



INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique


Max Goering and Lukas Koch

A note on improved differentiability for the Banach-space valued Finsler γ -Laplacian

Volume 361 (2023), p. 1091-1105

Published online: 24 October 2023

<https://doi.org/10.5802/crmath.474>

 This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



Les Comptes Rendus. Mathématique sont membres du
Centre Mersenne pour l'édition scientifique ouverte

www.centre-mersenne.org

e-ISSN : 1778-3569



Partial differential equations / *Équations aux dérivées partielles*

A note on improved differentiability for the Banach-space valued Finsler γ -Laplacian

Max Goering^a and Lukas Koch^a

^a MPI for Mathematics in the Sciences, Inselstrasse 22, 04177 Leipzig, Germany

E-mails: goering@mis.mpg.de, lkoch@mis.mpg.de

Abstract. We obtain improved fractional differentiability of solutions to the Banach-space valued Finsler γ -Laplacian defined on a σ -convex, τ -smooth Banach space. The operators we consider are non-linear and very degenerately elliptic. Our results are new already in the \mathbb{R} -valued setting.

Manuscript received 19 August 2022, revised 4 November 2022 and 11 January 2023, accepted 1 February 2023.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain and let (\mathbb{X}, ρ) be a σ -convex, τ -smooth Banach space. Consider $u: \Omega \rightarrow \mathbb{X}$ and let F be a norm on $\mathbb{X} \otimes \mathbb{R}^n$. We study solutions of the problem

$$\int_{\Omega} \langle F(\nabla u)^{\gamma-1} DF(\nabla u), \nabla \varphi \rangle_{\mathbb{X} \otimes \mathbb{R}^n} = \int_{\Omega} \langle f, \varphi \rangle_{\mathbb{X}} \quad \forall \varphi \in W_0^{1,\gamma}(\Omega; \mathbb{X}), \quad (\text{F}\gamma\text{L})$$

for sufficiently regular f . Here DF denotes the derivative of the map $F: \mathbb{X} \otimes \mathbb{R}^n \rightarrow \mathbb{R}$. We refer to (F γ L) as the Banach-space valued Finsler γ -Laplacian. We refer the reader to Section 2 for precise definitions and notation.

Given a norm ρ on \mathbb{R}^n the (scalar) Finsler Laplacian

$$\int_{\Omega} \langle \rho(\nabla u) D\rho(\nabla u), \nabla \varphi \rangle = 0 \quad \forall \varphi \in W_0^{1,2}(\Omega; \mathbb{R}) \quad (1)$$

is closely related to the anisotropic isoperimetric inequality, [4]. Under additional 2nd order ellipticity conditions, the Finsler Laplacian has also been used to study bubbling phenomena for volume constrained minimizers [16]. Some qualitative regularity for the Finsler Laplacian was first shown in [19], where no quantitative regularity or convexity of ρ was assumed.

Recently, extending results of [4], the Finsler γ -Laplacian

$$\int \langle \rho(\nabla u)^{\gamma-1} D\rho(\nabla u), \nabla \varphi \rangle = 0 \quad \forall \varphi \in W_0^{1,\gamma}(\Omega; \mathbb{R}) \quad (2)$$

has been studied in [18, 25, 28, 31] with a focus on spectral properties, due to applications to the isoperimetric inequality and Finslerian analogs of other classical inequalities in Riemannian geometric analysis. Concerning the regularity theory of solutions to (2), in the scalar Euclidean set-up standard theory applies, see [21, Chapter 7], and in particular Hölder regularity of solutions can

be recovered via Harnack’s inequality and maximum principles, see also [19] and [22] for further comments in this direction.

In order to relate well-studied (quantified) geometric properties of Banach spaces to regularity of solutions of the Finsler γ -Laplacian, we make use of duality methods and notation taken from Banach space geometry, see [32]. In particular, given a σ -convex and τ -smooth Banach space (\mathbb{Y}, F) , we make use of the duality mappings $j_F^\gamma(\xi) := F(\xi)^{\gamma-1}DF(\xi)$. Here we only mention that $j_F^\gamma(\nabla u)$ coincides with the stress functional, which is of recent interest in the calculus of variations. For the readers convenience, we include background for these tools in Section 2.1.

Duality techniques in the context of the calculus of variations already appear in [33] and were used in the context of integrands with linear growth in [30]. More recently, such ideas have been employed in the context of linear growth [26], standard growth [10–12], faster than exponential growth [6, 7], and (p, q) -growth [20]. To facilitate a clear analysis of how our results relate to the literature on quasi-linear elliptic PDEs, we state our main result precisely.

Theorem 1. Fix $1 < \gamma < \infty, \tau \in (1, 2]$, and $\sigma \in [2, \infty)$. Let (\mathbb{X}, ρ) be a Banach space, $\Omega \subset \mathbb{R}^n$ a domain, and F a σ -convex and τ -smooth norm on $\mathbb{X} \otimes \mathbb{R}^n$ satisfying for some norm $\|\cdot\|$ on \mathbb{R}^n

$$F(\xi_1, \dots, \xi_n) \lesssim \|(\rho(\xi_1), \dots, \rho(\xi_n))\|.$$

Set $\alpha = \max(\min(\gamma', \sigma'), \min(\gamma, \tau))$, and $\alpha_* = \alpha - 1$. Assume $f \in B_{\infty}^{\alpha_*, \gamma'}(\Omega)$. Suppose $u \in W^{1, \gamma}(\Omega; \mathbb{X})$ solves (F γ L). If

$$V(\nabla u) := F(\nabla u)^{\frac{\gamma-\sigma}{\sigma}} \nabla u, \quad \text{and} \quad V^*(j_F^\gamma(\nabla u)) := F_*(j_F^\gamma(\nabla u))^{\frac{\gamma'-\tau'}{\tau'}} j_F^\gamma(\nabla u)$$

respectively, denote the V - and dual V -functions for ∇u , then

$$V(\nabla u) \in B_{\infty, \text{loc}}^{\frac{\alpha}{\sigma}, \sigma}(\Omega) \quad \text{and} \quad V^*(j_F^\gamma(\nabla u)) \in B_{\infty, \text{loc}}^{\frac{\alpha}{\sigma}, \tau'}(\Omega). \tag{3}$$

In fact, for any $B_r \Subset B_s \Subset \Omega$,

$$\|V(\nabla u)\|_{B_{\infty}^{\frac{\alpha}{\sigma}, \sigma}(B_r)}^\sigma + \|V^*(j_F^\gamma(\nabla u))\|_{B_{\infty}^{\frac{\alpha}{\sigma}, \tau'}(B_r)}^{\tau'} \lesssim_{(s-r)} \|u\|_{W^{1, \gamma}(B_s)}^\gamma + \|f\|_{B_{\infty}^{\alpha_*, \gamma'}(B_s)}^{\gamma'}.$$

Moreover,

$$\nabla u \in B_{\infty, \text{loc}}^{\frac{\alpha}{\max\{\sigma, \gamma\}}, \gamma}(\Omega) \quad \text{and} \quad j_F^\gamma(\nabla u) \in B_{\infty, \text{loc}}^{\frac{\alpha}{\max\{\tau', \gamma'\}}, \gamma'}(\Omega). \tag{4}$$

A wealth of examples of functionals F satisfying the assumptions of Theorem 1 can be constructed as follows.

Example 2. When (\mathbb{X}, ρ) is a σ -convex and τ -smooth Banach space, we can identify $\mathbb{X} \otimes \mathbb{R}^n$ with the space of “matrices” with n columns, where each column is an elements in \mathbb{X} after some choice of basis. This is analogous to how $\mathbb{R}^m \otimes \mathbb{R}^n$ is often identified with the space of $m \times n$ matrices. Then, define $F : \mathbb{X} \otimes \mathbb{R}^n \rightarrow \mathbb{R}$ by setting for $1 < p < \infty$,

$$F(x_1, \dots, x_n) = \|(\rho(x_1), \dots, \rho(x_n))\|_{\ell^p(\mathbb{R}^n)} \quad \forall x_i \in \mathbb{X} \tag{5}$$

Then $(\mathbb{X} \otimes \mathbb{R}^n, F)$ is $\min(p, \tau)$ -smooth and $\max(p, \sigma)$ -convex. We prove this fact in Proposition 12 and give examples of σ -convex and τ -smooth Banach spaces in Example 8.

The V - and dual V -functions, denoted by V, V^* in Theorem 1 are specific choices from the family of V functions associated to a Banach space (\mathbb{Y}, F) and its dual (\mathbb{Y}^*, F_*) , defined by

$$V_{p, q}(\xi) = F(\xi)^{\frac{p-q}{q}} \xi \quad \text{and} \quad V_{p, q}^*(\xi^*) = F_*(\xi^*)^{\frac{p-q}{q}} \xi^*. \tag{6}$$

We choose to study (F γ L) in a general σ -smooth and τ -convex Banach space (\mathbb{X}, ρ) . This poses essentially no additional difficulty compared to working over \mathbb{R}^m , but enables the application of our theorem to interesting Banach spaces. The main additional difficulty is to reprove a technical estimate concerning the V -functional in this setting. Since this result may be of independent

interest in the study of Banach-space valued problems in the calculus of variations, we state it here.

