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Bigness of the tangent bundles of projective bundles over curves

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Abstract. In this short article, we determine the bigness of the tangent bundle T_X of the projective bundle $X = \mathbb{P}_C(E)$ associated to a vector bundle E on a smooth projective curve C.

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1. Introduction

In this article, all varieties are defined over the field of complex numbers $\mathbb{C}$. After Mori's proof of the Hartshorne conjecture on ample tangent bundles [17], it has been asked to characterize a smooth projective variety X with certain positivity of its tangent bundle T_X. For example, a conjecture proposed by Campana and Peternell asks whether the homogeneous varieties are the only smooth Fano varieties X with nef T_X, and the conjecture is settled for dimension three [3], four [4, 8, 16] (see also [19, Corollary 4.4]), and five [10, 24]. Recently, a series of works done by Höring, Liu, Shao [6], and Höring, Liu [5] investigates smooth Fano varieties X with big T_X as follows.

Theorem 1 ([5, 6]). *Let X be a smooth Fano variety.*

(1) *If X has dimension 2, then T_X is big if and only if $(-K_X)^2 \geq 5$.*
(2) *If X has dimension 3 and Picard number 1, then T_X is big if and only if $(-K_X)^3 \geq 40$.*
(3) *If X has Picard number 1, and if X contains a rational curve with trivial normal bundle, then T_X is not big unless X is isomorphic to the quintic del Pezzo threefold.*

The second statement is extended to the following case.

Theorem 2 ([11]). *Let X be a smooth Fano variety of dimension 3 and Picard number 2. Then T_X is big if and only if $(-K_X)^3 \geq 34$.*

These results make use of a special divisor on the projective bundle $\mathbb{P}_X(T_X)$, called the total dual VMRT $\check{\mathscr{C}}$ (see [9,21]). In [6], they find a formula for $\check{\mathscr{C}}$, which can be written as follows in the case where X attains a conic bundle structure $X \to Y$.

$$[\check{\mathscr{C}}] \sim \zeta + \Pi^* K_{X/Y}$$

where $\Pi : \mathbb{P}_X(T_X) \to X$ is the projection and ζ is the tautological divisor on $\mathbb{P}_X(T_X)$. In other words, $\check{\mathscr{C}}$ arises as the divisor on $\mathbb{P}_X(T_X)$ corresponding to the natural subsheaf $T_{X/Y} \to T_X$ of rank 1.

In this article, we deal with a question on the bigness of T_X in the case of the projective bundles $X = \mathbb{P}_C(E)$ over a smooth projective curve C. When E has rank 2, X becomes a ruled surface, and the classification of X with big T_X is a consequence of some known facts. Indeed, if E is semi-stable, then $h^0(S^k T_X)$ is bounded above by a sum of dimensions of certain families of curves on X, whose bound can be obtained from a remark of [22] (see Remark 10). Otherwise, if E is unstable, then the bigness of T_X easily follows from the formula introduced above (cf. [11, Remark 2.4]). However, when the rank of E gets larger, we cannot apply the formula because $X \to C$ is not a conic bundle.

In the case of higher ranks, when E is unstable, we can find a rank 1 subsheaf of $S^k T_X$ instead of T_X to conclude that T_X is big. Also, when E is semi-stable, by computing an upper bound of $h^0(S^k T_X)$, we can determine the bigness of T_X according to the stability of E as follows.

Main Theorem. *Let C be a smooth projective curve and E be a vector bundle on C. Then the projective bundle $X = \mathbb{P}_C(E)$ has big tangent bundle T_X if and only if E is unstable or $C = \mathbb{P}^1$.*

The proof is divided into two parts; the case E is semi-stable (Theorem 9), and the case E is unstable (Theorem 15). The exceptional case $C = \mathbb{P}^1$ is explained in Remark 11. It is worth noting that the result is no longer true for varieties other than curves; there exist stable bundles E of rank 2 on $\mathbb{P}^2$ such that one of E gives big T_X whereas another choice of E gives not big T_X for $X = \mathbb{P}_{\mathbb{P}^2}(E)$ (see No. 24, 27, and 32 of Table 1 in [11]; No. 24 is the only case with non-big T_X, and see also [23]).

