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Homological dimension based on a class of Gorenstein flat modules

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Abstract. In this paper, we study the relative homological dimension based on the class of projectively coresolved Gorenstein flat modules (PGF-modules), that were introduced by Saroch and Stovicek in [26]. The resulting PGF-dimension of modules has several properties in common with the Gorenstein projective dimension, the relative homological theory based on the class of Gorenstein projective modules. In particular, there is a hereditary Hovey triple in the category of modules of finite PGF-dimension, whose associated homotopy category is triangulated equivalent to the stable category of PGF-modules. Studying the finiteness of the PGF global dimension reveals a connection between classical homological invariants of left and right modules over the ring, that leads to generalizations of certain results by Jensen [24], Gedrich and Gruenberg [17] that were originally proved in the realm of commutative Noetherian rings.

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Introduction

The concept of G-dimension for commutative Noetherian rings that was introduced by Auslander and Bridger in [1] has been extended to modules over any ring R through the notion of a Gorenstein projective module. Such a module is, by definition, a syzygy of an acyclic complex of projective modules which remains acyclic when applying the functor $\mathrm{Hom}_R(\_\_, P)$ for any projective module P. The modules of finite Gorenstein projective dimension are defined in the standard way, using resolutions by Gorenstein projective modules. A Gorenstein flat module is a syzygy of an acyclic complex of flat modules which remains acyclic when applying the functor $I \otimes_R \_\_$ for any injective right module I. The modules of finite Gorenstein flat dimension are then defined using resolutions by Gorenstein flat modules. The standard reference for these notions is Holm's paper [22]. The relation between Gorenstein projective and Gorenstein flat modules

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remains somehow mysterious in general. As shown in [22], all Gorenstein projective modules are Gorenstein flat if the ground ring is right coherent and has finite left finitistic dimension (i.e. if there is an upper bound on the projective dimension of all modules that have finite projective dimension).

The projectively coresolved Gorenstein flat modules (PGF-modules, for short) were defined by Saroch and Stovicek in [26]; these are the syzygies of the acyclic complexes of projective modules that remain acyclic when applying the functor $I \otimes_R \_\_$ for any injective right module I. It is clear that PGF-modules are Gorenstein flat. As shown in [26, Theorem 4.4], PGF-modules are also Gorenstein projective. A schematic presentation of the classes $\mathtt{GProj}(R)$, $\mathtt{GFlat}(R)$ and $\mathtt{PGF}(R)$ of Gorenstein projective, Gorenstein flat and PGF-modules respectively is given below

$$\begin{array}{ccccccc}
\mathtt{GProj}(R) & & & & \mathtt{GFlat}(R) & & \\
& \nwarrow & & \nearrow & & \nwarrow & \\
& & \mathtt{PGF}(R) & & & & \mathtt{Flat}(R) \\
& & & \nwarrow & & \nearrow & \\
& & & & \mathtt{Proj}(R) & &
\end{array}$$

Here, $\mathtt{Proj}(R)$ and $\mathtt{Flat}(R)$ denote the classes of projective and flat modules respectively and all arrows are inclusions. Moreover, the class $\mathtt{Proj}(R)$ of projective modules is the intersection $\mathtt{PGF}(R) \cap \mathtt{Flat}(R)$ and all classes pictured above are projectively resolving; in fact, $\mathtt{GFlat}(R)$ is the smallest projectively resolving class of modules that contains both $\mathtt{PGF}(R)$ and $\mathtt{Flat}(R)$; these assertions are proved in [26].

In this paper, we study the relative homological dimension which is based on the class $\mathtt{PGF}(R)$ and define the PGF-dimension $\text{PGF-dim}_R M$ (independently introduced in [3]) of a module M as the minimal length of a resolution of M by PGF-modules (provided that such a resolution exists). The resulting class $\overline{\mathtt{PGF}}(R)$ of modules of finite PGF-dimension has many of the standard properties that one would expect. In particular, it is closed under direct sums, direct summands and has the 2-out-of-3 property for short exact sequences of modules. The PGF-dimension is a refinement of the ordinary projective dimension, whereas the Gorenstein projective dimension is a refinement of the PGF-dimension. In other words, if M is a module of finite projective dimension (resp. of finite PGF-dimension), then M has finite PGF-dimension (resp. finite Gorenstein projective dimension) and $\text{PGF-dim}_R M = \text{pd}_R M$ (resp. $\text{Gpd}_R M = \text{PGF-dim}_R M$). When restricted to the class $\mathtt{Flat}(R)$ of flat modules, the PGF-dimension coincides with the projective dimension. The modules of finite PGF-dimension can be approximated by modules of finite projective dimension and PGF-modules, in analogy with the case of modules of finite Gorenstein projective dimension. In particular, this leads to a description, up to triangulated equivalence, of the stable category of PGF-modules modulo projective modules, as the homotopy category of the exact model structure which is associated with a Hovey triple in the category $\overline{\mathtt{PGF}}(R)$. Using the analogous approximations of Gorenstein flat modules by PGF-modules and flat modules, that were obtained by Saroch and Stovicek in [26], we describe a similar Hovey triple in the category $\mathtt{GFlat}(R)$. Therefore, in order to realize the stable category of PGF-modules as the homotopy category of a Quillen model structure, it is sufficient to work on either subcategory $\overline{\mathtt{PGF}}(R)$ or $\mathtt{GFlat}(R)$ of the module category.

In order to present an application of the notion of PGF-dimension studied in this paper, we consider the invariants $\text{silp}\, R$ and $\text{spli}\, R$, which are defined as the suprema of the injective lengths of projective modules and the projective lengths of injective modules, respectively. It is easily seen that these invariants are equal, if they are both finite. Nevertheless, as Gedrich and Gruenberg point out in [17], it is not clear whether the finiteness of one of these implies the finiteness of the other, i.e. whether we always have an equality $\text{silp}\, R = \text{spli}\, R$. In the special case where R is an

Artin algebra, the equality $\operatorname{silp} R = \operatorname{spli} R$ is equivalent to the Gorenstein Symmetry Conjecture in representation theory; cf. [2, Conjecture 13], [4, §11] and [5, Chapter VII].

The study of the finiteness of the PGF global dimension reveals a connection between the silp and spli invariants for left and right modules over any ring, which may be itself used in order to show that:

If both $\operatorname{spli} R$ *and* $\operatorname{spli} R^{op}$ *are finite, then* $\operatorname{silp} R = \operatorname{spli} R$ *and* $\operatorname{silp} R^{op} = \operatorname{spli} R^{op}$.

Using the Hopf algebra structure of the group algebra kG of a group G with coefficients in a commutative ring k, Gedrich and Gruenberg proved in [17] that $\operatorname{silp} kG \leq \operatorname{spli} kG$, in the special case where the commutative ring k is Noetherian of finite self-injective dimension. It follows from the result displayed above that we actually have an inequality $\operatorname{silp} R \leq \operatorname{spli} R$ for any ring R which is isomorphic with its opposite ring R^{op}. In particular, the inequality holds for group algebras of groups over *any* commutative coefficient ring. On the other hand, Jensen has proved in [24, 5.9] that the equality $\operatorname{silp} R = \operatorname{spli} R$ holds for any commutative Noetherian ring R. The result displayed above, combined with earlier work in [14], shows that the equality $\operatorname{silp} R = \operatorname{spli} R$ actually holds for any commutative $\aleph_0$-Noetherian ring R, i.e. for any commutative ring R all of whose ideals are countably generated.

Notations and terminology

We work over a fixed unital associative ring R and, unless otherwise specified, all modules are left R-modules. We denote by R^{op} the opposite ring of R and do not distinguish between right R-modules and left R^{op}-modules. If $\lambda(R)$ is an invariant, which is defined in terms of a certain class of left R-modules, then we denote by $\lambda(R^{op})$ the corresponding invariant, which is defined for R in terms of the appropriate class of right R-modules. Finally, we say that a class $\mathscr{C}$ of modules is projectively resolving if $\texttt{Proj}(R) \subseteq \mathscr{C}$ and $\mathscr{C}$ is closed under extensions and kernels of epimorphisms.

1. Preliminary notions

In this section, we collect certain basic notions and preliminary results that will be used in the sequel. These involve basic concepts related to Gorenstein homological algebra in module categories and the theory of Hovey triples in exact additive categories.

1.1. *Gorenstein projective and Gorenstein flat modules*

An acyclic complex P_* of projective modules is said to be a complete projective resolution if the complex of abelian groups $\operatorname{Hom}_R(P_*, Q)$ is acyclic for any projective module Q. Then, a module is called Gorenstein projective if it is a syzygy of a complete projective resolution. Holm's paper [22] is the standard reference in Gorenstein homological algebra. The class $\texttt{GProj}(R)$ of Gorenstein projective modules is projectively resolving; it is also closed under direct sums and direct summands. The Gorenstein projective dimension $\operatorname{Gpd}_R M$ of a module M is the length of a shortest resolution of M by Gorenstein projective modules. If no such resolution of finite length exists, then we write $\operatorname{Gpd}_R M = \infty$. If M is a module of finite projective dimension, then M has finite Gorenstein projective dimension as well and $\operatorname{Gpd}_R M = \operatorname{pd}_R M$.