Lemma 3. *Let (\mathbb{Y}, F) be a Banach space. Then, for any $p, q > 0$,*

$$F(V_{p,q}(\xi) - V_{p,q}(\eta)) \sim (F(\xi) + F(\eta))^{\frac{p-q}{q}} F(\xi - \eta).$$

The implicit constants depend only on $\frac{p-q}{q}$.

While Lemma 3 is well known when F is the Euclidean norm on \mathbb{R}^n , our proof is elementary and offers insight even in this case.

Our method of proving Theorem 1 follows difference quotient arguments well-known in the study of solutions to elliptic systems with p -growth or even so-called non-standard growth, see [21] for an introduction. The main difficulty is the lack of ellipticity inherent in the Finsler γ -Laplacian. The Euclidean-norm is 2-smooth and 2-convex. Weakening these assumptions to τ -smoothness and σ -convexity fundamentally changes the ellipticity. In particular very degenerate elliptic behavior is possible (see Example 4). This also changes the precise form of growth bounds of the system (F γ L). σ -convexity and τ -smoothness have been studied since at least [14]. Nonetheless, a direct relationship between the regularity of solutions to any Finsler γ -Laplacian and the basic quantified geometric properties of the Finslerian norm have not previously been exploited. By using the so-called characteristic inequalities of [32], we are able to transfer difference quotient techniques to this setting.

We emphasize that we deal with a broad class of problems and do not impose structure conditions beyond σ -convexity and τ -smoothness, and a natural condition that ensures the norm F on the space of gradients $\mathbb{X} \otimes \mathbb{R}^n$ behaves well relative to the norm ρ on the space of partial derivatives \mathbb{X} . Our results also suggest new critical thresholds concerning the regularity of solutions. These thresholds depend on the relationship between the smoothness and convexity of F as well as the homogeneity γ of the equation. We give two examples to demonstrate the type of problems we are able to deal with and to contrast our results with the existing literature.

Example 4. The elliptic behavior of (F γ L) can be badly behaved, even in relatively simple cases, when $\mathbb{X} = \mathbb{R}$ and $\gamma = 2$. Consider $\rho = \|\cdot\|_{\ell^p}$ for some $p > 2$. Then, the Finsler Laplacian takes the strong form,

$$-\operatorname{div}(D|\nabla u(x)\| \cdot \|\cdot\|_{\ell^p}^2) = -\operatorname{div}(A(x)\nabla u(x)) = 0,$$

where

$$(A(x))_j^i = \|\nabla u(x)\|_{\ell^p}^{2-p} |\partial_i u(x)|^{p-2} \delta_j^i.$$

Thus when $\nabla u(x_0) = \lambda e_i$ for some standard basis direction, it follows $A(x_0)$ has precisely 1-nonnzero entry, no matter the size of $|\lambda| = |\nabla u(x_0)|$. Consequently, (2) falls outside the realm of PDEs studied in [8] and [15], where elliptic operators with degeneracies in a convex set were studied.

Example 5. If M is a rotation matrix, the Finsler p -Laplacian with norm $\rho(\cdot) = \|M \cdot\|_{\ell^p}$ takes weak form

$$\sum_i \int |(M\nabla u)^i|^{p-2} (M\nabla u)^i (M\nabla \varphi)^i = 0. \tag{7}$$

When M is the identity matrix (7) reduces to the (scalar) orthotropic p -Laplacian, which has been studied extensively, see [17], with substantial work performed on generalizations focused on preserving the orthotropic structure, see for instance [9]. However, due to the non-linear mixing of partial derivatives, one cannot hope to adapt techniques depending upon the orthotropic structure to study (7) for general rotations M . Hence, improved differentiability for rotations of the orthotropic p -Laplacian is an open question answered by Theorem 1. This answer is made explicit in Corollary 1 and the discussion following it. Regularity of functions solving (7) for a

fixed, but arbitrary, rotation M arise naturally in anisotropic geometric problems, for instance when studying the regularity of $\|\cdot\|_{\ell^p}$ -energy minimizing surfaces. For a further discussion, see [22].

To aid in comparing Theorem 1 to the existing literature, we state a simplified version in the case that F is either 2-smooth or 2-convex and simplify to finite-dimensional systems, i.e., the case that $\mathbb{X} = \mathbb{R}^m$.

Corollary 6. *Suppose $u \in W^{1,\gamma}(\Omega, \mathbb{R}^m)$ solves (F γ L). If F is 2-smooth and σ -convex, then whenever $\gamma \geq 2$,*

$$V^*(j_F^\gamma(\nabla u)) \in W_{\text{loc}}^{1,2}(\Omega) \quad \text{and} \quad j_F^\gamma(\nabla u) \in W_{\text{loc}}^{1,\gamma'}(\Omega). \tag{8}$$

If F is τ -smooth, and 2-convex, then whenever $\gamma \leq 2$,

$$V(\nabla u) \in W_{\text{loc}}^{1,2}(\Omega) \quad \text{and} \quad \nabla u \in W_{\text{loc}}^{1,\gamma}(\Omega). \tag{9}$$

The dual statement in Corollary 6, that is (8), follows from Theorem 1 when $\alpha = \min(\gamma, \tau) = 2$. The primal statement of Corollary 6, i.e. (9), follows from Theorem 1 when $\alpha = \min(\sigma', \gamma') = 2$.

Corollary 6 recovers a number of classical results. To recover standard results for the p -Laplacian from Corollary 6, note when $F = |\cdot|$ is the Euclidean norm, it is 2-smooth, 2-convex, and $\nabla F(z) = \frac{z}{F(z)}$. So, in this setting,

$$V(\nabla u) = V^*(j_F^\gamma(\nabla u)) = |\nabla u(x)|^{\frac{p-2}{2}} \nabla u(x).$$

Therefore, the regularity of $V(\nabla u)$ when $p \geq 2$ and of $V^*(j_F^\gamma(\nabla u))$ when $p \leq 2$ recovers the classical theorem due to Uhlenbeck, that $|\nabla u|^{\frac{p-2}{2}} \nabla u \in W_{\text{loc}}^{1,2}$ when u is p -harmonic, proven with difference quotient methods in Euclidean space in [5].

We can also recover results for the orthotropic p -Laplacian. Recall the orthotropic p -Laplacian is the Finsler γ -Laplacian when $F = \|\cdot\|_p$ and $\gamma = p$. When $p \geq 2$, $\|\cdot\|_p$ is 2-smooth and p -convex. In this case,

$$V^*(\nabla u) = F(j_F^p(\nabla u))^{\frac{p'-2}{2}} j_F^p(\nabla u) = \|\nabla u\|_{\ell^p}^{\frac{2-p}{2}} \sum_i |\partial_i u|^{p-2} \partial_i u e_i.$$

Then, Corollary 6 says that $V^*(j_F^\gamma(\nabla u)) \in W_{\text{loc}}^{1,2}(\Omega)$ when $p \geq 2$. In [9, Theorem 1.1.] a very similar theorem is proven for generalizations of the orthotropic p -Laplacian, which arise by focusing on the anisotropic structure. In fact, for the (scalar) orthotropic p -Laplacian [9, Theorem 1.1.] states that

$$\mathcal{V}^*(\nabla u) = \sum_{i=1}^n |\partial_i u|^{\frac{2-p}{2}} |\partial_i u|^{p-2} \partial_i u e_i \in W_{\text{loc}}^{1,2}(\Omega).$$

When $p \leq 2$, $\|\cdot\|_{\ell^p}$ is p' -smooth and 2-convex. So, when u is orthotropic p -harmonic, (9) of Corollary 6 states

$$V(\nabla u) = F(\nabla u)^{\frac{2-p'}{p'}} \nabla u = \|\nabla u\|_{\ell^p}^{\frac{p-2}{2}} \nabla u \in W_{\text{loc}}^{1,2}(\Omega) \quad \text{and} \quad u \in W_{\text{loc}}^{2,p}(\Omega).$$

We believe experts in the field may not be surprised by this result, and might already know it. But, we have not been able to find an explicit reference in the literature.

Finally, we remark that for any $1 < p < \infty$, Corollary 6 also applies to rotations of the orthotropic p -Laplacian, see Example 5, which cannot be addressed in the existing literature.

So far we have highlighted the relation of our results to classical regularity statements concerning ∇u , $V(\nabla u)$, and $V^*(j_F^\gamma(\nabla u))$. Recently improved regularity for the stress, which herein is denoted by $j_F^\gamma(\nabla u)$, has gained increasing attention. We do not aim to give an overview here, but highlight some related results to which we refer for further references. In [13] double-sided global estimates relating $\|DF(\nabla u)\|_{W^{1,2}(\Omega)}$ and $\|f\|_{L^2(\Omega)}$ in the case of elliptic equations with Uhlenbeck structure have been shown. [1] provides a non-linear Caldéron–Zygmund theory in the setting

of elliptic equations of p -growth with measure-valued right-hand side. The technique heavily relies on stress regularity. Moreover, regularity transfer (in terms of differentiability) from the right-hand side to the stress for the p -Laplace operator has been studied in [2]. We note that with regards to the regularity of the right-hand side, our results are weaker than those obtained in the classical setting. It would be interesting to see whether our results can be improved in order to capture (sharp) assumptions on the right-hand side. Finally, we remark that in [23] the question of what structure conditions on F are necessary in order to obtain $W_{loc}^{1,2}$ -regularity of the stress for minima of the functional $\int_{\Omega} F(\nabla u)$ are studied. The key assumption was found to be quasi-conformality of the map $z \rightarrow DF(z)$. It would be interesting to study whether such a result extends to our setting.