2. Preliminaries

Let X be a smooth projective variety of dimension $n > 0$ and V be a vector bundle of rank $r \geq 2$ on X. In this article, $\mathbb{P}_X(V)$ denotes the projective bundle with the projection $\Pi : \mathbb{P}_X(V) \to X$ in the sense of Grothendieck. That is, for the tautological line bundle $\mathscr{O}_{\mathbb{P}_X(V)}(1)$ on $\mathbb{P}_X(V)$, we have

$$\Pi_* \mathscr{O}_{\mathbb{P}_X(V)}(m) = \begin{cases} S^m V & \text{for } m \geq 0, \\ 0 & \text{for } m < 0 \end{cases}$$

where the 0-th power is taken to be $S^0 V = \mathscr{O}_X$ for convenience.

For an integer $m > -r$ and vector bundle W on X,

$$R^i \Pi_* (\Pi^* W \otimes \mathscr{O}_{\mathbb{P}_X(V)}(m)) = W \otimes R^i \Pi_* \mathscr{O}_{\mathbb{P}_X(V)}(m) = 0 \quad \text{for all } i > 0.$$

Thus, when $m > -r$,

$$H^i(\mathbb{P}_X(V), \Pi^* W \otimes \mathscr{O}_{\mathbb{P}_X(V)}(m)) \cong H^i(X, W \otimes \Pi_* \mathscr{O}_{\mathbb{P}_X(V)}(m)) \quad \text{for all } i \geq 0.$$

In particular, $H^0(\Pi^* W \otimes \mathscr{O}_{\mathbb{P}_X(V)}(-1)) = 0$.

2.1. *Bigness of Vector Bundle*

In this article, we define certain positivity of a vector bundle by the same positivity of the tautological line bundle on the projective bundle associated to the given vector bundle. The definition may differ, depending on the article; for example, there are distinct notions of bigness of vector bundles; L-big and V-big (see [2]).

Definition. *A vector bundle V is said to be* ample *(resp.,* nef, big, effective*, and* pseudo-effective*) on X if the tautological line bundle $\mathcal{O}_{\mathbb{P}_X(V)}(1)$ is ample (resp., nef, big, effective, and pseudo-effective) on $\mathbb{P}_X(V)$.*

Remark 3. Recall that a line bundle $L = \mathcal{O}_X(D)$ on X is big if and only if it satisfies one of the followings (see [13, Section 2.2]).

- $h^0(L^k) \sim k^n$ (which is the maximum possible).
- $mD \sim_{\text{lin}} A + E$ for some integer $m > 0$, ample divisor A, and effective divisor E on X.
- D lies in the interior of the closure $\overline{\text{Eff}}(X) \subseteq N^1(X)$ of the cone of effective divisors (as bigness is well-defined under numerical equivalence).

If L is a line bundle on X, then the following holds.

- L is big if and only if $L^{\otimes k}$ is big for some integer $k > 0$.

If V is a vector bundle on X, then the following holds (see also [14, Section 6.1]).

- V is big if and only if $h^0(S^k V) \sim k^{n+r-1}$ (which is the maximum possible). In particular, T_X is big if and only if $h^0(S^k T_X) \sim k^{2n-1}$.

We will denote by ζ the tautological divisor on $\mathbb{P}_X(V)$. Also, for a divisor B on X, we will denote by Bf the divisor $\Pi^* B$ on $\mathbb{P}_X(V)$.

Lemma 4 (cf. [6, Lemma 2.3]). *Let V be a vector bundle on a normal projective variety X. Let $k > 0$ and B be a divisor on X. If $k\zeta + (B - D)f$ is pseudo-effective for some big divisor D on X, then $k\zeta + Bf$ is big on $\mathbb{P}_X(V)$. In particular, if $S^k V \otimes \mathcal{O}_X(-D)$ is effective for some big divisor D on X, then V is big on X.*

Proof. Let $\zeta' = k\zeta + Bf$. Assume that $\zeta' - Df$ is pseudo-effective. Note that $mD \sim A + N$ for some $m > 0$, ample divisor A, and effective divisor N on X. By [12, Proposition 1.45], there exists an integer $n > 0$ such that $\zeta' + nAf$ is ample on X because ζ' is Π-ample. Then $(mn+1)\zeta'$ is written by a sum of ample and pseudo-effective divisors as

$$(mn+1)\zeta' = (\zeta' + nAf) + mn(\zeta' - Df) + nNf.$$

Thus $(mn+1)\zeta'$ is big, and it implies that $\zeta' = k\zeta + Bf$ is big on $\mathbb{P}_X(V)$. □

As an application of the lemma, we present a proof of the following fact.

