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Optimal weak estimates for Riesz potentials

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Abstract. In this note we prove a sharp reverse weak estimate for Riesz potentials

$$\|I_s(f)\|_{L^{\frac{n}{n-s},\infty}} \geq \gamma_s v_n^{\frac{n-s}{n}} \|f\|_{L^1} \text{ for } 0 < f \in L^1(\mathbb{R}^n),$$

where $\gamma_s = 2^{-s}\pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n-s}{2})}{\Gamma(\frac{s}{2})}$. We also consider the behavior of the best constant $\mathscr{C}_{n,s}$ of weak type estimate for Riesz potentials, and we prove $\mathscr{C}_{n,s} = O(\frac{\gamma_s}{s})$ as $s \to 0$.

Keywords. Riesz potentials, sharp constant, optimal estimate.

2020 Mathematics Subject Classification. 42B20.

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1. introduction

The Riesz potentials(fractional integral operators) I_s, which play an important part in Analysis, are defined by

$$I_s(f)(x) = \gamma_s \int_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-s}} \mathrm{d}y,$$

where $0 < s < n$ and $\gamma_s = 2^{-s}\pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n-s}{2})}{\Gamma(\frac{s}{2})}$. Such operators were first systematically investigated by M.Riesz [11]. The (L^p, L^q)-boundedness of Riesz potentials were proved by G.Hardy and J. Littlewood [6] when $n = 1$ and by S. Sobolev [12] when $n > 1$. The $(L^1, L^{\frac{n}{n-s},\infty})$ -boundedness were obtained by A. Zygmund [16]. More precisely, they established the following theorem.

Theorem 1. *Let $0 < s < n$ and let p, q satisfy $1 \leq p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{s}{n}$, then when $p > 1$,*

$$\|I_s(f)\|_{L^q(\mathbb{R}^n)} \leq C(n,p,s)\|f\|_{L^p(\mathbb{R}^n)}.$$

And when $p = 1$,

$$\|I_s(f)\|_{L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : |I_s f| > \lambda\}|^{\frac{n-s}{n}} \leq C(n,s)\|f\|_{L^1(\mathbb{R}^n)}.$$

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The best constant in the (L^p, L^q) inequality when $p = \frac{2n}{n+s}$, $q = \frac{2n}{n-s}$ was precisely calculated by E. Lieb [8] (see also [4]), and E. Lieb and M. Loss also offered an upper bound of $C(n,p,s)$(see [9, Chapter 4]).

Although the best constant of (L^p, L^q) estimate for Riesz potentials has been studied for decades, to the best of the authors' knowledge there is no result about the best constant of $(L^1, L^{\frac{n}{n-s},\infty})$ estimate for Riesz potentials. In this paper, we will provide some estimates for the best constant of the weak type inequality.

In [14] (see multilinear case in [15]), the second author setted up the following limiting weak-type behavior for Riesz potentials,

$$\lim_{\lambda\to 0}\lambda|\{x\in\mathbb{R}^n : |I_s f|>\lambda\}|^{\frac{n-s}{n}} = \gamma_s v_n^{\frac{n-s}{n}}\|f\|_{L^1(\mathbb{R}^n)} \quad \text{for } 0<f\in L^1(\mathbb{R}^n),$$

which implies a reverse weak estimate

$$\|I_s(f)\|_{L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)} \ge \gamma_s v_n^{\frac{n-s}{n}}\|f\|_{L^1(\mathbb{R}^n)} \quad \text{for } 0<f\in L^1(\mathbb{R}^n), \tag{1}$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$. So a natural question that arises here is whether the constant $\gamma_s v_n^{\frac{n-s}{n}}$ is sharp? In the paper, we will give an affirmative answer.

Let $\mathscr{C}_{n,s}$ be the best constant such that the $(L^1, L^{\frac{n}{n-s},\infty})$ estimate holds for Riesz potentials, i.e.

$$\mathscr{C}_{n,s} = \sup_{f\in L^1(\mathbb{R}^n)}\frac{\|I_s(f)\|_{L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)}}{\|f\|_{L^1(\mathbb{R}^n)}}.$$