An acyclic complex F_* of flat modules is said to be a complete flat resolution if the complex of abelian groups $I \otimes_R F_*$ is acyclic for any injective right module I. We say that a module is Gorenstein flat if it is a syzygy of a complete flat resolution. We let $\texttt{GFlat}(R)$ be the class of

Gorenstein flat modules. The Gorenstein flat dimension $\operatorname{Gfd}_R M$ of a module M is the length of a shortest resolution of M by Gorenstein flat modules. If no such resolution of finite length exists, then we write $\operatorname{Gfd}_R M = \infty$. If M is a module of finite flat dimension, then M has finite Gorenstein flat dimension as well and $\operatorname{Gfd}_R M = \operatorname{fd}_R M$.

Even though the relation between Gorenstein projective and Gorenstein flat modules is not fully understood, the notion of a projectively coresolved Gorenstein flat module (for short, PGF-module) defined in [26] sheds some light in and helps clarifying that relation. A PGF-module is a syzygy of an acyclic complex of projective modules P_*, which is such that the complex of abelian groups $I \otimes_R P_*$ is acyclic for any injective right module I. It is clear that the class $\mathtt{PGF}(R)$ of PGF-modules is contained in $\mathtt{GFlat}(R)$. The inclusion $\mathtt{PGF}(R) \subseteq \mathtt{GProj}(R)$ is proved in [26, Theorem 4.4]; in fact, it is shown that $\operatorname{Ext}^1_R(M,F) = 0$ for any PGF-module M and any flat module F. It is also proved in [26] that the classes $\mathtt{PGF}(R)$ and $\mathtt{GFlat}(R)$ are both projectively resolving, closed under direct sums and direct summands.

1.2. *Gorenstein global dimensions*

The existence of complete projective resolutions of modules (i.e. of complete projective resolutions that coincide in sufficiently large degrees with an ordinary projective resolution of the module) has been studied by Gedrich and Gruenberg [17], Cornick and Kropholler [12], in connection with the existence of complete cohomological functors in the category of modules. Even though they were mainly interested in the case where R is the integral group ring of a group, they were able to characterize those rings over which all modules admit complete projective resolutions, in terms of the finiteness of the invariants $\operatorname{spli} R$ and $\operatorname{silp} R$. Here, $\operatorname{spli} R$ is the supremum of the projective lengths (dimensions) of injective modules and $\operatorname{silp} R$ is the supremum of the injective lengths (dimensions) of projective modules. As shown by Holm [22], the existence of a complete projective resolution for a module M is equivalent to the finiteness of the Gorenstein projective dimension $\operatorname{Gpd}_R M$ of M. From this point of view, the above result by Cornick and Kropholler was alternatively proved by Bennis and Mahbou in [7], where the notion of the Gorenstein global dimension of the ring was introduced, in analogy with the classical notion of global dimension defined in [9, Chapter VI, §2]; see also [15, §4]. More precisely, the (left) Gorenstein global dimension $\operatorname{Ggl.dim} R$ of the ring R is defined by letting

$$\operatorname{Ggl.dim} R = \sup\{\operatorname{Gpd}_R M : M \text{ a left } R\text{-module}\}.$$

Then, the following conditions are equivalent:

(i) $\operatorname{Ggl.dim} R < \infty$,
(ii) $\operatorname{Gpd}_R M < \infty$ for any module M,
(iii) any module M admits a complete projective resolution,
(iv) the invariants $\operatorname{spli} R$ and $\operatorname{silp} R$ are finite.

If these conditions are satisfied, then $\operatorname{Ggl.dim} R = \operatorname{spli} R = \operatorname{silp} R$.

The corresponding characterization of the finiteness of the Gorenstein weak global dimension $\operatorname{Gwgl.dim} R$ of the ring R, which is defined by letting

$$\operatorname{Gwgl.dim} R = \sup\{\operatorname{Gfd}_R M : M \text{ a left } R\text{-module}\},$$

turned out to be more difficult to achieve. The relevant homological invariants here are $\operatorname{sfli} R$, the supremum of the flat lengths (dimensions) of injective modules, and its analogue $\operatorname{sfli} R^{op}$ for the opposite ring R^{op}. Using in an essential way results in [26], it was proved by Christensen, Estrada and Thompson in [10] that the following conditions are equivalent:

(i) $\operatorname{Gwgl.dim} R < \infty$,
(ii) $\operatorname{Gfd}_R M < \infty$ for any module M,

(iii) the invariants sfli R and sfli R^{op} are finite.

If these conditions are satisfied, then Gwgl. dim R = sfli R = sfli R^{op}.

1.3. *Cotorsion pairs and Hovey triples*

Let $\mathscr{A}$ be an exact additive category, in the sense of Quillen [8], and consider a full subcategory $\mathscr{B} \subseteq \mathscr{A}$. A morphism $f : B \longrightarrow A$ in $\mathscr{A}$ is called a $\mathscr{B}$-precover of the object $A \in \mathscr{A}$ if:

(i) $B \in \mathscr{B}$ and

(ii) the induced map $f_* : \mathrm{Hom}_{\mathscr{A}}(B', B) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(B', A)$ is surjective for any $B' \in \mathscr{B}$.

The reader is referred to [20] for a thorough and systematic study of precovers.

The Ext^1-pairing induces an orthogonality relation between subclasses of $\mathscr{A}$. If $\mathscr{B} \subseteq \mathscr{A}$, then we define the left orthogonal ${}^{\perp}\mathscr{B}$ of $\mathscr{B}$ as the class consisting of those objects $X \in \mathscr{A}$, which are such that $\mathrm{Ext}^1_{\mathscr{A}}(X, B) = 0$ for all $B \in \mathscr{B}$. Analogously, the right orthogonal $\mathscr{B}^{\perp}$ of $\mathscr{B}$ is the class consisting of those objects $Y \in \mathscr{A}$, which are such that $\mathrm{Ext}^1_{\mathscr{A}}(B, Y) = 0$ for all $B \in \mathscr{B}$. If $\mathscr{C}, \mathscr{D}$ are two subclasses of $\mathscr{A}$, then the pair $(\mathscr{C}, \mathscr{D})$ is a cotorsion pair in $\mathscr{A}$ (cf. [16]) if $\mathscr{C} = {}^{\perp}\mathscr{D}$ and $\mathscr{C}^{\perp} = \mathscr{D}$. The cotorsion pair is called hereditary if $\mathrm{Ext}^i_{\mathscr{A}}(C, D) = 0$ for all $i > 0$ and all objects $C \in \mathscr{C}$ and $D \in \mathscr{D}$. The cotorsion pair is complete if for any object $A \in \mathscr{A}$ there exist short exact sequences (conflations), usually called approximation sequences

$$0 \longrightarrow D \longrightarrow C \longrightarrow A \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow A \longrightarrow D' \longrightarrow C' \longrightarrow 0,$$

where $C, C' \in \mathscr{C}$ and $D, D' \in \mathscr{D}$. In that case, the morphism $C \longrightarrow A$ is a $\mathscr{C}$-precover of A.

A Hovey triple on $\mathscr{A}$ is a triple $(\mathscr{C}, \mathscr{W}, \mathscr{F})$ of subclasses of $\mathscr{A}$, which are such that the pairs $(\mathscr{C}, \mathscr{W} \cap \mathscr{F})$ and $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ are complete cotorsion pairs and the class $\mathscr{W}$ is closed under direct summands and satisfies the 2-out-of-3 property for short exact sequences (conflations) in $\mathscr{A}$. The fundamental work of Gillespie [18], which is based on work of Hovey [23], gives a bijection between Hovey triples on a (weakly) idempotent complete exact category $\mathscr{A}$ and certain, so-called exact, Quillen model structures on $\mathscr{A}$; cf. [18, Theorem 3.3]. In the context of Gillespie's bijection, it is proved in [18, Proposition 5.2] that for an exact model structure on $\mathscr{A}$ that has its associated complete cotorsion pairs hereditary, the class $\mathscr{C} \cap \mathscr{F}$ is a Frobenius exact category with projective-injective objects equal to $\mathscr{C} \cap \mathscr{W} \cap \mathscr{F}$. Then, a result of Happel [21] implies that the associated stable category, which is $\mathscr{C} \cap \mathscr{F}$ modulo its projective-injective objects, is triangulated. The upshot of this connection is that the (Quillen) homotopy category of an exact model structure is triangulated equivalent to the stable category of the Frobenius exact category $\mathscr{C} \cap \mathscr{F}$; cf. [18, Proposition 4.4 and Corollary 4.8].

2. Modules of finite PGF-dimension

In this section, we define the notion of PGF-dimension for a module and show that the resulting class $\overline{\mathtt{PGF}}(R)$ of modules of finite PGF-dimension has many standard closure properties.

We recall that the class $\mathtt{PGF}(R)$ is projectively resolving and closed under direct sums and direct summands. The following result is a formal consequence of these properties of $\mathtt{PGF}(R)$; cf. [1, Lemma 3.12] and [3, Lemma 3.2].