Proposition 5. *Let X and Y be smooth projective varieties with big tangent bundles T_X and T_Y. Then the tangent bundle $T_{X\times Y}$ of $X \times Y$ is big.*

Proof. Let B and D be big and effective divisors on X and Y, respectively. As T_X and T_Y are big, there exist integers $m, n > 0$ such that $S^m T_X(-B)$ and $S^n T_Y(-D)$ are effective by Kodaira's Lemma. Note that $T_{X\times Y} = p^* T_X \oplus q^* T_Y$ for the natural projections $p : X \times Y \to X$ and $q : X \times Y \to Y$, and $p^* B + q^* D$ is a big divisor on $X \times Y$. Since $S^{m+n} T_{X\times Y}$ contains $S^m p^* T_X \otimes S^n q^* T_Y$ as a direct summand, we have

$$H^0(S^{m+n} T_{X\times Y} \otimes \mathcal{O}_{X\times Y}(-p^*B - q^*D)) \supseteq H^0(S^m p^* T_X \otimes \mathcal{O}_{X\times Y}(-p^*B) \otimes S^n q^* T_Y \otimes \mathcal{O}_{X\times Y}(-q^*D)) \neq 0.$$

Thus $S^{m+n} T_{X\times Y} \otimes \mathcal{O}_{X\times Y}(-(p^*B + q^*D))$ is effective, and hence $T_{X\times Y}$ is big by Lemma 4. □

2.2. *Stability of Vector Bundle*

In this article, stability is defined in the sense of Mumford and Takemoto. For the definitions introduced in this section, we add a mild condition (torsion-freeness) from the definitions in the reference [7, Chapter 1].

Let Y be a smooth projective variety and E be a torsion-free coherent sheaf on Y. Then there exists an open dense subset $U \subseteq Y$ such that $Y \setminus U$ has codimension at least two and $E|_U$ is locally free. The *rank* of E is defined by $\operatorname{rank} E = \operatorname{rank} E|_U$.

Definition. *Fix an ample divisor H on Y. For a torsion-free coherent sheaf E on Y, the H-*slope *of E is defined by*

$$\mu_H(E) = \frac{\deg_H E}{\operatorname{rank} E}$$

where the H-degree of E is defined by $\deg_H E = c_1(E).H^{n-1}$.

Let E be a torsion-free coherent sheaf of rank $r > 0$ on Y. Then E is said to be μ_H-stable *(resp.,* μ_H-semi-stable*) if for every coherent subsheaf F of E with* $0 < \operatorname{rank} F < r$,

$$\mu_H(F) < \mu_H(E) \quad (\textit{resp., } \mu(F) \le \mu(E)).$$

Also, E is said to be μ_H-unstable *if it is not μ_H-semi-stable. If there is no confusion in the choice of H, then we denote it by μ-stable (resp. μ-semi-stable, μ-unstable), or stable (resp. semi-stable, unstable) in the case where Y is a curve.*

Remark 6. The followings are some known facts on the μ-stability and slope of vector bundles E and F on Y. For the proofs, we may refer [7, Chapter 3].

