}e^{(2a-b+2\alpha+\varepsilon)(s_{j+k}-s)}\right)\\ &\le 2D^2\beta\delta\|\xi-\bar{\xi}\|\left(\frac{1}{|2a-b+2\alpha+\varepsilon|}+\widetilde{r}^{|2a-b+2\alpha+\varepsilon|}\right),\end{aligned} \tag{49}$$

for fixed point s with $0<s<+\infty$, provided that δ is so small that $2a-b+2\alpha+\varepsilon<0$ and $2c-d+2\alpha+\varepsilon<0$.

Put (48) and (49) into (47) and provided that δ is sufficiently small, we have

$$\left\|\frac{\partial(\mathscr{J}\phi)}{\partial\xi}(s,\xi)-\frac{\partial(\mathscr{J}\phi)}{\partial\bar{\xi}}(s,\bar{\xi})\right\|\le L\|\xi-\bar{\xi}\|.$$

Next, we show that operator $\mathscr{J}$ is a contraction. Given $\phi,\varphi\in\mathscr{S}$ and $(s,\xi)\in(s_j,t_{j+1})\times E(s)$, let $u_\phi=u_\phi(\cdot,\xi)$ and $u_\varphi=u_\varphi(\cdot,\xi)$ be the unique functions given by Lemma 6 such that $u_\phi(s)=u_\varphi(s)=\xi$. From Lemma 7, we obtain

$$\begin{aligned}\widehat{\mathfrak{b}}(\tau,\xi) &:= \|f(\tau,u_\phi(\tau),\phi(\tau,u_\phi(\tau)))-f(\tau,u_\varphi(\tau),\varphi(\tau,u_\varphi(\tau)))\|\\ &\le \delta e^{-2\varepsilon|\tau|}(\|u_\phi(\tau)\|d(\phi,\varphi)+(1+L)\|u_\phi(\tau)-u_\varphi(\tau)\|)\\ &\le \delta(2D+(1+L)K)e^{\kappa_\rho(\tau)-2\varepsilon|\tau|}\|\xi\|d(\phi,\varphi),\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathfrak{b}}_i(s_i,\xi) &= \|g_i(s_i,u_\phi(t_i),\phi(t_i,u_\phi(t_i)))-g_i(s_i,u_\varphi(t_i),\varphi(t_i,u_\varphi(t_i)))\|\\ &\le \delta e^{-2\varepsilon|s_i|}(\|u_\phi(t_i)\|d(\phi,\varphi)+(1+L)\|u_\phi(t_i)-u_\varphi(t_i)\|)\\ &\le \delta(2D+(1+L)K)e^{\kappa_\rho(t_i)-2\varepsilon|s_i|}\|\xi\|d(\phi,\varphi),\ i\in\mathbb{Z}.\end{aligned}$$

From (37) and (43), we obtain

$$\|(\mathscr{J}\phi)(s,\xi)-(\mathscr{J}\varphi)(s,\xi)\| \le \|(\phi_1(s,\xi)-\varphi_1(s,\xi)|+|\phi_2(s,\xi)-\varphi_1(s,\xi))\| := J_5+J_6, \tag{50}$$

where

$$J_5 \le \int_{-\infty}^{s} \|W(s,\tau)Q_1(\tau)\|\|\widehat{\mathfrak{b}}(\tau,\xi)\|\mathrm{d}\tau + \sum_{k=1}^{r(s,0)} \|W(s,s_k)Q_1(s_k)\|\|\widehat{\mathfrak{b}}_k(s_k,\xi)\|$$
$$+ \sum_{k=1}^{+\infty} \|W(s,s_{-k})Q_1(s_{-k})\|\|\widehat{\mathfrak{b}}_{-k}(s_{-k},\xi)\|$$
$$\le D\delta(2D+(1+L)K)\|\xi\|d(\phi,\varphi)\bigg(\int_{-\infty}^{s} e^{(c-d+\alpha+\rho\ln(1+D\delta(1+L))+\varepsilon)(s-\tau)}\mathrm{d}\tau$$
$$+ \sum_{k=1}^{r(s,0)} e^{(c+\alpha+\rho\ln(1+D\delta(1+L)))(s_k-t_k)} + \sum_{k=1}^{+\infty} e^{(c-d+\alpha+\rho\ln(1+D\delta(1+L))+\varepsilon)(s-s_{-k})}\bigg), \tag{51}$$

and

$$J_6 \le \int_{s}^{+\infty} \|W(\tau,s)^{-1}Q_2(\tau)\|\|\widehat{\mathfrak{b}}(\tau,\xi)\|\mathrm{d}\tau + \sum_{k=1}^{+\infty} \|W(s_{j+k},s)^{-1}Q_2(s_{j+k})\|\|\widehat{\mathfrak{b}}_{j+k}(s_{j+k},\xi)\|$$
$$\le D\delta(2D+(1+L)K)\|\xi\|d(\phi,\varphi)\bigg(\int_{s}^{+\infty} e^{(a-b+\alpha+\rho\ln(1+D\delta(1+L))+\varepsilon)(\tau-s)}\mathrm{d}\tau$$
$$+ \sum_{k=1}^{+\infty} e^{(a-b+\alpha+\rho\ln(1+D\delta(1+L))+\varepsilon)(s_{j+k}-s)}\bigg). \tag{52}$$

