INSTITUT DE FRANCE
Académie des sciences

# Comptes Rendus
# Mathématique

Ye Zhang

**On the H.-Q. Li inequality on step-two Carnot groups**

Volume 361 (2023), p. 1107-1114

Published online: 24 October 2023

https://doi.org/10.5802/crmath.475



*Les* Comptes Rendus. Mathématique *sont membres du*
*Centre Mersenne pour l'édition scientifique ouverte*
www.centre-mersenne.org
e-ISSN : 1778-3569

*Comptes Rendus*
*Mathématique*
**2023**, Vol. 361, p. 1107-1114
https://doi.org/10.5802/crmath.475

Partial differential equations / *Équations aux dérivées partielles*

# On the H.-Q. Li inequality on step-two Carnot groups

**Ye Zhang** [a]

[a] Analysis on Metric Spaces Unit, Okinawa Institute of Science and Technology Graduate University, Okinawa 904-0495, Japan
*E-mail:* Ye.Zhang2@oist.jp

**Abstract.** In this note we show that the gradient estimate of the heat semigroup, or more precisely the H.-Q. Li inequality, is preserved under tensorization, some suitable group epimorphism, and central sum. We also establish the Riemannian counterpart of the H.-Q. Li inequality. As a byproduct, we provide a simpler proof of the fact that the constant in H.-Q. Li inequality is strictly larger than 1.

**2020 Mathematics Subject Classification.** 58J35, 22E30, 35H10, 35R03.

*Manuscript received 16 November 2022, revised 3 February 2023, accepted 6 February 2023.*

## 1. Introduction

The following equivalent characterization of the lower boundedness of the Ricci curvature is well-known (cf. [17, 27]):

**Theorem 1.** *On a complete Riemannian manifold $M$, the followings are equivalent:*

(I) $\mathrm{Ric} \geq \rho$;
(II) $|\nabla e^{t\Delta} f|^2 \leq e^{-2\rho t} e^{t\Delta}(|\nabla f|^2), \quad \forall\ t > 0, f \in C_c^\infty(M)$;
(III) $|\nabla e^{t\Delta} f| \leq e^{-\rho t} e^{t\Delta}(|\nabla f|), \quad \forall\ t > 0, f \in C_c^\infty(M)$.

*Here* Ric *denotes the Ricci curvature tensor and* $\rho$ *is a real number.*

For example, on Euclidean spaces we have $\rho = 0$ and the bounds in (II) and (III) are 1. However, when it comes to the case of nonabelian nilpotent Lie group with a left-invariant Riemannian metric, J. Milnor showed in [25] that there exists a direction of strictly negative Ricci curvature. This implies the bounds in (II) and (III) of Theorem 1 have exponential growth (in $t$), although we expect these spaces are flat in some appropriate sense.

Then on the first Heisenberg group $\mathbb{H}^1$, using Malliavin Calculus, B. K. Driver and T. Melcher established

$$|\nabla e^{t\Delta} f|^p \leq C_p e^{t\Delta}(|\nabla f|^p), \quad \forall\ t > 0, f \in C_c^\infty(\mathbb{H}^1) \tag{1}$$

with $p, C_p \in (1, \infty)$ in [8], where $\nabla$ and $\Delta$ denote the horizontal gradient and sub-Laplacian instead of the Riemannian counterparts. Comparing (1) with (II) of Theorem 1, we can say that

$\mathbb{H}^1$ (endowed with the sub-Riemannian structure) is in some sense flat. We believe that this kind of flatness can be justified by (or even equivalent to) some weak variant of the well-known curvature-dimension condition, such as the measure contraction property. Recall that on the first Heisenberg group $\mathbb{H}^1$ (endowed with the sub-Riemannian structure), no curvature-dimension bounds hold but it admits the measure contraction property MCP(0,5) (cf. [15]).

After [8], H.-Q. Li first proved (1) for the remaining case $p = 1$ in [18], which implies (1) for $p > 1$. Such inequality is called H.-Q. Li inequality in [1], where the authors gave more proofs of it. The corresponding generalizations to the H-type groups and nonisotropic Heisenberg groups can be found in [10,14,31].

To be more precise, all the proofs depend on the precise upper and lower bound estimates of the heat kernel and suitable upper bound of the gradient of the heat kernel, which are usually obtained by the heat kernel asymptotics thanks to the explicit formula of the heat kernel given by oscillatory integral. See for example [9,19,20,22] for related results. Although some general approach to obtain the heat kernel asymptotics on step-two Carnot groups is given in [23], to get a suitable expression of the heat kernel bound is still a challenging problem due to the complexity of the geometry of the underlying group.

The main aim of this work is to establish H.-Q. Li inequality without using the bounds of the heat kernel. More precisely, we show that H.-Q. Li inequality is preserved under tensorization, some suitable group epimorphism, and the central sum introduced in [21, Section 8.1]. This approach is motivated by the work [13] in which their target is to establish the logarithmic Sobolev inequality w.r.t. the heat kernel measure on nonisotropic Heisenberg groups. After that, we can construct a lot of concrete examples on which the H.-Q. Li inequality holds but the heat kernel asymptotics have not been obtained or are rather complicated. Moreover, we reduce the study of H.-Q. Li inequality to the case of free step-two Carnot groups.