- If E and F are μ-semi-stable and $\mu(E) < \mu(F)$, then $\operatorname{Hom}(F, E) = 0$.
- If E and F are μ-semi-stable, then $E \otimes F$ is μ-semi-stable.
- If E is μ-semi-stable, then $S^m E$ is μ-semi-stable for all $m > 0$.
- $\operatorname{rank}(S^m E) = \binom{m+r-1}{r-1}$, $c_1(S^m E) = c_1(E)^{\otimes \binom{m+r-1}{r}}$, and $\mu(S^m E) = m \cdot \mu(E)$.
- Assume that E fits into the following exact sequence of vector bundles on Y.

  $$0 \to F \to E \to Q \to 0$$

  If $\mu(F) = \mu(E) = \mu(Q)$, then E is μ-semi-stable if and only if both F and Q are μ-semi-stable.
- E is μ-semi-stable if and only if its dual $E^\vee$ is μ-semi-stable, and $\mu(E^\vee) = -\mu(E)$.

For a torsion-free coherent sheaf E on Y, there exists a canonical filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E,$$

which satisfies

- E_i / E_{i-1} is μ-semi-stable (also, torsion-free) for all $0 < i \le k$, and
- $\mu(E_{i+1}/E_i) < \mu(E_i/E_{i-1})$ for all $0 < i < k$.

This filtration is called the *Harder-Narasimhan filtration* of E. We call $F = E_1$ the *maximal destabilizing subsheaf* of E. When E is μ-unstable, we must have $\mu(F) > \mu(E)$. Also, it follows from the definition that E/F is torsion-free. In the case of curves $Y = C$, a coherent sheaf is torsion-free if and only if it is locally free, so we can further say that E/F is locally free.

3. Semi-Stable Case

In this section, let Y be a smooth projective variety of dimension $n > 0$, and fix an ample divisor H on Y. Let E be a vector bundle of rank $r > 0$ on Y. We denote by $X = \mathbb{P}_Y(E)$ the projective bundle associated to E with the projection $\pi : \mathbb{P}_Y(E) \to Y$, and by $\mathcal{O}_X(\xi)$ the tautological line bundle on X. Then, after taking symmetric powers to the relative Euler sequence

$$0 \to \mathcal{O}_X \to \pi^* E^\vee \otimes \mathcal{O}_X(\xi) \to T_{X/Y} \to 0,$$

we obtain the following exact sequence on X.

$$0 \to S^{m-1}\pi^* E^\vee \otimes \mathcal{O}_X((m-1)\xi) \to S^m \pi^* E^\vee \otimes \mathcal{O}_X(m\xi) \to S^m T_{X/Y} \to 0 \tag{1}$$

By pushing forward the exact sequence via π, we have the following exact sequence on Y.

$$0 \to S^{m-1}E^\vee \otimes S^{m-1}E \to S^m E^\vee \otimes S^m E \to \pi_* S^m T_{X/Y} \to 0$$

Lemma 7. *Let $X = \mathbb{P}_Y(E)$ and $\pi : \mathbb{P}_Y(E) \to Y$ be the projection. If E is μ-semi-stable, then $\pi_* S^m T_{X/Y}$ is a μ-semi-stable bundle of $\deg_H \pi_* S^m T_{X/Y} = 0$ on Y.*

Proof. Note that $S^m E^\vee \otimes S^m E$ is μ-semi-stable for all $m > 0$ because E is μ-semi-stable. Moreover, we have $\deg_H \pi_* S^m T_{X/Y} = 0$ due to the above exact sequence and

$$\deg_H(S^m E^\vee \otimes S^m E) = \operatorname{rank}(S^m E) \cdot \deg_H(S^m E^\vee) + \operatorname{rank}(S^m E^\vee) \cdot \deg_H(S^m E) = 0.$$

Since $\pi_* S^m T_{X/C}$ is a quotient of a μ-semi-stable bundle of the same H-slope, it is μ-semi-stable. □

Proposition 8. *Assume that T_Y is μ-semi-stable and $\deg_H T_Y < 0$. If E is μ-semi-stable, then the tangent bundle T_X of $X = \mathbb{P}_Y(E)$ is not big.*

Proof. Since the projection $\pi : X \to Y$ is a smooth morphism, there is the following exact sequence of vector bundles on X.

$$0 \to T_{X/Y} \to T_X \to \pi^* T_Y \to 0$$

From this exact sequence, we can find a bound of the dimension of the global sections of $S^k T_X$ as follows.

