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Representation theory / *Théorie des représentations*

On the symmetry of the finitistic dimension

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Dedicated to the memory of Helmut Lenzing.

Abstract. For any ring we propose the construction of a cover which increases the finitistic dimension on one side and decreases the finitistic dimension to zero on the opposite side. This complements recent work of Cummings.

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The finitistic dimension is a homological invariant of a ring which is conjectured to be finite when the ring is a finite dimensional algebra over a field [2]. In recent work [3] Cummings introduces for any finite dimensional algebra a related algebra; its purpose is to increase the finitistic dimension on one side and to decrease the finitistic dimension to zero on the opposite side. In this note we propose the construction of such an *asymmetric cover* for any ring and we establish the same properties. This specialises to Cummings' construction for finite dimensional algebras over an algebraically closed field and yields examples of rings such that the finitistic dimension is infinite while the finitistic dimension of the opposite ring is zero. We need to distinguish between the small and the big finitistic dimension but our results cover both cases.

Let A be an associative ring. We consider the category of (right) A-modules and identify (left) A-modules with modules over the opposite ring A^{op}. For a module M we write $\operatorname{rad} M$ for its *radical* and $\operatorname{soc} M$ for its *socle*. We set $\operatorname{top} M = M/\operatorname{rad} M$. The functor

$$(-)^* := \operatorname{Hom}_A(-, A)$$

yields a duality between right and left A-modules. We consider the *trivial extension*

$$T(A) := A \ltimes A^\natural$$

which is given by the bimodule $A^\natural := {}_A A_A$. This ring is by definition the abelian group $T(A) = A \oplus A^\natural$ with multiplication given by the formula

$$(x, y) \cdot \big(x', y'\big) = \big(xx', xy' + yx'\big).$$

Note that $A^\natural$ is a two-sided ideal with $T(A)/A^\natural \xrightarrow{\sim} A$.

Lemma 1. *Let A be a semisimple ring. Then*

$$\operatorname{rad} T(A) = A^\natural = \operatorname{soc} T(A) \quad \textit{and} \quad \operatorname{top} T(A) \cong \operatorname{soc} T(A).$$

Proof. The first assertion is clear. Left multiplication with $(0,1)$ gives a map $T(A) \to T(A)$ which induces an isomorphism $\operatorname{top} T(A) \xrightarrow{\sim} \operatorname{soc} T(A)$ □

For a ring A we denote by $\Sigma(A)$ the set of isomorphism classes of simple A-modules. Set

$$\bar{S} := \coprod_{S\in\Sigma(A)} S \quad\text{and}\quad \bar{A} := \prod_{S\in\Sigma(A)} T(\operatorname{End}_A(S)).$$

We view $\bar{S}$ as an $\bar{A}$-A-bimodule, with left action via $T(\operatorname{End}_A(S)) \twoheadrightarrow \operatorname{End}_A(S)$ for each S in $\Sigma(A)$, and consider the triangular matrix ring

$$\tilde{A} := \begin{bmatrix} A & 0 \\ \bar{S} & \bar{A} \end{bmatrix}.$$

The idempotents $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $f = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ provide an $\tilde{A}$-module decomposition $\tilde{A} = P \oplus Q$, where

$$P := e\tilde{A} \cong A \qquad\text{and}\qquad Q := f\tilde{A} \cong \bar{S} \oplus \bar{A}.$$

We call $\tilde{A}$ a *cover* of A because the idempotent $e \in \tilde{A}$ yields an isomorphism

$$\operatorname{End}_{\tilde{A}}(P) \cong e\tilde{A}e \cong A.$$

The following lemma expresses the distinct property of the cover $\tilde{A}$, namely that every simple $\tilde{A}$-module embeds into $\tilde{A}$.

Lemma 2. *We have $S^* \neq 0$ for every simple $\tilde{A}$-module S.*

Proof. We claim that each simple $\tilde{A}$-module arises as the image of a morphism $\tilde{A} = P \oplus Q \to Q$, using that

$$\operatorname{Hom}_{\tilde{A}}(P,Q) \cong e\tilde{A}f = \bar{S} \quad\text{and}\quad \operatorname{Hom}_{\tilde{A}}(Q,Q) \cong f\tilde{A}f = \bar{A}.$$

With Lemma 1 we compute $\operatorname{soc} Q$ and obtain a decomposition into simples:

$$\operatorname{soc} Q = \bar{S} \oplus \operatorname{soc} \bar{A} \cong \coprod_{S\in\Sigma(A)} \left(S \oplus \operatorname{End}_A(S)^{\natural}\right).$$

A simple $\tilde{A}$-module T comes either with a nonzero map $P \to T$ or a nonzero map $Q \to T$. In the first case T identifies with a simple A-module via the inclusion $A \to \tilde{A}$ given by $x \mapsto \left[\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and therefore with a summand of $\bar{S} \subseteq \operatorname{soc} Q$. In the second case T identifies with a simple $\bar{A}$-module via the inclusion $\bar{A} \to \tilde{A}$ given by $x \mapsto \left[\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right]$, and therefore with a summand of $\operatorname{soc} \bar{A} \subseteq \operatorname{soc} Q$. In any case one obtains a monomomorphism $T \to Q \hookrightarrow \tilde{A}$. □

Let $\mathcal{P}(A)$ denote the class of A-modules M that admit a finite resolution

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with all P_i finitely generated projective. We denote by

$$\operatorname{fin.dim} A := \sup\left\{\operatorname{proj.dim} M \,\middle|\, M \in \mathcal{P}(A)\right\}$$

the *small finitistic dimension* of A; this is a slight variation of the usual definition which seems natural as the modules in $\mathcal{P}(A)$ are precisley the ones which become compact (or perfect) when viewed as an object in the derived category of A.

