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Doubly slice knots and obstruction to Lagrangian concordance

Noeuds doublement bordant et obstruction aux concordances lagrangiennes

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Abstract. In this short note we observe that a result of Eliashberg and Polterovitch allows to use the doubly slice genus as an obstruction for a Legendrian knot to be a slice of a Lagrangian concordance from the trivial Legendrian knot with maximal Thurston–Bennequin invariant to itself. This allows to obstruct concordances from the Pretzel knot $P(3,-3,-m)$ when $m \geq 4$ to the unknot. Those examples are of interest because the Legendrian contact homology algebra cannot be used to obstruct such a concordance.

Résumé. Dans cette note, nous remarquons qu'un résultat d'Eliashberg et Polterovitch permet d'utiliser la notion de nœuds doublement bordant afin d'obstruer la possibilité pour un noeud legendrien d'apparaitre comme une tranche dans une concordance lagrangienne du noeud legendrien trivial d'invariant de Thurston–Bennequin maximal vers lui-même. Cela permet d'obstruer l'existence pour $m \geq 4$ de concordances du noeud pretzel $P(3,-3,-m)$ vers le noeud trivial. Ces exemples s'avèrent particulièrement intéressants car l'algèbre d'homologie de contact legendrienne ne permet pas d'obstruer une telle concordance.

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1. Introduction and results

Definition 1. *Let* Λ^-, Λ^+ *be two Legendrian knots in* S^3*. A Lagrangian concordance from* Λ^- *to* Λ^+ *is a Lagrangian submanifold* $\Sigma \subset \mathbb{R} \times S^3$ *diffeomorphic to a cylinder and such that for some* $T > 0$*,*

- $((-\infty, -T) \times S^3) \cap \Sigma = (-\infty, -T) \times \Lambda^-$ *and*
- $((T, +\infty) \times S^3) \cap \Sigma = (T, +\infty) \times \Lambda^+$

We denote $\Lambda^- \prec \Lambda^+$ *if there is a concordance from* Λ^- *to* Λ^+*.*

Up to now it is not known if this relation induces a partial order on the set of Legendrian isotopy classes of Legendrian knots. It is of course well defined, transitive and reflexive but it is not known if $\Lambda^- \prec \Lambda^+$ and $\Lambda^+ \prec \Lambda^-$ implies that Λ^- and Λ^+ are Legendrian isotopic. There is a related notion of *decomposable Lagrangian concordance* (denoted $\prec_{\text{dec}}$) that are concordances built from elementary combinatorial moves. The projection to $\mathbb{R}$ of such a concordance only has critical points of index 0 and 1 hence are ribbon. It follows from recent work of Agol [1] that if $\Lambda^- \prec_{\text{dec}} \Lambda^+$ and $\Lambda^+ \prec_{\text{dec}} \Lambda^-$ then Λ^- and Λ^+ are smoothly isotopic. Since Lagrangian concordances preserve both the Thurston–Bennequin number (TB) and the rotation number it implies that the relation $\prec_{\text{dec}}$ is a partial order on the class of Legendrian knots whose isotopy classes are simple.

In this note we are concerned with Legendrian knots Λ such that $\Lambda_0 \prec \Lambda \prec \Lambda_0$ where Λ_0 is the maximal TB Legendrian unknot. The Legendrian knot Λ_0 is fillable by an exact Lagrangian disc. Thus it follows from the fact that a Lagrangian disc can be perturbed to a symplectic disc (because it is an open manifold) and a combination of results of Rudolph [12] and Boileau–Orevkov [2] that the concordance realising $\Lambda_0 \prec \Lambda$ is ribbon. On the other hand in [4, Theorem 3.2] it is shown that the concordance realising $\Lambda \prec \Lambda_0$ cannot be decomposable, actually following the proof one observes that they prove that this concordance cannot be ribbon (which is also implied by a recent result of Zemke [15] or the above mentioned result from [1]). In [4] they give a list of Lagrangianly slice Legendrian knots for which the existence of a concordance to Λ_0 cannot be obstructed using Legendrian contact homology. The reason is that their Chekanov-Eliashberg algebras are stable tame isomorphic to the one of Λ_0 (see [6, Appendix A] for explicit computation). The family in question consists of some $TB = -1$ Legendrian realisations of pretzel knots $P(3, -3, -m)$, for $m \geq 4$. Denote Λ_m such a Legendrian knot, see Figure 1 for its front projection. As observed in [4], there is a Lagrangian concordance $\Lambda_0 \prec \Lambda_m$ for all $m \geq 4$. Such a concordance can be built starting with a pinch move along the red Reeb chord depicted in Figure 1. After some Reidemeister moves, one gets a cobordism from two unlinked copies of Λ_0 to Λ_m. Filling one of these Λ_0 by a Lagrangian disc leads to the desired concordance.

In this note we make the following observation.