Then from (1), one can directly obtain a lower bound for $\mathscr{C}_{n,s}$,

$$\mathscr{C}_{n,s} \ge \gamma_s v_n^{\frac{n-s}{n}}.$$

Our another goal in this paper is to provide upper and lower bounds of $\mathscr{C}_{n,s}$ and to study the behavior of $\mathscr{C}_{n,s}$ as $s\to 0$. Our approach depends on the weak $L^{\frac{n}{n-s}}$ norm $|||\cdot|||_{L^{\frac{n}{n-s},\infty}}$ which is defined by

$$|||f|||_{L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)} = \sup_{0<|E|<\infty}|E|^{-\frac{1}{r}+\frac{n-s}{n}}\left(\int_E |f|^r \mathrm{d}x\right)^{\frac{1}{r}}, \quad 0<r<\frac{n}{n-s}.$$

The norm $|||\cdot|||_{L^{\frac{n}{n-s},\infty}}$ is equivalent to $\|\cdot\|_{L^{\frac{n}{n-s},\infty}}$. In fact there holds(see Exercise 1.1.12 in [5])

$$\|f\|_{L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)} \le |||f|||_{L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)} \le \left(\frac{n}{n-r(n-s)}\right)^{\frac{1}{r}}\|f\|_{L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)}. \tag{2}$$

Closely related to the Riesz potentials is the centered fractional maximal function, which is defined by

$$M_s f(x) = \sup_{r>0}\frac{1}{|B(x,r)|^{1-\frac{s}{n}}}\int_{B(x,r)}|f(y)|\mathrm{d}y,\ 0<s<n.$$

M_s satisfies the same (L^p, L^q) and $(L^1, L^{\frac{n}{n-s},\infty})$ inequality as I_s does, see [1] and [10]. For any positive function f it is easy to see $M_s(f) \le 1/\gamma(s) v_n^{\frac{s-n}{n}} I_s(f)$. Although the reverse inequality dose not hold in general, B.Muckenhoupt and R.Wheeden [10] proved that the two quantities are comparable in L^p norm($1<p<\infty$) when f is nonnegative.

Now let us state our main results. First of all we consider the weak estimate of $I_s(f)$ and $M_s(f)$ under the norm $|||\cdot|||_{L^{\frac{n}{n-s},\infty}}$. Surprisingly identities for the weak type estimate of Riesz potentials and fractional maximal function can be established, which implies the two quantities are comparable in $L^{\frac{n}{n-s},\infty}$ (quasi)norm when $f\in L^1(\mathbb{R}^n)$ is nonnegative.

Theorem 2. *Let $0<s<n$ and $f\in L^1(\mathbb{R}^n)$. When $1\le r<\frac{n}{n-s}$,*

$$|||I_s(f)|||_{L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)} \le \gamma_s v_n^{\frac{n-s}{n}}\left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}}\|f\|_{L^1(\mathbb{R}^n)},$$

and

$$|||M_s(f)|||_{L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)} = \left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}} \|f\|_{L^1(\mathbb{R}^n)}.$$

Moreover if $0 < f \in L^1(\mathbb{R}^n)$*, then*

$$|||I_s(f)|||_{L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)} = \gamma_s v_n^{\frac{n-s}{n}} \left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}} \|f\|_{L^1(\mathbb{R}^n)}.$$

Remark 3. In fact, from the proof one can obtain the reverse weak estimate holds when $0 < r < \frac{n}{n-s}$. More precisely when $0 < r < \frac{n}{n-s}$,

$$|||I_s(f)|||_{L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)} \geq \gamma_s v_n^{\frac{n-s}{n}} \left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}} \|f\|_{L^1(\mathbb{R}^n)}, \quad \text{if } 0 < f \in L^1(\mathbb{R}^n),$$

and

$$|||M_s(f)|||_{L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)} \geq \left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}} \|f\|_{L^1(\mathbb{R}^n)}, \quad \text{if } f \in L^1(\mathbb{R}^n).$$

Then we prove the following sharp reverse weak estimates for Riesz potentials.

Theorem 4. *Let* $0 < f \in L^1(\mathbb{R}^n)$*, then*

$$\|I_s(f)\|_{L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)} \geq \gamma_s v_n^{\frac{n-s}{n}} \|f\|_{L^1(\mathbb{R}^n)}.$$

And the equality holds when $f = (\frac{a}{b+|x-x_0|^2})^{\frac{n+s}{2}}$*, where* $a, b > 0$ *and* $x_0 \in \mathbb{R}^n$.

As a corollary of Theorem 2 and Theorem 4, we can obtain the following sharp reverse inequality.