$$h^0(S^k T_X) \le \sum_{m=0}^{k} h^0(S^m T_{X/Y} \otimes S^{k-m} \pi^* T_Y)$$

By the assumption, $\left(S^{k-m} T_Y\right)^\vee$ is μ-semi-stable, and $\deg_H \left(S^{k-m} T_Y\right)^\vee > 0$. Due to Lemma 7, $\pi_* S^m T_{X/Y}$ is μ-semi-stable, and $\deg_H S^m T_{X/Y} = 0$. So we have

$$h^0(S^m T_{X/Y} \otimes S^{k-m} \pi^* T_Y) = h^0(\pi_* S^m T_{X/Y} \otimes S^{k-m} T_Y) = \dim \operatorname{Hom}(\left(S^{k-m} T_Y\right)^\vee, \pi_* S^m T_{X/Y}) = 0$$

whenever $0 \le m < k$. Thus

$$h^0(S^k T_X) \le h^0(S^k T_{X/Y}) = O(k^{n+r-2}).$$

That is, T_X is not big. □

Theorem 9. *Let C be a smooth projective curve of genus $g > 0$ and E be a vector bundle on C. If E is semi-stable, then the tangent bundle T_X of $X = \mathbb{P}_C(E)$ is not big.*

Proof. If $g \ge 2$, then $\deg T_C < 0$ and T_C is stable as every line bundle is stable. So T_X is not big by Proposition 8. Otherwise, if $g = 1$, then $T_C = \mathcal{O}_C$. By [1, Lemma 15], $h^0(S^m T_{X/C}) = h^0(\pi_* S^m T_{X/C})$ is bounded above by the number of indecomposable direct summands of $\pi_* S^m T_{X/C}$ as $\pi_* S^m T_{X/C}$ is a semi-stable bundle of degree 0 on C. Thus we have

$$h^0(S^m T_{X/C}) \le \operatorname{rank}(\pi_* S^m T_{X/C}) = \operatorname{rank}(S^m E^\vee \otimes S^m E) - \operatorname{rank}(S^{m-1} E^\vee \otimes S^{m-1} E).$$

After telescoping, we can conclude that

$$h^0(S^k T_X) \le \sum_{m=0}^{k} h^0(S^m T_{X/C}) \le \operatorname{rank}(S^k E^\vee \otimes S^k E) = O(k^{2r-2}).$$

That is, T_X is not big as X has dimension $r = \operatorname{rank} E$. □

Remark 10. Let E be a semi-stable bundle of rank 2 on a smooth projective curve C of genus $g > 0$. Then $T_{X/C}$ is a line bundle on $X = \mathbb{P}_C(E)$, and

$$S^m T_{X/C} = {T_{X/C}}^{\otimes m} \cong \mathcal{O}_X(2mC_0)$$

for some $\mathbb{Q}$-divisor C_0 on X with ${C_0}^2 = 0$. For a divisor $\mathfrak{b}$ on C, we denote by $\mathfrak{b}f$ the divisor $\pi^*\mathfrak{b}$ on X.

Let $D \sim 2mC_0 + \mathfrak{b}f$ for some $m > 0$. If $\deg \mathfrak{b} < 0$, then $h^0(\mathscr{O}_X(D)) = 0$ because there is no effective divisor D on X with $D^2 < 0$ (cf. [13, Section 1.5.A]). Assume that $\deg \mathfrak{b} = 0$ and D is effective. If D is integral, then it is known from [22, Remark in p. 122] that $h^0(\mathscr{O}_X(D)) = 1$. Otherwise, if D is not integral, then D is written in a sum of effective divisors linearly equivalent to $kC_0 + \mathfrak{a}f$ for some $k > 0$ and $\deg \mathfrak{a} = 0$. In this case, we can find an upper bound of $h^0(\mathscr{O}_X(2mC_0))$ by the same remark and the fact that E splits once we have $h^0(\mathscr{O}_X(C_0 + \mathfrak{a}f)) \geq 2$ for some divisor $\mathfrak{a}$ on C with $\deg \mathfrak{a} = 0$ [20, Lemma 5.4].