In Section 2, we provide some background results on Banach space geometry and then prove Lemma 3 in Section 3. We finally prove our main Theorem in Section 4.

2. Preliminaries

Throughout, given any number $p \in (1, \infty)$, we let p' denote its Hölder conjugate. For a given function u on Ω and $h \in \mathbb{X}$, denote $\Delta_h u(x) = u(x) - u(x + h)$. If $x \notin \Omega$ or $x + h \notin \Omega$, we understand $\Delta_h u(x) \equiv 0$. We denote by $|\cdot|$ the usual L^2 -based norm on \mathbb{R}^n .

We refer to [24] or [27] for the theory of vector-valued Sobolev and Besov spaces. For our purposes we only say that if $\Omega \subset \mathbb{R}^n$ and \mathbb{Y} is a reflexive Banach space and $\alpha \in (0, 1)$, the Besov space $B_{\infty}^{\alpha, \gamma}(\Omega; \mathbb{Y})$ is characterized by difference quotients, i.e.,

$$u \in B_{\infty}^{\alpha, \gamma}(\Omega; \mathbb{Y}) \iff \exists C > 0 \text{ so that } \|\Delta_h u\|_{L^p(\omega)} \leq C|h|^{\alpha} \quad \forall \omega \Subset \Omega. \tag{10}$$

Further, $u \in B_{\infty, loc}^{\alpha, \gamma}(\Omega; \mathbb{Y})$ if C is allowed to depend on ω . If $\alpha = 1$, the same definition gives the usual Sobolev space $W^{1,p}(\Omega)$ and its local variant $W_{loc}^{1,p}(\Omega)$.

2.1. Banach space geometry

Let (\mathbb{X}, ρ) be a real Banach space and (\mathbb{X}^*, ρ_*) denote its dual. The modulus of convexity

$$\sigma_{\rho}(\epsilon) = \inf \left\{ 1 - \rho \left(\frac{x+y}{2} \right) : x, y \in \{\rho = 1\}, \rho(x-y) \geq \epsilon \right\}$$

and the modulus of smoothness

$$\tau_{\rho}(\epsilon) = \frac{1}{2} \sup \{ \rho(x+y) + \rho(x-y) - 2 : x \in \{\rho = 1\}, \rho(y) \leq \epsilon \}$$

are fundamental in understanding the geometry of \mathbb{X} .

For $\tau \in (1, 2]$ and $\sigma \in [2, \infty)$, we define a Banach space (\mathbb{X}, ρ) to be σ -convex if

$$\sigma_{\rho}(\epsilon) \gtrsim \epsilon^{\sigma} \quad \forall \epsilon \in [0, 2]$$

and (\mathbb{X}, ρ) is called τ -smooth if

$$\tau_{\rho}(\epsilon) \lesssim \epsilon^{\tau} \quad \forall \epsilon \in [0, \infty).$$

For any $\gamma > 1$, the duality mapping $j_{\rho}^{\gamma} : \mathbb{X} \rightarrow \mathbb{X}^*$ defined by

$$j_{\rho}^{\gamma}(x) := \{x^* : \rho(x^*) = \rho(x)^{\gamma-1} \text{ and } \langle x, x^* \rangle = \rho(x)^{\gamma}\}$$

coincides with the single-valued function

$$j_{\rho}^{\gamma}(x) = \rho(x)^{\gamma-1} \nabla \rho(x), \tag{11}$$

whenever \mathbb{X} is τ -smooth and σ -convex for some $\tau \in (1, 2]$ and some $\sigma \in [2, \infty)$. See for example [32, Properties (J2, J3)]. Moreover a Banach space is reflexive if and only if it is uniformly smooth or uniformly convex [29, Proposition 1.e.3]. In particular, if (\mathbb{Y}, F) is τ -smooth and σ -convex, it is reflexive and (10) holds. We recall,

Proposition 7. *A Banach space (\mathbb{X}, ρ) is σ -convex and τ -smooth if and only if its dual (\mathbb{X}^*, ρ_*) is τ' -convex and σ' -smooth.*

It is also well-known that if $\tau \in (1, 2]$ and $\sigma \in [2, \infty)$ then

$$j_\rho^\gamma \circ j_{\rho_*}^{\gamma'} = \text{Id}_{\mathbb{X}_*} \quad \text{and} \quad j_{\rho_*}^{\gamma'} \circ j_\rho^\gamma = \text{Id}_{\mathbb{X}}, \tag{12}$$

see for instance [3].

Example 8. We note the following examples of σ -smooth and τ -convex spaces.

- (1) $\|\cdot\|_{\ell^p}$ is $\max(p, 2)$ -convex and $\min(p, 2)$ -smooth for $p \in (1, \infty)$.
- (2) $\|\cdot\|_{L^p}$ is $\max(p, 2)$ -convex and $\min(p, 2)$ -smooth for $p \in (1, \infty)$.
- (3) Note that for $p \in (1, \infty)$, $W^{m,p}(\Omega, \mathbb{V}^n)$, equipped with the usual L^1 -based norm is not τ -smooth or σ -convex. However, due to Proposition 12, if endowed with the equivalent norm

$$\|u\|_{W^{m,p}(\Omega, \mathbb{V})} = \left(\sum_{|\alpha| \leq m} \|\nabla^\alpha u\|_{L^p(\Omega, \mathbb{V})}^2 \right)^{\frac{1}{2}}$$

it is $\max(p, 2)$ -convex and $\min(p, 2)$ -smooth. So, one must be careful since equivalent norms do not preserve smoothness and convexity of Banach spaces.

We now recall the main results of [32] which are relevant to this article, restated in the special case of σ -convex and τ -smooth Banach spaces.

Theorem 9. *Let (\mathbb{X}, ρ) be a Banach space. The following are equivalent:*

- (1) ρ is σ -convex
- (2) For all $\gamma > 1$ and every $x, y \in \mathbb{X}$

$$\langle j_\rho^\gamma(x) - j_\rho^\gamma(y), x - y \rangle \gtrsim_\gamma (\rho(x) + \rho(y))^{\gamma-\sigma} \rho(x - y)^\sigma. \tag{13}$$

- (3) There exists a $\gamma > 1$ so that (13) holds.

Alternatively, the following are equivalent:

- (a) ρ is τ -smooth
- (b) For all $\gamma > 1$ and every $x, y \in \mathbb{X}$

$$\rho_*(j_\rho^\gamma(x) - j_\rho^\gamma(y)) \lesssim_\gamma (\rho(x) + \rho(y))^{\gamma-\tau} \rho(x - y)^{\tau-1}. \tag{14}$$

- (c) There exists a $\gamma > 1$ so that (14) holds.

Remark 10. In light of Proposition 7 and Theorem 9, we note the following dual characterization of σ -convexity and τ -smoothness of (\mathbb{X}, ρ) . Namely, ρ is τ -smooth if and only if for every $\gamma > 1$

$$(\rho_*(x^*) + \rho_*(y^*))^{\gamma'-\tau'} \rho_*(x^* - y^*)^{\tau'} \lesssim_\gamma \langle j_{\rho_*}^{\gamma'}(x^*) - j_{\rho_*}^{\gamma'}(y^*), x^* - y^* \rangle \quad \forall x^*, y^* \in \mathbb{X}^*, \tag{15}$$

while ρ is σ -smooth if and only if for every $\gamma > 1$,

$$\rho(j_{\rho_*}^{\gamma'}(x^*) - j_{\rho_*}^{\gamma'}(y^*)) \lesssim (\rho_*(x^*) + \rho_*(y^*))^{\gamma'-\sigma'} \rho_*(x^* - y^*)^{\sigma'-1}. \tag{16}$$

We now demonstrate a straightforward relationship between 2-smoothness and 2-convexity and more standard notions of smoothness and ellipticity in the calculus of variations.

Lemma 11. *Suppose $F \in C^2(\mathbb{R}^m \otimes \mathbb{R}^n \setminus \{0\}, \mathbb{R})$ is a norm on $\mathbb{R}^m \otimes \mathbb{R}^n$. Then $(\mathbb{R}^m \otimes \mathbb{R}^n, F)$ is 2-smooth. On the other hand, if $\rho \in C^1(\mathbb{R}^m \otimes \mathbb{R}^n \setminus \{0\}, \mathbb{R})$ is a norm and $F = \rho^2$ is uniformly elliptic in the sense that*

$$\langle \nabla F(x) - \nabla F(y), x - y \rangle \gtrsim |x - y|^2, \tag{17}$$

then ρ is 2-convex.

Proof. First suppose $F \in C^2(\mathbb{R}^m \otimes \mathbb{R}^n \setminus \{0\}, \mathbb{R})$ is a norm. Let $x \in \{F = 1\}$ and $F(y) < 1/2$. Then $F(x - y), F(x + y) \in [1/2, 3/2]$. From the regularity of F and the fundamental theorem of calculus,

$$\begin{aligned} F(x + y) + F(x - y) - 2 &= F(x + y) - F(x) + F(x - y) - F(x) \\ &= \int_0^1 \langle DF(x + ty) - DF(x - ty), y \rangle dt \\ &= 2 \int_0^1 \int_0^1 \langle D^2 F(x - ty + 2sy)y, y \rangle s ds dt \\ &\lesssim \|D^2 F\|_{L^\infty(B_{3/2} \setminus B_{1/2})} F(y)^2. \end{aligned}$$

The last inequality holds due to equivalence of norms on $\mathbb{R}^m \otimes \mathbb{R}^n$.