Another target of this work is to use H.-Q. Li inequality to improve the bound in (III) of Theorem 1 on step-two Carnot groups endowed with the canonical left-invariant Riemannian structure. As a byproduct, we give a simpler proof of the fact that the constant in the H.-Q. Li inequality is strictly larger than 1.

Finally we remark that as consequences of the H.-Q. Li inequality, the logarithmic Sobolev inequality and Poincaré inequality w.r.t. the heat kernel measure also hold. For more details and more functional inequalities we refer to [1, Section 6] and will not go into details.

This paper is organized as follows. After recalling the background settings and notations in Section 2, we illustrate tensorization, suitable group epimorphism, and central sum preserve the H.-Q. Li inequality in Sections 3–5 respectively. After that, we study the counterpart of H.-Q. Li inequality for the canonical left-invariant Riemannian structure and prove the constant in H.-Q. Li inequality is strictly larger than 1 in Section 6. Concrete examples are provided in Section 7.

## 2. Preliminaries and notations

Via the exponential map, any step-two Carnot group (or step-two group, in short) $\mathbb{G}$ can be identified with the direct product of Euclidean spaces $\mathbb{R}^q \times \mathbb{R}^m$, $q, m \in \mathbb{N}^* = \{1,2,3,\dots\}$, with the group law

$$(x,u)\cdot(x',u') = \left(x + x', u + u' + \frac{1}{2}\langle \mathbb{U}x, x'\rangle\right),$$

where $\langle \mathbb{U}x, x'\rangle := (\langle U^{(1)}x, x'\rangle, \dots, \langle U^{(m)}x, x'\rangle) \in \mathbb{R}^m$. See for example [4, Section 3.2]. Here $\mathbb{U} = \{U^{(1)}, \dots, U^{(m)}\}$ is an $m$-tuple of linearly independent $q \times q$ real skew-symmetric matrices and $\langle \cdot,\cdot\rangle$ denotes the usual inner product on the Euclidean space. We call $q$ and $m$ the rank and corank of $\mathbb{G}$ respectively. We also use $g := (x,u)$ to write an element of $\mathbb{G}$. In particular, the identity element is denoted by $e := (0,0)$. The left translation by $g$ is denoted by $L_g^{\mathbb{G}}$ and the Haar measure

$\mathrm{d}g$ coincides with the $(q+m)$-dimensional Lebesgue measure on $\mathbb{G}$. Let $U^{(j)} = (U^{(j)}_{l,k})_{1\le l,k\le q}$ $(1 \le j \le m)$. The horizontal gradient and canonical sub-Laplacian are given by

$$\nabla_{\mathbb{G}} := (\mathrm{X}_1, \dots, \mathrm{X}_q), \ \ \Delta_{\mathbb{G}} := \sum_{l=1}^{q} \mathrm{X}_l^2, \ \text{ with } \mathrm{X}_l(g) := (\mathrm{d}L_g^{\mathbb{G}})_e \left(\frac{\partial}{\partial x_l}\right) = \frac{\partial}{\partial x_l} + \frac{1}{2}\sum_{j=1}^{m}\left(\sum_{k=1}^{q} U^{(j)}_{l,k}\, x_k\right)\frac{\partial}{\partial u_j}$$

respectively. Note that $\Delta_{\mathbb{G}}$ is not elliptic but hypoelliptic, corresponding to the left-invariant sub-Riemannian structure w.r.t. which the horizontal distribution is given by $\mathscr{H} := \mathrm{span}\{\mathrm{X}_1, \dots, \mathrm{X}_q\}$ and $\{\mathrm{X}_1, \dots, \mathrm{X}_q\}$ forms an orthonormal basis at each point.

It is well-known that the heat semigroup $(e^{t\Delta_{\mathbb{G}}})_{t>0}$ has a convolution kernel $p_t^{\mathbb{G}}$, in the sense that

$$(e^{t\Delta_{\mathbb{G}}} f)(g) = f * p_t^{\mathbb{G}}(g) = \int_{\mathbb{G}} f(g')\, p_t^{\mathbb{G}}((g')^{-1}\cdot g)\,\mathrm{d}g' \quad \text{for suitable function } f.$$

By H.-Q. Li inequality, we mean the following inequality for the gradient of the heat semigroup:

$$|\nabla_{\mathbb{G}} e^{t\Delta_{\mathbb{G}}} f|(g) \le C e^{t\Delta_{\mathbb{G}}}(|\nabla_{\mathbb{G}} f|)(g), \quad \forall\ t>0,\ g\in\mathbb{G},\ f\in C_c^\infty(\mathbb{G}),$$

which is denoted by $\mathbf{HQLI}(\mathbb{G}, C)$ in this work. Here and in the following, $|\cdot| := \sqrt{\langle\cdot,\cdot\rangle}$ always denotes the Euclidean norm.