Corollary 5. *Let* $f \in L^1(\mathbb{R}^n)$*, then*

$$\|M_s f\|_{L^{\frac{n}{n-s},\infty}} \geq \|f\|_{L^1}.$$

And the equality holds when $f(x) = h(|x-x_0|)$ *where h is a radial decreasing function.*

At last we offer an upper and a lower bound for $\mathscr{C}_{n,s}$, which implies that the behavior of the best constant $\mathscr{C}_{n,s}$ for small s is optimal, i.e. $\mathscr{C}_{n,s} = O(\frac{\gamma_s}{s}) = O(1)$ as $s \to 0$.

Theorem 6. *When* $n > 2$ *and* $0 < s < \frac{n-2}{4}$,

$$\gamma_s v_n^{\frac{n-s}{n}} \frac{n-2-4s}{2s(n-2-s)} \leq \mathscr{C}_{n,s} \leq \gamma_s v_n^{\frac{n-s}{n}} \frac{n}{s}.$$

Remark 7. Besides using the rearrangement inequality to obtain an upper bound $\gamma_s v_n^{\frac{n-s}{n}} \frac{n}{s}$, we can take the heat-diffusion semi-group as a tool (see the Appendix), which was used by E. Stein and J. Strömberg in [13] to study the $(L^1, L^{1,\infty})$ bound for centered maximal function, to obtain another upper bound which is equal to $O(\gamma_s v_n^{\frac{n-s}{n}} \frac{n}{s}) = O(1)$ as $(s, n) \to (0, \infty)$.

2. The identity for $I_s(f)$ and $M_s(f)$ in $|||\cdot|||_{L^{\frac{n}{n-s},\infty}}$

In this section, we will prove Theorem 2. Without loss of generality let us assume $\|f\|_{L^1(\mathbb{R}^n)} = 1$. Since $I_s(f) \leq I_s(|f|)$ and $r \geq 1$, using Minkowski's inequality one have for any measurable set E with $|E| < \infty$,

$$|E|^{-\frac{1}{r}+\frac{n-s}{n}} \left[\int_E |I_s f(x)|^r \mathrm{d}x\right]^{\frac{1}{r}} \leq \gamma_s |E|^{-\frac{1}{r}+\frac{n-s}{n}} \int_{\mathbb{R}^n} \left[\int_E \frac{\mathrm{d}x}{|x-y|^{(n-s)r}}\right]^{\frac{1}{r}} |f(y)| \mathrm{d}y. \tag{3}$$

Then by Hardy Littlewood rearrangement inequality, there holds

$$\int_E \frac{\mathrm{d}x}{|x-y|^{(n-s)r}} \leq \int_{E^*} \frac{\mathrm{d}x}{|x|^{(n-s)r}} = v_n^{\frac{n-s}{n}r} \frac{n}{n-(n-s)r} |E|^{1-\frac{n-s}{n}r}, \tag{4}$$

where E^* is the symmetric rearrangement of the set E, i.e. E^* is an open ball centered at the origin whose volume is $|E|$. Therefore by (3) and (4) one can obtain

$$|||I_s(f)|||_{L^{\frac{n}{n-s},\infty}} \le \gamma_s v_n^{\frac{n-s}{n}} \left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}} \|f\|_{L^1}.$$

Next, let us prove when $0 \le f \in L^1(\mathbb{R}^n)$ and $0 < r < \frac{n}{n-s}$,

$$|||I_s(f)|||_{L^{\frac{n}{n-s},\infty}} \ge \gamma_s v_n^{\frac{n-s}{n}} \left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}} \|f\|_{L^1}. \tag{5}$$

For any $\epsilon > 0$, choose R large enough such that $\int_{B_R(0)} f(y)dy = 1-\epsilon$. Let $E = B_{lR}(0)$. Since

$$\int_{B_R(0)} \frac{f(y)}{|x-y|^{n-s}}\mathrm{d}y \ge \int_{B_R(0)} \frac{f(y)}{(|x|+R)^{n-s}}\mathrm{d}y = (1-\epsilon)(|x|+R)^{s-n},$$