Lemma 1. *Let M be an R-module, n a non-negative integer and*

$$0 \longrightarrow K \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$
$$0 \longrightarrow K' \longrightarrow G'_{n-1} \longrightarrow \cdots \longrightarrow G'_0 \longrightarrow M \longrightarrow 0$$

two exact sequences of modules with $G_0, \ldots, G_{n-1}, G'_0, \ldots, G'_{n-1} \in \mathtt{PGF}(R)$. Then, $K \in \mathtt{PGF}(R)$ if and only if $K' \in \mathtt{PGF}(R)$.

Proposition 2. *The following conditions are equivalent for an R-module M and a non-negative integer n:*

(i) *There exists an exact sequence of modules*

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

with $G_0, \ldots, G_{n-1}, G_n \in \mathtt{PGF}(R)$.

(ii) *For any exact sequence of modules*

$$0 \longrightarrow K \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

with $G_0, \ldots, G_{n-1} \in \mathtt{PGF}(R)$, we also have $K \in \mathtt{PGF}(R)$.

Proof. The implication (i) ⇒ (ii) is a consequence of Lemma 1 (see also [3, Lemma 2.2] for a special case), whereas the implication (ii) ⇒ (i) follows by considering a truncated projective resolution of M. □

If the equivalent conditions in Proposition 2 are satisfied, then we say that the module M has a PGF-resolution of length n and write $\text{PGF-dim}_R M \le n$. In the case where $\text{PGF-dim}_R M \le n$ and M has no PGF-resolution of length $< n$, we say that M has PGF-dimension equal to n and write $\text{PGF-dim}_R M = n$. Finally, we say that M has infinite PGF-dimension and write $\text{PGF-dim}_R M = \infty$, if M has no PGF-resolution of finite length.

We now consider the class $\overline{\mathtt{PGF}}(R)$ of all modules of finite PGF-dimension and describe certain closure properties of that class.

Proposition 3. *Let $(M_i)_i$ be a family of modules and $M = \bigoplus_i M_i$ the corresponding direct sum. Then, $\text{PGF-dim}_R M = \sup_i \text{PGF-dim}_R M_i$. In particular, the class $\overline{\mathtt{PGF}}(R)$ is closed under finite direct sums and direct summands.*

Proof. In order to show that $\text{PGF-dim}_R M \le \sup_i \text{PGF-dim}_R M_i$, it suffices to consider the case where $\sup_i \text{PGF-dim}_R M_i = n < \infty$. Then, $\text{PGF-dim}_R M_i \le n$ and hence M_i has a PGF-resolution of length n for all i. Since $\mathtt{PGF}(R)$ is closed under direct sums, the direct sum of these resolutions is a PGF-resolution of M of length n, so that $\text{PGF-dim}_R M \le n$.

It remains to show that we also have $\sup_i \text{PGF-dim}_R M_i \le \text{PGF-dim}_R M$. To that end, assume that $\text{PGF-dim}_R M = n < \infty$ and consider for any i an exact sequence

$$0 \longrightarrow K_i \longrightarrow G_{i,n-1} \longrightarrow \cdots \longrightarrow G_{i,0} \longrightarrow M_i \longrightarrow 0,$$

with $G_{i,0}, \ldots, G_{i,n-1} \in \mathtt{PGF}(R)$. Since $\mathtt{PGF}(R)$ is closed under direct sums, the exactness of the direct sum of these exact sequences

$$0 \longrightarrow \bigoplus_i K_i \longrightarrow \bigoplus_i G_{i,n-1} \longrightarrow \cdots \longrightarrow \bigoplus_i G_{i,0} \longrightarrow M \longrightarrow 0$$

and our assumption on the PGF-dimension of M imply that $\bigoplus_i K_i$ is a PGF-module. Since $\mathtt{PGF}(R)$ is closed under direct summands, it follows that K_i is a PGF-module for all i. Then, M_i has a PGF-resolution of length n and hence $\text{PGF-dim}_R M_i \le n$ for all i, as needed. □

Proposition 4. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of modules. Then:*

(i) $\text{PGF-dim}_R M \le \max\{\text{PGF-dim}_R M', \text{PGF-dim}_R M''\}$,

(ii) $\text{PGF-dim}_R M' \le \max\{\text{PGF-dim}_R M, \text{PGF-dim}_R M''\}$,

(iii) $\text{PGF-dim}_R M'' \le 1 + \max\{\text{PGF-dim}_R M', \text{PGF-dim}_R M\}$.

In particular, the class $\overline{\mathtt{PGF}}(R)$ has the 2-out-of-3 property: if two out of the three modules that appear in a short exact sequence have finite PGF-dimension, then so does the third.

Proof. (i). Assume that $\max\{\text{PGF-dim}_R M', \text{PGF-dim}_R M''\} = n$ and consider two projective resolutions $P'_* \longrightarrow M' \longrightarrow 0$ and $P''_* \longrightarrow M'' \longrightarrow 0$ of M' and M'' respectively. Then, we may construct by the standard step-by-step process a projective resolution $P_* \longrightarrow M \longrightarrow 0$ of M, such that $P_i = P'_i \oplus P''_i$ and the corresponding syzygy module $\Omega_i M$ is an extension of $\Omega_i M''$ by $\Omega_i M'$ for all i. Since both M' and M'' have PGF-dimension $\leq n$, the modules $\Omega_n M'$ and $\Omega_n M''$ are both PGF-modules. Then, the short exact sequence

$$0 \longrightarrow \Omega_n M' \longrightarrow \Omega_n M \longrightarrow \Omega_n M'' \longrightarrow 0$$

and the closure of $\texttt{PGF}(R)$ under extensions show that $\Omega_n M$ is a PGF-module as well. Then, the exact sequence

$$0 \longrightarrow \Omega_n M \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_o \longrightarrow M \longrightarrow 0$$

is a PGF-resolution of M of length n and hence $\text{PGF-dim}_R M \leq n$, as needed.

(ii). We can prove this assertion by using the same argument as the one used in order to prove assertion (i) above, by invoking the closure of $\texttt{PGF}(R)$ under kernels of epimorphisms.

(iii). We fix a short exact sequence

$$0 \longrightarrow K \longrightarrow P \stackrel{p}{\longrightarrow} M'' \longrightarrow 0, \qquad (1)$$

where P is a projective module, and consider the pullback of the short exact sequence given in the statement of the Proposition along p

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & K & = & K & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & X & \longrightarrow & P & \longrightarrow & 0 \\ & & \| & & \downarrow & & p\downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Since P is projective, the horizontal short exact sequence in the middle of the diagram splits and hence $X \simeq P \oplus M'$. We now invoke Proposition 3 and conclude that $\text{PGF-dim}_R X = \text{PGF-dim}_R M'$. Then, the vertical short exact sequence in the middle of the diagram and assertion (ii) above show that

$$\text{PGF-dim}_R K \leq \max\{\text{PGF-dim}_R X, \text{PGF-dim}_R M\} = \max\{\text{PGF-dim}_R M', \text{PGF-dim}_R M\}.$$

Since we may splice any PGF-resolution of K of length $\text{PGF-dim}_R K$ with the short exact sequence (1) and obtain a PGF-resolution of M'' of length $1 + \text{PGF-dim}_R K$, it follows that

$$\text{PGF-dim}_R M'' \leq 1 + \text{PGF-dim}_R K \leq 1 + \max\{\text{PGF-dim}_R M', \text{PGF-dim}_R M\},$$

as needed. □

As a consequence of the equality $\texttt{PGF}(R) \cap \texttt{Flat}(R) = \texttt{Proj}(R)$, we obtain the following result on the relation between the projective dimension and the PGF-dimension of flat modules and, analogously, the relation between the projective dimension and the flat dimension of PGF-modules.

Proposition 5.

(i) *If M is a flat module, then* $\text{pd}_R M = \text{PGF-dim}_R M$.

(ii) *If M is a PGF-module, then* $\text{pd}_R M = fd_R M$.

Proof. **(i).** Since $\mathtt{Proj}(R) \subseteq \mathtt{PGF}(R)$, we always have PGF-$\dim_R M \leq \mathrm{pd}_R M$. In order to prove the reverse inequality, it suffices to assume that PGF-$\dim_R M = n < \infty$. Then, the truncation of a projective resolution of M provides us with an exact sequence

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where $P_0, \ldots, P_{n-1}$ are projective modules and $K \in \mathtt{PGF}(R)$. Since M is flat, it follows that K is also flat and hence $K \in \mathtt{PGF}(R) \cap \mathtt{Flat}(R) = \mathtt{Proj}(R)$. We conclude that M admits a projective resolution of length n and hence $\mathrm{pd}_R M \leq n =$ PGF-$\dim_R M$, as needed.

