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Generalized H-fold sumset and Subsequence sum

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Abstract. Let A and H be nonempty finite sets of integers and positive integers, respectively. The *generalized H-fold sumset*, denoted by $H^{(r)}A$, is the union of the sumsets $h^{(r)}A$ for $h \in H$ where, the sumset $h^{(r)}A$ is the set of all integers that can be represented as a sum of h elements from A with no summand in the representation appearing more than r times. In this paper, we find the optimal lower bound for the cardinality of $H^{(r)}A$, i.e., for $|H^{(r)}A|$ and the structure of the underlying sets A and H when $|H^{(r)}A|$ is equal to the optimal lower bound in the cases A contains only positive integers and A contains only nonnegative integers. This generalizes recent results of Bhanja. Furthermore, with a particular set H, since $H^{(r)}A$ generalizes *subsequence sum* and hence *subset sum*, we get several results of subsequence sums and subset sums as special cases.

Keywords. sumset, subset sum, subsequence sum.

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1. Introduction

Let $\mathbb{N}$ be the set of positive integers. Let $A = \{a_1, \dots, a_k\}$ be a nonempty finite set of integers and h be a positive integer. The *h-fold sumset*, denoted by hA, and the *restricted h-fold sumset*, denoted by $h^\wedge A$ of A, are defined, respectively, by

$$hA := \left\{ \sum_{i=1}^{k} \lambda_i a_i : \lambda_i \in \mathbb{N} \cup \{0\} \text{ for } i = 1, \dots, k \text{ with } \sum_{i=1}^{k} \lambda_i = h \right\},$$

$$h^\wedge A := \left\{ \sum_{i=1}^{k} \lambda_i a_i : \lambda_i \in \{0,1\} \text{ for } i = 1, \dots, k \text{ with } \sum_{i=1}^{k} \lambda_i = h \right\}.$$

Mistri and Pandey [6] generalized hA and $h^\wedge A$, into the generalized h-fold sumset, denoted by $h^{(r)}A$, as follows:

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Let r be a positive integer such that $1 \leq r \leq h$. The *generalized h-fold sumset* $h^{(r)}A$, is defined by
$$h^{(r)}A := \left\{ \sum_{i=1}^{k} \lambda_i a_i : 0 \leq \lambda_i \leq r \text{ for } i = 1, \ldots, k \text{ with } \sum_{i=1}^{k} \lambda_i = h \right\}.$$
So, the generalized h-fold sumset $h^{(r)}A$ is the set of all sums of h elements of A, in which every summand can repeat at most r times. Therefore, hA and $h^\wedge A$ are particular cases of $h^{(r)}A$ for $r = h$ and $r = 1$, respectively.

For a finite set H of positive integers, Bajnok [1] introduced the sumset
$$HA := \bigcup_{h \in H} hA,$$
and the restricted sumset
$$H^\wedge A := \bigcup_{h \in H} h^\wedge A.$$
In a recent article, Bhanja and Pandey [5] considered a generalization of HA and $H^\wedge A$, the *generalized H-fold sumset*, denoted by $H^{(r)}A$, defined by
$$H^{(r)}A := \bigcup_{h \in H} h^{(r)}A.$$
Observed that, if $r \geq \max(H)$, then $H^{(r)}A = HA$ and if $r = 1$, then $H^{(r)}A = H^\wedge A$. The sumset $H^{(r)}A$ becomes more important as it also generalizes *subset sums* and *subsequence sums*.

1.1. *Subset sum and Subsequence sum*

Let A be a finite set of integers. The sum of all the elements of a given subset B of A is called *subset sum* and it is denoted by $s(B)$. That is,
$$s(B) = \sum_{b \in B} b.$$
The set of all nonempty subset sum of A, denoted by $\sum(A)$, that is
$$\sum(A) = \left\{ s(B) : \emptyset \neq B \subseteq A \right\}.$$
Also we define, for $1 \leq \alpha \leq k$
$$\sum_{\alpha}(A) = \left\{ s(B) : \emptyset \neq B \subseteq A \text{ and } |B| \geq \alpha \right\}.$$
Similarly, we define subsequence sum of a given sequence of integers. Let $A = \{a_1, a_2, \ldots, a_k\}$ be a set of k integers and r be a positive integer, with $a_1 < a_2 < \cdots < a_k$. Then we define a sequence associated with A as
$$\mathbb{A} = (\underbrace{a_1, \ldots, a_1}_{r-\text{times}}, \underbrace{a_2, \ldots, a_2}_{r-\text{times}}, \ldots, \underbrace{a_k, \ldots, a_k}_{r-\text{times}}) = (a_1, a_2, \ldots, a_k)_r \text{(say)}.$$
Let $\mathbb{B}$ be a subsequence of $\mathbb{A}$. Then
$$\mathbb{B} = (\underbrace{a_1, \ldots, a_1}_{r_1-\text{times}}, \underbrace{a_2, \ldots, a_2}_{r_2-\text{times}}, \ldots, \underbrace{a_k, \ldots, a_k}_{r_k-\text{times}}) \text{ with } 0 \leq r_i \leq r.$$
Given any subsequence $\mathbb{B}$ of $\mathbb{A}$, the sum of all terms of the subsequence $\mathbb{B}$ is called the *subsequence sum*, is denoted by $s(\mathbb{B})$ and we write
$$s(\mathbb{B}) = \sum_{b \in \mathbb{B}} b.$$
The set of all subsequence sums of a given sequence $\mathbb{A}$ is the set
$$\sum(\mathbb{A}) = \left\{ s(\mathbb{B}) : \mathbb{B} \text{ is subsequence of } \mathbb{A} \text{ of length} \geq 1 \right\}.$$

For $1 \le \alpha \le kr$, define

$$\sum_{\alpha}(\mathbb{A}) = \left\{s(\mathbb{B}) : \mathbb{B} \text{ is subsequence of } \mathbb{A} \text{ of length} \ge \alpha\right\}.$$

Note that, we can write

$$h^{(r)}A = \left\{s(\mathbb{B}) : \mathbb{B} \text{ is subsequence } \mathbb{A} \text{ of length } h\right\}.$$

With suitable sets H, we can express $\sum(A), \sum_{\alpha}(A), \sum(\mathbb{A})$ and $\sum_{\alpha}(\mathbb{A})$ in terms of $H^{\wedge}A$ and $H^{(r)}A$, as follows:

- If $H = \{1, 2, \ldots, k\}$, then $H^{\wedge}A = \bigcup_{h=1}^{k} h^{\wedge}A = \sum(A)$.
- If $H = \{\alpha, \alpha+1, \ldots, k\}$, then $H^{\wedge}A = \bigcup_{h=\alpha}^{k} h^{\wedge}A = \sum_{\alpha}(A)$.
- If $H = \{1, 2, \ldots, kr\}$, then $H^{(r)}A = \bigcup_{h=1}^{kr} h^{(r)}A = \sum(\mathbb{A})$.
- If $H = \{\alpha, \alpha+1, \ldots, kr\}$, then $H^{(r)}A = \bigcup_{h=\alpha}^{kr} h^{(r)}A = \sum_{\alpha}(\mathbb{A})$.

Let $A = \{a_1, a_2, \ldots, a_k\}$ be a nonempty set of integers with $a_1 < a_2 < \cdots < a_k$. For an integer c, we write $c * A = \{ca : a \in A\}$ and for integers a and b with $a < b$, we write $[a, b] = \{a, a+1, \ldots, b\}$. For a nonempty set $S = \{s_1, s_2, \ldots, s_{n-1}, s_n\}$, we let $\max(S), \min(S), \max_{-}(S), \min_{+}(S)$ be the largest, smallest, second largest and second smallest elements of S, respectively. For a given real number x, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote, floor function and ceiling function of x, respectively. We assume $\sum_{i=1}^{t} f(i) = 0$ if $t < 1$.

Two standard problems associated with a sumset in additive number theory are to find best possible lower bound for the cardinality of sumset when the set A is known (called the direct problem) and to find the structure of the underlying set A when the size of the sumset attains its lower bound (called the inverse problem). These two types of problems have been solved for the sumsets in various types of groups. We have several classical results on sumsets for the case when A is a subset of group of integers, (see [1, 3, 6, 8–11]), and for subsequence sums and subset one may refer to [2, 4, 5, 7]. We mention now, some of these results that are applied in this paper.

Theorem 1 ([10, Theorem 1.3, Theorem 1.6]). *Let $h \ge 1$, and let A be a nonempty finite set of integers. Then*

$$|hA| \ge h|A| - h + 1.$$

This lower bound is best possible. Furthermore, if $|hA|$ attains this lower bound with $h \ge 2$, then A is an arithmetic progression.

Theorem 2 ([9, 10, Theorem 1, Theorem 2]). *Let A be a nonempty finite set of integers, and let $1 \le h \le |A|$. Then*

$$\left|h^{\wedge}A\right| \ge h|A| - h^2 + 1.$$

This lower bound is best possible. Furthermore, if $|h^{\wedge}A|$ attains this lower bound with $|A| \ge 5$ and $2 \le h \le |A| - 2$, then A is an arithmetic progression.

Mistri and Pandey [6] generalized above results as follows:

Theorem 3 ([6, Theorem 2.1]). *Let A be a nonempty finite set of k integers. Let r and h be integers such that $1 \le r \le h \le kr$. Set $m = \lfloor h/r \rfloor$. Then*

$$|h^{(r)}A| \ge mr(k-m) + (h-mr)(k-2m-1) + 1.$$

This lower bound is best possible.

Theorem 4 ([6, Theorem 3.1, Theorem 3.2]). *Let $k \ge 3$. Let r and $h \ge 2$ be integers such that $1 \le r \le h \le kr - 2$ and $(k, h, r) \ne (4, 2, 1)$. Set $m = \lfloor h/r \rfloor$. If A is a finite set of k integers such that*

$$|h^{(r)}A| = mr(k-m) + (h-mr)(k-2m-1) + 1,$$

then A is an arithmetic progression.