then

$$\begin{aligned}|||I_s(f)|||_{L^{\frac{n}{n-s},\infty}} &\ge \gamma_s |E|^{-\frac{1}{r}+\frac{n-s}{n}} \left[\int_E \left(\int_{B_R(0)} \frac{f(y)}{|x-y|^{n-s}}\mathrm{d}y\right)^r \mathrm{d}x\right]^{\frac{1}{r}} \\ &\ge \gamma_s |E|^{-\frac{1}{r}+\frac{n-s}{n}} (1-\epsilon) \left[\int_E \frac{\mathrm{d}x}{(|x|+R)^{(n-s)r}}\right]^{\frac{1}{r}} \\ &= \gamma_s v_n^{\frac{n-s}{n}} n^{\frac{1}{r}} (1-\epsilon) l^{-\frac{n}{r}+n-s} \left[\int_0^l \frac{t^{n-1}}{(t+1)^{(n-s)r}}\mathrm{d}t\right]^{\frac{1}{r}}.\end{aligned}$$

By the fact that this inequality holds for any $l > 0$, then letting $l \to \infty$, one obtain

$$|||I_s(f)|||_{L^{\frac{n}{n-s},\infty}} \ge \gamma_s v_n^{\frac{n-s}{n}} (1-\epsilon) \left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}},$$

which implies (5). And we finish the proof of the identity for Riesz potential.

For fractional maximal function M_s, since

$$\begin{aligned}M_s(f)(x) &\ge \frac{1}{v_n^{\frac{n-s}{n}}(|x|+R)^{n-s}} \int_{|y-x|\le R+|x|} |f(y)|\mathrm{d}y \\ &\ge \frac{1}{v_n^{\frac{n-s}{n}}(|x|+R)^{n-s}} \int_{|y|\le R} |f(y)|\mathrm{d}y = \frac{1-\epsilon}{v_n^{\frac{n-s}{n}}(|x|+R)^{n-s}},\end{aligned}$$

then one can use the same method to get

$$|||M_s(f)|||_{L^{\frac{n}{n-s},\infty}} \ge \left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}} \quad \text{when } 0 < r < \frac{n}{n-s}.$$

On the other hand,

$$|||M_s(f)|||_{L^{\frac{n}{n-s},\infty}} \le |||1/\gamma(s) v_n^{\frac{s-n}{n}} I_s(|f|)|||_{L^{\frac{n}{n-s},\infty}} = \left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}}.$$

Thus one can obtain the desired identity for M_s.

3. The sharp reverse weak estimate for I_s and M_s

In this section, first we prove the sharp reverse weak estimate for Riesz potentials I_s. By (2) and Theorem 2, there holds

$$\|I_s(f)\|_{L^{\frac{n}{n-s},\infty}} \ge \gamma_s v_n^{\frac{n-s}{n}} \|f\|_{L^1},\ 0 < f \in L^1(\mathbb{R}^n).$$

Next, we will prove that the equality can be attained by the function $g(x) = \left(\frac{a}{b+|x-x_0|^2}\right)^{\frac{n+s}{2}}$, where $a, b > 0$ and $x_0 \in \mathbb{R}^n$. Since the translation and dilation of g do not change the ratio $\|I_s(g)\|_{L^{\frac{n}{n-s},\infty}}/\|g\|_{L^1}$, we only need to consider $g(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n+s}{2}}$. In our calculus we will use the stereographic projection, so we will introduce some notations about the stereographic projection here.

The inverse stereographic projection $\mathscr{S}:\mathbb{R}^n\to\mathbb{S}^n\setminus\{S\}$, where $S=-e_{n+1}$ denotes the south-pole, is given by

$$(\mathscr{S}(x))_i=\frac{2x_i}{1+|x|^2},\quad i=1,\dots,n,\quad (\mathscr{S}(x))_{n+1}=\frac{1-|x|^2}{1+|x|^2}.$$

Correspondingly, the stereographic projection is given by $\mathscr{S}^{-1}:\mathbb{S}^n\setminus\{S\}\to\mathbb{R}^n$,

$$(\mathscr{S}^{-1}(\xi))_i=\frac{\xi_i}{1+\xi_{n+1}},\quad i=1,\dots,n.$$

And the Jacobian of the (inverse) stereographic projection are

$$\mathscr{J}_{\mathscr{S}}(x)=\left(\frac{2}{1+|x|^2}\right)^n\quad\text{and}\quad \mathscr{J}_{\mathscr{S}^{-1}}(\xi)=(1+\xi_{n+1})^{-n}.$$