(ii). Since projective modules are flat, we always have $\mathrm{fd}_R M \leq \mathrm{pd}_R M$. In order to prove the reverse inequality, it suffices to assume that $\mathrm{fd}_R M = n < \infty$. Then, the truncation of a projective resolution of M provides us with an exact sequence

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where $P_0, \ldots, P_{n-1}$ are projective modules and K is flat. Since M is a PGF-module and the class $\mathtt{PGF}(R)$ is projectively resolving, it follows that K is also a PGF-module. Then, $K \in \mathtt{PGF}(R) \cap \mathtt{Flat}(R) = \mathtt{Proj}(R)$ and hence M admits a projective resolution of length n, i.e. $\mathrm{pd}_R M \leq n = \mathrm{fd}_R M$. □

Remark 6. If we denote by $\overline{\mathtt{Proj}}(R)$ and $\overline{\mathtt{Flat}}(R)$ the classes of modules of finite projective dimension and finite flat dimension respectively, then $\overline{\mathtt{PGF}}(R) \cap \overline{\mathtt{Flat}}(R) = \overline{\mathtt{Proj}}(R)$. Indeed, the inclusion $\overline{\mathtt{Proj}}(R) \subseteq \overline{\mathtt{PGF}}(R) \cap \overline{\mathtt{Flat}}(R)$ is clear, since any projective resolution of finite length is both a PGF-resolution and a flat resolution of finite length. Conversely, if M is a module contained in $\overline{\mathtt{PGF}}(R) \cap \overline{\mathtt{Flat}}(R)$, then the n-th syzygy module $\Omega_n M$ in a projective resolution of M is a flat and PGF-module for $n \gg 0$. Since $\mathtt{PGF}(R) \cap \mathtt{Flat}(R) = \mathtt{Proj}(R)$, it follows that $\Omega_n M$ is projective for $n \gg 0$ and hence $M \in \overline{\mathtt{Proj}}(R)$.

3. Approximation sequences

In this section, we show that the finiteness of PGF-dimension can be detected by the existence of suitable approximation sequences, in analogy with the case of the finiteness of Gorenstein projective dimension.

The next result is akin to [22, Theorem 2.10]; see also [3, Lemma 3.3].

Proposition 7. *Let M be a module with* PGF-$\dim_R M = n$. *Then, there exists a short exact sequence*

$$0 \longrightarrow K \longrightarrow G \xrightarrow{\pi} M \longrightarrow 0,$$

where G is a PGF-module and $\mathrm{pd}_R K = n - 1$. *(If $n = 0$, this is understood to mean $K = 0$.) In particular, π is a* $\mathtt{PGF}(R)$*-precover of M.*

Proof. The result is clear if $n = 0$ and hence we may assume that $n \geq 1$. Since PGF-$\dim_R M = n$, there exists an exact sequence

$$0 \longrightarrow N \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where $P_0, \ldots, P_{n-1}$ are projective modules and $N \in \mathtt{PGF}(R)$. Then, there exists another exact sequence

$$0 \longrightarrow N \longrightarrow Q_0 \longrightarrow Q_{-1} \longrightarrow \cdots \longrightarrow Q_{-n+1} \longrightarrow G \longrightarrow 0,$$

where $Q_0, \ldots, Q_{-n+1}$ are projective modules and $G \in \mathtt{PGF}(R)$. Since all kernels of the latter exact sequence are PGF-modules as well, it follows from [26, Corollary 4.5] that the exact sequence remains exact after applying the functor $\mathrm{Hom}_R(\_\_, P)$ for any projective module P. We conclude that there exists a morphism of complexes

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & N & \longrightarrow & Q_0 & \longrightarrow \cdots \longrightarrow & Q_{-n+1} & \longrightarrow & G & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & P_{n-1} & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

The unlabelled vertical arrows induce a quasi-isomorphism between the corresponding complexes and hence we may consider the associated mapping cone, which is an acyclic complex

$$0 \longrightarrow Q_0 \longrightarrow Q_{-1} \oplus P_{n-1} \longrightarrow \cdots \longrightarrow G \oplus P_0 \xrightarrow{\pi} M \longrightarrow 0. \tag{2}$$

Note that $G \oplus P_0$ is a PGF-module and the module $K = \ker \pi$ has projective dimension $\leq n-1$. In fact, our assumption that PGF-$\dim_R M = n$ implies that the inequality $\mathrm{pd}_R K \leq n-1$ cannot be strict, i.e. $\mathrm{pd}_R K = n-1$. Since $K \in \overline{\mathtt{Proj}}(R) \subseteq \mathtt{GProj}(R)^{\perp} \subseteq \mathtt{PGF}(R)^{\perp}$, where the latter inclusion is a consequence of the inclusion $\mathtt{PGF}(R) \subseteq \mathtt{GProj}(R)$, we conclude that π is indeed a $\mathtt{PGF}(R)$-precover of M. □

Corollary 8. *If M is a module with* PGF-$\dim_R M \leq 1$, *then the following conditions are equivalent:*

(i) $M \in \mathtt{PGF}(R)$,
(ii) $\mathrm{Ext}^1_R(M, F) = 0$ *for any flat module F,*
(iii) $\mathrm{Ext}^1_R(M, P) = 0$ *for any projective module P.*

Proof. The implication (i) ⇒ (ii) follows from [26, Theorem 4.4], whereas the implication (ii) ⇒ (iii) is obvious. In order to prove that (iii) ⇒ (i), we use Proposition 7 and note that the hypothesis PGF-$\dim_R M \leq 1$ implies the existence of a short exact sequence

$$0 \longrightarrow P \longrightarrow G \longrightarrow M \longrightarrow 0,$$

where P is projective and $G \in \mathtt{PGF}(R)$. By our assumption, the group $\mathrm{Ext}^1_R(M, P)$ is trivial and hence the exact sequence splits. It follows that M is a direct summand of G. Since the class $\mathtt{PGF}(R)$ is closed under direct summands, we conclude that $M \in \mathtt{PGF}(R)$ as well. □

Corollary 9. *If $M \in \overline{\mathtt{PGF}}(R)$, then the following conditions are equivalent:*

(i) $M \in \mathtt{PGF}(R)$,
(ii) $\mathrm{Ext}^i_R(M, F) = 0$ *for any $i > 0$ and any flat module F,*
(iii) $\mathrm{Ext}^i_R(M, P) = 0$ *for any $i > 0$ and any projective module P.*

Proof. The implication (i) ⇒ (ii) follows from [26, Corollary 4.5], whereas the implication (ii) ⇒ (iii) is obvious. In order to prove that (iii) ⇒ (i), we consider a PGF-resolution of M of finite length

$$0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

and argue by induction on n. The case where $n = 0$ is trivial. Assume that $n > 0$ and let K be the kernel of the map $G_0 \longrightarrow M$, so that there is a short exact sequence

$$0 \longrightarrow K \longrightarrow G_0 \longrightarrow M \longrightarrow 0.$$

Since $G_0 \in \mathtt{PGF}(R)$, the group $\mathrm{Ext}^i_R(G_0, P)$ is trivial and hence $\mathrm{Ext}^i_R(K, P) = \mathrm{Ext}^{i+1}_R(M, P)$ is also trivial for all $i > 0$ and all projective modules P. The module K admits a PGF-resolution of length $n-1$ and our induction hypothesis implies that $K \in \mathtt{PGF}(R)$. Therefore, it follows that PGF-$\dim_R M \leq 1$. Since $\mathrm{Ext}^1_R(M, P) = 0$ for any projective module P, we finish the proof by invoking Corollary 8. □

In view of Proposition 4 (iii), the existence of a short exact sequence as in the statement of Proposition 7 is equivalent to the finiteness of the PGF-dimension of M. In fact, we may complement this assertion and prove the following result. Here, condition (iii) is analogous to [11, Lemma 2.17] (see also [25, Lemma 1.9]) and conditions (iv) and (v) are inspired by the Remark following [26, Theorem 4.11].

Theorem 10. *The following conditions are equivalent for a module M and a non-negative integer n:*

(i) PGF-$\dim_R M = n$.

(ii) *There exists a short exact sequence*

$$0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0,$$

where G is a PGF-module and $\mathrm{pd}_R K = n-1$. *If $n = 0$, this is understood to mean $K = 0$. If $n = 1$, we also require that the exact sequence be non-split.*

(iii) *There exists a short exact sequence*

$$0 \longrightarrow M \longrightarrow K \longrightarrow G \longrightarrow 0,$$

where G is a PGF-module and $\mathrm{pd}_R K = n$.

(iv) *There exists a projective module P, such that the module $M' = M \oplus P$ fits into an exact sequence*

$$0 \longrightarrow G \longrightarrow M' \longrightarrow K \longrightarrow 0,$$

which remains exact after applying the functor $\mathrm{Hom}_R(\_\_, Q)$ *for any module* $Q \in \mathtt{PGF}(R)^{\perp}$, *where G is a PGF-module and* $\mathrm{pd}_R K = n$.

(v) *There exists a PGF-module P, such that the module $M' = M \oplus P$ fits into an exact sequence*

$$0 \longrightarrow G \longrightarrow M' \longrightarrow K \longrightarrow 0,$$

where G is a PGF-module and $\mathrm{pd}_R K = n$. *If $n = 1$, we also require that the exact sequence remain exact after applying the functor* $\mathrm{Hom}_R(\_\_, Q)$ *for any projective module Q.*

Proof. (i)⇒(ii). The existence of the short exact sequence follows from Proposition 7. If $n = 1$, then the exact sequence cannot split. (Indeed, if the short exact sequence were split, then M would be a direct summand of the PGF-module G and hence M would be itself a PGF-module; this is absurd, since PGF-$\dim_R M = 1$.)