Further generalization of $h^{(r)}A$ was considered in [6] for which the direct and inverse results were proved by Yang and Chen [11]. Direct results for $h^{(r)}A$ when A is a subset of the group of residual classes modulo a prime and A is a subset of a finite cyclic group were given, respectively, by Monopoli [8] and Bhanja [3].

The direct and inverse theorems for the sumsets HA and $H^{\wedge}A$ proved by Bhanja [2] are the following:

Theorem 5 ([2, Theorem 3]). *Let A be a set of k positive integers. Let H be a set of t positive integers with* $\max(H) = h_t$. *Then*

$$|HA| \geq h_t(k-1) + t.$$

This lower bound is optimal.

Theorem 6 ([2, Theorem 5]). *Let A be a set of* $k \geq 2$ *positive integers and H be a set of* $t \geq 2$ *positive integers with* $\max(H) = h_t$. *If*

$$|HA| = h_t(k-1) + t,$$

then H is an arithmetic progression with common difference d and A is an arithmetic progression with common difference $d * \min(A)$.

Theorem 7 ([2, Theorem 6, Corollary 7]). *Let A be a set of k nonnegative integers and* $H = \{h_1, h_2, \ldots, h_t\}$ *be a set ofpositive integers with* $h_1 < h_2 < \cdots < h_t$. *Set* $h_0 = 0$. *If* $0 \notin A$ *and* $h_t \leq k$, *then*

$$\left|H^{\wedge}A\right| \geq \sum_{i=1}^{t} (h_i - h_{i-1})(k - h_i) + t.$$

If $0 \in A$ *and* $h_t \leq k-1$, *then*

$$\left|H^{\wedge}A\right| \geq h_1 + \sum_{i=1}^{t} (h_i - h_{i-1})(k - h_i - 1) + t.$$

The lower bounds are optimal.

Theorem 8 ([2, Theorem 9, Corollary 10]). *Let A be a set of k nonnegative integers. Let* $H = \{h_1, h_2, \ldots, h_t\}$ *be a set ofpositive integers with* $h_1 < h_2 < \cdots < h_t$. *Set* $h_0 = 0$. *If* $0 \notin A$, $k \geq 6$, $h_t \leq k-1$, *and*

$$\left|H^{\wedge}A\right| = \sum_{i=1}^{t} (h_i - h_{i-1})(k - h_i) + t,$$

then $H = h_1 + [0, t-1]$ *and* $A = \min(A) * [1, k]$.

If $0 \in A$, $k \geq 7$, $h_t \leq k-2$, *and*

$$\left|H^{\wedge}A\right| = h_1 + \sum_{i=1}^{t} (h_i - h_{i-1})(k - h_i - 1) + t,$$

then $H = h_1 + [0, t-1]$ *and* $A = \min(A \setminus \{0\}) * [0, k-1]$.

In this paper, we prove similar direct and inverse results for the sumset $H^{(r)}A$ when A is a finite nonempty set of positive integers. In Sections 2 and 3, we prove our main theorems, Theorem 9 and Theorem 14, the direct and inverse theorems for sumset $H^{(r)}A$, when A is a finite set of positive integers. Consequentaly we prove direct and inverse theorems when A contains nonnegative integers with $0 \in A$.

To state our main results we need some notation that are used throughout the paper. Let $H = \{h_1, h_2, \ldots, h_t\}$ be a set of positive integers with $0 = h_0 < h_1 < h_2 < \cdots < h_t$ and r be a positive integer. If $t = 1$, then $H^{(r)}A = h_1^{(r)}A$. So, we assume $t \geq 2$. If $r > h_t$, then $h_i^{(r)}A = h_iA$ for $1 \leq i \leq t$, giving $H^{(r)}A = HA$. So we assume that $r \leq h_t$. There always exists a unique positive integer l such that $h_{l-1} < r \le h_l$, where $1 \le l \le t$. For $i = 1,2,\dots,t$, let $h_i = m_i r + \epsilon_i$, where $0 \le \epsilon_i \le r-1$. For given set H of positive integers and set of integers A with $|H| = t$ and $|A| = k$, let

$$\mathscr{L}(H^{(r)}A) = h_{l-1}(k-1) + (l-1) + \sum_{i=l}^{t} r(m_i - m_{i-1})(k-m_i) + \sum_{i=l}^{t}\Big((\epsilon_i - \epsilon_{i-1})(k-m_i-1) - \max\{\epsilon_i,\epsilon_{i-1}\}(m_i - m_{i-1}) + 1\Big). \quad (1)$$

Note that, if $0 \le i \le l-1$, then $m_i = 0$ and $\epsilon_i = h_i$. So, we can also write

$$\mathscr{L}(H^{(r)}A) = \sum_{i=1}^{t}\Big(r(m_i - m_{i-1})(k-m_i) + (\epsilon_i - \epsilon_{i-1})(k-m_i-1) - \max\{\epsilon_i,\epsilon_{i-1}\}(m_i - m_{i-1}) + 1\Big).$$

For $i = 1,\dots,t$, define

$$M_i = \left\lfloor \frac{h_i - h_{i-1}}{r} \right\rfloor$$

and for $j = 0,\dots,t-1$, define

$$N_j = \begin{cases} \left\lceil \frac{h_j}{r} \right\rceil & \text{if } l-1 \le j \le t-1 \\ 0 & \text{otherwise.} \end{cases}$$

Also, let $\{0\}^{(r)}A = \{0\}$.

2. Direct Theorems

Theorem 9. *Let A be a nonempty finite set of $k \ge 3$ positive integers. Let r be a positive integer and H be a set of $t \ge 2$ positive integers with $1 \le r \le \max(H) \le (k-1)r - 1$. Then*

$$\left|H^{(r)}A\right| \ge \mathscr{L}(H^{(r)}A). \quad (2)$$

This lower bound is best possible.

Proof. Let $A = \{a_1, a_2, \dots, a_k\}$ and $H = \{h_1, h_2, \dots, h_t\}$ be such that

$$0 < a_1 < a_2 < \cdots < a_k \quad \text{and} \quad 0 = h_0 < h_1 < h_2 < \cdots < h_t.$$

For $i = 0,1,\dots,t$, write $h_i = m_i r + \epsilon_i$, where $0 \le \epsilon_i \le r-1$. Then, we have

$$0 = m_0 \le m_1 \le m_2 \le \cdots \le m_t \le k-2.$$

Since l is a positive integer satisfying $h_{l-1} < r \le h_l$, we have $m_i = 0$ and $\epsilon_i = h_i$ for $i = 0,\dots,l-1$. Set $S_0 = \emptyset$. Define

$$S_i = (h_i - h_{i-1})^{(r)}A_i + \max\{h_{i-1}^{(r)}A\} \quad \text{for } i = 1,2,\dots,t,$$

where

$$A_i = \{a_1,\dots,a_{k-N_{i-1}}\} \quad \text{for } i = 1,2,\dots,t.$$

Note that $S_i \subseteq h_i^{(r)}A \subseteq H^{(r)}A$ and $\max(S_i) < \min(S_{i+1})$ for all $i \in [1,t-1]$. We shall define sets $T_i \subseteq h_i^{(r)}A$ that satistfy $T_i \cap S_i = \emptyset$ for $i \in [0,t]$. Let $R_i = S_i \cup T_i \subseteq h_i^{(r)}A$, for $i = 0,1,\dots,t$. If $i \in [0,l-1]$, then define $T_i = \emptyset$. So, $|R_0| = 0$ for $l \ge 1$, and by Theorem 1, we have $|R_i| = |S_i| \ge (h_i - h_{i-1})(k-1) + 1$ for $l \ge 2$ and $i \in [1,l-1]$. If $i \in [l,t]$, then we define T_i for every possible values of ϵ_{i-1} and ϵ_i, and consequently find $|R_i|$.

Let $i \in [l,t]$ be such that $\epsilon_{i-1} = 0$ and $\epsilon_i \ge 0$. Then $M_i = m_i - m_{i-1}$ and $N_{i-1} = m_{i-1}$. Let $T_i = \emptyset$ in this case. Then by Theorem 3, we have

$$\begin{aligned}|R_i| &= |S_i| + |T_i| \\ &\ge M_i r(k - N_{i-1} - M_i) + (h_i - h_{i-1} - M_i r)(k - N_{i-1} - 2M_i - 1) + 1 \\ &= r(m_i - m_{i-1})(k-m_i) + (\epsilon_i - \epsilon_{i-1})(k-m_i-1) - \epsilon_i(m_i - m_{i-1}) + 1.\end{aligned}$$

Let $i \in [l,t]$ be such that $\epsilon_i > \epsilon_{i-1} > 0$ and $m_i = m_{i-1}$. Then $M_i = m_i - m_{i-1} = 0$ and $N_{i-1} = m_{i-1} + 1$. For $j = 0, 1, \ldots, \epsilon_i - \epsilon_{i-1}$, define

$$T_{i,j}^0 = (\epsilon_i - \epsilon_{i-1} - j)a_{k-m_i-1} + (\epsilon_{i-1} + j)a_{k-m_i} + \sum_{p=1}^{m_i} r a_{k-m_i+p}.$$

Then, we have $\max(S_i) = T_{i,0}^0 < T_{i,1}^0 < \cdots < T_{i,\epsilon_i-\epsilon_{i-1}}^0 = \max(h_i^{(r)}A) < \min(S_{i+1})$. Let

$$T_i = \left\{T_{i,j}^0 : j = 1, \ldots, \epsilon_i - \epsilon_{i-1}\right\}. \tag{3}$$

Then by Theorem 3 and (3), we have

$$\begin{aligned} |R_i| &= |S_i| + |T_i| \\ &\geq M_i r(k - N_{i-1} - M_i) + (h_i - h_{i-1} - M_i r)(k - N_{i-1} - 2M_i - 1) + 1 + (\epsilon_i - \epsilon_{i-1}) \\ &= r(m_i - m_{i-1})(k - m_i) + (\epsilon_i - \epsilon_{i-1})(k - m_i - 1) - \epsilon_i(m_i - m_{i-1}) + 1. \end{aligned}$$