Therefore, setting $\widehat{\kappa}_\rho = c-d+\alpha+\rho\ln(1+D\delta(1+L))+\varepsilon$ and $\widetilde{\kappa}_\rho = a-b+\alpha+\rho\ln(1+D\delta(1+L))+\varepsilon$, provided that δ is so small that $\widehat{\kappa}_\rho < 0$ and $\widetilde{\kappa}_\rho < 0$.

Let

$$K_1 = (2D+(1+L)K)\|\xi\|\bigg(\frac{1}{|\widehat{\kappa}_\rho|} + r^{|\widehat{\kappa}_\rho|} + \frac{1}{|\widetilde{\kappa}_\rho|} + \widetilde{r}^{|\widetilde{\kappa}_\rho|} + \sum_{k=1}^{r(s,0)} e^{(c+\alpha+\rho\ln(1+D\delta(1+L)))(s_k-t_k)}\bigg).$$

Put (51) and (52) into (50) and provided that δ is sufficiently small, then the operator $\mathscr{J}$ is a contraction in the complete metric space $\mathscr{S}$. Hence, there exists a unique function $\phi \in \mathscr{S}$ such that $\mathscr{J}\phi = \phi$. The proof is complete. □

The following center manifold theorem is in the sense that we have the unique graph of the form $\mathcal{V}_\phi^c$ (for some function $\phi \in \mathscr{S}$) which is invariant under the semiflow.

Theorem 10. *Assume that* (1) *admits a nonuniform exponential trichotomy. If* (16) *and* (34) *hold, then for $\delta > 0$ is sufficiently small, there exists a unique function $\phi \in \mathscr{S}$ such that*

$$\Psi_t(s,\xi,\phi(s,\xi)) \in \mathcal{V}_\phi^c, \quad \textit{for every } t \in \mathbb{R}. \tag{53}$$

Furthermore, for every $s \in (s_j, t_{j+1})$, $\xi, \bar{\xi} \in E(s)$ and $j \in \mathbb{N}$, if $t \ge s$, then

$$\|\Psi_t(s,\xi,\phi(s,\xi)) - \Psi_t(s,\bar{\xi},\phi(s,\bar{\xi}))\| \le 2D(1+L)e^{(a+2D\delta)(t-s)+\varepsilon|s|}\|\xi-\bar{\xi}\|,$$

and if $t \le s$, then

$$\|\Psi_t(s,\xi,\phi(s,\xi)) - \Psi_t(s,\bar{\xi},\phi(s,\bar{\xi}))\| \le 2D(1+L)e^{(c+2D\delta)(s-t)+\varepsilon|s|}\|\xi-\bar{\xi}\|.$$

Proof. From Lemma 6, for each $(s,\xi) \in (s_j, t_{j+1}) \times E(s)$ with $\xi \neq 0$, $j \in \mathbb{N}$ and $\phi \in \mathscr{S}$, there exists a unique function $u_\phi \in \mathbb{B}$. Using Lemma's 8 and 9, for each $(s,\xi) \in (s_j, t_{j+1}) \times E(s)$, there exists a unique function $\phi \in \mathscr{S}$ such that (35) and (36) hold. This shows that (53) holds, for any sufficiently small δ.

Let $u_\phi = u_\phi(\cdot,\xi)$ and $\bar{u}_\phi(\cdot,\bar{\xi})$ be the unique functions given by Lemma 6 such that $u_\phi(s,\xi) = \xi$ and $\bar{u}_\phi(s,\xi) = \bar{\xi}$. From Lemma 6 (see (29)), we have

$$\|\Psi_t(s,\xi,\phi(s,\xi)) - \Psi_t(s,\bar{\xi},\phi(s,\bar{\xi}))\| = \|(t,u_\phi(t),\phi(t,u_\phi(t))) - (t,\bar{u}_\phi(t),\phi(t,\bar{u}_\phi(t)))\|$$
$$\le (1+L)\|u_\phi(t) - \bar{u}_\phi(t)\|$$
$$\le 2D(1+L)e^{\kappa(t)}\|\xi-\bar{\xi}\|.$$

□

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