## 3. The direct product

We begin with the simple tensorization property of H.-Q. Li inequality. Assume we have two Carnot groups $\mathbb{G} \cong \mathbb{R}^q \times \mathbb{R}^m$ and $\mathbb{G}' \cong \mathbb{R}^{q'} \times \mathbb{R}^{m'}$. Then we consider their direct product $\mathbb{G} \times \mathbb{G}'$. In this work we adopt the convention that we write an element of $\mathbb{G}\times\mathbb{G}'$ as $(g, g')$ with $g \in \mathbb{G}$ and $g' \in \mathbb{G}'$ instead of an element of $\mathbb{R}^{q+q'} \times \mathbb{R}^{m+m'}$. It is easy to see

$$\nabla_{\mathbb{G}\times\mathbb{G}'} = (\nabla_{\mathbb{G}}, \nabla_{\mathbb{G}'}), \qquad \Delta_{\mathbb{G}\times\mathbb{G}'} = \Delta_{\mathbb{G}} + \Delta_{\mathbb{G}'},$$

and the following theorem holds.

**Theorem 2.** *In the setting of step-two groups, if we have* $\mathbf{HQLI}(\mathbb{G}, C)$ *and* $\mathbf{HQLI}(\mathbb{G}', C')$*, then* $\mathbf{HQLI}(\mathbb{G}\times\mathbb{G}', \max(C, C'))$ *holds.*

**Proof.** For every $(g, g') \in \mathbb{G}\times\mathbb{G}'$, $(\alpha, \beta) \in \mathbb{R}^q \times \mathbb{R}^{q'}$ with $|\alpha|^2 + |\beta|^2 = 1$, $t > 0$, and $f \in C_c^\infty(\mathbb{G}\times\mathbb{G}')$, we have

$$\begin{aligned}
\langle \nabla_{\mathbb{G}\times\mathbb{G}'}(e^{t\Delta_{\mathbb{G}\times\mathbb{G}'}} f)(g,g'), (\alpha,\beta)\rangle &= \langle \nabla_{\mathbb{G}}(e^{t\Delta_{\mathbb{G}'}} e^{t\Delta_{\mathbb{G}}} f)(g,g'), \alpha\rangle + \langle \nabla_{\mathbb{G}'}(e^{t\Delta_{\mathbb{G}}} e^{t\Delta_{\mathbb{G}'}} f)(g,g'), \beta\rangle \\
&= \langle e^{t\Delta_{\mathbb{G}'}}[\nabla_{\mathbb{G}}(e^{t\Delta_{\mathbb{G}}} f)](g,g'), \alpha\rangle + \langle e^{t\Delta_{\mathbb{G}}}[\nabla_{\mathbb{G}'}(e^{t\Delta_{\mathbb{G}'}} f)](g,g'), \beta\rangle \\
&\le C e^{t\Delta_{\mathbb{G}'}} e^{t\Delta_{\mathbb{G}}}(|\alpha||\nabla_{\mathbb{G}} f|)(g,g') + C' e^{t\Delta_{\mathbb{G}}} e^{t\Delta_{\mathbb{G}'}}(|\beta||\nabla_{\mathbb{G}'} f|)(g,g') \\
&\le \max(C, C') e^{t\Delta_{\mathbb{G}\times\mathbb{G}'}}(|\nabla_{\mathbb{G}\times\mathbb{G}'} f|)(g,g').
\end{aligned}$$

Taking the supremum w.r.t. $(\alpha, \beta)$ we prove the theorem. □

## 4. The group epimorphism

Before proceeding further, we digress a little bit to consider the situation that $\mathbb{G} \cong \mathbb{R}^q \times \mathbb{R}^m$, $\mathbb{G}' \cong \mathbb{R}^q \times \mathbb{R}^{m'}$ (or equivalently they have the same rank), and there exists a group epimorphism between $\mathbb{G}$ and $\mathbb{G}'$ of the following type:

$$\pi_B(x, u) := (x, Bu), \tag{2}$$

where $B$ is a surjective linear map between $\mathbb{R}^m$ and $\mathbb{R}^{m'}$. This kind of assumption is closely related to the Rothschild–Stein lifting theorem. See Subsection 7.5 for detailed explanation. The idea of this section comes from [13] and we generalize it to a slightly more general setting. To be more precise, we first establish the following lemma.

**Lemma 3.** *Assume there exists a group epimorphism between* $\mathbb{G}$ *and* $\mathbb{G}'$ *of type* (2). *Then we have*

$$\nabla_{\mathbb{G}}(f\circ\pi_B)=(\nabla_{\mathbb{G}'}f)\circ\pi_B,\quad \Delta_{\mathbb{G}}(f\circ\pi_B)=(\Delta_{\mathbb{G}'}f)\circ\pi_B,\quad \forall\, f\in C_c^\infty(\mathbb{G}'),\tag{3}$$

$$(\pi_B)_\#(p_t^{\mathbb{G}}(g)\mathrm{d}g)=p_t^{\mathbb{G}'}(g')\mathrm{d}g',\ \textit{where \# denotes the measure push-forward,}\tag{4}$$

$$e^{t\Delta_{\mathbb{G}}}(f\circ\pi_B)=(e^{t\Delta_{\mathbb{G}'}}f)\circ\pi_B,\quad \forall\, t>0, f\in C_c^\infty(\mathbb{G}').\tag{5}$$