Let $i \in [l,t]$ be such that $\epsilon_i > \epsilon_{i-1} > 0$ and $m_i = m_{i-1} + 1$. Then $M_i = m_i - m_{i-1} = 1$ and $N_{i-1} = m_{i-1} + 1 = m_i$. For $j = 0, \ldots, \epsilon_i - \epsilon_{i-1} - 1$, define

$$T_{i,j}^0 = (\epsilon_i - \epsilon_{i-1} - j)a_{k-m_{i-1}-2} + r a_{k-m_{i-1}-1} + (\epsilon_{i-1} + j)a_{k-m_{i-1}} + \sum_{p=1}^{m_{i-1}} r a_{k-m_{i-1}+p},$$

$$T_{i,j}^1 = (\epsilon_i - \epsilon_{i-1} - j)a_{k-m_{i-1}-2} + (r-1)a_{k-m_{i-1}-1} + (\epsilon_{i-1} + j + 1)a_{k-m_{i-1}} + \sum_{p=1}^{m_{i-1}} r a_{k-m_{i-1}+p}$$

and for $j = 0, 1, \ldots, r - \epsilon_i$,

$$U_{i,j}^0 = (r - j)a_{k-m_{i-1}-1} + (\epsilon_i + j)a_{k-m_{i-1}} + \sum_{p=1}^{m_{i-1}} r a_{k-m_{i-1}+p}.$$

Then, we have $\max(S_i) = T_{i,0}^0 < T_{i,0}^1 < T_{i,1}^0 < T_{i,1}^1 < \cdots < T_{i,\epsilon_i-\epsilon_{i-1}-1}^0 < T_{i,\epsilon_i-\epsilon_{i-1}-1}^1 < U_{i,0}^0 < U_{i,1}^0 < \cdots < U_{i,r-\epsilon_i}^0 = \max(h_i^{(r)}A) < \min(S_{i+1})$. Assume $\left\{T_{i,j}^0 : j = 1, \ldots, \epsilon_i - \epsilon_{i-1} - 1\right\} = \emptyset$, if $\epsilon_i - \epsilon_{i-1} = 1$. Let

$$T_i = \left\{T_{i,j}^0 : j = 1, \ldots, \epsilon_i - \epsilon_{i-1} - 1\right\} \cup \left\{T_{i,j}^1 : j = 0, \ldots, \epsilon_i - \epsilon_{i-1} - 1\right\} \cup \left\{U_{i,j}^0 : j = 0, \ldots, r - \epsilon_i\right\}. \tag{4}$$

Then by Theorem 3 and (4), we have

$$\begin{aligned} |R_i| &= |S_i| + |T_i| \\ &\geq M_i r(k - N_{i-1} - M_i) + (h_i - h_{i-1} - M_i r)(k - N_{i-1} - 2M_i - 1) + 1 + 2(\epsilon_i - \epsilon_{i-1}) + (r - \epsilon_i) \\ &= r(m_i - m_{i-1})(k - m_i) + (\epsilon_i - \epsilon_{i-1})(k - m_i - 1) - \epsilon_i(m_i - m_{i-1}) + 1. \end{aligned}$$

Let $i \in [l,t]$ be such that $\epsilon_i > \epsilon_{i-1} > 0$ and $m_i > m_{i-1} + 1$. Then $M_i = m_i - m_{i-1} \geq 2$ and $N_{i-1} = m_{i-1} + 1$. For $j = 0, \ldots, \epsilon_i - \epsilon_{i-1} - 1$ and $q = 1, \ldots, m_i - m_{i-1}$, define

$$T_{i,j}^0 = (\epsilon_i - \epsilon_{i-1} - j)a_{k-m_i-1} + \left(\sum_{p=1}^{m_i-m_{i-1}} r a_{k-m_i-1+p}\right) + (\epsilon_{i-1} + j)a_{k-m_{i-1}} + \sum_{p=1}^{m_{i-1}} r a_{k-m_{i-1}+p},$$

$$\begin{aligned} T_{i,j}^q = (\epsilon_i - \epsilon_{i-1} - j)a_{k-m_i-1} + \left(\sum_{p=1,\ p\neq m_i-m_{i-1}+1-q}^{m_i-m_{i-1}} r a_{k-m_i-1+p}\right) + (r-1)a_{k-m_{i-1}-q} \\ + (\epsilon_{i-1} + j + 1)a_{k-m_{i-1}} + \sum_{p=1}^{m_{i-1}} r a_{k-m_{i-1}+p}. \end{aligned}$$

For $j = 0, \ldots, r - \epsilon_i - 1$ and $q = 1, \ldots, m_i - m_{i-1} - 1$, define

$$U_{i,j}^0 = (r - j)a_{k-m_i} + \sum_{p=1}^{m_i-m_{i-1}-1} r a_{k-m_i+p} + (\epsilon_i + j)a_{k-m_{i-1}} + \sum_{p=1}^{m_{i-1}} r a_{k-m_{i-1}+p},$$

$$U_{i,j}^{q}=(r-j)a_{k-m_i}+\left(\sum_{p=1,\ p\neq m_i-m_{i-1}-q}^{m_i-m_{i-1}-1} ra_{k-m_i+p}\right)+(r-1)a_{k-m_{i-1}-q}+(\epsilon_i+j+1)a_{k-m_{i-1}} + \sum_{p=1}^{m_{i-1}} ra_{k-m_{i-1}+p}.$$

Furthermore, define

$$U_{i,r-\epsilon_i}^{0}=\epsilon_i a_{k-m_i}+\sum_{p=k-m_i+1}^{k} ra_p.$$

Then $U_{i,r-\epsilon_i}^{0}=\max(h_i^{(r)}A)<\min(S_{i+1})$ and

$$\begin{array}{ccccccc}
\max(S_i)=T_{i,0}^{0} & < & T_{i,0}^{1} & <\cdots< & T_{i,0}^{m_i-m_{i-1}} & < \\
T_{i,1}^{0} & < & T_{i,1}^{1} & <\cdots< & T_{i,1}^{m_i-m_{i-1}} & < \\
\vdots & \vdots & \vdots & \vdots\ \ \vdots & \vdots & \vdots \\
T_{i,\epsilon_i-\epsilon_{i-1}-1}^{0} & < & T_{i,\epsilon_i-\epsilon_{i-1}-1}^{1} & <\cdots< & T_{i,\epsilon_i-\epsilon_{i-1}-1}^{m_i-m_{i-1}} & < \\
U_{i,0}^{0} & < & U_{i,0}^{1} & <\cdots< & U_{i,0}^{m_i-m_{i-1}-1} & < \\
U_{i,1}^{0} & < & U_{i,1}^{1} & <\cdots< & U_{i,1}^{m_i-m_{i-1}-1} & < \\
\vdots & \vdots & \vdots & \vdots\ \ \vdots & \vdots & \vdots \\
U_{i,r-\epsilon_i-1}^{0} & < & U_{i,r-\epsilon_i-1}^{1} & <\cdots< & U_{i,r-\epsilon_i-1}^{m_i-m_{i-1}-1} & < U_{i,r-\epsilon_i}^{0}.
\end{array}$$

Assume $\{T_{i,j}^{0}: j=1,\ldots,\epsilon_i-\epsilon_{i-1}-1\}=\emptyset$, if $\epsilon_i-\epsilon_{i-1}=1$. Let

$$\begin{aligned}T_i=&\left\{T_{i,j}^{0}: j=1,\ldots,\epsilon_i-\epsilon_{i-1}-1\right\}\cup\left\{T_{i,j}^{q}: j=0,\ldots,\epsilon_i-\epsilon_{i-1}-1; q=1,\ldots,m_i-m_{i-1}\right\}\\ &\cup\left\{U_{i,j}^{0}: j=0,\ldots,r-\epsilon_i\right\}\cup\left\{U_{i,j}^{q}: j=0,\ldots,r-\epsilon_i-1; q=1,\ldots,m_i-m_{i-1}-1\right\}.\end{aligned}\tag{5}$$

Then by Theorem 3 and (5), we have

$$\begin{aligned}|R_i|&=|S_i|+|T_i|\\ &\geq M_i r(k-N_{i-1}-M_i)+(h_i-h_{i-1}-M_i r)(k-N_{i-1}-2M_i-1)+1\\ &\quad+(\epsilon_i-\epsilon_{i-1})(m_i-m_{i-1}+1)+(r-\epsilon_i)(m_i-m_{i-1})\\ &=r(m_i-m_{i-1})(k-m_i)+(\epsilon_i-\epsilon_{i-1})(k-m_i-1)-\epsilon_i(m_i-m_{i-1})+1.\end{aligned}$$

Let $i\in[l,t]$ be such that $\epsilon_i=\epsilon_{i-1}>0$ and $m_i=m_{i-1}+1$. Then $M_i=m_i-m_{i-1}=1$ and $N_{i-1}=m_{i-1}+1$. For $j=0,\ldots,r-\epsilon_i$, define

$$U_{i,j}^{0}=(r-j)a_{k-m_{i-1}-1}+(\epsilon_i+j)a_{k-m_{i-1}}+\sum_{p=1}^{m_{i-1}} ra_{k-m_{i-1}+p}.$$

Then $\max(S_i)=U_{i,0}^{0}<U_{i,1}^{0}<\cdots<U_{i,r-\epsilon_i}^{0}=\max(h_i^{(r)}A)<\min(S_{i+1})$. Let

$$T_i=\left\{U_{i,j}^{0}: j=1,\ldots,r-\epsilon_i\right\}.\tag{6}$$

Then by Theorem 3 and (6), we have

$$\begin{aligned}|R_i|&=|S_i|+|T_i|\\ &\geq M_i r(k-N_{i-1}-M_i)+(h_i-h_{i-1}-M_i r)(k-N_{i-1}-2M_i-1)+1+(r-\epsilon_i)\\ &=r(m_i-m_{i-1})(k-m_i)+(\epsilon_i-\epsilon_{i-1})(k-m_i-1)-\epsilon_i(m_i-m_{i-1})+1.\end{aligned}$$