The following lemma is [5, Lemma 7.2.8] and its proof is sketched for the convenience of the reader; cf. the discussion in [2, § 5].

Lemma 3. *For a ring A we have* $\operatorname{fin.dim} A = 0$ *if and only if $M^* \neq 0$ for every finitely presented A^{op}-module M.*

Proof. Let $\operatorname{proj} A$ denote the category of finitely generated projective A-modules. The condition $\operatorname{fin.dim} A = 0$ means that every monomorphism in $\operatorname{proj} A$ splits. The duality

$$(-)^* \colon \left(\operatorname{proj} A\right)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{proj}\left(A^{\mathrm{op}}\right)$$

translates this into the condition on finitely presented A^{op}-modules. □

Theorem 4. *For a ring A we have*

$$\operatorname{fin.dim} \tilde{A} \geq \operatorname{fin.dim} A \qquad \textit{and} \qquad \operatorname{fin.dim} \tilde{A}^{\mathrm{op}} = 0.$$

Proof. The idempotent $e \in \tilde{A}$ with $e\tilde{A}e \cong A$ gives rise to a fully faithful functor

$$- \otimes_A e\tilde{A} \colon \operatorname{Mod} A \longrightarrow \operatorname{Mod} \tilde{A}$$

which is exact and maps projectives to projectives; also it preserves finite generation. This yields the first assertion. The second assertion follows from Lemma 3 because we have $M^* \neq 0$ for every finitely presented $\tilde{A}$-module M by Lemma 2. □

There is a somewhat more natural construction of a cover when the ring A is semilocal. Recall that A is *semilocal* if the ring $A/\operatorname{rad} A$ is semisimple. In this case we have an idempotent $\varepsilon \in A/\operatorname{rad} A$ and a Morita equivalence

$$A/\operatorname{rad} A \sim \varepsilon(A/\operatorname{rad} A)\varepsilon \cong \prod_{S \in \Sigma(A)} \operatorname{End}_A(S).$$

We set

$$\widetilde{A} := \begin{bmatrix} A & 0 \\ A/\operatorname{rad} A & T(A/\operatorname{rad} A) \end{bmatrix}$$

and this is closely related to the cover $\tilde{A}$ via the idempotent $\tilde{\varepsilon} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & (\varepsilon,0) \end{smallmatrix}\right]$.

Lemma 5. *For a semilocal ring A we have a Morita equivalence*

$$\widetilde{A} \sim \tilde{\varepsilon}\widetilde{A}\tilde{\varepsilon} \cong \tilde{A}.$$

Proof. We use some general facts. Let $\Lambda = \left[\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right]$ be a triangular matrix ring and $e \in B$ an idempotent such that B and eBe are Morita equivalent. Set $\bar{e} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & e \end{smallmatrix}\right]$. Then Λ and $\bar{e}\Lambda\bar{e} = \left[\begin{smallmatrix} A & 0 \\ eM & eBe \end{smallmatrix}\right]$ are Morita equivalent. Also, the trivial extensions $T(B)$ and $T(eBe) = (e,0)T(B)(e,0)$ are Morita equivalent. □

We may identify $\widetilde{A}$ with $\tilde{A}$, as we are mostly interested in homological properties. In fact, $\widetilde{A} \cong \tilde{A}$ when A is semiperfect and basic. Note that the definition of $\widetilde{A}$ does not depend on any choices. In particular, we have an identity $\widetilde{A^{\mathrm{op}}} = \widetilde{A}^{\mathrm{op}}$.

Next we discuss some ring theoretic properties which are preserved under the passage from A to its cover $\tilde{A}$. Recall that a ring is *semiprimary* if it is semilocal and its radical is a nilpotent ideal.

Remark 6. If A is semilocal then $\tilde{A}$ is semilocal. This follows from the isomorphism

$$\operatorname{top} \tilde{A} = \operatorname{top} P \oplus \operatorname{top} Q \xrightarrow{\sim} A/\operatorname{rad} A \oplus (A/\operatorname{rad} A)^{\natural} = \operatorname{soc} Q$$

which is induced by the morphism $\tilde{A} = P \oplus Q \to Q \hookrightarrow \tilde{A}$ given by left multiplication with $\left[\begin{smallmatrix} 0 & 0 \\ 1 & (0,1) \end{smallmatrix}\right]$. Moreover, in this case the inclusion $A \hookrightarrow \tilde{A}$ yields the identity $(\operatorname{rad} A)^n = (\operatorname{rad} \tilde{A})^n$ for all $n > 1$.