By a change of variables,

$$\begin{aligned}\|g\|_{L^1}&=\int_{\mathbb{R}^n}\left(\frac{2}{1+|x|^2}\right)^{\frac{n+s}{2}}\mathrm{d}x=\int_{\mathbb{S}^n}\left(\frac{2}{1+|\mathscr{S}^{-1}(\xi)|^2}\right)^{\frac{s-n}{2}}\mathrm{d}\xi\\&=\int_{\mathbb{S}^n}(1+\xi_{n+1})^{\frac{s-n}{2}}\mathrm{d}\xi=\left|\mathbb{S}^{n-1}\right|\int_{-1}^{1}(1+t)^{\frac{s-2}{2}}(1-t)^{\frac{n-2}{2}}\mathrm{d}t\\&=\pi^{n/2}2^{\frac{s+n}{2}}\frac{\Gamma(s/2)}{\Gamma(\frac{s+n}{2})}.\end{aligned}\tag{6}$$

Denote

$$c_{n,s}=\pi^{n/2}2^{\frac{s+n}{2}}\frac{\Gamma(s/2)}{\Gamma(\frac{s+n}{2})}.$$

Since

$$|\mathscr{S}^{-1}(\xi)-\mathscr{S}^{-1}(\eta)|^2=\mathscr{J}_{\mathscr{S}^{-1}}(\xi)^{\frac{1}{n}}|\xi-\eta|^2\mathscr{J}_{\mathscr{S}^{-1}}(\eta)^{\frac{1}{n}},\quad\text{for any }\ \xi,\eta\in\mathbb{S}^n,$$

and

$$\int_{\mathbb{S}^n}\frac{\mathrm{d}\eta}{|\xi-\eta|^{n-s}}=\frac{2^s\pi^{n/2}\Gamma(s/2)}{\Gamma(\frac{n+s}{2})}=\frac{c_{n,s}}{2^{\frac{n-s}{2}}},\quad\text{for any }\ \eta\in\mathbb{S}^n\text{ (see [5, D.4])},$$

one can obtain

$$\begin{aligned}I_s(g)(x)&=\gamma(s)\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-s}}\left(\frac{2}{1+|y|^2}\right)^{\frac{n+s}{2}}\mathrm{d}y\\&=\gamma(s)\int_{\mathbb{S}^n}\frac{1}{|\mathscr{S}^{-1}(\xi)-\mathscr{S}^{-1}(\eta)|^{n-s}}\left(\frac{2}{1+|\mathscr{S}^{-1}(\eta)|^2}\right)^{\frac{s-n}{2}}\mathrm{d}\eta\\&=\gamma(s)\int_{\mathbb{S}^n}\frac{1}{|\xi-\eta|^{n-s}|\mathscr{J}_{\mathscr{S}^{-1}}(\xi)|^{\frac{n-s}{2n}}|\mathscr{J}_{\mathscr{S}^{-1}}(\eta)|^{\frac{n-s}{2n}}}\left(\frac{2}{1+|\mathscr{S}^{-1}(\eta)|^2}\right)^{\frac{s-n}{2}}\mathrm{d}\eta\\&=\gamma(s)\int_{\mathbb{S}^n}\frac{\mathrm{d}\eta}{|\xi-\eta|^{n-s}}(1+\xi_{n+1})^{\frac{n-s}{2}}=\gamma(s)\frac{c_{n,s}}{(1+|x|^2)^{\frac{n-s}{2}}}.\end{aligned}$$

Thus for any $\lambda>0$,

$$|\{I_s(g)>\lambda\}|=v_n\left(\left(\frac{\gamma(s)c_{n,s}}{\lambda}\right)^{\frac{2}{n-s}}-1\right)^{\frac{n}{2}}.\tag{7}$$

Therefore combining (6) and (7) one has

$$\frac{\|I_s(g)\|_{L^{\frac{n}{n-s},\infty}}}{\|g\|_{L^1}}=v_n^{\frac{n-s}{n}}\sup_{\lambda>0}\left(\gamma(s)^{\frac{2}{n-s}}-\left(\frac{\lambda}{c_{n,s}}\right)^{\frac{2}{n-s}}\right)^{\frac{n-s}{2}}=\gamma(s)v_n^{\frac{n-s}{n}}.$$