Let $i \in [l, t]$ be such that $\epsilon_i = \epsilon_{i-1} > 0$ and $m_i > m_{i-1} + 1$. Then $M_i = m_i - m_{i-1} \geq 2$ and $N_{i-1} = m_{i-1} + 1$. For $j = 0, \ldots, r - \epsilon_i - 1$ and $q = 1, \ldots, m_i - m_{i-1} - 1$, define

$$U^0_{i,j} = (r-j)a_{k-m_i} + \left(\sum_{p=1}^{m_i - m_{i-1}-1} r a_{k-m_i+p}\right) + (\epsilon_i + j)a_{k-m_{i-1}} + \sum_{p=1}^{m_{i-1}} r a_{k-m_{i-1}+p}$$

and

$$U^q_{i,j} = (r-j)a_{k-m_i} + \left(\sum_{p=1,\ p\neq m_i - m_{i-1}-q}^{m_i - m_{i-1}-1} r a_{k-m_i+p}\right) + (r-1)a_{k-m_{i-1}-q} + (\epsilon_i + j + 1)a_{k-m_{i-1}} + \sum_{p=1}^{m_{i-1}} r a_{k-m_{i-1}+p}.$$

Furthermore, define

$$U^0_{i,r-\epsilon_i} = \epsilon_i a_{k-m_i} + \sum_{p=1}^{m_i} r a_{k-m_i+p}.$$

It is easy to see that $U^0_{i,r-\epsilon_i} = \max(h_i^{(r)} A) < \min(S_{i+1})$ and

$$\begin{array}{ccccccc} \max(S_i) = U^0_{i,0} & < & U^1_{i,0} & < \cdots < & U^{m_i-m_{i-1}-1}_{i,0} & < & \\ U^0_{i,1} & < & U^1_{i,1} & < \cdots < & U^{m_i-m_{i-1}-1}_{i,1} & < & \\ \vdots & \vdots & \vdots & \vdots\ \ \vdots & \vdots & \vdots & \\ U^0_{i,r-\epsilon_i-1} & < & U^1_{i,r-\epsilon_i-1} & < \cdots < & U^{m_i-m_{i-1}-1}_{i,r-\epsilon_i-1} & < & U^0_{i,r-\epsilon_i}. \end{array}$$

Let

$$T_i = \{U^0_{i,j} : j = 1, \ldots, r - \epsilon_i\} \cup \{U^q_{i,j} : j = 0, \ldots, r - \epsilon_i - 1 \text{ and } q = 1, \ldots, m_i - m_{i-1} - 1\}. \tag{7}$$

Then by Theorem 3 and (7), we have

$$\begin{aligned} |R_i| &= |S_i| + |T_i| \\ &\geq M_i r(k - N_{i-1} - M_i) + (h_i - h_{i-1} - M_i r)(k - N_{i-1} - 2M_i - 1) + 1 \\ &\quad + (r - \epsilon_i)(m_i - m_{i-1}) \\ &= r(m_i - m_{i-1})(k - m_i) + (\epsilon_i - \epsilon_{i-1})(k - m_i - 1) - \epsilon_i(m_i - m_{i-1}) + 1. \end{aligned}$$

Let $i \in [l, t]$ be such that $\epsilon_i < \epsilon_{i-1}$ and $m_i = m_{i-1} + 1$. Then $M_i = m_i - m_{i-1} - 1 = 0$ and $N_{i-1} = m_{i-1} + 1 = m_i$. For $j = 0, \ldots, r - \epsilon_{i-1}$, define

$$T^0_{i,j} = (r + \epsilon_i - \epsilon_{i-1} - j)a_{k-m_{i-1}-1} + (\epsilon_{i-1} + j)a_{k-m_{i-1}} + \sum_{p=1}^{m_{i-1}} r a_{k-m_{i-1}+p}.$$

Then $\max(S_i) = T^0_{i,0} < T^0_{i,1} < \cdots < T^0_{i,r-\epsilon_{i-1}} = \max(h_i^{(r)} A) < \min(S_{i+1})$. Let

$$T_i = \{T^0_{i,j} : j = 1, \ldots, r - \epsilon_{i-1}\}. \tag{8}$$

Then by Theorem 3 and (8), we have

$$\begin{aligned} |R_i| &= |S_i| + |T_i| \\ &\geq M_i r(k - N_{i-1} - M_i) + (h_i - h_{i-1} - M_i r)(k - N_{i-1} - 2M_i - 1) + 1 + (r - \epsilon_{i-1}) \\ &= r(m_i - m_{i-1})(k - m_i) + (\epsilon_i - \epsilon_{i-1})(k - m_i - 1) - \epsilon_{i-1}(m_i - m_{i-1}) + 1. \end{aligned}$$

Let $i \in [l,t]$ be such that $\epsilon_i < \epsilon_{i-1}$ and $m_i > m_{i-1}+1$. Then $M_i = m_i - m_{i-1} - 1 \geq 1$ and $N_{i-1} = m_{i-1}+1$. For $j = 0,\ldots,r-\epsilon_{i-1}-1$ and $q = 1,\ldots,m_i - m_{i-1}-1$, define

$$T^0_{i,j} = (r+\epsilon_i - \epsilon_{i-1} - j)a_{k-m_i} + \left(\sum_{p=1}^{m_i-m_{i-1}-1} ra_{k-m_i+p}\right) + (\epsilon_{i-1}+j)a_{k-m_{i-1}} + \sum_{p=1}^{m_{i-1}} ra_{k-m_{i-1}+p},$$

$$T^q_{i,j} = (r+\epsilon_i - \epsilon_{i-1} - j)a_{k-m_i} + \left(\sum_{p=1,\ p\neq m_i-m_{i-1}-q}^{m_i-m_{i-1}-1} ra_{k-m_i+p}\right) + (r-1)a_{k-m_{i-1}-q}$$
$$+ (\epsilon_{i-1}+j+1)a_{k-m_{i-1}} + \sum_{p=1}^{m_{i-1}} ra_{k-m_{i-1}+p}.$$

Define also

$$T^0_{i,r-\epsilon_{i-1}} = \epsilon_i a_{k-m_i} + \sum_{p=1}^{m_i} ra_{k-m_i+p}.$$

It is easy to see that $T^0_{i,r-\epsilon_{i-1}} = \max(h_i^{(r)}A) < \min(S_{i+1})$ and

$$\begin{array}{ccccccc}
\max(S_i) = T^0_{i,0} & < & T^1_{i,0} & < \cdots < & T^{m_i-m_{i-1}-1}_{i,0} & < & \\
T^0_{i,1} & < & T^1_{i,1} & < \cdots < & T^{m_i-m_{i-1}-1}_{i,1} & < & \\
\vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots & \vdots & \\
T^0_{i,r-\epsilon_{i-1}-1} & < & T^1_{i,r-\epsilon_{i-1}-1} & < \cdots < & T^{m_i-m_{i-1}-1}_{i,r-\epsilon_{i-1}-1} & < & T^0_{i,r-\epsilon_{i-1}} < \min(S_{i+1}).
\end{array}$$

Let

$$T_i = \left\{T^0_{i,j} : j = 1,\ldots,r-\epsilon_{i-1}\right\} \cup \left\{T^q_{i,j} : j = 0,\ldots,r-\epsilon_{i-1}-1 \text{ and } q = 1,\ldots,m_i-m_{i-1}-1\right\}. \quad (9)$$

Then by Theorem 3 and (9), we have

$$\begin{aligned}
|R_i| &= |S_i| + |T_i| \\
&\geq M_i r(k - N_{i-1} - M_i) + (h_i - h_{i-1} - M_i r)(k - N_{i-1} - 2M_i - 1) + 1 \\
&\quad + (r - \epsilon_{i-1})(m_i - m_{i-1}) \\
&> r(m_i - m_{i-1})(k - m_i) + (\epsilon_i - \epsilon_{i-1})(k - m_i - 1) - \epsilon_{i-1}(m_i - m_{i-1}) + 1.
\end{aligned}$$

Hence

$$\begin{aligned}
|H^{(r)}A| &\geq \sum_{i=0}^{t}|R_i| = \sum_{i=0}^{l-1}|S_i| + \sum_{i=l}^{t}|S_i \cup T_i| \\
&\geq \sum_{i=1}^{l-1}(h_i - h_{i-1})(k-1) + 1 \\
&\quad + \sum_{i=l}^{t} r(m_i - m_{i-1})(k - m_i) + (\epsilon_i - \epsilon_{i-1})(k - m_i - 1) - \max\{\epsilon_i, \epsilon_{i-1}\}(m_i - m_{i-1}) + 1 \\
&= h_{l-1}(k-1) + (l-1) \\
&\quad + \sum_{i=l}^{t} r(m_i - m_{i-1})(k - m_{i)} + (\epsilon_i - \epsilon_{i-1})(k - m_i - 1) - \max\{\epsilon_i, \epsilon_{i-1}\}(m_i - m_{i-1}) + 1 \\
&= \mathscr{L}(H^{(r)}A).
\end{aligned}$$

This proves (2). Next, we show that this bound is best possible. Let $H = [1,(k-1)r-1]$, $A = \{1,2,\ldots,k\}$. Then $H^{(r)}A \subseteq [1, 2(r-1)+3r+\cdots+kr]$. So $|H^{(r)}A| \leq \frac{rk(k+1)}{2} - r - 2$. On the other hand, we have by (2), $|H^{(r)}A| \geq \frac{rk(k+1)}{2} - r - 2$. This completes the proof of Theorem 9. □

Remark 10. Following the notation from Theorem 9.

(a) If $0 = h_0 < h_1 < \cdots < h_{t_0-1} < (k-1)r \le h_{t_0} < \cdots < h_t \le kr$ with $t_0 \ge 2$, then we have

$$\max(h_{t_0-1}^{(r)}A) < \max_{-}(h_{t_0}^{(r)}A) < \max(h_{t_0}^{(r)}A) < \max(h_{t_0+1}^{(r)}A) < \cdots < \max(h_t^{(r)}A).$$

So $|H^{(r)}A| \ge |H_{t_0-1}^{(r)}A| + t - t_0 + 2 \ge \mathscr{L}(H_{t_0-1}^{(r)}A) + t - t_0 + 2$, where $H_{t_0-1} = \{h_1, \ldots, h_{t_0-1}\}$, $t_0 \ge 2$. This lower bound is best possible, as that can be verified with $A = [1,k]$ and $H = [1, rk]$. Clearly, we have $|H^{(r)}A| = \frac{rk(k+1)}{2}$.