**Proof.** In the proof we use $g':=(y,v)\in\mathbb{R}^q\times\mathbb{R}^{m'}$ to denote an element in $\mathbb{G}'$. Suppose the horizontal gradient and canonical sub-Laplacian on $\mathbb{G}'$ are given by

$$\nabla_{\mathbb{G}'}:=(\mathrm{Y}_1,\dots,\mathrm{Y}_q),\quad \Delta_{\mathbb{G}'}:=\sum_{l=1}^q\mathrm{Y}_l^2,\quad\text{with}\quad \mathrm{Y}_l(g'):=(\mathrm{d}L_{g'}^{\mathbb{G}'})_e\left(\frac{\partial}{\partial y_l}\right)\quad\text{respectively.}$$

Recalling that $\pi_B$ is a group homomorphism, we have

$$L_{\pi_B(g)}^{\mathbb{G}'}\circ\pi_B=\pi_B\circ L_g^{\mathbb{G}},\qquad \forall\, g\in\mathbb{G}.$$

Taking differential at the identity $e$, we obtain

$$(\mathrm{d}L_{\pi_B(g)}^{\mathbb{G}'})_e\circ(\mathrm{d}\pi_B)_e=(\mathrm{d}\pi_B)_g\circ(\mathrm{d}L_g^{\mathbb{G}})_e,$$

which implies

$$(\mathrm{d}\pi_B)_g\circ(\mathrm{X}_l(g))=(\mathrm{d}\pi_B)_g\circ(\mathrm{d}L_g^{\mathbb{G}})_e\left(\frac{\partial}{\partial x_l}\right)=(\mathrm{d}L_{\pi_B(g)}^{\mathbb{G}'})_e\circ(\mathrm{d}\pi_B)_e\left(\frac{\partial}{\partial x_l}\right)=(\mathrm{d}L_{\pi_B(g)}^{\mathbb{G}'})_e\left(\frac{\partial}{\partial y_l}\right)=\mathrm{Y}_l(\pi_B(g)).$$

Applying the last equation to the function $f\in C_c^\infty(\mathbb{G}')$ we obtain (3). Now we are in a position to prove (4). In the following we use $\mu_t^{\mathbb{G}}:=p_t^{\mathbb{G}}(g)\mathrm{d}g$ to denote the heat kernel measure on $\mathbb{G}$. Then the second equation of (3), combining with [13, Proposition 2.6] or [7, Theorem 6.15], yields $(\pi_B)_\#\mu_t^{\mathbb{G}}=\mu_t^{\mathbb{G}'}$. We include the theorem below for the sake of completeness. Now we are left with (5):

$$\begin{aligned}e^{t\Delta_{\mathbb{G}}}(f\circ\pi_B)(g)&=\int_{\mathbb{G}}f(\pi_B(g^*))\,p_t^{\mathbb{G}}((g^*)^{-1}\cdot g)\mathrm{d}g^*=\int_{\mathbb{G}}f(\pi_B(g\cdot g^*))\,p_t^{\mathbb{G}}(g^*)\mathrm{d}g^*\\&=\int_{\mathbb{G}}f(\pi_B(g)\cdot\pi_B(g^*))\mathrm{d}\mu_t^{\mathbb{G}}(g^*)=\int_{\mathbb{G}'}f(\pi_B(g)\cdot g')\mathrm{d}\mu_t^{\mathbb{G}'}(g')=(e^{t\Delta_{\mathbb{G}'}}f)(\pi_B(g)),\end{aligned}$$

where we have used the fact that $p_t^{\mathbb{G}}(g)=p_t^{\mathbb{G}}(g^{-1})$ (see for example [12, G of Chapter I]) in the second equality and $(\pi_B)_\#\mu_t^{\mathbb{G}}=\mu_t^{\mathbb{G}'}$ in the penultimate equality. This ends the proof of Lemma 3. □

**Theorem 4 ([13, Proposition 2.6] or [7, Theorem 6.15]).** *The* $\{\mu_t^{\mathbb{G}}:=p_t^{\mathbb{G}}(g)\mathrm{d}g\}_{t>0}$ *is the unique family of probability measures on* $\mathbb{G}$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{G}}f\mathrm{d}\mu_t^{\mathbb{G}}=\int_{\mathbb{G}}\Delta_{\mathbb{G}}f\mathrm{d}\mu_t^{\mathbb{G}},\quad\textit{and}\quad \lim_{t\to0^+}\int_{\mathbb{G}}f\mathrm{d}\mu_t^{\mathbb{G}}=f(e),\quad\forall\, f\in C_c^\infty(\mathbb{G}).$$

**Remark 5.** Although we only state that the H.-Q. Li inequality, as well as Lemma 3, holds for the smooth functions with compact support, we can extend these results without difficulties to a suitably larger class. For instance, (5) is true for all continuous functions with compact support after an approximation process by $C_c^\infty$ functions. In the following, we adopt the convention that we have extended these results to some larger class and apply these results to functions which may not belong to $C_c^\infty$ without further explanations.