On the other hand, when $F(y) \geq 1/2$,

$$F(x + y) + F(x - y) - 2 \leq F(x) + F(y) + F(x) + F(y) - 2 = 2F(y) \leq 4F(y)^2.$$

In particular, for all $\epsilon \in [0, \infty)$, $\tau_F(\epsilon) \lesssim (1 + \|D^2 F\|_{L^\infty(B_{3/2} \setminus B_{1/2})})\epsilon^2$ verifying F is 2-smooth.

Now suppose $\rho \in C^1(\mathbb{R}^m \otimes \mathbb{R}^n \setminus \{0\}, \mathbb{R})$, $F = \rho^2$, and (17) holds. Then, the 2-convexity of ρ follows from Theorem 9 (3). Indeed, (17) is precisely (13) when $\gamma = \sigma = 2$. \square

We next study the convexity and smoothness properties of the spaces $(\mathbb{X} \otimes \mathbb{R}^n, F)$, when F is defined via (5).

Proposition 12. *Let (\mathbb{X}, ρ) be a Banach space and define $F : \mathbb{X} \otimes \mathbb{R}^n \rightarrow \mathbb{R}$ by (5). Then*

- (i) $(\mathbb{X} \otimes \mathbb{R}^n, F)$ is a Banach space and $(\mathbb{X}^* \otimes \mathbb{R}^n, F_*)$ is its dual. Here we have defined $F_*(x^*) = (\sum_i \rho_*(x_i^*))^{1/p'}$.
- (ii) $(\mathbb{X} \otimes \mathbb{R}^n, F)$ is reflexive if and only if (\mathbb{X}, ρ) is reflexive.
- (iii) If ρ is τ -smooth then F is $\min(p, \tau)$ -smooth.
- (iv) If ρ is σ -convex then F is $\max(p, \sigma)$ -convex.
- (v) $F(x \otimes \xi) = \rho(x) \|\xi\|_{\ell^p}$.

Proof. (i) follows from the linearity of the tensor product and (ii) follows from (i).

We next check (iii). Let $x = (x_1, \dots, x_n) \in \mathbb{X} \otimes \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{X} \otimes \mathbb{R}^n$. Note by direct calculation that

$$j_F^p(x) = (j_\rho^p(x_1), \dots, j_\rho^p(x_n)).$$

Hence, by (i),

$$F_*(j_F^p(x) - j_F^p(y)) = \left(\sum_i \rho_*(j_\rho^p(x_i) - j_\rho^p(y_i))^{p'} \right)^{\frac{1}{p'}}.$$

Since ρ is τ -smooth, Theorem 9 then implies,

$$F_*(j_F^p(x) - j_F^p(y)) \leq \left(\sum_i ((\rho(x_i) + \rho(y_i))^{p-\tau} \rho(x_i - y_i)^{\tau-1})^{p'} \right)^{\frac{1}{p'}}. \tag{18}$$

When $p \leq \tau$, the triangle inequality says $(\rho(x_i) + \rho(y_i))^{p-\tau} \leq \rho(x_i - y_i)^{p-\tau}$. So (18) implies

$$F_*(j_F^p(x) - j_F^p(y)) \leq \left(\sum_i \rho(x_i - y_i)^p \right)^{\frac{p-1}{p}} = F(x - y)^{p-1}.$$

By Theorem 9 (c), this implies F is p -smooth when $p = \min(p, \tau)$. On the other hand, when $\tau \leq p$, it follows from (18), monotonicity of $t \mapsto t^{p-\tau}$, and Jensen's inequality

$$\begin{aligned} F_* (j_F^p(x) - j_F^p(y)) &\leq (F(x) + F(y))^{p-\tau} \left(\sum_i \rho(x_i - y_i)^{p \left(\frac{\tau-1}{p-1} \right)} \right)^{\frac{p-1}{p}} \\ &\lesssim (F(x) + F(y))^{p-\tau} \left(\sum_i \rho(x_i - y_i)^p \right)^{\frac{\tau-1}{p}} \\ &= (F(x) + F(y))^{p-\tau} F(x - y)^{\tau-1}. \end{aligned}$$

Now, Theorem 9 (c) verifies τ -smoothness of F when $\tau = \min(p, \tau)$. This verifies (iii).

To prove (iv), note that by Proposition 7, ρ is σ -convex if and only if ρ_* is σ' -smooth. Thus by part (i) and (iii) of this Proposition, F_* is $\min(p', \sigma')$ -smooth. Since $p' \leq \sigma' \iff \sigma \leq p$, another application of Proposition 7 implies $(F_*)_* = F$ is $\max(p, \sigma)$ -convex as desired.

Finally (v) is a straightforward computation, since $x \otimes \xi = (\xi_1 x, \dots, \xi_n x)$. □

3. A V -functional estimate in Banach spaces

In this section, we prove Lemma 3. In the case when $\mathbb{X} = \mathbb{R}^m$ and ρ is the Euclidean norm this result is well-known. However, even in this set-up our elementary proof is new. For the convenience of the reader, we restate the lemma here. Recall the definition of $V_{p,q}$ from (6).

Lemma 13. *Let (\mathbb{Y}, F) be a Banach space and fix $p, q > 1$. Then, for any $\xi, \eta \in \mathbb{Y}$,*

$$F(V_{p,q}(\xi) - V_{p,q}(\eta)) \sim (F(\xi) + F(\eta))^{\frac{p-q}{q}} F(\xi - \eta). \tag{19}$$

Proof.

Case 1: $p - q \geq 0$. For $\xi, \eta \in \mathbb{Y}$, convexity of F implies the difference quotient

$$t \mapsto \frac{F(\xi + t\eta) - F(\xi)}{t}$$

is a non-decreasing function for $t \in [0, \infty)$. Suppose without loss of generality, $F(\xi) \geq F(\eta)$ and define $\kappa = \frac{F(\eta)}{F(\xi)} \leq 1$. Since $p - q \geq 0$, $\kappa^{\frac{p-q}{q}} \leq 1$. Using $F(\xi)\kappa = F(\eta)$ and convexity of F , it follows

$$\begin{aligned} F(V_{p,q}(\xi) - V_{p,q}(\eta)) &= F(\xi)^{\frac{p-q}{q}} \kappa^{\frac{p-q}{q}} \frac{F\left(\xi - \kappa^{\frac{p-q}{q}} \eta\right) - F(\xi)}{\kappa^{\frac{p-q}{q}}} + F(\xi)^{\frac{p}{q}} \\ &\leq F(\eta)^{\frac{p-q}{q}} [F(\xi - \eta) - F(\xi)] + F(\xi)^{\frac{p}{q}} \\ &= F(\eta)^{\frac{p-q}{q}} F(\xi - \eta) + \left(F(\xi)^{\frac{p-q}{q}} - F(\eta)^{\frac{p-q}{q}} \right) F(\xi) \\ &\lesssim F(\eta)^{\frac{p-q}{q}} F(\xi - \eta) + F(\xi)^{\frac{p-q}{q}} F(\xi - \eta), \end{aligned}$$

where the last line follows by the mean-value theorem applied to $t \mapsto t^{\frac{p-q}{q}}$ on $[F(\eta), F(\xi)]$. This verifies

$$F(V_{p,q}(\xi) - V_{p,q}(\eta)) \leq \left(F(\xi)^{\frac{p-q}{q}} + F(\eta)^{\frac{p-q}{q}} \right) F(\xi - \eta)$$

which suffices since for $a, b > 0$ and any $\alpha > 0$, $a^\alpha + b^\alpha \leq 2(a + b)^\alpha$. To confirm the reverse inequality, assume again without loss of generality, that $F(\xi) \geq F(\eta)$. Then, using the reverse triangle inequality,

$$\begin{aligned} F(V_{p,q}(\xi) - V_{p,q}(\eta)) &= F\left(F(\xi)^{\frac{p-q}{q}}(\xi - \eta) - \left(F(\eta)^{\frac{p-q}{q}} - F(\xi)^{\frac{p-q}{q}}\right)\xi\right) \\ &\geq F(\eta)^{\frac{p-q}{q}}F(\xi - \eta) - \left(F(\eta)^{\frac{p-q}{q}} - F(\xi)^{\frac{p-q}{q}}\right)F(\xi) \\ &\geq F(\eta)^{\frac{p-q}{q}}F(\xi - \eta) \\ &\gtrsim (F(\eta) + F(\xi))^{\frac{p-q}{q}}F(\xi - \eta) \end{aligned}$$

This is the desired inequality.