(b) If $0 = h_0 < (k-1)r \le h_1 < \cdots < h_t \le kr$, then

$$H^{(r)}A \supseteq h_1^{(r)}A \cup \{\max(h_i^{(r)}A) : i = 2, \ldots, t\}.$$

Therefore

$$\left|H^{(r)}A\right| \ge \left|h_1^{(r)}A\right| + t - 1 \ge m_1 r(k - m_1) + (h_1 - m_1 r)(k - 2m_1 - 1) + t,$$

where $m_1 = \lfloor h_1/r \rfloor$. To check, this bound is best possible, we take $A = [1,k]$ and $H = [(k-1)r, kr]$. Then $H^{(r)}A = [r + 2r + \cdots + (k-1)r, r + 2r + \cdots + kr]$ and hence $|H^{(r)}A| = kr + 1$.

Corollary 11. *Let A be a nonempty finite set of $k \ge 4$ nonnegative integers with $0 \in A$. Let r be a positive integer and H be a set of $t \ge 2$ positive integers with $1 \le r \le \max(H) \le (k-2)r - 1$. Let $m = \lceil \min(H)/r \rceil$ and $m_1 = \lfloor \min(H)/r \rfloor$. Then*

$$\left|H^{(r)}A\right| \ge m_1 r(m - m_1 + 1) + (\min(H) - m_1 r)(m - 2m_1) + \mathscr{L}\big(H^{(r)}(A \setminus \{0\})\big). \qquad (10)$$

This lower bound is best possible.

Proof. Let $A = \{0, a_1, \ldots, a_{k-1}\}$ be a set of nonnegative integers with $0 < a_1 < \cdots < a_{k-1}$ and $H = \{h_1, h_2, \ldots, h_t\}$ be a set of positive integers with $0 = h_0 < h_1 = \min(H) < h_2 < \cdots < h_t = \max(H)$. Consider $A' = A \setminus \{0\}$. Then $H^{(r)}A' \subseteq H^{(r)}A$.

Let $m = \lceil h_1/r \rceil$, $h_1 = m_1 r + \epsilon_1$, where $0 \le \epsilon_1 \le r - 1$ and $B = \{0, a_1, \ldots, a_m\}$. Then

$$h_1^{(r)}B \subseteq H^{(r)}A$$

and $h_1^{(r)}B \cap H^{(r)}A' = \max(h_1^{(r)}B) = \min(H^{(r)}A') = ra_1 + \cdots + ra_{m_1} + \epsilon_1 a_{m_1+1}$. Hence by Theorem 3 and Theorem 9, we have

$$\begin{aligned}\left|H^{(r)}A\right| &\ge \left|h_1^{(r)}B\right| + \left|H^{(r)}A'\right| - 1\\ &\ge m_1 r(m - m_1 + 1) + (\min(H) - m_1 r)(m - 2m_1) + \mathscr{L}\big(H^{(r)}(A \setminus \{0\})\big).\end{aligned}$$

This proves the Corollary. To check optimallity of the bound, take $A = [0, k-1]$ and $H = [1, (k-2)r - 1]$. Then $H^{(r)}A \subseteq [0, 2(r-1) + 3r + \cdots + (k-1)r]$ and $|H^{(r)}A| \le \frac{rk(k-1)}{2} - r - 1$. From (10), we have $|H^{(r)}A| \ge \frac{rk(k-1)}{2} - r - 1$. □

Remark 12. Following the notation from Corollary 11.

(a) If $0 = h_0 < h_1 < \cdots < h_{t_0-1} < (k-2)r \le h_{t_0} < \cdots < h_t \le (k-1)r$ with $t_0 \ge 2$, then we have

$$\max\big(h_{t_0-1}^{(r)}A\big) < \max_{-}\big(h_{t_0}^{(r)}A\big) < \max\big(h_{t_0}^{(r)}A\big) < \max\big(h_{t_0+1}^{(r)}A\big) < \cdots < \max\big(h_t^{(r)}A\big).$$

So $|H^{(r)}A| \ge m_1 r(m - m_1 + 1) + (\min(H) - m_1 r)(m - 2m_1) + \mathscr{L}\big(H^{(r)}(A \setminus \{0\})\big) + t - t_0 + 2$. This lower bound is best possible, as that can be verified with $A = [0, k-1]$ and $H = [1, (k-1)r]$. Clearly, we have $|H^{(r)}A| = \frac{rk(k-1)}{2} + 1$. Also, if we take $H = [1, (k-1)r] \cup X$, where $X \subseteq [(k-1)r + 1, kr]$, then again $|H^{(r)}A| = \frac{rk(k-1)}{2} + 1$.

(b) If $0 = h_0 < (k-2)r \le h_1 < \cdots < h_t \le (k-1)r$, then

$$H^{(r)}A \supseteq h_1^{(r)}A \cup \{\max(h_i^{(r)}A) : i = 2, \ldots, t\}.$$

Therefore

$$|H^{(r)}A| \ge |h_1^{(r)}A| + t - 1 \ge m_1 r(k - m_1) + (h_1 - m_1 r)(k - 2m_1 - 1) + t,$$

where $m_1 = \lfloor h_1/r \rfloor$. To check, this bound is best possible, we take $A = [0, k-1]$ and $H = [(k-2)r, (k-1)r]$. Then $H^{(r)}A = [r+2r+\cdots+(k-3)r, r+2r+\cdots+(k-1)r]$ and hence $|H^{(r)}A| = (2k-3)r+1$.

Remark 13.

(a) For $r = \max(H) = h_t$, Theorem 5 follows from Theorem 9 as a consequence.
(b) For $r = 1$, Theorem 7 follows from Remark 10 and Remark 12 as a consequence.

3. Inverse problem

This section deals with the inverse theorems associated with the sumset $H^{(r)}A$. In this section, we charaterize the sets A and H, when the cardinality of $H^{(r)}A$ is equal to its optimal lower bound. There are some cases in which either A or H or both may not be arithmetic progression but size of $H^{(r)}A$ is equal to the optimal lower bound (called extremal set). See some extremal sets in [6, Section 3] and [4, Section 2.2]. Here we give some more example of extremal sets.

(1) Let A be a set of $k\,(\geq 3)$ integers and r be a positive integer. If $H = \{1, rk\}$ or $H = \{rk-1, rk\}$, then $|H^{(r)}A| = k+1$.
(2) Let $A = \{a_1, a_2, a_1+a_2\}$ with $0 < a_1 < a_2$ and $H \subseteq \{1,2,3\}$ with $r = 1$; or $A = \{0, a_1, a_2, a_1+a_2,\}$ with $H \subseteq \{1,2,3\}$ and $r = 1$. Then the sets A are extremal sets.

We now present the main inverse results associated with $H^{(r)}A$.

Theorem 14. *Let $r \geq 1$ be a positive integer, A be a nonempty finite set of $k \geq 6$ positive integers and H be a set of $t \geq 2$ positive integers with $1 \leq r \leq \max(H) \leq (k-1)r-1$. If*

$$\left|H^{(r)}A\right| = \mathscr{L}\left(H^{(r)}A\right),$$

then H is an arithmetic progression with common difference $d \leq r$ and A is an arithmetic progression with common difference $d * \min(A)$.

Proof. Let $A = \{a_1, a_2, \ldots, a_k\}$ and $H = \{h_1, h_2, \ldots, h_t\}$ be such that

$$0 < a_1 < a_2 < \cdots < a_k \text{ and } 0 = h_0 < h_1 < h_2 < \cdots < h_t.$$

For $i = 1, \ldots, t$, let $h_i = m_i r + \epsilon_i$, where $0 \leq \epsilon_i \leq r-1$. Let l be a positive integer such that $h_{l-1} < r \leq h_l$, where $1 \leq l \leq t$. Since $|H^{(r)}A|$ is equal to its lower bound given in (2), we have, from the proof of Theorem 9 that, $|H^{(r)}A| = \sum_{i=1}^{t}|R_i|$. This implies that

$$|R_1| = \left|h_1^{(r)}A\right| = m_1 r(k-m_1) + (h_1 - m_1 r)(k - 2m_1 - 1) + 1$$

and $|R_i| = |S_i| + |T_i| = r(m_i - m_{i-1})(k - m_i) + (\epsilon_i - \epsilon_{i-1})(k - m_i - 1) - \max\{\epsilon_i, \epsilon_{i-1}\}(m_i - m_{i-1}) + 1$, for $i = 2, \ldots, t$. If $h_1 > 1$, then by Theorem 4, the set A is an arithmetic progression. Let $h_1 = 1$ and $h_2 > 2$. Then we have

$$R_1 = A \text{ and } R_2 = S_2 \cup T_2.$$

Therefore $|S_2|$ is minimum and hence $A_2 = \{a_1, a_2, \ldots, a_{k-1}\}$ is an arithmetic progression. Now we show that $a_{k-1} - a_{k-2} = a_k - a_{k-1}$. Let $m_2 \leq k-3$. We have

$$\begin{aligned}
a_{m_2+1} &< \min\left((h_2-1)^{(r)}A_2\right) + a_{m_2+1} \\
&< \min\left((h_2-1)^{(r)}A_2\right) + a_{m_2+2} \\
&\;\;\vdots \\
&< \min\left((h_2-1)^{(r)}A_2\right) + a_{k-1} \\
&< \min\left((h_2-1)^{(r)}A_2\right) + a_k = \min(R_2).
\end{aligned}$$

We also have $R_1 = A$ and $a_1 < a_2 < \cdots < a_{m_2+1} < a_{m_2+2} < \cdots < a_k < \min((h_2-1)^{(r)}A_2) + a_k = \min(R_2)$. So $\min((h_2-1)^{(r)}A_2) + a_{m_2+i} = a_{m_2+i+1}$ for $i = 1,2,\ldots,k-m_2-1$. This gives $a_{k-1} - a_{k-2} = a_k - a_{k-1}$.