**Remark 6.** Equation (4) implies

$$\int_{\mathbb{G}}f(\pi_B(g))p_t^{\mathbb{G}}(g)\mathrm{d}g=\int_{\mathbb{G}'}f(g')p_t^{\mathbb{G}'}(g')\mathrm{d}g',\qquad\forall\, f\in C_c^\infty(\mathbb{G}').\tag{6}$$

From the definition of $\pi_B$, it is linear and thus for every $g'\in\mathbb{G}'$, $\pi_B^{-1}(g')$ is an affine space. To be more precise, we can split $\mathbb{G}$ into $\ker\pi_B\oplus(\ker\pi_B)^\perp\cong\ker\pi_B\times(\ker\pi_B)^\perp$, and $\nu_B:=\pi_B|_{(\ker\pi_B)^\perp}$ is an isomorphism. Thus $\pi_B^{-1}(g')$ can be identified with $\ker\pi_B\times\{(\nu_B)^{-1}(g')\}$. Applying the change of variables formula (cf. [11, Theorem 3.11]) to the LHS of (6), and combining with the identification above, we obtain

$$p_t^{\mathbb{G}'}(g') = C_{\pi_B} \int_{\ker \pi_B} p_t^{\mathbb{G}}(u, (\nu_B)^{-1}(g'))\mathrm{d}u, \quad \forall\, g' \in \mathbb{G}',$$

for some constant $C_{\pi_B} > 0$. This provides us with a new formula for the heat kernel and may help to obtain the heat kernel asymptotics on $\mathbb{G}'$, which generalizes the idea of [6, Section 5].

Combining Lemma 3 with **HQLI**$(\mathbb{G}, C)$ for $f \circ \pi_B$, we obtain the following theorem immediately.

**Theorem 7.** *Assmue $\mathbb{G}$ and $\mathbb{G}'$ are two step-two groups which have the same rank and there exists a group epimorphism between $\mathbb{G}$ and $\mathbb{G}'$ of type* (2). *If* **HQLI**$(\mathbb{G}, C)$ *holds, then* **HQLI**$(\mathbb{G}', C)$ *holds as well.*

## 5. The central sum introduced in [21]

Recall that in [21, Section 8.1], the author gave a method to construct a new step-two group from two step-two groups with the same corank other than the direct product, which extends the one from two copies of $\mathbb{H}^1$ to $\mathbb{H}^2$ (the Heisenberg group in higher dimensions). Via this construction, he showed that every step-two group can be embedded into an uncountable number of GM-groups (a special subclass of step-two groups). Then in [24, Appendix C], some properties of this construction are studied.

To be more precise, we first recall the construction, which is called central sum in this work. Throughout this section we assume that $\mathbb{G} \cong \mathbb{R}^q \times \mathbb{R}^m$, $\mathbb{G}' \cong \mathbb{R}^{q'} \times \mathbb{R}^m$ (or equivalently they have the same corank), and the group structure of $\mathbb{G}'$ is determined by $\mathbb{V} := \{V^{(1)}, \ldots, V^{(m)}\}$, that is, on $\mathbb{G}'$ we have

$$(y, v) \cdot (y', v') = \left( y + y', v + v' + \frac{1}{2} \langle \mathbb{V} y, y' \rangle \right).$$

Then the central sum of two step-two Carnot groups $\mathbb{G}$ and $\mathbb{G}'$, denoted by $\mathbb{G} \# \mathbb{G}' \cong \mathbb{R}^{q+q'} \times \mathbb{R}^m$, is defined by the group law:

$$((x, y), u) \cdot ((x', y'), u') = \left( (x + x', y + y'), u + u' + \frac{1}{2} \langle \mathbb{U} x, x' \rangle + \frac{1}{2} \langle \mathbb{V} y, y' \rangle \right).$$

We now show that central sum shares a similar property with the direct product for the H.-Q. Li inequality from the results of the previous two sections.

**Theorem 8.** *Under the circumstance of step-two groups, if* **HQLI**$(\mathbb{G}, C)$ *and* **HQLI**$(\mathbb{G}', C')$ *hold, then* **HQLI**$(\mathbb{G} \# \mathbb{G}', \max(C, C'))$ *also holds.*