Case 2: $p - q < 0$. Suppose $F(\xi) \geq F(\eta)$ and $F(\xi)\kappa = F(\eta)$. Then, $\kappa^{\frac{q-p}{q}} \leq 1$. Akin to the previous case, we estimate,

$$\begin{aligned} F(V_{p,q}(\xi) - V_{p,q}(\eta)) &= F(\eta)^{\frac{p-q}{q}}\kappa^{\frac{q-p}{q}}\frac{F(\kappa^{\frac{q-p}{q}}\xi - \eta) - F(\eta)}{\kappa^{\frac{q-p}{q}}} + F(\eta)^{\frac{p}{q}} \\ &\leq F(\xi)^{\frac{p-q}{q}}[F(\xi - \eta) - F(\eta)] + F(\eta)^{\frac{p}{q}} \\ &= F(\xi)^{\frac{p-q}{q}}F(\xi - \eta) + F(\eta)^{\frac{p}{q}} - F(\eta)F(\xi)^{\frac{p-q}{q}}. \end{aligned}$$

Since $F(\xi) \geq F(\eta)$,

$$F(\eta)^{\frac{p}{q}} - F(\eta)F(\xi)^{\frac{p-q}{q}} \leq F(\xi)^{\frac{p-q}{q}}|F(\xi) - F(\eta)| \leq F(\xi)^{\frac{p-q}{q}}F(\xi - \eta).$$

Recalling $p - q < 0$, combining the two previous estimates with the observation that when $|a| \geq |b|$ and $\alpha < 0$, $|a|^\alpha \leq 2^{-\alpha}|a + b|^\alpha$ finishes the first inequality. We now show the opposite. Assume without loss of generality $F(\xi) > F(\eta) > 0$, noting that if $F(\eta) = 0$ or $F(\xi) = F(\eta)$, there is nothing to prove. Then, using the reverse triangle inequality,

$$\begin{aligned} F(V_{p,q}(\eta) - V_{p,q}(\xi)) &= F\left(F(\eta)^{\frac{p-q}{q}}(\eta - \xi) - \left(F(\xi)^{\frac{p-q}{q}} - F(\eta)^{\frac{p-q}{q}}\right)\xi\right) \\ &\geq F(\eta)^{\frac{p-q}{q}}F(\eta - \xi) - \left(F(\xi)^{\frac{p-q}{q}} - F(\eta)^{\frac{p-q}{q}}\right)F(\xi) \\ &\geq F(\eta)^{\frac{p-q}{q}}F(\eta - \xi) \\ &\gtrsim (F(\eta) + F(\xi))^{\frac{p-q}{q}}F(\eta - \xi) \end{aligned}$$

Since $F(\xi) = F(-\xi)$ for all $\xi \in \mathbb{Y}$, this is the desired inequality. □

We explicitly state a consequence of Lemma 13 in our set-up.

Corollary 14. *Let (\mathbb{Y}, F) be a Banach space. Suppose $u \in W_{\text{loc}}^{1,p}(\Omega, \mathbb{Y})$. Then for all $p, q \geq 0$,*

$$F(\Delta_h V_{p,q}(\nabla u)) \sim (F(\nabla u) + F(\nabla u_h))^{\frac{p-q}{q}} F(\Delta_h \nabla u). \tag{20}$$

In particular, if $p \geq q$,

$$F(\Delta_h \nabla u)^p \lesssim F(\Delta_h V_{p,q}(\nabla u))^q, \tag{21}$$

and when $p \leq q$, if $\eta \in C_c(\Omega)$ and $A = \text{spt } \eta$, $\beta \geq 0$,

$$\int \eta^\beta F(\Delta_h \nabla u)^p \lesssim \left(\int \eta^{\frac{q\beta}{p}} F(\Delta_h V_{p,q}(\nabla u))^q\right)^{\frac{p}{q}} \left(\int_A (F(\nabla u) + F(\nabla u_h))^p\right)^{1-\frac{p}{q}} \tag{22}$$

Proof. Applying Lemma 13 with $\xi = \nabla u$ and $\eta = \nabla u_h$ confirms (20).

Suppose $p \geq q$. Then $F(\Delta_h \nabla u)^{p-q} \leq (F(\nabla u) + F(\nabla u_h))^{p-q}$. So (20) implies

$$F(\Delta_h V_{p,q}(\nabla u))^q \sim (F(\nabla u) + F(\nabla u_h))^{p-q} F(\Delta_h \nabla u)^q \geq F(\Delta_h u)^p,$$

confirming (21).

Suppose $p \leq q$. $\eta \in C_c(\Omega)$ and $A = \text{spt} \eta$. Choosing $\tau = \frac{p-q}{q}$, (20) implies

$$F(\Delta_h \nabla u)^p \sim F(\Delta_h V_{p,q}(\nabla u))^p (F(\nabla u) + F(\nabla u_h))^{\frac{p(q-p)}{q}}.$$

Now (22) follows by applying Hölder's inequality with exponents $\frac{q}{p}$ and $\frac{q}{q-p}$ to

$$\int \eta^\beta F(\Delta_h \nabla u)^p \lesssim \int \eta^\beta F(\Delta_h V_{p,q}(\nabla u))^p (F(\nabla u) + F(\nabla u_h))^{\frac{p(q-p)}{q}}. \quad \square$$

4. Proof of main theorem

We restate the main theorem for the readers convenience.

Theorem 15. Fix $1 < \gamma < \infty, \tau \in (1, 2]$, and $\sigma \in [2, \infty)$. Let (\mathbb{X}, ρ) be a Banach space, $\Omega \subset \mathbb{R}^n$ a domain, and F a σ -convex and τ -smooth norm on $\mathbb{X} \otimes \mathbb{R}^n$ satisfying for some norm $\|\cdot\|$ on \mathbb{R}^n

$$F(\xi_1, \dots, \xi_n) \lesssim \|(\rho(\xi_1), \dots, \rho(\xi_n))\|, \tag{23}$$

Set $\alpha = \max(\min(\gamma', \sigma'), \min(\gamma, \tau))$, and $\alpha_* = \alpha - 1$. Assume $f \in B_{\infty}^{\alpha_*, \gamma'}(\Omega)$. Suppose $u \in W^{1, \gamma}(\Omega; \mathbb{X})$ solves (F γ L). Then

$$V(\nabla u) \in B_{\infty, \text{loc}}^{\frac{\alpha}{\sigma}, \sigma}(\Omega) \quad \text{and} \quad V^*(j_F^\gamma(\nabla u)) \in B_{\infty, \text{loc}}^{\frac{\alpha}{\tau'}, \tau'}(\Omega) \tag{24}$$

In fact, for any $B_r \Subset B_s \Subset \Omega$,

$$\|V(\nabla u)\|_{B_{\infty}^{\frac{\alpha}{\sigma}, \sigma}(B_r)}^\sigma + \|V^*(j_F^\gamma(\nabla u))\|_{B_{\infty}^{\frac{\alpha}{\tau'}, \tau'}(B_r)}^{\tau'} \lesssim_{(s-r)} \|u\|_{W^{1, \gamma}(B_s)}^\gamma + \|f\|_{B_{\infty}^{\alpha_*, \gamma'}(B_s)}^{\gamma'}. \tag{25}$$

Moreover,

$$\nabla u \in B_{\infty, \text{loc}}^{\frac{\alpha}{\max\{\sigma, \gamma\}}, \gamma}(\Omega) \quad \text{and} \quad j_F^\gamma(\nabla u) \in B_{\infty, \text{loc}}^{\frac{\alpha}{\max\{\tau', \gamma'\}}, \gamma'}(\Omega). \tag{26}$$

A key tool in the proof is the following Lemma:

Lemma 16. Let $1 \leq p < \infty$. Suppose $f \in L_{\text{loc}}^p(\Omega)$. Then for all $\phi \in W_0^{1, p'}(\Omega)$ and $h = |h|e$ with $|e| = 1$,

$$\int_{\Omega} \phi \Delta_h f = -|h| \int_{\Omega} \frac{\partial \phi}{\partial e} \left(\int_0^1 f(x + th) dt \right).$$

Proof. The identity holds for smooth functions, since

$$\Delta_h f = \frac{\partial}{\partial e} \int_0^1 f(x + th) dt.$$

The result follows by approximation. □

Proof of Theorem 15. Due to equivalence of norm on \mathbb{R}^n , we deduce from (23),

$$F(z \otimes \xi) \lesssim \rho(z)|\xi| \quad \forall z \in \mathbb{X} \quad \forall \xi \in \mathbb{R}^n. \tag{27}$$

In fact, due to equivalence of norms on \mathbb{R}^n ,

$$F(z \otimes \xi) \lesssim \|\xi_1 \rho(z), \dots, \xi_n \rho(z)\| \lesssim \rho(z)|\xi|. \tag{28}$$

Suppose for some $r < s$, $B_r \Subset B_s \Subset \Omega$. Let $\eta \in C_c^\infty(\Omega; \mathbb{R})$ be a smooth non-negative function supported on $r + \frac{s-r}{3}$ with $|\nabla \eta| \lesssim (s-r)^{-1}, |\nabla^2 \eta| \lesssim (s-r)^{-2}$ such that $\eta = 1$ on B_r . Let $h = |h|e \in \mathbb{R}^n$ with $|e| = 1$ and write $u_h(x) = u(x + h)$ for $|h| < \frac{s-r}{3}$.