Let $m_2 = k-2$ and $\epsilon_2 = 0$. Then

$$\begin{aligned} a_{k-2} &< ra_1 + \cdots + ra_{k-2} \\ &< ra_1 + \cdots + ra_{k-3} + (r-1)a_{k-2} + a_{k-1} \\ &< ra_1 + \cdots + ra_{k-3} + (r-1)a_{k-2} + a_k = \min(R_2). \end{aligned}$$

This implies that $ra_1 + \cdots + ra_{k-3} + (r-1)a_{k-2} = a_{k-1} - a_{k-2} = a_k - a_{k-1}$. Let $m_2 = k-2$ and $\epsilon_2 \geq 1$. Then $r \geq 2$, $m_1 = 0$ and $a_{k-1} < \min(h_2^{(r)}A)) < \min(R_2)$. Note that

$$|R_2| = 2r(k-2) - \epsilon_2(k-3)$$

and by Theorem 3

$$\left|h_2^{(r)}A\right| \geq 2r(k-2) - \epsilon_2(k-3) + 1.$$

Let y be an element of $h_2^{(r)}A$, which is different from $\min(h_2^{(r)}A)$. If $y \notin R_2$, then

$$H^{(r)}A \supseteq \left\{a_1, a_2, \ldots, a_{k-1}\right\} \cup \left\{\min(h_2^{(r)}A), y\right\} \cup \left(\bigcup_{i=2}^{t} R_i\right).$$

This gives $|H^{(r)}A| > \sum_{i=1}^{t} |R_i|$, which is not possible. Therefore $y \in R_2$. This gives that $h_2^{(r)}A = R_2 \cup \left\{\min(h_2^{(r)}A)\right\}$ and

$$\left|h_2^{(r)}A\right| = 2r(k-2) - \epsilon_2(k-3) + 1$$

and so by Theorem 4, A is an arithmetic progression.

Let $h_1 = 1$ and $h_2 = 2$. Then $R_1 = A$. Consider $R_1' = \{a_1 + a_i : i = 2, \ldots, k-1\}$, a subset of $h_2^{(r)}A$. Then $\max(R_1') < \min(R_2) = a_1 + a_k$. Therefore $R_1' \subseteq R_1 = A$. This gives that $a_1 + a_i = a_{i+1}$ for $i = 2, \ldots, k-1$. Also $a_k = a_1 + a_{k-1} < a_2 + a_{k-1} < a_2 + a_k = \min_+(R_2)$ and $a_k < \min(R_2) < \min_+(R_2)$ give $a_2 + a_{k-1} = a_1 + a_k$. Hence A is an arithmetic progression.

Let $A = a_1 + d_1 \cdot [0, k-1]$, where d_1 is the common difference of A. We show that H is an arithmetic progression with common difference d and $d_1 = da_1$. Note that, for all $i \in [1, t-1]$, we have

$$\max\nolimits_-(R_i) < \min\left\{(h_{i+1} - h_i)^{(r)}A_{i+1}\right\} + \max\nolimits_-(R_i) < \min(R_{i+1}).$$

But we already know that

$$\max\nolimits_-(R_i) < \max(R_i) < \min(R_{i+1}).$$

So

$$\min\left\{(h_{i+1} - h_i)^{(r)}A_{i+1}\right\} + \max\nolimits_-(R_i) = \max(R_i).$$

This implies that

$$\min\left\{(h_{i+1} - h_i)^{(r)}A_{i+1}\right\} = \max(R_i) - \max\nolimits_-(R_i) = a_{s+1} - a_s = a_2 - a_1 \text{ for some } s. \tag{11}$$

Consider the following cases:

(a) Let $i \in [1, t-1]$ be such that $\epsilon_i = \epsilon_{i+1}$. Then $m_{i+1} > m_i$. If $m_{i+1} - m_i \geq 2$, then $\min\{(h_{i+1} - h_i)^{(r)}A_{i+1}\} = ra_1 + \cdots + ra_{m_{i+1}-m_i} > a_2 > a_2 - a_1$, which contradicts (11). Hence $m_{i+1} - m_i = 1$ and $ra_1 = (h_{i+1} - h_i)a_1 = a_2 - a_1$.

(b) Let $i \in [1, t-1]$ be such that $\epsilon_i < \epsilon_{i+1}$. Then $m_{i+1} \geq m_i$. If $m_{i+1} - m_i \geq 1$, then $\min\{(h_{i+1} - h_i)^{(r)}A_{i+1}\} = ra_1 + \cdots + ra_{m_{i+1}-m_i} + (\epsilon_{i+1} - \epsilon_i)a_{m_{i+1}-m_i+1} > a_2 > a_2 - a_1$, which contradicts (11). Hence $m_{i+1} = m_i$ and $(\epsilon_{i+1} - \epsilon_i)a_1 = (h_{i+1} - h_i)a_1 = a_2 - a_1$.

(c) Let $i \in [1, t-1]$ be such that $\epsilon_i > \epsilon_{i+1}$. Then $m_{i+1} > m_i$. If $m_{i+1} > m_i + 1$, then $\min\{(h_{i+1} - h_i)^{(r)}A_{i+1}\} = ra_1 + \cdots + ra_{m_{i+1}-m_i-1} + (r + \epsilon_{i+1} - \epsilon_i)a_{m_{i+1}-m_i} > a_2 > a_2 - a_1$, which contradicts (11). Hence $m_{i+1} = m_i + 1$ and $(r + \epsilon_{i+1} - \epsilon_i)a_1 = (h_{i+1} - h_i)a_1 = a_2 - a_1$.

Hence, $(h_{i+1}-h_i)a_1 = a_2 - a_1 = d_1$ for each $i = 1,\dots,t-1$. So H is an arithmetic progresion with common difference $d \le r$ and $d_1 = da_1$. This completes the proof. □

Corollary 15. *Let $r \ge 1$ and $t > t_0 \ge 2$ be integers. Let A be a nonempty finite set of $k \ge 6$ positive integers and $H = \{h_1, h_2, \dots, h_t\}$ be a set of t positive integers with $h_1 < h_2 < \cdots < h_{t_0-1} \le (k-1)r - 1 < h_{t_0} < \cdots < h_t < kr$. If $(t_0, h_1) \ne (2,1)$ and $|H^{(r)}A| = \mathscr{L}(H^{(r)}_{t_0-1}A) + t - t_0 + 2$, then H is an arithmetic progression with common difference $d \le r$ and A is an arithmetic progression with common difference $d * \min(A)$, where $H_{t_0-1} = \{h_1, h_2, \dots, h_{t_0-1}\}$.*

Proof. Note that

$$\max\big(H^{(r)}_{t_0-1}A\big) = \max\big(h^{(r)}_{t_0-1}A\big) < \max_-\big(h^{(r)}_{t_0}A\big) < \max\big(h^{(r)}_{t_0}A\big) < \max\big(h^{(r)}_{t_0+1}A\big) < \cdots < \max\big(h^{(r)}_{t}A\big)$$

and

$$H^{(r)}A \supseteq H^{(r)}_{t_0-1}A \cup \big\{\max_-\big(h^{(r)}_{t_0}A\big)\big\} \cup \big\{\max\big(h^{(r)}_{i}A\big) : i = t_0,\dots,t\big\}.$$

Therefore $\big|H^{(r)}_{t_0-1}A\big| = \mathscr{L}(H^{(r)}_{t_0-1}A)$. If $t_0 \ge 3$, then by Theorem 14, H_{t_0-1} is an arithmetic progression with common difference d and A is an arithmetic progression with common difference $d * \min(A)$. Since $(t_0, h_1) \ne (2,1)$, so if $t_0 = 2$, then $h_1 > 1$. So by Theorem 4, A is an arithmetic progression.

Claim. If $t_0 \ge 2$, $t \ge t_0 + 1$, and A is an arithmetic progression with common difference d_1, then

(1) $\epsilon_{t_0} < \epsilon_{t_0-1}$,
(2) $m_{t_0-1} = k-2$,
(3) $h_i - h_{i-1} = d$ for $i = t_0,\dots,t$ and the common difference of A is $d_1 = da_1$.

Proof of the claim. Note that $m_{t_0-1}r + \epsilon_{t_0-1} = h_{t_0-1} \le (k-1)r - 1 = (k-2)r + r - 1$. Hence $m_{t_0-1} \le k-2$. Also $h_{t_0} \ge (k-1)r$ and $h_{t_0} < h_t \le kr - 1$, i.e., $h_{t_0} \le kr - 2 = (k-1)r + r - 2$. Thus $(k-1)r \le h_{t_0} \le (k-1)r + r - 2$. Hence $m_{t_0} = k-1$ and $0 \le \epsilon_{t_0} \le r-2$. Note also that

$$\max\big(h^{(r)}_{t_0}A\big) = \epsilon_{t_0}a_1 + ra_2 + \cdots + ra_k,$$
$$\max_-\big(h^{(r)}_{t_0}A\big) = (\epsilon_{t_0}+1)a_1 + (r-1)a_2 + \cdots + ra_k.$$

(1). If $\epsilon_{t_0} \ge \epsilon_{t_0-1}$, then

$$\max\big(h^{(r)}_{t_0-1}A\big) < y = ra_1 + \cdots + ra_{k-m_{t_0-1}-1} + \epsilon_{t_0}a_{k-m_{t_0-1}} + ra_{k-(m_{t_0-1}-1)} + \cdots + ra_k < \max_-\big(h^{(r)}_{t_0}A\big),$$

and $y \in h^{(r)}_{t_0}A$, which is a contradiction. Hence $\epsilon_{t_0} < \epsilon_{t_0-1}$.