**Proof.** It suffices to observe that there exists a group epimorphism from $\mathbb{G} \times \mathbb{G}'$ to $\mathbb{G} \# \mathbb{G}'$:

$$((x, u), (y, v)) \mapsto ((x, y), u + v).$$

Then Theorem 8 follows from Theorems 2 and 7. □

## 6. The Riemannian counterpart

On the step-two Carnot group $\mathbb{G}$, we also consider the following Riemannian gradient and full-Laplacian

$$\nabla_{\mathbb{G}}^{\mathrm{R}} := (\nabla_{\mathbb{G}}, \nabla_{\mathrm{Z}}), \quad \Delta_{\mathbb{G}}^{\mathrm{R}} := \Delta_{\mathbb{G}} + \Delta_{\mathrm{Z}}, \quad \text{with} \quad \nabla_{\mathrm{Z}} := (\mathrm{Z}_1, \ldots, \mathrm{Z}_m), \quad \Delta_{\mathrm{Z}} := \sum_{k=1}^m \mathrm{Z}_k^2, \quad \mathrm{Z}_k(g) := \frac{\partial}{\partial u_k}$$

instead of $\nabla_{\mathbb{G}}$ and $\Delta_{\mathbb{G}}$. This corresponds to the left-invariant Riemannian structure w.r.t. which $\{\mathrm{X}_1, \ldots, \mathrm{X}_q, \mathrm{Z}_1, \ldots, \mathrm{Z}_m\}$ forms the orthonormal basis at every point. In this case, we use

**HQLRI**$(\mathbb{G}, C)$ to denote the following counterpart of H.-Q. Li inequality in the Riemannian setting:

$$|\nabla^{\mathrm{R}}_{\mathbb{G}} e^{t\Delta^{\mathrm{R}}_{\mathbb{G}}} f|(g) \le C e^{t\Delta^{\mathrm{R}}_{\mathbb{G}}}(|\nabla^{\mathrm{R}}_{\mathbb{G}} f|)(g), \quad \forall\, t>0, g\in\mathbb{G}, f\in C_c^\infty(\mathbb{G}).$$

**Theorem 9.** *In the framework of step-two groups, if* **HQLI**$(\mathbb{G}, C)$ *holds, then so does* **HQLRI**$(\mathbb{G}, C)$.

**Proof.** First note that $\nabla_{\mathrm{Z}}$ and $\Delta_{\mathrm{Z}}$ are nothing but the Euclidean gradient and Laplacian in the second layer of $\mathbb{G}$. They commute with each other, and also with $\nabla_{\mathbb{G}}$ and $\Delta_{\mathbb{G}}$. As a result, we have $e^{t\Delta^{\mathrm{R}}_{\mathbb{G}}} = e^{t\Delta_{\mathbb{G}}} e^{t\Delta_{\mathrm{Z}}} = e^{t\Delta_{\mathrm{Z}}} e^{t\Delta_{\mathbb{G}}}$. This can also be justified by the expressions of the heat kernels of $(e^{t\Delta^{\mathrm{R}}_{\mathbb{G}}})_{t>0}$ and $(e^{t\Delta_{\mathbb{G}}})_{t>0}$ respectively. See for example [5, Section 10] for more details. Then it is very similar to the case of the direct product. The argument used in the proof of Theorem 2, combining with the fact that the corresponding gradient bound in the Euclidean case is 1, yields **HQLRI**$(\mathbb{G}, \max(C,1))$ without difficulties.

Now we are left with the proof of $C > 1$. We argue by contradiction. If we assume on the contrary that $C \le 1$, then **HQLRI**$(\mathbb{G}, 1)$ holds, which implies $\mathrm{Ric} \ge 0$ by Theorem 1. But it contradicts with [25, Theorem 2.4], which says that there exists a direction of strictly negative Ricci curvature. □

From the proof above, we can deduce the following corollary easily. In other words, we provide a different approach to prove the constant in H.-Q. Li inequality is strictly larger than 1.

**Corollary 10.** *In the setting of step-two groups, if* **HQLI**$(\mathbb{G}, C)$ *holds, then* $C > 1$.

## 7. Concrete examples

### 7.1. *The first Heisenberg group* $\mathbb{H}^1 \cong \mathbb{C}\times\mathbb{R}$

In the rest of this work we always identify $\mathbb{C}$ with $\mathbb{R}^2$ in the canonical way. Recall that the first Heisenberg group $\mathbb{H}^1 \cong \mathbb{C}\times\mathbb{R}$ is the simplest example of step-two Carnot group, which also plays an important role in several branches of mathematics (see for example [29,30]). The group structure on $\mathbb{H}^1$ is given by

$$(z,u)\cdot(z',u') = \left(z+z', u+u'+\frac{1}{2}\operatorname{Im}(z\bar{z}')\right).$$

In [1, 18], the authors established **HQLI**$(\mathbb{H}^1, C_0)$ for some constant $C_0 > 1$. Then it follows from Theorem 9 that **HQLRI**$(\mathbb{H}^1, C_0)$ is also true on $\mathbb{H}^1$.

### 7.2. *The step-two group associated to quadratic CR manifolds* $\mathbb{H}(\mathbf{A})$

In [26], the authors introduced the group $\mathbb{H}(\mathbf{A}) \cong \mathbb{C}^n\times\mathbb{R}^m$ with the group multiplication:

$$(z,u)\cdot(z',u') = \left(z+z', u+u'+\frac{1}{2}\sum_{j=1}^n A_j \operatorname{Im}(z_j\bar{z}'_j)\right)$$

with $A_1,\dots,A_n \in \mathbb{R}^m$ (regarding as column vectors), $\mathbf{A} = (A_1,\dots,A_n)$, and $\operatorname{rank}\mathbf{A} = m$. The sub-Riemannian geometry of $\mathbb{H}(\mathbf{A})$ is studied in [24] and the abnormal set can be very complicated on such group, which implies the heat kernel asymptotic on $\mathbb{H}(\mathbf{A})$ can also be very involved.