Remark 7. If the ring A is left or right perfect then the same holds for $\tilde{A}$. This follows from Remark 6, since A is right perfect if and only if A is semilocal and $\operatorname{rad} A$ is right T-nilpotent.

There is an analogue of Theorem 4 for the *big finitistic dimension*

$$\operatorname{Fin.dim} A := \sup\left\{\operatorname{proj.dim} M \mid M \in \operatorname{Mod} A,\ \operatorname{proj.dim} M < \infty\right\}.$$

We use the following fact which is a slight variation of [2, Theorem 6.3].

Proposition 8. *For a ring A we have* $\operatorname{Fin.dim} A = 0$ *if and only if A is right perfect and* $\operatorname{fin.dim} A = 0$.

Proof. Suppose that A is right perfect and $\operatorname{fin.dim} A = 0$. We need to show that every monomorphism $\phi\colon M \to N$ between projective A-modules splits. This holds when M and N are finitely generated since $\operatorname{fin.dim} A = 0$. Because A is right perfect, any projective A-module decomposes into a direct sum of finitely generated modules and can therefore be written as a filtered colimit of finitely generated direct summands. Choose such a presentation $M = \operatorname{colim}_i M_i$. Then ϕ is a filtered colimit of split monomorphisms $\phi_i\colon M_i \to N$, and therefore $\operatorname{colim}_i \operatorname{Coker}\phi_i \cong \operatorname{Coker}\phi$ is a filtered colimit of projectives. Thus $\operatorname{Coker}\phi$ is projective and ϕ splits. For the other implication we refer to [2]. □

Theorem 9. *For a ring A we have*

$$\operatorname{Fin.dim} \tilde{A} \geq \operatorname{Fin.dim} A.$$

Moreover, $\operatorname{Fin.dim} \tilde{A}^{\mathrm{op}} = 0$ *if and only if A is left perfect.*

Proof. The first assertion is easily checked as in the proof of Theorem 4. If $\operatorname{Fin.dim} \tilde{A}^{\mathrm{op}} = 0$, then $\tilde{A}$ is left perfect by Proposition 8, and this implies that A is left perfect. For the converse suppose that A is left perfect. Then $\tilde{A}$ is left perfect by Remark 7. Thus $\operatorname{Fin.dim} \tilde{A}^{\mathrm{op}} = 0$ by Theorem 4 and Proposition 8. □

The preceding results demonstrate a failure of symmetry for the notion of 'finite finitistic dimension', as pointed out in the recent work of Cummings [3]. In particular we have the following examples.

Recall that a noetherian ring is *regular* if all its finitely generated modules have finite projective dimension.

Corollary 10. *Let A be a commutative noetherian ring that is regular of infinite Krull dimension. Then*

$$\operatorname{fin.dim} \tilde{A} = \infty \quad \textit{and} \quad \operatorname{fin.dim} \tilde{A}^{\mathrm{op}} = 0.$$

Proof. The finitistic dimension $\operatorname{fin.dim} A$ is infinite by [1, Theorem 1.6 and Corollary 1.7]. Thus the assertion follows from Theorem 4. □

Specific examples of regular rings of infinite Krull dimension have been constructed by Nagata; cf. [5, Example 7.2.20].

We continue with an example due to Kirkman and Kuzmanovich [4]. Let k be a field and consider the quotient $\Lambda = kQ/I$ of the path algebra kQ given by the quiver

$$Q\colon \qquad \circ \underset{b_i}{\overset{a_i}{\rightleftarrows}} \circ \quad (i \in \mathbb{N})$$

(with k-basis given by the paths in Q and multiplication induced by the composition of paths, where for any pair of paths α, β we write $\beta\alpha$ for the composite when the terminal vertex of α equals the initial vertex of β) modulo the ideal I that is generated by the elements

$$b_t a_s b_r \quad (r,s,t \in \mathbb{N}) \qquad b_t a_s - a_t b_t \quad (t > s) \qquad a_r b_r \quad (r \in \mathbb{N}).$$

Note that Λ is a semiprimary ring with $(\operatorname{rad} \Lambda)^4 = 0$.

Corollary 11. *The ring $\tilde{\Lambda}$ is semiprimary satisfying*

$$\operatorname{Fin.dim} \tilde{\Lambda} = \operatorname{fin.dim} \tilde{\Lambda} = \infty \qquad \textit{and} \qquad \operatorname{Fin.dim} \tilde{\Lambda}^{\mathrm{op}} = \operatorname{fin.dim} \tilde{\Lambda}^{\mathrm{op}} = 0.$$

Proof. From Remark 6 it follows that the ring $\tilde{\Lambda}$ is semilocal with $(\operatorname{rad} \tilde{\Lambda})^4 = 0$. Thus $\tilde{\Lambda}$ is semiprimary. In [4] it is shown that $\operatorname{fin.dim} \Lambda = \infty$. Then the assertion follows from Theorem 9, using that Λ is left perfect. □

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