(2). If $m_{t_0-1} \le k-3$, then

$$\begin{aligned}&\max\big(h^{(r)}_{t_0-1}A\big)\\ &< ra_1 + \cdots + ra_{k-m_{t_0-1}-2} + (r-(\epsilon_{t_0-1}-\epsilon_{t_0}))a_{k-m_{t_0-1}-1} + \epsilon_{t_0-1}a_{k-m_{t_0-1}} + ra_{k-m_{t_0-1}+1} + \cdots + ra_k\\ &< \max_-\big(h^{(r)}_{t_0}A\big),\end{aligned}$$

which is a contradiction. Hence $\epsilon_{t_0} < \epsilon_{t_0-1}$ and $m_{t_0-1} = k-2$. Consequently, we can write

$$\max\big(h^{(r)}_{t_0-1}A\big) < (r-(\epsilon_{t_0-1}-\epsilon_{t_0}))a_1 + \epsilon_{t_0-1}a_2 + ra_3 + \cdots + ra_k < \max\big(h^{(r)}_{t_0}A\big).$$

But we already know that

$$\max\big(h^{(r)}_{t_0-1}A\big) < \max_-\big(h^{(r)}_{t_0}A\big) < \max\big(h^{(r)}_{t_0}A\big).$$

This implies that

$$(r-(\epsilon_{t_0-1}-\epsilon_{t_0}))a_1 + \epsilon_{t_0-1}a_2 + ra_3 + \cdots + ra_k = \max_-\big(h^{(r)}_{t_0}A\big),$$

which gives $\epsilon_{t_0-1} = r-1$. Therefore $h_{t_0} - h_{t_0-1} = (k-1)r + \epsilon_{t_0} - (k-2)r - (r-1) = \epsilon_{t_0} + 1$. Now we have

$$\max_-\big(h^{(r)}_{t_0-1}A\big) < (\epsilon_{t_0}+1)a_1 + ra_2 + (r-1)a_3 + ra_4 + \cdots + ra_k < \max_-\big(h^{(r)}_{t_0}A\big).$$

We also have
$$\max_-\left(h_{t_0-1}^{(r)}A\right)<\max\left(h_{t_0-1}^{(r)}A\right)<\max_-\left(h_{t_0}^{(r)}A\right).$$
Therefore
$$(\epsilon_{t_0}+1)a_1+ra_2+(r-1)a_3+ra_4+\cdots+ra_k=\max\left(h_{t_0-1}^{(r)}A\right).$$
This gives
$$(\epsilon_{t_0}+1)a_1=a_3-a_2=d_1.$$
This implies that a_1 divides d_1, so $d_1=da_1$ where $d=\epsilon_{t_0}+1$. Hence $h_{t_0}-h_{t_0-1}=d$.

(3). Now we show that $h_i-h_{i-1}=d$ for $i=t_0+1,\ldots,t$.

Note that
$$\max_-\left(h_{t_0}^{(r)}A\right)<\max_-\left(h_{t_0+1}^{(r)}A\right)<\cdots<\max_-\left(h_{t}^{(r)}A\right)<\max\left(h_{t}^{(r)}A\right).$$
We already have
$$\max_-\left(h_{t_0}^{(r)}A\right)<\max\left(h_{t_0}^{(r)}A\right)<\max\left(h_{t_0+1}^{(r)}A\right)\cdots<\max\left(h_{t}^{(r)}A\right).$$
Therefore
$$\max\left(h_{i}^{(r)}A\right)=\max_-\left(h_{i+1}^{(r)}A\right),$$
which gives $(\epsilon_{i+1}-\epsilon_i)a_1=a_2-a_1=d_1$ for $i=t_0,t_0+1,\ldots,t-1$. Hence, H is an arithmetic progression with common difference d and A is an arithmetic progression with common difference da_1. □

Now we discuss the case when $t=t_0$.

Corollary 16. *Let $r\geq 1$ and $t\geq 2$ be positive integers. Let A be a nonempty finite set of $k\geq 6$ positive integers and $H=\{h_1,h_2,\ldots,h_t\}$ be a set of t positive integers with $h_1<\cdots<h_{t-1}\leq(k-1)r-1<h_t<kr$. If $(t,h_1)\neq(2,1)$ and $|H^{(r)}A|=\mathscr{L}((H\setminus\{h_t\})^{(r)}A)+2$, then H is an arithmetic progression with common difference $d\leq r$ and A is an arithmetic progression with common difference $d*\min(A)$.*

Proof. Note that
$$\max\left((H\setminus\{h_t\})^{(r)}A\right)=\max\left(h_{t-1}^{(r)}A\right)<\max_-\left(h_{t}^{(r)}A\right)<\max\left(h_{t}^{(r)}A\right)$$
and
$$H^{(r)}A\supseteq(H\setminus\{h_t\})^{(r)}A\cup\left\{\max_-\left(h_{t}^{(r)}A\right),\max\left(h_{t}^{(r)}A\right)\right\}.$$
Therefore $\left|H_{t-1}^{(r)}A\right|=\mathscr{L}((H\setminus\{h_t\})^{(r)}A)$. Also, if $t=2$ and $h_1>1$, then by Theorem 4, A is an arithmetic progression.

Claim. If $t\geq 2$, then

(1) $h_{t-1}>r$,
(2) $\epsilon_t\leq\epsilon_{t-1}$,
(3) $m_{t-1}=k-2$.

Proof of the claim.

(1). If $h_{t-1}\leq r$, then $\max(h_{t-1}^{(r)}A)=h_{t-1}a_k$. Note that
$$h_{t-1}a_k<(\epsilon_t+1)a_1+ra_2+(r-1)a_3+\cdots+ra_k<(\epsilon_t+1)a_1+(r-1)a_2+\cdots+ra_k=\max_-\left(h_{t}^{(r)}A\right),$$
which is a contradiction. Hence $h_{t-1}>r$ and so $m_{t-1}\geq 1$.

Note that $m_{t-1}r+\epsilon_{t-1}=h_{t-1}\leq(k-1)r-1=(k-2)r+r-1$. Hence $m_{t-1}\leq k-2$. Also $(k-1)r\leq h_t\leq kr-1$. Hence $m_t=k-1$. Note also that
$$\max\left(h_{t-1}^{(r)}A\right)=\epsilon_{t-1}a_{k-m_{t-1}}+ra_{k-m_{t-1}+1}+\cdots+ra_k,$$
$$\max\left(h_{t}^{(r)}A\right)=\epsilon_t a_1+ra_2+\cdots+ra_k,$$
$$\max_-h_{t}^{(r)}A=(\epsilon_t+1)a_1+(r-1)a_2+\cdots+ra_k.$$

(2). Let $\epsilon_t > \epsilon_{t-1}$. Then

$$\begin{aligned}\max\big(h_{t-1}^{(r)}A\big) < x &= ra_1 + \cdots + ra_{k-m_{t-1}-1} + \epsilon_t a_{k-m_{t-1}} + ra_{k-(m_{t-1}-1)} + \cdots + ra_k \\ &\le y = ra_1 + \epsilon_t a_2 + ra_3 + \cdots + ra_k \\ &\le (\epsilon_t+1)a_1 + (r-1)a_2 + \cdots + ra_k = \max\nolimits_-\big(h_t^{(r)}A\big),\end{aligned}$$

and $x, y \in h_t^{(r)}A$. If $x < y$ or $y < \max_-(h_t^{(r)}A)$, then we get a contradiction. So we assume that $x = y = \max_-(h_t^{(r)}A)$. This implies that $\epsilon_t = r-1$ and $m_{t-1} = k-2$. Since $\epsilon_t > \epsilon_{t-1}$, we have $\epsilon_{t-1} \le r-2$. Now consider $z = ra_1 + ra_2 + (r-1)a_3 + ra_4 + \cdots + ra_k \in h_t^{(r)}A$. Then we have $\max(h_{t-1}^{(r)}A) < z < \max_-(h_t^{(r)}A)$, which is again a contradiction. Hence $\epsilon_t \le \epsilon_{t-1}$.

(3). If $m_{t-1} \le k-3$, then

$$\begin{aligned}&\max\big(h_{t-1}^{(r)}A\big)\\ &< ra_1 + \cdots + ra_{k-m_{t-1}-2} + (r-(\epsilon_{t-1}-\epsilon_t))a_{k-m_{t-1}-1} + \epsilon_{t-1}a_{k-m_{t-1}} + ra_{k-m_{t-1}+1} + \cdots + ra_k\\ &< \max\nolimits_-\big(h_t^{(r)}A\big),\end{aligned}$$

which is a contradiction. Hence $\epsilon_t \le \epsilon_{t-1}$ and $m_{t-1} = k-2$. Consequently, we can write

$$\max\big(h_{t-1}^{(r)}A\big) < (r-(\epsilon_{t-1}-\epsilon_t))a_1 + \epsilon_{t-1}a_2 + ra_3 + \cdots + ra_k < \max\big(h_t^{(r)}A\big).$$

But we already know

$$\max\big(h_{t-1}^{(r)}A\big) < \max\nolimits_-\big(h_t^{(r)}A\big) < \max\big(h_t^{(r)}A\big).$$

This implies

$$(r-(\epsilon_{t-1}-\epsilon_t))a_1 + \epsilon_{t-1}a_2 + ra_3 + \cdots + ra_k = \max\nolimits_-\big(h_t^{(r)}A\big).$$

This gives $\epsilon_{t-1} = r-1$. Therefore $h_t - h_{t-1} = (k-1)r + \epsilon_t - (k-2)r - (r-1) = \epsilon_t + 1$. We have

$$\max\nolimits_-\big(h_{t-1}^{(r)}A\big) < (\epsilon_t+1)a_1 + ra_2 + (r-1)a_3 + ra_4 + \cdots + ra_k < \max\nolimits_-\big(h_t^{(r)}A\big).$$

We also have

$$\max\nolimits_-\big(h_{t-1}^{(r)}A\big) < \max(h_{t-1}^{(r)}A) < \max\nolimits_-\big(h_t^{(r)}A\big).$$

Therefore

$$(\epsilon_t+1)a_1 + ra_2 + (r-1)a_3 + ra_4 + \cdots + ra_k = \max\big(h_{t-1}^{(r)}A\big).$$