Next let us prove the sharp reverse weak estimate for M_s. By the identity in Theorem 2 for M_s and (2) one can find for any $f \in L^1$,

$$\|M_s(f)\|_{L^{\frac{n}{n-s},\infty}} \geq \|f\|_{L^1}. \tag{8}$$

On the other hand, since $M_s(f) \leq 1/\gamma(s) v_n^{\frac{s-n}{n}} I_s(f)$ and we already proved that the function $g = \left(\frac{a}{b+|x-x_0|^2}\right)^{\frac{n+s}{2}}$ satisfies $\|I_s(g)\|_{L^{\frac{n}{n-s},\infty}} = \gamma(s) v_n^{\frac{n-s}{n}} \|g\|_{L^1}$, then by (8) the following equality holds

$$\|M_s(g)\|_{L^{\frac{n}{n-s},\infty}} = \|g\|_{L^1}. \tag{9}$$

In fact, one can prove that (9) holds for any L^1 function $f(x) = h(|x-x_0|)$, where h is a radial decreasing function, by using an approach from [2]. First assume $\|f\|_{L^1} = 1$. Let δ_{x_0} denote the Dirac delta mass placed at x_0. It is easy to check that

$$M(\delta_{x_0})(x) = \frac{1}{|B(x,|x|)|},$$

where M is the centered Hardy–Littlewood maximal function. Hence, for every $\lambda > 0$, there holds

$$\lambda |\{x : M(\delta_{x_0})(x) > \lambda\}|^{\frac{n}{n-s}} = 1.$$

Since h is a radial decreasing function with $\|h\|_{L^1} = 1$, then by Lemma 2.1 in [2], one has

$$M(f)(x) \leq M(\delta_{x_0})(x) \text{ for every } x \in \mathbb{R}^n.$$

Then for any $r > 0$ and $x \in \mathbb{R}^n$,

$$\frac{1}{|B(x,r)|^{1-\frac{s}{n}}} \int_{B(x,r)} f(y)\mathrm{d}y \leq \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\mathrm{d}y \|f\|_{L^1}^{\frac{s}{n-s}}\right)^{\frac{n-s}{n}} \leq (M(\delta_{x_0})(x))^{\frac{n-s}{n}},$$

which implies that

$$\|M_s f\|_{L^{\frac{n}{n-s},\infty}} \leq 1 = \|f\|_{L^1}. \tag{10}$$

Combining this inequality with (8), one can obtain the desired result for M_s.

What is noteworthy at the end of the section is that this result is also true for the centered Hardy–Littlewood maximal function, that is because, using the same method, one can prove (10) when $s = 0$, i.e. (10) is true for the centered Hardy–Littlewood maximal function. On the other hand, using the limiting weak type behavior for the maximal function in [7], (8) is also true for the centered Hardy–Littlewood maximal function.

4. The upper and lower bounds of $\mathscr{C}_{n,s}$

In this section, we will provide an upper and a lower bound for $\mathscr{C}_{n,s}$. Using Theorem 2 and (2), we can get an upper bound

$$\mathscr{C}_{n,s} \leq \gamma_s \frac{n}{s} v_n^{\frac{n-s}{n}}.$$

To obtain the lower bound, we will use the following formula (see [9, Section 5.10]). Let $0 < \alpha < n$, $0 < s < n$ and $\alpha + s < n$, then

$$\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-s}} \frac{1}{|y|^{n-\alpha}} \mathrm{d}y = C_{n,\alpha,s} \frac{1}{|x|^{n-s-\alpha}} \tag{11}$$

with

$$C_{n,\alpha,s} = \pi^{\frac{n}{2}} \frac{\Gamma(\frac{s}{2})\Gamma(\frac{\alpha}{2})\Gamma(\frac{n-s-\alpha}{2})}{\Gamma(\frac{n-s}{2})\Gamma(\frac{n-\alpha}{2})\Gamma(\frac{s+\alpha}{2})}.$$

Now assume $n-2 > 4s$. Choose $f(y) = \frac{1}{|y|^{n-2}} \chi_{(|y| \leq 1)}$ and let us prove

$$\|I_s f\|_{\frac{n}{n-s},\infty} \geq \gamma_s \frac{v_n^{\frac{n-s}{n}}}{s} \frac{n-2-4s}{2(n-2-s)} \|f\|_{L^1}.$$