Theorem 2. *There is no concordance* $\Lambda_m \prec \Lambda_0$*.*

Theorem 2 follows from a result of Eliashberg-Polterovitch [5] that we recall here (rephrased in the language that fits our purpose):

Theorem 3. *Let* D *be a filling of* Λ_0*, then* D *is Hamiltonian isotopic to the standard filling of* Λ_0*.*

Indeed this result leads to obstructions for Legendrian knots to exist as middle slices of a concordance from Λ_0 to Λ_0. The symplectic flavour of the result leads to obstructions from Legendrian contact homology and variations of it, this has been used in [3, 11] and [14] for instance. But the unknottedness of the disc in [5] has topological implications that to our knowledge have not been used in order to study Lagrangian cobordisms. Recall that a smooth knot K is *doubly slice* if there is an unknotted 2-sphere S in $\mathbb{R}^4$ such that $S \cap S^3 = K$. As an immediate corollary of Theorem 3 we obtain:

Theorem 4. *If there are concordances $\Lambda_0 \prec \Lambda$ and $\Lambda \prec \Lambda_0$, then Λ is doubly slice.*

Now the main computation of this note shows that:

Theorem 5. *The pretzel knots $P(3,-3,-m)$ for $m \geq 4$ are not doubly slice.*

This implies Theorem 2.

Remark 6. For the case $m = 3$, the knot Λ_3 has topological type $m(9_{46})$ and it has been shown previously by the first author in [3] that there is no concordance from Λ_3 to Λ_0. In this case computations using Legendrian contact homology or other modern tools for the case are still necessary because it turns out that the knot $m(9_{46})$ is doubly slice. The fact that the only one to be doubly slice in this family happens to be the only one to have a rich Legendrian contact homology algebra is both fortunate and puzzling.

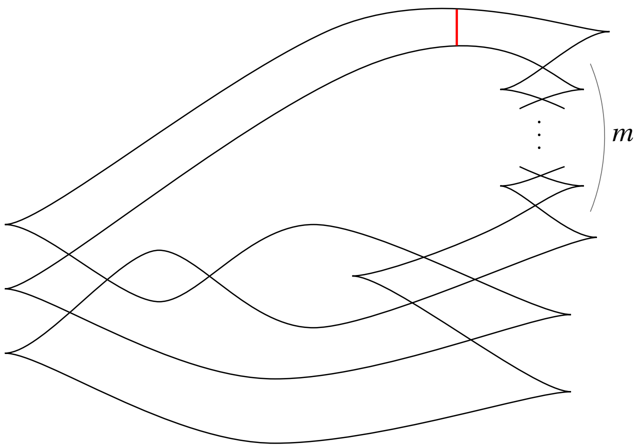

Figure 1. Front projection of the Legendrian representative Λ_m of the pretzel knot $P(3,-3,-m)$. The red line indicates a Reeb chord that can be pinched in order to produce a concordance $\Lambda_0 \prec \Lambda_m$.

Remark 7. In [9] the authors give an (almost complete) classification of doubly slice knots up to 12 crossings. From their work, the pretzel knots $P(3,-3,-4) = m(10_{140})$, $P(3,-3,-5) = 11n_{139}$ and $P(3,-3,-6) = 12n_{582}$ are not doubly slice.

2. Proof of Theorem 5

2.1. *Case $P(3,-3,-2k-1)$, $k \geq 2$*

In this case the pretzel knots are odd because they have an odd number of parameters which are all odd. McDonald in [10], using work of Issa and McCoy in [7], proves the following result:

Theorem 8 ([10, Theorem 3]). *For K an odd pretzel knot, the following are equivalent:*

- $\Sigma_2(S^3, K)$ *embeds in* S^4,
- K *is a doubly slice pretzel knot,*
- K *is a mutant of* $P(a,-a,a,-a,\ldots,a)$ *for some odd* a.

It is noted in [10] that it follows from the proof of [7, Theorem 1.11] that an odd pretzel knot which is a mutant of $P(a,-a,a,-a,\ldots,a)$ for some odd a has for parameters a permutation of $(a,-a,a,-a,\ldots,a)$. One concludes that the pretzel knots $P(3,-3,-2k-1)$ for $k \geq 2$ are not doubly slice.

2.2. *Case* $P(3,-3,-2k)$, $k \geq 2$

For these knots we obstruct doubly sliceness using the signature function as in [9] for $P(3,-3,-6)$. Given a knot K and a Seifert matrix A for K, the Levine-Tristram signature of K is a function $\sigma_K : S^1 \to \mathbb{Z}$ whose value at $\omega \in S^1$ is given by the signature of the matrix

$$(1-\omega)A + (1-\bar{\omega})A^T$$

where the signature is the number of positive eigenvalues minus the number of negative eigenvalues. If K is slice, then $\sigma_K(\omega) = 0$ away of the roots of the Alexander polynomial of K. Moreover, if K is doubly slice then $\sigma_K(\omega)$ vanishes for all $\omega \in S^1$. This is discussed in [9, Proposition 2.2] and follows from the fact that when K is doubly slice then there exists a Seifert matrix for K that is hyperbolic (see [13]). Seifert matrices for the knots $P(3,-3,-2k)$ have been computed in [8, Section 3]. We denote A_k a Seifert matrix for $P(3,-3,-2k)$. We have:

$$A_k = \begin{pmatrix} -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -k \end{pmatrix}.$$

It turns out that the Alexander polynomial of these knots is the same, given by $(t^2-t+1)^2$ and has for roots the sixth root of unity and its conjugate. One can then compute that $\sigma_K(\omega) \neq 0$ for $K = P(3,-3,-2k)$ and $\omega = e^{i\frac{\pi}{3}}$. Indeed a direct computation shows that the matrix $(1-\omega)A_k + (1-\bar{\omega})A_k^T$ has rank 5 and hence cannot have even signature (one can compute that its signature is actually -1). Thus K is not doubly slice which concludes both the proofs of Theorem 5 and Theorem 2.

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