(ii)⇒(iii). Consider a short exact sequence as in (ii). Since $G \in \mathtt{PGF}(R)$, there exists a short exact sequence

$$0 \longrightarrow G \longrightarrow P \longrightarrow G' \longrightarrow 0,$$

where P is a projective module and $G' \in \mathtt{PGF}(R)$. By considering the pushout of that short exact sequence along the given epimorphism $G \longrightarrow M$, we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & K' & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & G' & = & G' & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

We claim that the rightmost vertical exact sequence is of the required type. Indeed, if $n = 0$, then $K = 0$ and hence $K' = P$ is a projective module. If $n = 1$, then K is projective and the monomorphism $K \longrightarrow P$ is not split. (Indeed, if that monomorphism were split, then the monomorphism $K \longrightarrow G$ would be split as well, contradicting our assumption.) It follows that the module $K' = \mathrm{coker}(K \longrightarrow P)$ is not projective and hence $\mathrm{pd}_R K' = 1$. If $n \geq 2$, then $\mathrm{pd}_R K =$

$n-1>0$, so that $\operatorname{Ext}^n_R(K',\_\_) = \operatorname{Ext}^{n-1}_R(K,\_\_) \neq 0$ and $\operatorname{Ext}^{n+1}_R(K',\_\_) = \operatorname{Ext}^n_R(K,\_\_) = 0$; it follows that $\operatorname{pd}_R K' = n$.

(iii)⇒(iv). Consider a short exact sequence as in (iii) and let

$$0 \longrightarrow K' \longrightarrow P \longrightarrow K \longrightarrow 0$$

be a short exact sequence, where P is a projective module and $\operatorname{pd}_R K' = n-1$. (If $n=0$, then K is projective and we choose $P=K$ and $K'=0$.) By considering the pullback of that short exact sequence along the given monomorphism $M \longrightarrow K$, we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K' & = & K' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & G' & \longrightarrow & P & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & K & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since the class $\mathtt{PGF}(R)$ is projectively resolving, the horizontal short exact sequence in the middle shows that G' is a PGF-module. Then, the definition of the pullback and the surjectivity of the map $P \longrightarrow K$ imply that there is a short exact sequence

$$0 \longrightarrow G' \longrightarrow M \oplus P \longrightarrow K \longrightarrow 0.$$

In order to show that this short exact sequence has the required additional property, we note that for any module $Q \in \mathtt{PGF}(R)^\perp$ the two horizontal short exact sequences in the diagram above induce a commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}_R(G,Q) & \longrightarrow & \operatorname{Hom}_R(P,Q) & \longrightarrow & \operatorname{Hom}_R(G',Q) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \operatorname{Hom}_R(G,Q) & \longrightarrow & \operatorname{Hom}_R(K,Q) & \longrightarrow & \operatorname{Hom}_R(M,Q) & \longrightarrow & 0
\end{array}$$

It follows readily that there is an induced sequence of abelian groups

$$0 \longrightarrow \operatorname{Hom}_R(K,Q) \longrightarrow \operatorname{Hom}_R(M,Q) \oplus \operatorname{Hom}_R(P,Q) \longrightarrow \operatorname{Hom}_R(G',Q) \longrightarrow 0,$$

as needed.

(iv)⇒(v). This is immediate, since projective modules are contained in both classes $\mathtt{PGF}(R)$ and $\mathtt{PGF}(R)^\perp$.

(v)⇒(i). Consider an exact sequence as in (v) and note that Proposition 3 implies that $\text{PGF-dim}_R M' = \text{PGF-dim}_R M$. Therefore, it suffices to prove that $\text{PGF-dim}_R M' = n$. Since G is a PGF-module and $\text{PGF-dim}_R K \le \operatorname{pd}_R K = n$, we may invoke Proposition 4 (i) and conclude that $\text{PGF-dim}_R M' \le n$. It remains to show that the latter inequality cannot be strict. Indeed, let us assume that $n \ge 1$ and $\text{PGF-dim}_R M' \le n-1$.

If $n=1$, then M' is a PGF-module and hence $\text{PGF-dim}_R K \le 1$. Since the short exact sequence is assumed to remain exact after applying the functor $\operatorname{Hom}_R(\_\_,Q)$ for any projective module Q and $M' \in \mathtt{PGF}(R) \subseteq {}^\perp\mathtt{Proj}(R)$, it follows that the abelian group $\operatorname{Ext}^1_R(K,Q)$ is trivial for any projective module Q. Then, Corollary 8 implies that $K \in \mathtt{PGF}(R)$; in particular, $K \in \mathtt{GProj}(R)$. As shown in [22, Proposition 2.27], any Gorenstein projective module of finite projective dimension is necessarily projective. We therefore conclude that the module K is projective.[1] This is absurd, since $\operatorname{pd}_R K = 1$.

[1] Alternatively, the projectivity of K follows since $\overline{\mathtt{Proj}}(R) \subseteq \mathtt{PGF}(R)^\perp$ and $\mathtt{PGF}(R) \cap \mathtt{PGF}(R)^\perp = \mathtt{Proj}(R)$; cf. [26].

We now consider the case where $n > 1$. Since the PGF-module G is Gorenstein projective, the functor $\operatorname{Ext}^{n-1}_R(G, \_\_)$ vanishes on projective modules. Since PGF-$\dim_R M' \leq n-1$ and $\mathtt{PGF}(R) \subseteq \mathtt{GProj}(R)$, we also have $\operatorname{Gpd}_R M' \leq n-1$. Therefore, [22, Theorem 2.20] implies that the functor $\operatorname{Ext}^n_R(M', \_\_)$ vanishes on projective modules as well. It follows that the functor $\operatorname{Ext}^n_R(K, \_\_)$ vanishes on projective modules. This contradicts our assumption that $\operatorname{pd}_R K = n$; indeed, if we consider a projective resolution $P_* \longrightarrow K \longrightarrow 0$ of length n, then the monomorphism $P_n \longrightarrow P_{n-1}$ is not split and hence $\operatorname{Ext}^n_R(K, P_n) \neq 0$. □

Remarks 11.

(i) In the case where $n = 1$, it is necessary to impose some restrictions on the short exact sequences appearing in Theorem 10 (ii),(v). Indeed, if P is any non-zero projective module and $M \in \mathtt{PGF}(R)$, then the (split) short exact sequence

$$0 \longrightarrow P \longrightarrow P \oplus M \longrightarrow M \longrightarrow 0$$

is of the type appearing in Theorem 10 (ii), but PGF-$\dim_R M = 0 \neq 1$. On the other hand, if K is a module with $\operatorname{pd}_R K = 1$, then a projective resolution of K provides an exact sequence

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow K \longrightarrow 0$$

of the type appearing in Theorem 10 (v), but PGF-$\dim_R P_0 = 0 \neq 1$.

(ii) It is clear from the proof of Theorem (10) that the analogues of conditions (iv) and (v) therein for Gorenstein projective modules are equivalent to the analogues of conditions (i), (ii) and (iii) for such modules, thereby complementing the characterizations of the finiteness of the Gorenstein projective dimension given in [22, Theorem 2.10] and [11, Lemma 2.17].

The next result is a characterization of modules of finite PGF-dimension, that parallels the characterization of modules of finite Gorenstein projective dimension in [22, Theorem 2.20].

Proposition 12. *The following conditions are equivalent for a module M of finite PGF-dimension and a non-negative integer n:*

(i) PGF-$\dim_R M \leq n$.
(ii) $\operatorname{Ext}^i_R(M, F) = 0$ *for all $i > n$ and any flat module F.*
(ii′) $\operatorname{Ext}^i_R(M, P) = 0$ *for all $i > n$ and any projective module P.*
(iii) $\operatorname{Ext}^i_R(M, F) = 0$ *for all $i > n$ and any module F of finite flat dimension.*
(iii′) $\operatorname{Ext}^i_R(M, P) = 0$ *for all $i > n$ and any module P of finite projective dimension.*

Proof. (i) ⇒ (ii). We consider a PGF-resolution of length n

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

and fix a flat module F. Since the functors $\operatorname{Ext}^j_R(\_\_, F)$ vanish on the class of PGF-modules for all $j > 0$ (cf. [26, Corollary 4.5]), we may deduce the desired vanishing by dimension shifting.

(ii) ⇒ (i). Let

$$0 \longrightarrow K \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

be an exact sequence, where $G_0, \ldots, G_{n-1} \in \mathtt{PGF}(R)$. Since the modules $M, G_0, \ldots, G_{n-1}$ are of finite PGF-dimension, an iterated application of Proposition 4 (ii) shows that the module K has finite PGF-dimension as well. On the other hand, our hypothesis and the dimension shifting argument employed in the proof of the implication (i) ⇒ (ii) above show that the functors $\operatorname{Ext}^i_R(K, \_\_)$ vanish on flat modules for all $i > 0$. Invoking Corollary 9, we conclude that $K \in \mathtt{PGF}(R)$, as needed.