Since $|x| \le \frac{1}{2}$, $|y| > 1$ implies $|y - x| \ge \frac{|y|}{2}$, using (11) with $\alpha = 2$ one have

$$\begin{aligned}\frac{1}{\gamma_s} I_s(f)(x) &= \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-s}} f(y)\mathrm{d}y = \int_{|y|\le 1} \frac{1}{|x-y|^{n-s}} \frac{1}{|y|^{n-2}}\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-s}} \frac{1}{|y|^{n-2}}\mathrm{d}y - \int_{|y|>1} \frac{1}{|x-y|^{n-s}} \frac{1}{|y|^{n-2}}\mathrm{d}y \\ &\ge \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-s}} \frac{1}{|y|^{n-2}}\mathrm{d}y - \int_{|y|>1} \frac{2^{n-s}}{|y|^{2n-2-s}}\mathrm{d}y \\ &= \frac{c}{|x|^{n-s-2}} - d, \end{aligned} \tag{12}$$

where

$$c = \frac{4\pi^{n/2}}{(n-s-2)\Gamma(n/2-1)s} \quad \text{and} \quad d = \frac{2^{n-s+1}\pi^{n/2}}{(n-s-2)\Gamma(n/2)}.$$

Choose $\lambda_0 = \gamma_s(2^{n-s-2}c - d)$, since $\frac{c}{d} = \frac{n-2}{s}\frac{1}{2^{n-s}} > \frac{1}{2^{n-s-2}}$, then λ_0 is positive. Thus by (12), there holds

$$|\{I_s f > \lambda_0\}| \ge \left|\left\{|x| \le 1/2, \frac{c}{|x|^{n-s-\alpha}} - d > \frac{\lambda_0}{\gamma_s}\right\}\right| = v_n(\frac{1}{2})^n. \tag{13}$$

Using the fact $\|f\|_{L^1(\mathbb{R}^n)} = \frac{\omega_{n-1}}{2}$ and (13) one can obtain

$$\frac{\|I_s f\|_{\frac{n}{n-s},\infty}}{\|f\|_{L^1}} \ge \frac{\lambda_0 |\{I_s f > \lambda_0\}|^{\frac{n-s}{n}}}{\|f\|_{L^1}} = \lambda_0 v_n^{\frac{n-s}{n}} \frac{\Gamma(n/2)}{2^{n-s}\pi^{n/2}} = \gamma_s \frac{v_n^{\frac{n-s}{n}}}{s} \frac{n-2-4s}{2(n-2-s)}.$$

So we complete the proof of Theorem 6.

Appendix

In this Appendix, we give an alternative approach to prove the $(L^1, L^{\frac{n}{n-s},\infty})$ estimate for Riesz potentials, and at the same time this approach also provide an upper bound for $\mathscr{C}_{n,s}$, which have the same behavior with $\gamma_s v_n^{(n-s)/n} n/s$ as $(s,n) \to (0,\infty)$. First, we state a lemma (see [13, Section 3], also see the Hopf abstract maximal ergodic theorem in [3]) about the weak estimate of the average of the heat-diffusion semi-group $T^t(f) = P_t * f$, where

$$P_t = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{t}}.$$

Lemma 8. *For any $f \in L^1(\mathbb{R}^n)$, there holds*

$$\left|\left\{x \in \mathbb{R}^n : \sup_{s>0} \frac{1}{s}\int_0^s P_t f(x)\mathrm{d}t > \lambda\right\}\right| \le \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}, \quad \lambda > 0.$$

Now let prove the $(L^1, L^{\frac{n}{n-s},\infty})$ estimate for Riesz potentials $I_s(f)$, which also can be presented by the following formula related to $T^t(f)$,

$$I_s(f)(x) = \frac{1}{\Gamma(s/2)}\int_0^\infty t^{\frac{s}{2}-1} P_t * f(x)\mathrm{d}t.$$

We divide the integral into two parts

$$\int_0^\infty t^{\frac{s}{2}-1} P_t * f(x)\mathrm{d}t = J_1(f)(x) + J_2(f)(x),$$

where

$$J_1(f)(x) = \int_0^R t^{\frac{s}{2}-1} P_t * f(x)\mathrm{d}t,$$
$$J_2(f)(x) = \int_R^\infty t^{\frac{s}{2}-1} P_t * f(x)\mathrm{d}t,$$

for some R to be determined later.