Remark 11. If $g = 0$ and E is semi-stable, then $C = \mathbb{P}^1$ and $E = \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus r}$ for some $a \in \mathbb{Z}$. Thus $X = \mathbb{P}_C(E) \cong \mathbb{P}^{r-1} \times \mathbb{P}^1$ and T_X is big by Lemma 5.

Remark 12. Using the result on curves, we can state the non-bigness of T_X under some special assumptions on Y and E (cf. Proposition 8). Assume that Y has a fibration $p : Y \to B$ over a smooth base B whose general fiber f is a smooth curve of genus $g > 0$. If $E|_f$ is semi-stable on a general fiber f, then the tangent bundle T_X of $X = \mathbb{P}_Y(E)$ is not big.

Suppose that T_X is big. Let $Z = \mathbb{P}_f(E|_f)$ and $\pi_f : Z \to f$ be the induced projection. Then, for general $Z = \mathbb{P}_f(E|_f)$, $T_X|_Z$ is big, and it implies that T_Z is big. Indeed, from the exact sequence

$$0 \to T_Z \to T_X|_Z \to N_{Z|X} \to 0,$$

we have $N_{Z|X} \cong {\pi_f}^* N_{f|Y} \cong {\pi_f}^* \mathscr{O}_f^{\oplus n-1} \cong \mathscr{O}_Z^{\oplus n-1}$, and it gives the following bound.

$$h^0(S^k T_X|_Z) \leq \sum_{m=0}^{k} h^0(S^m T_Z \otimes S^{k-m}(\mathscr{O}_f^{\oplus n-1})) = \sum_{m=0}^{k} O(k^{n-2}) \cdot h^0(S^m T_Z) = O(k^{n-1}) \cdot h^0(S^k T_Z)$$

However, as $E|_f$ is assumed to be semi-stable, $Z = \mathbb{P}_f(E|_f)$ cannot have big T_Z due to Theorem 9. By the contradiction, T_X is not big.

4. Unstable Case

In this section, we concentrate on the case where Y is a smooth projective curve C of genus $g \geq 0$. We continue to use the notation in the previous section.

Proposition 13. *If E is unstable, then $S^m E$ is* unstabilized *by a line subbundle for some $m > 0$; there exists a line subbundle L of $S^m E$ with $\mu(L) > \mu(S^m E)$.*

Proof. Let F be the maximal destabilizing subbundle of E. Then $\mu(F) > \mu(E)$ as E is unstable. Also, the quotient Q of E by F is locally free, so we obtain the following exact sequence of vector bundles on C.

$$0 \to F \to E \to Q \to 0$$

By taking symmetric powers to the exact sequence,

$$0 \to S^m F \to S^m E \to S^{m-1} E \otimes Q \to S^{m-2} E \otimes \wedge^2 Q \to \cdots \to S^{m-\operatorname{rank} Q} E \otimes \wedge^{\operatorname{rank} Q} Q \to 0,$$

we can observe that $S^m F$ is a subbundle of $S^m E$. Note that $\mu(S^m F) - \mu(S^m E) = m \cdot (\mu(F) - \mu(E)) > 0$.

According to [18], for each $m > 0$, there exists a line subbundle L of $S^m F$ satisfying

$$\mu(S^m F) - \mu(L) \leq \frac{\operatorname{rank}(S^m F) - \operatorname{rank}(L)}{\operatorname{rank}(S^m F) \cdot \operatorname{rank}(L)} \cdot g < g.$$

So we can find a line subbundle L of $S^m F$ such that

$$\mu(L) - \mu(S^m E) = \{\mu(S^m F) - \mu(S^m E)\} - \{\mu(S^m F) - \mu(L)\} > m \cdot (\mu(F) - \mu(E)) - g > 0$$

by taking $m > 0$ large enough. As $S^m F$ is a subbundle of $S^m E$, L is also a nonzero subbundle of $S^m E$. Hence we obtain a line subbundle L of $S^m E$ satisfying $\mu(L) > \mu(S^m E)$ for some $m > 0$. □

Lemma 14. *Let $m>0$ and $\mathfrak{b}$ be a divisor on C with $b=\deg\mathfrak{b}$. If $m\mu(E)+b>0$, then $m\xi+\mathfrak{b}f$ is big on $\mathbb{P}_C(E)$.*

Proof. Assume that $m\mu(E)+b>0$. If E is semi-stable, then $m\xi+\mathfrak{b}f$ is ample on $\mathbb{P}_C(E)$ by [15, Theorem 3.1], and hence $m\xi+\mathfrak{b}f$ is big on $\mathbb{P}_C(E)$.