Recall that we denote $\Delta_h u(x) = u(x) - u(x + h)$. Let $\beta > 0$ to be determined at a later stage and test (F γ L) against $\Delta_{-h}(\eta^\beta \Delta_h u)$. Using discrete integration by parts, we find

$$I := \int_{\Omega} \langle \Delta_h j_F^\gamma(\nabla u), \nabla(\eta^\beta \Delta_h u) \rangle_{\mathbb{X} \otimes \mathbb{R}^n} = \int_{\Omega} \eta^\alpha \langle \Delta_h u, \Delta_h f \rangle_{\mathbb{X}} =: II. \tag{29}$$

We write

$$I = \int_{\Omega} \langle \Delta_h j_F^\gamma(\nabla u), \eta^\beta \Delta_h \nabla u \rangle_{\mathbb{X} \otimes \mathbb{R}^n} + \beta \int_{\Omega} \eta^{\beta-1} \langle \Delta_h j_F^\gamma(\nabla u), \Delta_h u \otimes \nabla \eta \rangle =: A_1 + A_2.$$

Note that due to (12),

$$\langle \Delta_h j_F^\gamma(\nabla u), \Delta_h \nabla u \rangle_{\mathbb{X} \otimes \mathbb{R}^n} = \langle \Delta_h j_F^\gamma(\nabla u), \Delta_h j_{F_*}^{\gamma'}(j_F^\gamma(\nabla u)) \rangle_{\mathbb{X} \otimes \mathbb{R}^n}.$$

Therefore, we bound A_1 below by (13) and (15), as

$$\begin{aligned} A_1 &\gtrsim \int_{\Omega} \eta^\beta (F(\nabla u) + F(\nabla u_h))^{\gamma-\sigma} F(\Delta_h \nabla u)^\sigma \\ &\quad + \int_{\Omega} \eta^\beta (F_*(j_F^\gamma(\nabla u)) + F_*(j_F^\gamma(\nabla u_h)))^{\gamma'-\tau'} F_*(\Delta_h j_F^\gamma(\nabla u))^{\tau'}. \end{aligned} \tag{30}$$

It follows from (20) that (30) is equivalent to

$$A_1 \gtrsim \int_{\Omega} \eta^\beta F(\Delta_h V(\nabla u))^\sigma + \int_{\Omega} \eta^\beta F_*(\Delta_h V^*(j_F^\gamma(\nabla u)))^{\tau'}. \tag{31}$$

On the other hand, consecutively using Fenchel, Cauchy–Schwarz, (10) and Young’s, we obtain

$$\begin{aligned} |II| &\lesssim \int_{\Omega} \eta^\beta \rho(\Delta_h u) \rho_*(\Delta_h f) \lesssim \|\rho(\Delta_h u)\|_{L^\gamma(B_{r+1/3(s-r)})} \|\rho_*(\Delta_h f)\|_{L^{\gamma'}(B_{r+1/3(s-r)})} \\ &\lesssim |h|^{1+\alpha_*} \left(\|u\|_{W^{1,\gamma}(B_s)}^\gamma + \|f\|_{B_{\infty}^{\alpha_*,\gamma'}(B_s)}^{\gamma'} \right). \end{aligned} \tag{32}$$

The claimed regularity statement will follow from combining the observations made so far with various estimates on A_2 : a primal estimate, carried out separately in the regime $\gamma \leq \sigma$ and $\gamma \geq \sigma$ and similar dual estimates in the regime $\gamma' \leq \tau'$ and $\gamma' \geq \tau'$.

Case 1.1: A primal estimate if $\gamma \leq \sigma$. Applying Lemma 16 with $f = j_F^\gamma(\nabla u)$ and $\phi = \eta^{\beta-1} \Delta_h u \otimes \nabla \eta$, we get

$$\begin{aligned} |A_2| &\lesssim |h| \int_{\Omega} \left\langle \frac{\partial}{\partial e} \left(\eta^{\beta-1} \Delta_h u \otimes \nabla \eta \right), \int_0^1 j_F^\gamma(\nabla u(x + th)) dt \right\rangle \\ &= |h| \int_{\Omega} \left\langle \Delta_h u \otimes \frac{\partial}{\partial e} (\eta^{\beta-1} \nabla \eta), \int_0^1 j_F^\gamma(\nabla u(x + th)) dt \right\rangle \\ &\quad + |h| \int_{\Omega} \left\langle \frac{\partial}{\partial e} (\Delta_h u) \otimes (\eta^{\beta-1} \nabla \eta), \int_0^1 j_F^\gamma(\nabla u(x + th)) dt \right\rangle = A_{21} + A_{22} \end{aligned}$$

Using $F_*(j_F^\gamma(\xi)) = F(\xi)^{\gamma-1}$, (27), and (10), implies

$$\begin{aligned} |A_{21}| &\lesssim |h| \int_{\Omega} F \left(\Delta_h u \otimes \frac{\partial}{\partial e} (\eta^{\beta-1} \nabla \eta) \right) F_* \left(\int_0^1 j_F^\gamma(\nabla u(x + th)) dt \right) \\ &\leq |h| \int_{\Omega} \left\{ \rho(\Delta_h u) \left| \frac{\partial}{\partial e} (\eta^{\alpha-1} \nabla \eta) \right| \int_0^1 F_*(j_F^\gamma(\nabla u(x + th))) dt \right\} \\ &\lesssim |h| \|u\|_{W^{1,\gamma}(B_s)} |h| \left(\int_{B_{r+\frac{2}{3}(s-r)}} \int_0^1 F(\nabla u)^\gamma dt \right)^{\frac{1}{\gamma}} \\ &\lesssim |h|^2 \|u\|_{W^{1,\gamma}(B_s)} \left(\int_{B_s} F(\nabla u)^\gamma \right)^{\frac{1}{\gamma}} \lesssim |h|^2 \|u\|_{W^{1,\gamma}(B_s)}^\gamma. \end{aligned} \tag{33}$$

Choosing $\beta = \sigma'$ and arguing similarly, using (27) and (23)

$$\begin{aligned}
 |A_{22}| &\lesssim |h| \int_{\Omega} F \left(\frac{\partial}{\partial e} (\Delta_h u) \otimes (\eta^{\sigma'-1} \nabla \eta) \right) \int_0^1 F_* (j_F^{\gamma} (\nabla u(x+th))) dt \\
 &\lesssim |h| \int_{\Omega} \rho \left(\Delta_h \frac{\partial u}{\partial e} \right) |\eta^{\sigma'-1} \nabla \eta| \int_0^1 F(\nabla u(x+th))^{\gamma-1} dt \\
 &\lesssim |h| \int_{\Omega} F(\Delta_h \nabla u) \eta^{\sigma'-1} \int_0^1 F(\nabla u(x+th))^{\gamma-1} dt \\
 &\lesssim |h| \left(\int_{\Omega} F(\Delta_h \nabla u)^{\gamma} \eta^{\gamma(\sigma'-1)} \right)^{\frac{1}{\gamma}} \left(\int_{B_{\rho_2}} \int_0^1 F(\nabla u(x+th))^{\gamma} dt \right)^{\frac{1}{\gamma}}. \tag{34}
 \end{aligned}$$

Using (22) with $p = \gamma \leq \sigma = q$ we find,

$$\left(\int_{\Omega} F(\Delta_h u)^{\gamma} \eta^{\gamma(\sigma'-1)} \right)^{\frac{1}{\gamma}} \lesssim \left(\int \eta^{\sigma'} F(\Delta_h V)^{\sigma} \right)^{\frac{1}{\sigma}} \left(\int_{B_{\rho_2}} (F(\nabla u) + F(\nabla u_h))^{\gamma} \right)^{\frac{\sigma-\gamma}{\sigma\gamma}}.$$

Plugging this into (34) and applying Young's inequality yields for any $\varepsilon > 0$,

$$\begin{aligned}
 |A_{22}| &\lesssim \varepsilon \int \eta^{\sigma'} F(\Delta_h V)^{\sigma} + C(\varepsilon) |h|^{\sigma'} \|\nabla u\|_{L^{\gamma}(B_s)}^{\sigma'(\gamma-1)} \|\nabla u\|_{L^{\gamma}(B_s)}^{\frac{\sigma-\gamma}{\gamma(\sigma-1)}} \\
 &= \varepsilon \int \eta^{\sigma'} F(\Delta_h V)^{\sigma} + C(\varepsilon) |h|^{\sigma'} \|\nabla u\|_{L^{\gamma}(B_s)}^{\gamma} \tag{35}
 \end{aligned}$$

From the PDE, we have $A_1 + A_2 = I \leq |III|$. In particular $A_1 - |A_{22}| \leq |A_{21}| + |III|$. Estimating each term respectively by (30), (35), (33), and (32) yields, choosing ε sufficiently small,

$$\int_{\Omega} \eta^{\beta} F(\Delta_h V)^{\sigma} + \int_{\Omega} \eta^{\beta} F_* (\Delta_h V^*)^{\tau'} \lesssim \varepsilon \left(|h|^{\sigma'} + |h|^2 + |h|^{1+\alpha_*} \right) \left(\|u\|_{W^{1,\gamma}(B_s)}^{\gamma} + \|f\|_{B_{\infty}^{\alpha_*,\gamma'}(B_s)}^{\gamma'} \right). \tag{36}$$

Noting $\alpha_* = \sigma'$ and dividing by $|h|^{\sigma'}$ implies

$$\|V(\nabla u)\|_{B_{\infty}^{\sigma'-1,\sigma}(B_r)} + \|V^*(j_F^{\gamma}(\nabla u))\|_{B_{\infty}^{\frac{\sigma'}{\gamma},\tau'}(B_r)} \lesssim_{(s-r)} \|u\|_{W^{1,\gamma}(B_s)}^{\gamma} + \|f\|_{B_{\infty}^{\sigma'-1,\gamma'}(B_s)}^{\gamma'}.$$