To show **HQLI**$(\mathbb{H}(\mathbf{A}), C_0)$ holds with the same $C_0$ in the first Heisenberg group case, we first consider the case $\mathbf{A} = \mathbb{I}_n$. In fact, if $\mathbf{A} = \mathbb{I}_n$, then it is not hard to show that $\mathbb{H}(\mathbb{I}_n) \cong \underbrace{\mathbb{H}^1\times\ldots\times\mathbb{H}^1}_{n\text{ times}}$ and it follows from Theorem 2 that $\mathbf{HQLI}(\mathbb{H}(\mathbb{I}_n), C_0)$ holds. For general $\mathbf{A}$, we construct the following group epimorphism from $\mathbb{H}(\mathbb{I}_n) \cong \underbrace{\mathbb{H}^1 \times \ldots \times \mathbb{H}^1}_{n \text{ times}}$ to $\mathbb{H}(\mathbf{A})$:

$$((z_1, u_1), \ldots, (z_n, u_n)) \mapsto \left( (z_1, \ldots, z_n), \sum_{k=1}^{n} u_k A_k \right),$$

which implies $\mathbf{HQLI}(\mathbb{H}(\mathbf{A}), C_0)$ and $\mathbf{HQLRI}(\mathbb{H}(\mathbf{A}), C_0)$ hold by Theorems 7 and 9 respectively. Finally we remark that the case $m = 1$ corresponds to the higher dimensional Heisenberg groups $\mathbb{H}^n := \mathbb{H}(1_n)$ with $1_n := (1, \ldots, 1)_{1\times n}$ and the nonisotropic Heisenberg groups (cf. [3]).

### 7.3. *H-type groups*

Recall that a step-two Carnot group is a H-type group if and only if $\mathbb{U} = \{U^{(1)}, \ldots, U^{(m)}\}$ satisfies

$$U^{(i)} U^{(j)} + U^{(j)} U^{(i)} = -2\delta_{ij} \mathbb{I}_q, \qquad \forall\, 1 \le i, j \le m. \tag{7}$$

See [16] for the original definition. On an H-type group $\mathbb{H}$, $\mathbf{HQLI}(\mathbb{H}, C_*)$ was established for some constant $C_* > 1$ in [10, 14], which implies $\mathbf{HQLRI}(\mathbb{H}, C_*)$ by Theorem 9.

### 7.4. *Generalized H-type groups*

If $\mathbb{I}_q$ is replaced with some symmetric, positive definite matrix $S$ in (7), then we call such group a generalized H-type group. See [2] for the definition. After decomposing $\mathbb{R}^q$ into the eigenspaces of $S$, we can construct a group epimorphism as we did in Subsection 7.2 and $\mathbf{HQLI}(\mathbb{G}, C_*)$ as well as $\mathbf{HQLRI}(\mathbb{G}, C_*)$ holds on $\mathbb{G}$. Details are left to the interested reader.

### 7.5. *Rank $k (k \ge 3)$ step-two Carnot groups*

The free step-two Carnot group with rank $k$, $N_{k,2} \cong \mathbb{R}^k \times \mathfrak{so}_k$, is defined by the group law:

$$(x, \Lambda) \cdot (x', \Lambda') = \left( x + x', \Lambda + \Lambda' + \frac{1}{2} x \wedge x' \right),$$

where $x \wedge x' := x(x')^{\mathrm{T}} - x' x^{\mathrm{T}}$ (regarding $x, x'$ as column vectors). Here $\mathfrak{so}_k$ denotes the linear space of all skew-symmetric $k \times k$ matrices, which can be identified with $\mathbb{R}^{\frac{k(k-1)}{2}}$. See [4, Section 14] for more details about free Carnot groups.

By the well-known Rothschild–Stein lifting theorem (cf. [28] or [4, Section 17]), combining with Theorems 7 and 9, if $\mathbb{G}$ has rank $k$ and $\mathbf{HQLI}(N_{k,2}, C_k)$ holds, then so do $\mathbf{HQLI}(\mathbb{G}, C_k)$ and $\mathbf{HQLRI}(\mathbb{G}, C_k)$. In conclusion, we reduce the study of the H.-Q. Li inequality on step-two groups to the one on free step-two Carnot groups.

### *Acknowledgements*

The author would like to thank Prof. H.-Q. Li for helpful discussions during the preparation of this work.

## References


[1] D. Bakry, F. Baudoin, M. Bonnefont, D. Chafaï, "On gradient bounds for the heat kernel on the Heisenberg group", *J. Funct. Anal.* **255** (2008), no. 8, p. 1905-1938.

[2] D. Barilari, L. Rizzi, "Sharp measure contraction property for generalized H-type Carnot groups", *Commun. Contemp. Math.* **20** (2018), no. 6, article no. 1750081 (24 pages).