The implication (ii) ⇒ (iii) follows by induction on the flat dimension of the module F, whereas the implication (iii) ⇒ (ii) is immediate.

Finally, the implications (i) $\Longleftrightarrow$ (ii′) $\Longleftrightarrow$ (iii′) that involve projective modules can be proved by using exactly the same arguments as those used above for the implications that involve flat modules. □

An immediate consequence of the characterization above is that the PGF-dimension is a refinement of the ordinary projective dimension, whereas the Gorenstein projective dimension is a refinement of the PGF-dimension; cf. [3, Lemma 3.1].

Corollary 13. *Let M be a module.*

(i) *If* $\text{pd}_R M < \infty$, *then* $\text{PGF-dim}_R M = \text{pd}_R M$.

(ii) *If* $\text{PGF-dim}_R M < \infty$, *then* $\text{Gpd}_R M = \text{PGF-dim}_R M$.

Proof. (i). If $\text{pd}_R M = n$, then the functors $\text{Ext}^i_R(M, \_\_)$ vanish for all $i > n$ and $\text{Ext}^i_R(M, P) \neq 0$ for a suitable projective module P. Since $\text{PGF-dim}_R M \leq n$, the equality $\text{PGF-dim}_R M = n$ follows from Proposition 12.

(ii). Since $\text{Gpd}_R M \leq \text{PGF-dim}_R M < \infty$, the equality $\text{Gpd}_R M = \text{PGF-dim}_R M$ follows from Proposition 12 and [22, Theorem 2.20]. □

Since $\texttt{PGF}(R) \subseteq \texttt{GProj}(R)$, it follows from [22, Theorem 2.20] that $\text{Ext}^1_R(M, P) = 0$ whenever $M \in \texttt{PGF}(R)$ and $P \in \overline{\texttt{Proj}}(R)$; this is precisely the assertion of Proposition 12 (iii′) in the case where $n = 0$ therein. In fact, this vanishing provides a characterization of PGF-modules and modules of finite projective dimension, if we restrict to modules of finite PGF-dimension.

Proposition 14. *Let N be a module of finite PGF-dimension. Then:*

(i) $N \in \texttt{PGF}(R)$ *if and only if* $\text{Ext}^1_R(N, P) = 0$ *for any* $P \in \overline{\texttt{Proj}}(R)$.

(ii) $N \in \overline{\texttt{Proj}}(R)$ *if and only if* $\text{Ext}^1_R(M, N) = 0$ *for any* $M \in \texttt{PGF}(R)$.

Proof. (i). As we noted above, the Ext-group is trivial if $N \in \texttt{PGF}(R)$. Conversely, assume that N is a module of finite PGF-dimension contained in ${}^{\perp}\overline{\texttt{Proj}}(R)$. Proposition 7 implies the existence of a short exact sequence

$$0 \longrightarrow K \longrightarrow G \longrightarrow N \longrightarrow 0,$$

where $G \in \texttt{PGF}(R)$ and $K \in \overline{\texttt{Proj}}(R)$. In view of our assumption on N, this sequence splits and hence N is a direct summand of the PGF-module G. Since the class $\texttt{PGF}(R)$ is closed under direct summands, we conclude that N is a PGF-module.

(ii). As we noted above, the Ext-group is trivial if $N \in \overline{\texttt{Proj}}(R)$. Conversely, assume that N is module of finite PGF-dimension contained in $\texttt{PGF}(R)^{\perp}$. Then, Theorem 10 (iii) implies the existence of a short exact sequence

$$0 \longrightarrow N \longrightarrow K \longrightarrow G \longrightarrow 0,$$

where $G \in \texttt{PGF}(R)$ and $K \in \overline{\texttt{Proj}}(R)$. In view of our assumption on N, this sequence splits and hence N is a direct summand of K. Then, $\text{pd}_R N \leq \text{pd}_R K < \infty$ and hence $N \in \overline{\texttt{Proj}}(R)$, as needed. □

We now examine the special case of Gorenstein flat modules and show that the values of their PGF-dimension are controlled by the values of the projective dimension of flat modules. We let $\text{splf}\, R$ be the supremum of the projective lengths (dimensions) of flat modules.

Proposition 15. *We have an equality* $\sup\{\text{PGF-dim}_R M : M \in \texttt{GFlat}(R)\} = \text{splf}\, R$. *In particular,* $\texttt{Flat}(R) \subseteq \overline{\texttt{Proj}}(R)$ *if and only if* $\texttt{GFlat}(R) \subseteq \overline{\texttt{PGF}}(R)$.

Proof. Let $s = \sup\{\text{PGF-dim}_R M : M \in \mathtt{GFlat}(R)\}$. If M is any flat module, then Proposition 5 (i) implies that $\text{pd}_R M = \text{PGF-dim}_R M \le s$. It follows that $\text{splf} R \le s$. In order to prove the reverse inequality, it suffices to assume that $\text{splf} R < \infty$, so that any flat module has finite projective dimension. If M is any Gorenstein flat module, then [26, Theorem 4.11] implies that there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow F \longrightarrow G \longrightarrow 0,$$

where F is flat and $G \in \mathtt{PGF}(R)$. Since F has finite projective dimension, Theorem 10 (iii) implies that $\text{PGF-dim}_R M = \text{pd}_R F \le \text{splf} R$. We conclude that $s \le \text{splf} R$, as needed.

Considering the projective dimension of direct sums of flat modules, it is easily seen that $\mathtt{Flat}(R) \subseteq \overline{\mathtt{Proj}}(R)$ if and only if $\text{splf} R < \infty$. In the same way, we may consider the PGF-dimension of direct sums of Gorenstein flat modules (cf. Proposition 3) and conclude that $\mathtt{GFlat}(R) \subseteq \overline{\mathtt{PGF}}(R)$ if and only if $s < \infty$. Therefore, the final statement in the Proposition follows from the equality $s = \text{splf} R$. □

We may complement the characterization of the finiteness of PGF-dimension given in Theorem 10, in the case of a Gorenstein flat module M, by requiring that the module K that appears in assertions (ii), (iii), (iv) and (v) therein be also flat. To that end, we note that any Gorenstein flat module of finite projective dimension is necessarily flat. Indeed, such a module must also have finite flat dimension and its flatness follows then from [6, §2]; see also [15, Remark 1.5].

Proposition 16. *The following conditions are equivalent for a Gorenstein flat module M and a non-negative integer n:*

(i) $\text{PGF-dim}_R M = n$.

(ii) *There exists a short exact sequence*

$$0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0,$$

where G is a PGF-module and K is a flat module with $\text{pd}_R K = n-1$. *If $n = 0$, this is understood to mean $K = 0$. If $n = 1$, we also require that the exact sequence be non-split.*

(iii) *There exists a short exact sequence*

$$0 \longrightarrow M \longrightarrow K \longrightarrow G \longrightarrow 0,$$

where G is a PGF-module and K is a flat module with $\text{pd}_R K = n$.

(iv) *There exists a projective module P, such that the module $M' = M \oplus P$ fits into an exact sequence*

$$0 \longrightarrow G \longrightarrow M' \longrightarrow K \longrightarrow 0,$$

which remains exact after applying the functor $\text{Hom}_R(\_\_, Q)$ *for any module $Q \in \mathtt{PGF}(R)^{\perp}$, where G is a PGF-module and K is a flat module with* $\text{pd}_R K = n$.

(v) *There exists a PGF-module P, such that the module $M' = M \oplus P$ fits into an exact sequence*

$$0 \longrightarrow G \longrightarrow M' \longrightarrow K \longrightarrow 0,$$

where G is a PGF-module and K is a flat module with $\text{pd}_R K = n$. *If $n = 1$, we also require that the exact sequence remain exact after applying the functor* $\text{Hom}_R(\_\_, Q)$ *for any projective module Q.*

Proof. We proceed as in the proof of Theorem 10, showing that (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) ⇒ (i). Since M is Gorenstein flat, $\mathtt{PGF}(R) \subseteq \mathtt{GFlat}(R)$ and the class of Gorenstein flat modules is projectively resolving (cf. [26, Corollary 4.12]), the module K appearing in (ii) and (iii) is a Gorenstein flat module of finite projective dimension; as noted above, this forces K to be flat. We also note that the argument in the proof of the implication (iii) ⇒ (iv) in Theorem 10 provides a short exact sequence as in (iv) with K being *the same* module K that appears in (iii). □

4. Hovey triples on $\overline{\mathtt{PGF}}(R)$ and $\mathtt{GFlat}(R)$

We shall now relate the results obtained in the previous section to the theory of exact model structures and describe a hereditary Hovey triple in the exact category $\overline{\mathtt{PGF}}(R)$ of modules of finite PGF-dimension, which is such that the homotopy category of the associated exact model structure is equivalent as a triangulated category to the stable category of PGF-modules. We shall also describe the stable category of PGF-modules, up to triangulated equivalence, as the homotopy category of the exact model structure associated with a similar Hovey triple in the exact category $\mathtt{GFlat}(R)$ of Gorenstein flat modules.