Case 1.2: A primal estimate if $\gamma \geq \sigma$. We return to (34), choose $\beta = \gamma \geq \sigma \geq 2$ and estimate using Hölder's inequality and Young's inequality,

$$\begin{aligned}
 |A_{22}| &\lesssim |h|(s-r)^{-1} \left(\int_{\Omega} F(\Delta_h \nabla u)^{\gamma} \eta^{\gamma} \right)^{\frac{1}{\gamma}} \left(\int_{B_{r+\frac{s-r}{3}}} \eta^{(\gamma-2)\gamma'} \int_0^1 F(\nabla u(x+th))^{\gamma} dt \right)^{\frac{1}{\gamma'}} \\
 &\lesssim \varepsilon \int_{\Omega} \eta^{\gamma} F(\Delta_h \nabla u)^{\gamma} + C(\varepsilon) |h|^{\gamma'} \int_{B_s} F(\nabla u)^{\gamma}.
 \end{aligned}$$

Applying (21) yields

$$|A_{22}| \lesssim \varepsilon \int_{\Omega} F(\Delta_h V)^{\sigma} + C(\varepsilon) |h|^{\gamma'} \int_{B_s} F(\nabla u)^{\gamma}. \tag{37}$$

Now estimating each term in $A_1 - |A_{22}| \leq |A_{21}| + |III|$, respectively by (30), (37), (33), and (32), yields

$$\int_{\Omega} \eta^{\beta} F(\Delta_h V)^{\sigma} + \int_{\Omega} \eta^{\beta} F_* (\Delta_h V^*)^{\tau'} \lesssim \varepsilon \left(|h|^{\gamma'} + |h|^2 + |h|^{1+\alpha_*} \right) \left(\|u\|_{W^{1,\gamma}(B_s)}^{\gamma} + \|f\|_{B_{\infty}^{\alpha_*,\gamma'}(B_s)}^{\gamma'} \right). \tag{38}$$

Noting that $\alpha_* + 1 = \gamma'$, dividing by $|h|^{\gamma'}$ yields

$$\|V(\nabla u)\|_{B_{\infty}^{\frac{\gamma'}{\sigma},\sigma}(B_r)} + \|V^*(j_F^{\gamma}(\nabla u))\|_{B_{\infty}^{\frac{\gamma'}{\gamma},\tau'}(B_r)} \lesssim_{(s-r)} \|u\|_{W^{1,\gamma}(B_s)}^{\gamma} + \|f\|_{B_{\infty}^{\gamma'-1,\gamma'}(B_s)}^{\gamma'}.$$

Case 2.1.: A dual estimate if $\tau' \geq \gamma'$. Choose $\beta = \tau$. From Fenchel, Hölder, and (10) it follows

$$\begin{aligned}
 |A_2| &= \tau \left| \int_{\Omega} \eta^{\tau-1} \langle \Delta_h j_F^\gamma(\nabla u), \Delta_h u \otimes \nabla \eta \rangle \right| \\
 &\lesssim \int_{\Omega} \eta^{\tau-1} F(\Delta_h u \otimes \nabla \eta) F_* (\Delta_h j_F^\gamma(\nabla u)) \\
 &\lesssim \left(\int_{B_{\rho_1}} \rho(\Delta_h u)^\gamma |\nabla \eta|^\gamma \right)^{\frac{1}{\gamma}} \left(\int \eta^{\gamma'(\tau-1)} F_* (\Delta_h j_F^\gamma(\nabla u))^{\gamma'} \right)^{\frac{1}{\gamma'}} \\
 &\leq |h| \|u\|_{W^{1,\gamma}(B_s)} \left(\int \eta^{\gamma'(\tau-1)} F_* (\Delta_h j_F^\gamma(\nabla u))^{\gamma'} \right)^{\frac{1}{\gamma'}} \tag{39}
 \end{aligned}$$

Since $V^*(j_F^\gamma(\nabla u)) = V_{\gamma',\tau'}^*(j_F^\gamma(\nabla u))$, up to replacing F with F_* and ∇u with $j_F^\gamma(\nabla u)$, it follows from (22) with $p = \gamma'$ and $q = \tau'$, that

$$\begin{aligned}
 &\left(\int \eta^{\gamma'(\tau-1)} F_* (\Delta_h j_F^\gamma(\nabla u))^{\gamma'} \right)^{\frac{1}{\gamma'}} \\
 &\leq \left(\int \eta^\tau F_* (\Delta_h V^*(j_F^\gamma(\nabla u)))^{\tau'} \right)^{\frac{1}{\tau'}} \left(\int_{B_{\rho_2}} (F_*(j_F^\gamma(\nabla u)) + F_*(j_F^\gamma(\nabla u)))^{\gamma'} \right)^{\frac{\tau'-\gamma'}{\tau'\gamma'}} \\
 &\lesssim \left(\int \eta^\tau F_* (\Delta_h V^*(j_F^\gamma(\nabla u)))^{\tau'} \right)^{\frac{1}{\tau'}} \|j_F^\gamma(\nabla u)\|_{L^{\gamma'}(B_s)}^{1-\frac{\gamma'}{\tau}}. \tag{40}
 \end{aligned}$$

Recalling $F_*(j_F^\gamma(\xi)) = F(\xi)^{\gamma-1}$, we note $\|j_F^\gamma(\nabla u)\|_{L^{\gamma'}(B_s)}^{1-\frac{\gamma'}{\tau}} = \|\nabla u\|_{L^\gamma(B_s)}^{(\gamma-1)(1-\frac{\gamma'}{\tau})} = \|\nabla u\|_{L^\gamma(B_s)}^{\frac{\gamma-\tau}{\tau}}$. Then plugging (40) into (39) and applying Young's inequality yields for any $\varepsilon > 0$,

$$\begin{aligned}
 |A_2| &\lesssim \varepsilon \int \eta^\tau F_* (\Delta_h V^*(j_F^\gamma(\nabla u)))^{\tau'} + C(\varepsilon) |h|^\tau \|\nabla u\|_{L^\gamma(B_s)}^{\gamma-\tau} \|u\|_{W^{1,\gamma}(B_s)}^\tau \\
 &\lesssim \varepsilon \int \eta^\tau F_* (\Delta_h V^*(j_F^\gamma(\nabla u)))^{\tau'} + C(\varepsilon) |h|^\tau \|u\|_{W^{1,\gamma}(B_s)}^\gamma \tag{41}
 \end{aligned}$$

Since the PDE implies $A_1 \leq |A_2| + |II|$, choosing ε small enough, and estimating each term consecutively by (31), (41), and (32) yields

$$\int_{\Omega} \eta^\beta F(\Delta_h V)^\sigma + \int_{\Omega} \eta^\beta F_* (\Delta_h V^*)^{\tau'} \lesssim (|h|^\tau + |h|^2 + |h|^{1+\alpha_*}) \left(\|u\|_{W^{1,\gamma}(B_s)}^\gamma + \|f\|_{B_{\infty^*,\gamma'}^\alpha(B_s)}^{\gamma'} \right). \tag{42}$$

Noting $\alpha_* - 1 = \tau$, dividing by $|h|^\tau$ verifies

$$\|V(\nabla u)\|_{B_{\infty^*,\sigma}^{\frac{\tau}{\sigma}}(B_r)} + \|V^*(j_F^\gamma(\nabla u))\|_{B_{\infty^*}^{\tau-1,\tau'}(B_r)} \lesssim_{(s-r)} \|u\|_{W^{1,\gamma}(B_s)}^\gamma + \|f\|_{B_{\infty^*}^{\tau-1,\gamma'}(B_s)}^{\gamma'}.$$

Case 2.2.: A dual estimate if $\gamma \leq \tau$. We return to estimating $|A_2|$. Choosing $\beta = \gamma$ and using Hölder's then Young's inequalities,

$$\begin{aligned}
 |A_2| &\lesssim \int_{\Omega} \eta^{\gamma-1} F(\Delta_h u \otimes \nabla \eta) F_* (\Delta_h j_F^\gamma(\nabla u)) \\
 &\lesssim \left(\int_{\Omega} F(\Delta_h u \otimes \nabla \eta)^\gamma \right)^{\frac{1}{\gamma}} \left(\int_{\Omega} \eta^\gamma F_* (\Delta_h j_F^\gamma(\nabla u))^{\gamma'} \right)^{\frac{1}{\gamma'}} \\
 &\lesssim C(\varepsilon) |h|^\gamma \|u\|_{W^{1,\gamma}(B_{\rho_2})}^\gamma + \varepsilon \int_{\Omega} \eta^\gamma F_* (\Delta_h j_F^\gamma(\nabla u))^{\gamma'}.
 \end{aligned}$$

Since $\gamma' \geq \tau'$, up to replacing F with F_* , $V_{p,q}$ with $V_{\gamma',\tau'}^*$, and u with $j_F^\gamma(\nabla u)$, (21) reads

$$F_* (\Delta_h j_F^\gamma(\nabla u))^{\gamma'} \leq F_* (\Delta_h V^*(j_F^\gamma(\nabla u)))^{\tau'}.$$

Hence,

$$|A_2| \lesssim \epsilon \int_{\Omega} \eta^\gamma F_* (\Delta_h V^* (j_F^\gamma(\nabla u)))^{\tau'} + C(\epsilon) |h|^\gamma \|u\|_{W^{1,\gamma}(B_s)}. \tag{43}$$