[3] R. Beals, B. Gaveau, P. C. Greiner, "Hamilton-Jacobi theory and the heat kernel on Heisenberg groups", *J. Math. Pures Appl.* **79** (2000), no. 7, p. 633-689.

[4] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Monographs in Mathematics, Springer, 2007, xxvi+800 pages.

[5] O. Calin, D.-C. Chang, K. Furutani, C. Iwasaki, *Heat kernels for elliptic and sub-elliptic operators. Methods and techniques*, Applied and Numerical Harmonic Analysis, Birkhäuser, 2011, xviii+433 pages.

[6] J. Cygan, "Heat kernels for class 2 nilpotent groups", *Stud. Math.* **64** (1979), no. 3, p. 227-238.

[7] B. K. Driver, L. Gross, L. Saloff-Coste, "Growth of Taylor coefficients over complex homogeneous spaces", *Tôhoku Math. J.* **62** (2010), no. 3, p. 427-474.

[8] B. K. Driver, T. Melcher, "Hypoelliptic heat kernel inequalities on the Heisenberg group", *J. Funct. Anal.* **221** (2005), no. 2, p. 340-365.

[9] N. Eldredge, "Precise estimates for the subelliptic heat kernel on $H$-type groups", *J. Math. Pures Appl.* **92** (2009), no. 1, p. 52-85.

[10] ———, "Gradient estimates for the subelliptic heat kernel on $H$-type groups", *J. Funct. Anal.* **258** (2010), no. 2, p. 504-533.

[11] L. C. Evans, R. F. Gariepy, *Measure theory and fine properties of functions*, revised ed., Textbooks in Mathematics, CRC Press, 2015, xiv+299 pages.

[12] G. B. Folland, E. M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press; University of Tokyo Press, 1982, xii+285 pages.

[13] M. Gordina, L. Luo, "Logarithmic Sobolev inequalities on non-isotropic Heisenberg groups", *J. Funct. Anal.* **283** (2022), no. 2, article no. 109500 (33 pages).

[14] J.-Q. Hu, H.-Q. Li, "Gradient estimates for the heat semigroup on H-type groups", *Potential Anal.* **33** (2010), no. 4, p. 355-386.

[15] N. Juillet, "Geometric inequalities and generalized Ricci bounds in the Heisenberg group", *Int. Math. Res. Not.* (2009), no. 13, p. 2347-2373.

[16] A. Kaplan, "Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms", *Trans. Am. Math. Soc.* **258** (1980), no. 1, p. 147-153.

[17] M. Ledoux, "The geometry of Markov diffusion generators", *Ann. Fac. Sci. Toulouse, Math.* **9** (2000), no. 2, p. 305-366.

[18] H.-Q. Li, "Estimation optimale du gradient du semi-groupe de la chaleur sur le groupe de Heisenberg", *J. Funct. Anal.* **236** (2006), no. 2, p. 369-394.

[19] ———, "Estimations asymptotiques du noyau de la chaleur sur les groupes de Heisenberg", *C. R. Math. Acad. Sci. Paris* **344** (2007), no. 8, p. 497-502.

[20] ———, "Estimations optimales du noyau de la chaleur sur les groupes de type Heisenberg", *J. Reine Angew. Math.* **646** (2010), p. 195-233.

[21] ———, "The Carnot–Carathéodory distance on 2-step groups", 2021, https://arxiv.org/abs/2112.07822.

[22] H.-Q. Li, Y. Zhang, "Revisiting the heat kernel on isotropic and nonisotropic Heisenberg groups", *Commun. Partial Differ. Equations* **44** (2019), no. 6, p. 467-503.

[23] ———, "A complete answer to the Gaveau–Brockett problem", 2021, https://arxiv.org/abs/2112.07927.

[24] ———, "Sub-Riemannian geometry on some step-two Carnot groups", 2021, https://arxiv.org/abs/2102.09860.

[25] J. Milnor, "Curvatures of left invariant metrics on Lie groups", *Adv. Math.* **21** (1976), no. 3, p. 293-329.

[26] A. Nagel, F. Ricci, E. M. Stein, "Singular integrals with flag kernels and analysis on quadratic CR manifolds", *J. Funct. Anal.* **181** (2001), no. 1, p. 29-118.

[27] M.-K. von Renesse, K.-T. Sturm, "Transport inequalities, gradient estimates, entropy, and Ricci curvature", *Commun. Pure Appl. Math.* **58** (2005), no. 7, p. 923-940.

[28] L. P. Rothschild, E. M. Stein, "Hypoelliptic differential operators and nilpotent groups", *Acta Math.* **137** (1976), no. 3-4, p. 247-320.

[29] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, 1993, with the assistance of Timothy S. Murphy, xiv+695 pages.

[30] S. Thangavelu, *Harmonic analysis on the Heisenberg group*, Progress in Mathematics, vol. 159, Birkhäuser, 1998, xiv+192 pages.

[31] Y. Zhang, "A note on gradient estimates for the heat semigroup on nonisotropic Heisenberg groups", 2021, https://arxiv.org/abs/2109.12590.