It is easily seen that $\mathtt{PGF}(R)$ is an exact Frobenius category with projective-injective objects given by the projective modules. The proof of the latter claim is essentially identical to the proof of the corresponding claim for the class of Gorenstein projective modules, which can be found for instance in [13, Proposition 2.2].

The category $\overline{\mathtt{PGF}}(R)$ of modules of finite PGF-dimension is an extension closed subcategory of the abelian category of all modules (cf. Proposition 4 (i)), which is also closed under direct summands (cf. Proposition 3). Therefore, $\overline{\mathtt{PGF}}(R)$ is an idempotent complete exact additive category [8]. The following result is an analogue of [13, Theorem 3.7]. The idea is that in order to realize the stable category of PGF-modules as the homotopy category of a Quillen model structure, it suffices to work on the subcategory $\overline{\mathtt{PGF}}(R)$ of modules of finite PGF-dimension. We note that the class $\overline{\mathtt{Proj}}(R)$ of modules of finite projective dimension is closed under direct summands and has the 2-out-of-3 property for short exact sequences.

Theorem 17. *The triple* $\big(\mathtt{PGF}(R), \overline{\mathtt{Proj}}(R), \overline{\mathtt{PGF}}(R)\big)$ *is a hereditary Hovey triple in the idempotent complete exact category* $\overline{\mathtt{PGF}}(R)$*. The homotopy category of the associated exact model structure is equivalent, as a triangulated category, to the stable category of PGF-modules.*

Proof. We need to prove that the pairs

$$\big(\mathtt{PGF}(R), \overline{\mathtt{Proj}}(R) \cap \overline{\mathtt{PGF}}(R)\big) \quad \text{and} \quad \big(\mathtt{PGF}(R) \cap \overline{\mathtt{Proj}}(R), \overline{\mathtt{PGF}}(R)\big)$$

are complete and hereditary cotorsion pairs in the exact category $\overline{\mathtt{PGF}}(R)$. Since any PGF-module is Gorenstein projective, we conclude that

$$\mathtt{PGF}(R) \cap \overline{\mathtt{Proj}}(R) \subseteq \mathtt{GProj}(R) \cap \overline{\mathtt{Proj}}(R) = \mathtt{Proj}(R),$$

where the latter equality follows from [22, Proposition 2.27]. On the other hand, projective modules are contained in both classes $\mathtt{PGF}(R)$ and $\overline{\mathtt{Proj}}(R)$ and hence $\mathtt{PGF}(R) \cap \overline{\mathtt{Proj}}(R) = \mathtt{Proj}(R)$.[2] Thus, the two pairs displayed above become

$$\big(\mathtt{PGF}(R), \overline{\mathtt{Proj}}(R)\big) \quad \text{and} \quad \big(\mathtt{Proj}(R), \overline{\mathtt{PGF}}(R)\big).$$

We begin by considering the pair $\big(\mathtt{PGF}(R), \overline{\mathtt{Proj}}(R)\big)$ and note that Proposition 14 states precisely that this is indeed a cotorsion pair in $\overline{\mathtt{PGF}}(R)$. Theorem 10 provides the approximations referring to completeness, whereas Proposition 12 (iii′), applied to the case where $n = 0$, shows that the cotorsion pair is hereditary.

We now consider the pair $\big(\mathtt{Proj}(R), \overline{\mathtt{PGF}}(R)\big)$ and note that $\overline{\mathtt{PGF}}(R)$ is obviously the right orthogonal of $\mathtt{Proj}(R)$ within $\overline{\mathtt{PGF}}(R)$. In order to prove that $\mathtt{Proj}(R)$ is the left orthogonal of $\overline{\mathtt{PGF}}(R)$ within $\overline{\mathtt{PGF}}(R)$, we let M be a module of finite PGF-dimension which is also contained in ${}^{\perp}\overline{\mathtt{PGF}}(R)$ and consider a short exact sequence

$$0 \longrightarrow M' \longrightarrow P \longrightarrow M \longrightarrow 0,$$

where P is projective. Then, Proposition 4 (ii) implies that M' has also finite PGF-dimension and hence $\mathrm{Ext}^1_R(M, M') = 0$. In particular, the exact sequence above splits. It follows that M is

[2] Alternatively, the equality $\mathtt{PGF}(R) \cap \overline{\mathtt{Proj}}(R) = \mathtt{Proj}(R)$ follows since $\mathtt{Proj}(R) \subseteq \overline{\mathtt{Proj}}(R) \subseteq \mathtt{PGF}(R)^{\perp}$ and $\mathtt{PGF}(R) \cap \mathtt{PGF}(R)^{\perp} = \mathtt{Proj}(R)$; cf. [26].

a direct summand of P and hence M is projective. The cotorsion pair $\big(\texttt{Proj}(R), \overline{\texttt{PGF}}(R)\big)$ in $\overline{\texttt{PGF}}(R)$ is hereditary (since all higher Ext's with a projective first argument vanish) and complete (since the class $\overline{\texttt{PGF}}(R)$ is projectively resolving).

The rest of the statement follows from [18, Proposition 4.4 and Corollary 4.8]. ☐

The category $\texttt{GFlat}(R)$ of Gorenstein flat modules is also closed under extensions and direct summands; this follows from [26, Corollary 4.12]. Hence, $\texttt{GFlat}(R)$ is an idempotent complete exact category as well. As shown in [26, Theorem 4.4], the group $\text{Ext}^1_R(M,F)$ is trivial whenever M is a PGF-module and F is flat. This vanishing actually provides a characterization of PGF-modules and flat modules, if we restrict to Gorenstein flat modules.

Proposition 18. *Let N be a Gorenstein flat module. Then:*

(i) $N \in \texttt{PGF}(R)$ *if and only if* $\text{Ext}^1_R(N,F) = 0$ *for any flat module* F.
(ii) N *is flat if and only if* $\text{Ext}^1_R(M,N) = 0$ *for any* $M \in \texttt{PGF}(R)$.

Proof. **(i).** As we noted above, the Ext-group is trivial if N is a PGF-module. Conversely, assume that N is a Gorenstein flat module contained in ${}^{\perp}\texttt{Flat}(R)$. Then, there exists a short exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow N \longrightarrow 0,$$

where G is a PGF-module and F is flat; cf. [26, Theorem 4.11(2)]. In view of our assumption on N, this short sequence splits and hence N is a direct summand of the PGF-module G. Since the class $\texttt{PGF}(R)$ is closed under direct summands, we conclude that $N \in \texttt{PGF}(R)$.

(ii). As we noted above, the Ext-group is trivial if N is flat. Conversely, assume that N is a Gorenstein flat module contained in $\texttt{PGF}(R)^{\perp}$. Then, there exists a short exact sequence

$$0 \longrightarrow N \longrightarrow F \longrightarrow G \longrightarrow 0,$$

where G is a PGF-module and F is flat; cf. [26, Theorem 4.11(4)]. In view of our assumption on N, this short sequence splits and hence N is a direct summand of the flat module F. Therefore, N is flat. ☐

We note that the class $\texttt{Flat}(R)$ of flat modules is closed under direct summands and has the 2-out-of-3 property within the class of Gorenstein flat modules. Of course, $\texttt{Flat}(R)$ is closed under extensions and kernels of epimorphisms. Moreover, if the cokernel of a monomorphism between flat modules is Gorenstein flat, then that cokernel is necessarily flat.[3] The proof of the following result is very similar to the proof of Theorem 17.

Theorem 19. *The triple* $(\texttt{PGF}(R), \texttt{Flat}(R), \texttt{GFlat}(R))$ *is a hereditary Hovey triple in the idempotent complete exact category* $\texttt{GFlat}(R)$. *The homotopy category of the associated exact model structure is equivalent, as a triangulated category, to the stable category of PGF-modules.*

Proof. We need to prove that the pairs

$$(\texttt{PGF}(R), \texttt{Flat}(R) \cap \texttt{GFlat}(R)) \quad \text{and} \quad (\texttt{PGF}(R) \cap \texttt{Flat}(R), \texttt{GFlat}(R))$$

are complete and hereditary cotorsion pairs in the exact category $\texttt{GFlat}(R)$. Since $\texttt{PGF}(R) \cap \texttt{Flat}(R) = \texttt{Proj}(R)$, the two pairs displayed above become

$$(\texttt{PGF}(R), \texttt{Flat}(R)) \quad \text{and} \quad \big(\texttt{Proj}(R), \texttt{GFlat}(R)\big).$$

We begin by considering the pair $(\texttt{PGF}(R), \texttt{Flat}(R))$ and note that Proposition 18 states precisely that this is indeed a cotorsion pair in the exact category $\texttt{GFlat}(R)$. Completeness of the

[3] We have pointed out in the discussion preceding Proposition 16 that any Gorenstein flat module of finite flat dimension is necessarily flat.

cotorsion pair follows from the exact sequences in [26, Theorem 4.11(2),(4)], whereas Proposition 12 (ii), applied to the case where $n = 0$, shows that the cotorsion pair is hereditary.