Otherwise, if E is unstable, then there exists the maximal destabilizing subbundle F of E. Note that F is semi-stable and $\mu(F)>\mu(E)$. Let $\eta=\mathcal{O}_{\mathbb{P}_C(F)}(1)$ be the tautological line bundle on $\mathbb{P}_C(F)$. Then $m\eta+\mathfrak{b}f$ is big on $\mathbb{P}_C(F)$ by the previous argument, and so $km\eta+(k\mathfrak{b}-P)f$ is effective for some $k>0$ and $P\in C$ by Kodaira's Lemma. Thus $S^{km}F\otimes\mathcal{O}_C(k\mathfrak{b}-P)$ is effective, and from the inclusion

$$S^{km}F\otimes\mathcal{O}_C(k\mathfrak{b}-P)\to S^{km}E\otimes\mathcal{O}_C(k\mathfrak{b}-P),$$

we can observe that $S^{km}E\otimes\mathcal{O}_C(k\mathfrak{b}-P)$ is effective as well. That is, $km\xi+(k\mathfrak{b}-P)f$ is effective for some $k>0$ and $P\in C$. Therefore, $km\xi+k\mathfrak{b}f$ is big by Lemma 4, and it implies that $m\xi+\mathfrak{b}f$ is big on $\mathbb{P}_C(E)$. □

Theorem 15. *If E is unstable, then the tangent bundle T_X of $X=\mathbb{P}_C(E)$ is big.*

Proof. Since $E^\vee$ is also unstable, there exists an integer $m>0$ such that $S^mE^\vee$ has a line subbundle $L\to S^mE^\vee$ with $\mu(L)>\mu(S^mE^\vee)=-m\mu(E)$ by Proposition 13. By twisting $\mathcal{O}_X(m\xi)$ after pulling-back the inclusion $L\to S^mE^\vee$ via π, it gives a nonzero subbundle

$$\pi^*L\otimes\mathcal{O}_X(m\xi)\to\pi^*S^mE^\vee\otimes\mathcal{O}_X(m\xi). \tag{2}$$

Note that there cannot exist a nonzero morphism $\pi^*L\otimes\mathcal{O}_X(m\xi)\to S^{m-1}\pi^*E^\vee\otimes\mathcal{O}_X((m-1)\xi)$ as $\pi^*(S^{m-1}E^\vee\otimes L^{-1})\otimes\mathcal{O}_X(-\xi)$ never has a global section. Thus (2) induces a nonzero subsheaf

$$\pi^*L\otimes\mathcal{O}_X(m\xi)\to S^mT_{X/C} \tag{3}$$

via (1) as follows.

$$\begin{array}{ccccccccc} & & & & \pi^*L\otimes\mathcal{O}_X(m\xi) & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 & \to & S^{m-1}\pi^*E^\vee\otimes\mathcal{O}_X((m-1)\xi) & \to & S^m\pi^*E^\vee\otimes\mathcal{O}_X(m\xi) & \to & S^mT_{X/C} & \to & 0 \end{array}$$

Because $S^mT_{X/C}$ is a subbundle of S^mT_X, (3) induces a nonzero subsheaf

$$\pi^*L\otimes\mathcal{O}_X(m\xi)\to S^mT_X,$$

and hence $S^mT_X\otimes\mathcal{O}_X(-m\xi-\mathfrak{b}f)$ becomes effective for $\mathcal{O}_X(\mathfrak{b})=L$. Since $m\mu(E)=\mu(S^mE)+b>0$ for $b=\deg\mathfrak{b}=\mu(L)$, the divisor $m\xi+\mathfrak{b}f$ is big on X by Lemma 14. Thus, by applying Lemma 4, we can conclude that T_X is big. □

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