By writing $A_1 \leq |A_2| + |II|$ using respectively (31), (43), and (32) we obtain

$$\int_{\Omega} \eta^\beta F (\Delta_h V(\nabla u))^\sigma + \int_{\Omega} \eta^\beta F_* (\Delta_h V^* (j_F^\gamma(\nabla u)))^{\tau'} \lesssim (|h|^\gamma + |h|^2 + |h|^{1+\alpha_*}) \left(\|u\|_{W^{1,\gamma}(B_s)}^\gamma + \|f\|_{B_\infty^{\alpha_*,\gamma'}(B_s)}^{\gamma'} \right). \tag{44}$$

Noting $\alpha_* + 1 = \gamma$, dividing by $|h|^\gamma$ implies

$$\|V(\nabla u)\|_{B_\infty^{\frac{\gamma'}{\sigma},\sigma}(B_r)} + \|V^*(j_F^\gamma(\nabla u))\|_{B_\infty^{\frac{\gamma'}{\tau'},\tau'}(B_r)} \lesssim_{(s-r)} \|u\|_{W^{1,\gamma}(B_s)}^\gamma + \|f\|_{B_\infty^{\tau,\gamma'}(B_s)}^{\gamma'}.$$

Concluding the proof. One confirms (25), and consequently (24) using Case 1.1, 1.2, 2.1, and 2.2, respectively when $\alpha = \sigma', \gamma', \tau$, and γ . To deduce (26) from (24), we apply Corollary 14 with $p = \gamma, q = \sigma$ to obtain

$$\int_{B_r} F(\Delta_h V(\nabla u))^\sigma \gtrsim \begin{cases} \int_{B_r} F(\Delta_h \nabla u)^\gamma & \gamma \geq \sigma \\ \left(\int_{B_r} F(\Delta_h \nabla u)^\gamma \right)^{\frac{\sigma}{\gamma}} \left(\int_{B_r} (F(\nabla u) + F(\nabla u_h))^\gamma \right)^{1-\frac{\sigma}{\gamma}} & \gamma \leq \sigma. \end{cases}$$

Similarly with $p = \gamma', q = \tau', F = F_*$ and $u = j_F^\gamma(\nabla u)$, Corollary 14 says

$$\int_{B_r} F_* (\Delta_h V^* (j_F^\gamma(\nabla u)))^{\tau'} \gtrsim \begin{cases} \int_{B_r} F_* (\Delta_h j_F^\gamma(\nabla u))^{\gamma'} & \gamma' \geq \tau' \\ \left(\int_{B_r} F_* (\Delta_h j_F^\gamma(\nabla u))^{\gamma'} \right)^{\frac{\tau'}{\gamma'}} \|\nabla u\|_{L^{\frac{\gamma'-\tau'}{\gamma'}}(B_s)}^{\frac{\gamma'-\tau'}{\gamma'}} & \gamma' \leq \tau'. \end{cases}$$

Using these estimates to produce further lower bounds for (36), (38), (42), (44) and then dividing by $|h|^\alpha$ verifies (26). □

References

- [1] B. Avelin, T. Kuusi, G. Mingione, “Nonlinear Calderón-Zygmund Theory in the Limiting Case”, *Arch. Ration. Mech. Anal.* **227** (2018), no. 2, p. 663-714.
- [2] A. K. Balci, L. Diening, M. Weimar, “Higher Order Regularity Shifts for the p -Poisson Problem”, in *Numerical Mathematics and Advanced Applications ENUMATH 2019*, Lecture Notes in Computational Science and Engineering, vol. 139, Springer, 2021, p. 1147-1155.
- [3] C. Bargetz, E. Medjic, “On the rate of convergence of iterated bregman projections and of the alternating algorithm”, *J. Math. Anal. Appl.* **481** (2020), no. 1, article no. 123482 (23 pages).
- [4] M. Belloni, V. Ferone, B. Kawohl, “Isoperimetric inequalities, wulff shape and related questions for strongly nonlinear elliptic operators”, *Z. Angew. Math. Phys.* **54** (2003), no. 5, p. 771-783.
- [5] B. Bojarski, T. Iwaniec, “ p -harmonic equation and quasiregular mappings”, in *Partial differential equations*, Banach Center Publications, vol. 19, PWN - Polish Scientific Publishers, 1987, p. 25-38.
- [6] G. Bonfanti, A. Cellina, “The nonoccurrence of the Lavrentiev phenomenon for a class of variational functionals”, *SIAM J. Control Optim.* **51** (2013), no. 2, p. 1639-1650.
- [7] G. Bonfanti, A. Cellina, M. Mazzola, “The higher integrability and the validity of the Euler-Lagrange equation for solutions to variational problems”, *SIAM J. Control Optim.* **50** (2012), no. 2, p. 888-899.
- [8] L. Brasco, G. Carlier, F. Santambrogio, “Congested traffic dynamics, weak flows and very degenerate elliptic equations”, *J. Math. Pures Appl.* **93** (2010), no. 6, p. 652-671.
- [9] L. Brasco, C. Leone, G. Pisante, A. Verde, “Sobolev and lipschitz regularity for local minimizers of widely degenerate anisotropic functionals”, *Nonlinear Anal., Theory Methods Appl.* **153** (2017), p. 169-199.

- [10] M. Carozza, J. Kristensen, A. Passarelli di Napoli, "Regularity of minimisers of autonomous convex variational integrals", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **13** (2014), no. 4, p. 1065-1089.
- [11] M. Carozza, G. MoscarIELLO, A. Passarelli di Napoli, "Regularity results via duality for minimizers of degenerate functionals", *Asymptotic Anal.* **44** (2005), no. 3-4, p. 221-235.
- [12] M. Carozza, A. Passarelli di Napoli, "Regularity for minimizers of degenerate elliptic functionals", *J. Nonlinear Convex Anal.* **7** (2006), no. 3, p. 375-383.
- [13] A. Cianchi, V. G. Maz'ya, "Second-Order Two-Sided Estimates in Nonlinear Elliptic Problems", *Arch. Ration. Mech. Anal.* **229** (2018), no. 2, p. 569-599.
- [14] J. A. Clarkson, "Uniformly convex spaces", *Trans. Am. Math. Soc.* **40** (1936), no. 3, p. 396-414.
- [15] M. Colombo, "Regularity results for very degenerate elliptic equations", in *Flows of Non-smooth Vector Fields and Degenerate Elliptic Equations*, Springer, 2017, p. 119-157.
- [16] M. G. Delgadino, F. Maggi, C. Mihaila, R. Neumayer, "Bubbling with L^2 -almost constant mean curvature and an alexandrov-type theorem for crystals", *Arch. Ration. Mech. Anal.* **230** (2018), no. 3, p. 1131-1177.
- [17] F. Demengel, "Lipschitz interior regularity for the viscosity and weak solutions of the pseudo p -laplacian equation", *Adv. Differ. Equ.* **21** (2016), no. 3-4, p. 373-400.
- [18] G. Di Blasio, P. D. Lamberti, "Eigenvalues of the finsler p -laplacian on varying domains", *Mathematika* **66** (2020), no. 3, p. 765-776.
- [19] V. Ferone, B. Kawohl, "Remarks on a Finsler-Laplacian", *Proc. Am. Math. Soc.* **137** (2009), no. 1, p. 247-253.
- [20] C. d. Filippis, L. Koch, J. Kristensen, "Regularity in relaxed convex problems", in preparation.
- [21] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific, 2003.
- [22] M. Goering, "A harnack inequality for $W^{1,\gamma}$ -solutions of anisotropic pdes with non-standard growth", 2020, <https://arxiv.org/abs/2008.08028>.
- [23] U. Guarnotta, S. Mosconi, "A general notion of uniform ellipticity and the regularity of the stress field for elliptic equations in divergence form", 2021, <https://arxiv.org/abs/2105.12546>.
- [24] T. Hytönen, J. van Neerven, M. Veraar, L. Weis, *Analysis in Banach spaces. Volume I: Martingales and Littlewood-Paley theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 63, Springer, 2016.
- [25] B. Kawohl, M. Novaga, "The p -laplace eigenvalue problem as $p \rightarrow 1$ and Cheeger sets in a Finsler metric", *J. Convex Anal.* **15** (2008), no. 3, p. 623-634.
- [26] L. Koch, J. Kristensen, "On the validity of the Euler-Lagrange system without growth assumptions", 2022, <https://arxiv.org/abs/2203.00333>.
- [27] M. Kreuter, "Sobolev spaces of vector-valued functions", PhD Thesis, University of Ulm, 2015.
- [28] A. Kristály, Z. Shen, L. Yuan, W. Zhao, "Nonlinear spectrums of finsler manifolds", *Math. Z.* **300** (2022), no. 1, p. 81-123.
- [29] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces II: function spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 97, Springer, 2013.
- [30] G. Seregin, "Differential properties of solutions of variational problems for functionals of linear growth", *J. Sov. Math.* **64** (1993), no. 6, p. 1256-1277.
- [31] Q. Xia, "Sharp spectral gap for the Finsler p -Laplacian", *Sci. China, Math.* **62** (2019), no. 8, p. 1615-1644.
- [32] Z.-B. Xu, G. F. Roach, "Characteristic inequalities of uniformly convex and uniformly smooth banach spaces", *J. Math. Anal. Appl.* **157** (1991), no. 1, p. 189-210.
- [33] V. V. Zhikov, S. M. Kozlov, O. A. Olejnik, *Homogenization of differential operators and integral functionals*, Springer, 1994.