We now consider the pair $(\mathtt{Proj}(R), \mathtt{GFlat}(R))$ and note that $\mathtt{GFlat}(R)$ is obviously the right orthogonal of $\mathtt{Proj}(R)$ within $\mathtt{GFlat}(R)$. In order to prove that $\mathtt{Proj}(R)$ is the left orthogonal of $\mathtt{GFlat}(R)$ within $\mathtt{GFlat}(R)$, we let M be a Gorenstein flat module which is also contained in $^{\perp}\mathtt{GFlat}(R)$ and consider a short exact sequence

$$0 \longrightarrow M' \longrightarrow P \longrightarrow M \longrightarrow 0,$$

where P is projective. Since the class $\mathtt{GFlat}(R)$ is projectively resolving (cf. [26, Corollary 4.12]), we deduce that M' is also Gorenstein flat. Therefore, $\mathrm{Ext}^1_R(M, M') = 0$ and the exact sequence above splits. It follows that M is a direct summand of P and hence M is projective. The cotorsion pair $(\mathtt{Proj}(R), \mathtt{GFlat}(R))$ in $\mathtt{GFlat}(R)$ is hereditary (since all higher Ext's with a projective first argument vanish) and complete (since the class of Gorenstein flat modules is projectively resolving).

The final statement follows from [18, Proposition 4.4 and Corollary 4.8]. □

Remark 20. Another model for the stable category of PGF-modules can be obtained from the Hovey triple $(\mathtt{PGF}(R), \mathtt{PGF}(R)^{\perp}, R\text{-Mod})$ on the category R-Mod of all modules; cf. [26, Theorem 4.9] and [19, Proposition 37]. A possible advantage of the Hovey triples presented in this section is that the classes of modules that are involved herein admit a more manageable description.

5. The finiteness of the PGF global dimension

In this section, we characterize those rings over which all modules have finite PGF-dimension, in terms of classical homological invariants. As a consequence of this description, we generalize a result by Jensen [24] (on commutative Noetherian rings) and another result by Gedrich and Gruenberg [17] (on group rings of groups over a commutative Noetherian coefficient ring).

We define the (left) PGF global dimension $\text{PGF-gl.dim}\, R$ of the ring R, by letting

$$\text{PGF-gl.dim}\, R = \sup\{\text{PGF-dim}_R M : M \text{ a left } R\text{-module}\}.$$

Using the characterization of the finiteness of the Gorenstein global dimension and the Gorenstein weak global dimension, we may characterize the finiteness of $\text{PGF-gl.dim}\, R$, as follows:

Theorem 21. *The following conditions are equivalent for a ring R:*

(i) $\text{PGF-gl.dim}\, R < \infty$,
(ii) $\text{PGF-dim}_R M < \infty$ *for any module M,*
(iii) $\text{spli}\, R = \text{silp}\, R < \infty$ *and* $\text{sfli}\, R = \text{sfli}\, R^{op} < \infty$,
(iv) $\text{spli}\, R < \infty$ *and* $\text{sfli}\, R^{op} < \infty$.

If these conditions are satisfied, then $\text{PGF-gl.dim}\, R = \text{spli}\, R = \text{silp}\, R (= \text{Ggl.dim}\, R)$.

Proof. It is clear that (i) ⇒ (ii), whereas the implication (ii) ⇒ (i) is an immediate consequence of Proposition 3.

(ii) ⇒ (iii). Since $\mathtt{PGF}(R)$ is contained in both classes $\mathtt{GProj}(R)$ and $\mathtt{GFlat}(R)$, our hypothesis implies that any module M has both finite Gorenstein projective dimension and finite Gorenstein flat dimension. (In fact, both $\text{Gpd}_R M$ and $\text{Gfd}_R M$ are bounded by $\text{PGF-dim}_R M < \infty$.) Then, assertion (iii) follows from the characterization of the finiteness of the Gorenstein global dimension and the Gorenstein weak global dimension of R; cf. Section 1.2.

(iii) ⇒ (iv). This is straightforward.

(iv)⇒(ii). Assume that $\operatorname{spli} R = n < \infty$ and fix a module M. Then, the construction by Gedrich and Gruenberg in [17, §4] provides us with an acyclic complex of projective modules

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow Q_{n-1} \longrightarrow Q_{n-2} \longrightarrow \cdots, \tag{3}$$

which coincides in degrees $\geq n$ with a projective resolution

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of M. Since the acyclic complex (3) consists of projective (and hence flat) modules, it remains acyclic by applying the functor $L \otimes_R \_\_$ for any right module L of finite flat dimension; this follows easily by induction on the flat dimension of L. In particular, our assumption about the finiteness of $\operatorname{sfli} R^{op}$ implies that the complex (3) remains acyclic by applying the functor $I \otimes_R \_\_$ for any injective right module I. Therefore, the module $K = \operatorname{coker}(P_{n+1} \longrightarrow P_n)$ is a PGF-module. Then, the exact sequence

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

shows that $\text{PGF-dim}_R M \leq n < \infty$, as needed.

The final claim in the statement of the Theorem is an immediate consequence of Corollary 13 (ii), which implies that $\text{PGF-gl.dim}\, R = \text{Ggl.dim}\, R$, if $\text{PGF-gl.dim}\, R$ is finite. □

As an immediate consequence of the equivalence between assertions (iii) and (iv) in Theorem 21 above, we obtain the following result.

Corollary 22. *Let R be a ring, such that both invariants* $\operatorname{spli} R$ *and* $\operatorname{sfli} R^{op}$ *are finite. Then,* $\operatorname{silp} R = \operatorname{spli} R$.

We may obtain a left-right symmetric assertion, as follows.

Proposition 23. *Let R be a ring, such that both invariants* $\operatorname{spli} R$ *and* $\operatorname{spli} R^{op}$ *are finite. Then, we have* $\operatorname{silp} R = \operatorname{spli} R$ *and* $\operatorname{silp} R^{op} = \operatorname{spli} R^{op}$.

Proof. Since projective (left or right) modules are flat, we have

$$\operatorname{sfli} R \leq \operatorname{spli} R < \infty \quad \text{and} \quad \operatorname{sfli} R^{op} \leq \operatorname{spli} R^{op} < \infty.$$

Then, the result follows by applying Corollary 22 for the ring R and its opposite R^{op}. □

Corollary 24. *If R is a ring which is isomorphic with its opposite R^{op}, then* $\operatorname{silp} R \leq \operatorname{spli} R$ *with equality if* $\operatorname{spli} R < \infty$.

Proof. The inequality is obvious if $\operatorname{spli} R = \infty$ and hence it suffices to consider the case where $\operatorname{spli} R < \infty$. Then, $\operatorname{spli} R^{op} = \operatorname{spli} R$ is also finite and we may invoke Proposition 23. □

We recall that a ring R is called left (resp. right) $\aleph_0$-Noetherian if any left (resp. right) ideal of R is countably generated. For example, countable rings and countably generated algebras over fields are both left and right $\aleph_0$-Noetherian.

Remarks 25.

(i) Let k be a commutative ring, G a group and $R = kG$ the associated group algebra. Then, R is isomorphic with its opposite R^{op} and hence Corollary 24 implies that $\operatorname{silp} R \leq \operatorname{spli} R$. In the special case where the coefficient ring k is Noetherian of finite self-injective dimension, this inequality was proved by Gedrich and Gruenberg in [17, Theorem 2.4], using the Hopficity of the group algebra R.

(ii) Let k be a commutative $\aleph_0$-Noetherian ring, G a group and $R = kG$ the associated group algebra. Then, we may invoke [14, Proposition 4.3] and conclude that the inequality in (i) above is actually an equality, i.e. $\operatorname{silp} R = \operatorname{spli} R$. In this way, we extend the main result of [14] from the case of commutative Noetherian rings of finite self-injective dimension to any commutative $\aleph_0$-Noetherian ring of coefficients.

Proposition 26. *If R is a ring which is both left and right $\aleph_0$-Noetherian, then the following conditions are equivalent:*

(i) *The invariants* $\operatorname{spli} R$ *and* $\operatorname{spli} R^{op}$ *are finite.*
(ii) *The invariants* $\operatorname{silp} R$ *and* $\operatorname{silp} R^{op}$ *are finite.*

If these conditions are satisfied, then $\operatorname{silp} R = \operatorname{spli} R < \infty$ *and* $\operatorname{silp} R^{op} = \operatorname{spli} R^{op} < \infty$.

Proof. The implication (i) ⇒ (ii) follows from Proposition 23, whereas the implication (ii) ⇒ (i) is proved in [14, Theorem 3.6] using the $\aleph_0$-Noetherian hypothesis. □

Corollary 27. *Let R be a ring which is isomorphic with its opposite R^{op}. If R is left (and hence right) $\aleph_0$-Noetherian, then* $\operatorname{silp} R = \operatorname{spli} R$.

Corollary 28. *If R is a commutative $\aleph_0$-Noetherian ring, then* $\operatorname{silp} R = \operatorname{spli} R$.

Remark 29. In the special case where R is a commutative Noetherian ring, the equality in Corollary 28 was proved by Jensen in [24, 5.9].

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