Denote $\mathscr{M}^0 f(x) = \sup_{r>0} \frac{1}{r}\int_0^r P_t * f(x)dt$, then we have

$$\begin{aligned}
J_1(f)(x) &= \sum_{i=1}^{\infty}\int_{2^{-i}R}^{2^{-i+1}R} t^{\frac{s}{2}-1} P_t * f(x)\mathrm{d}t \\
&\le \sum_{i=1}^{\infty}\int_{2^{-i}R}^{2^{-i+1}R} (2^{-i}R)^{\frac{s}{2}-1} P_t * f(x)\mathrm{d}t \\
&\le \sum_{i=1}^{\infty} (2^{-i}R)^{\frac{s}{2}-1} 2^{-i+1}R\left(\frac{1}{2^{-i+1}R}\int_0^{2^{-i+1}R} P_t * f(x)\mathrm{d}t\right) \\
&\le 2R^{\frac{s}{2}} \frac{2^{-\frac{s}{2}}}{1-2^{-\frac{s}{2}}}\mathscr{M}^0 f(x). \qquad (14)
\end{aligned}$$

On the other hand, by direct computation, we obtain that

$$\begin{aligned}
J_2(f)(x) &\le \int_R^{\infty} t^{\frac{s}{2}-1}\|P_t\|_{L^\infty}\|f\|_{L^1}\mathrm{d}t \\
&\le \frac{2}{n-s}(4\pi)^{-\frac{n}{2}} R^{\frac{s}{2}-\frac{n}{2}}\|f\|_{L^1}. \qquad (15)
\end{aligned}$$

Combining (14) and (15), we obtain that

$$I_s(f)(x) \le \frac{1}{\Gamma(s/2)}\left(2R^{\frac{s}{2}}\frac{2^{-\frac{s}{2}}}{1-2^{-\frac{s}{2}}}\mathscr{M}^0 f(x) + \frac{2}{n-s}(4\pi)^{-\frac{n}{2}}R^{\frac{s}{2}-\frac{n}{2}}\|f\|_{L^1}\right) \qquad (16)$$

for all $R>0$. The choice of

$$R = \left(\frac{(4\pi)^{-\frac{n}{2}}\|f\|_{L^1}}{\frac{s}{2^{\frac{s}{2}-1}-1}\mathscr{M}^0 f(x)}\right)^{\frac{2}{n}}$$

minimizes the right side of the expression in (16). Thus

$$I_s(f)(x) \le \tau_s(\mathscr{M}^0 f(x))^{\frac{n-s}{n}}\|f\|_{L^1}^{\frac{s}{n}}, \qquad (17)$$

where

$$\tau_s = 2(4\pi)^{-\frac{s}{2}}(2^{\frac{s}{2}}-1)^{\frac{s-n}{n}}\frac{n}{n-s}\left(\frac{1}{s}\right)^{\frac{s}{n}}\frac{1}{\Gamma(s/2)}.$$

Now using Lemma 8 one can see that

$$\begin{aligned}
\lambda|\{I_s f>\lambda\}|^{\frac{n-s}{n}} &\le \lambda\left|\left\{\tau_s\left(\mathscr{M}^0 f(x)\right)^{\frac{n-s}{n}}\|f\|_{L^1}^{\frac{s}{n}}>\lambda\right\}\right|^{\frac{n-s}{n}} \\
&\le \lambda\left[\left(\frac{\tau_s\|f\|_{L^1}^{\frac{s}{n}}}{\lambda}\right)^{\frac{n}{n-s}}\|f\|_{L^1}\right]^{\frac{n-s}{n}} \\
&\le \tau_s\|f\|_{L^1}.
\end{aligned}$$

Notice that

$$2^{\frac{s}{2}}-1 > \frac{\ln 2}{2}s \quad \text{for } s>0,$$

thus,

$$\tau_s \le \frac{2}{\ln 2}\left(\frac{1}{4\pi}\right)^{-\frac{s}{2}}\frac{1}{\Gamma(\frac{s}{2}+1)}\frac{n}{n-s}.$$

So by this approach, one can obtain that $\mathscr{C}_{n,s} \le \frac{2}{\ln 2}(\frac{1}{4\pi})^{-\frac{s}{2}}\frac{1}{\Gamma(\frac{s}{2}+1)}\frac{n}{n-s}$ and it is easy to check that when $(s,n)\to(0,\infty)$,

$$\frac{2}{\ln 2}\left(\frac{1}{4\pi}\right)^{-\frac{s}{2}}\frac{1}{\Gamma(\frac{s}{2}+1)}\frac{n}{n-s} = O\Big(\gamma_s v_n^{\frac{n-s}{n}}\frac{n}{s}\Big).$$

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