This gives

$$(\epsilon_t+1)a_1 = a_3 - a_2. \tag{12}$$

If $t \ge 3$, by Theorem 14, $H \setminus \{h_t\}$ is an arithmetic progression with common difference $d \le r$ and A is an arithmetic progression with common difference $d * \min(A)$. Therefore

$$(\epsilon_t+1)a_1 = a_3 - a_2 = da_1,$$

which implies $h_t - h_{t-1} = \epsilon_2 + 1 = d$. Hence, if $t \ge 3$, H is an arithmetic progression with common difference $d \le r$ and A is an arithmetic progression with common difference $d * \min(A)$. If $t = 2$, then $H = \{h_1, h_2\}$ is an arithmetic progression with common difference $d = h_2 - h_1 = \epsilon_t + 1 \le r$. Since $h_1 > 1$ and $\big|H_1^{(r)}A\big| = \mathscr{L}((H\setminus\{h_2\})^{(r)}A) = |h_1^{(r)}A| = m_1r(k-m_1) + \epsilon_1(k-2m_1-1)+1$, we have from Theorem 4 that A is an arithmetic progression with common difference da_1 from (12). □

Corollary 17. *Let $r \ge 2$ be a positive integer and A be a nonempty finite set of $k \ge 6$ positive integers and H be a set of $t \ge 2$ positive integers with $(k-1)r - 1 < \min(H) < \max(H) < kr$. Let $m_1 = \lfloor \min(H)/r \rfloor$. If*

$$\big|H^{(r)}A\big| = m_1r(k-m_1) + (h_1 - m_1r)(k-2m_1-1) + t,$$

then H is an arithmetic progression with common difference $d \le r-1$ and A is an arithmetic progression with common difference $d * \min(A)$.

Proof. Note that

$$\max\left(h_2^{(r)}A\right) < \max\left(h_3^{(r)}A\right) < \cdots < \max\left(h_t^{(r)}A\right)$$

and

$$H^{(r)}A \supseteq h_1^{(r)}A \cup \left\{\max\left(h_i^{(r)}A\right) : 2 \le i \le t\right\}.$$

Therefore $\left|h_1^{(r)}A\right| = m_1 r(k-m_1) + (h_1 - m_1 r)(k-2m_1-1)+1$ and by Theorem 4, A is an arithmetic progression. Assume d_1 is the common difference of A. Note that

$$\max_-\left(h_1^{(r)}A\right) < \max_-\left(h_2^{(r)}A\right) < \cdots < \max_-\left(h_t^{(r)}A\right) < \max\left(h_t^{(r)}A\right).$$

We already have

$$\max_-\left(h_1^{(r)}A\right) < \max\left(h_1^{(r)}A\right) < \max\left(h_2^{(r)}A\right) \cdots < \max\left(h_t^{(r)}A\right).$$

Therefore

$$\max\left(h_i^{(r)}A\right) = \max_-\left(h_{i+1}^{(r)}A\right),$$

which gives $(\epsilon_{i+1} - \epsilon_i)a_1 = a_2 - a_1 = d_1$ for $i = 1, 2, \ldots, t-1$. Hence, set H is an arithmetic progression with common difference $d \le r-1$ and set A is an arithmetic progression with common difference $d * \min(A)$. □

Corollary 18. *Let r be a positive integer, A be a finite set of $k \ge 7$ nonnegative integers with $0 \in A$, and H be a set of $t \ge 2$ positive integers with $1 \le r \le \max(H) \le (k-2)r - 1$. Let $m = \lceil \min(H)/r \rceil$ and $m_1 = \lfloor \min(H)/r \rfloor$. If*

$$\left|H^{(r)}A\right| = m_1 r(m - m_1 + 1) + (\min(H) - m_1 r)(m - 2m_1) + \mathscr{L}\left(H^{(r)}(A \setminus \{0\})\right), \tag{13}$$

then H is an arithmetic progression with common difference $d \le r$ and A is an arithmetic progression with common difference $d * \min(A \setminus \{0\})$. Moreover, if $\min(H) > 1$, then $d = 1$.

Proof. Let $A = \{0, a_1, \ldots, a_{k-1}\}$ be a set of nonnegative integers with $0 < a_1 < \cdots < a_{k-1}$ and $H = \{h_1, h_2, \ldots, h_t\}$ be set of positive integer such that $h_1 < h_2 < \cdots < h_t$. From (13) and Corollary 11, we have

$$\left|h_1^{(r)}B\right| = m_1 r(m - m_1 + 1) + (h_1 - m_1 r)(m - 2m_1) + 1$$

and

$$\left|H^{(r)}A'\right| = \mathscr{L}\left(H^{(r)}(A')\right),$$

where $A' = \{a_1, a_2, \ldots, a_{k-1}\}$ and $B = \{0, a_1, \ldots, a_m\}$ with $m = \lceil h_1/r \rceil$. Then by Theorem 14, H is an arithmetic progression with common difference $d \le r$ and A' is an arithmetic progression with common difference $d * \min(A')$. Now, we show that $d = 1$, if $h_1 > 1$. To show that $d = 1$, it is sufficient to prove that the common difference of arithmetic progression A is a_1. If $r = 1$, then $d = 1$. Assume $r \ge 2$. Now, define $R_i = S_i \cup T_i$ for the set A' as it was defined in Theorem 9. So

$$R_1 = S_1 = h_1^{(r)}A' \subseteq h_1^{(r)}A.$$

Now $\max(h_1^{(r)}A') = \max(h_1^{(r)}A)$ implies that $h_1^{(r)}A \cap R_2 = \emptyset$. We write

$$\begin{aligned}\left|H^{(r)}A\right| = m_1 r(m - m_1 + 1) &+ (h_1 - m_1 r)(m - 2m_1) + \left|h_1^{(r)}A'\right| \\ &+ \sum_{i=2}^{t}\Big(r(m_i - m_{i-1})(k - m_i - 1) + (\epsilon_i - \epsilon_{i-1})(k - m_i - 2) \\ &\qquad - (\max\{\epsilon_i, \epsilon_{i-1}\})(m_i - m_{i-1}) + 1\Big).\end{aligned} \tag{14}$$

If $h_1 = m_1 r + \epsilon_1$ with $m_1 \ge 1$ and $\epsilon_1 \ge 1$, then $m = m_1 + 1$ and so $|B| \ge 3$. Since $\left|h_1^{(r)}B\right| = m_1 r(m - m_1 + 1) + (h_1 - m_1 r)(m - 2m_1) + 1$, Theorem 4 implies that B is an arithmetic progression with common difference a_1. Furthermore, as A' is also an arithmetic progression, we have $A = B \cup A'$ is an arithmetic progression with common difference a_1.

If $h_1 = m_1 r + \epsilon_1$ with $m_1 = 0$ or $\epsilon_1 = 0$, then $m = 1$ or $m_1 = m$. Since

$$h_1^{(r)} B \cup h_1^{(r)} A' \subseteq h_1^{(r)} A \text{ and } h_1^{(r)} A \cap R_2 = \emptyset,$$

we have from (14) that

$$\begin{aligned}\left|h_1^{(r)} A\right| &= \left|h_1^{(r)} B\right| + \left|h_1^{(r)} A'\right| - 1\\ &= m_1 r(m - m_1 + 1) + (h_1 - m_1 r)(m - 2m_1) + m_1 r(k - m_1 - 1) + (h_1 - m_1 r)(k - 2m_1 - 2) + 1\\ &\leq m_1 r + (h_1 - m_1 r) + m_1 r(k - m_1 - 1) + (h_1 - m_1 r)(k - 2m_1 - 2) + 1\\ &= m_1 r(k - m_1) + (h_1 - m_1 r)(k - 2m_1 - 1) + 1.\end{aligned}$$

This gives

$$\left|h_1^{(r)} A\right| = m_1 r(k - m_1) + (h_1 - m_1 r)(k - m_1 - 1) + 1.$$

So, by Theorem 4, A is an arithmetic progression with common difference a_1. This implies that $d = 1$. Hence,

$$H = h_1 + [0, t-1] \quad \text{and} \quad A = \min(A \setminus \{0\}) * [0, k-1]. \qquad \square$$

As a consequence of Corollary 15, Corollary 16, Corollary 17 and Corollary 18, we have the following Corollaries.

Corollary 19. *Let $r \geq 1$ and $t > t_0 \geq 2$ be integers. Let A be a nonempty finite set of $k \geq 7$ nonnegative integers with $0 \in A$ and $H = \{h_1, h_2, \ldots, h_t\}$ be a set of t positive integers with $h_1 < h_2 < \cdots < h_{t_0-1} \leq (k-2)r - 1 < h_{t_0} < \cdots < h_t < (k-1)r$. If $(t_0, h_1) \neq (2,1)$ and $|H^{(r)}A| \geq m_1 r(m - m_1 + 1) + (\min(H) - m_1 r)(m - 2m_1) + \mathscr{L}\big(H^{(r)}(A \setminus \{0\})\big) + t - t_0 + 2$, then H is an arithmetic progression with common difference $d \leq r$ and A is an arithmetic progression with common difference $d * \min(A \setminus \{0\})$. Moreover, if $\min(H) > 1$, then $d = 1$.*

Corollary 20. *Let $r \geq 1$ and $t \geq 2$ be integers. Let A be a nonempty finite set of $k \geq 7$ nonnegative integers with $0 \in A$ and $H = \{h_1, h_2, \ldots, h_t\}$ be a set of t positive integers with $h_1 < h_2 < \cdots < h_{t-1} \leq (k-2)r - 1 < h_t < (k-1)r$. If $(t, h_1) \neq (2,1)$ and $|H^{(r)}A| \geq m_1 r(m - m_1 + 1) + (\min(H) - m_1 r)(m - 2m_1) + \mathscr{L}\big(H^{(r)}(A \setminus \{0\})\big) + 2$, then H is an arithmetic progression with common difference $d \leq r$ and A is an arithmetic progression with common difference $d * \min(A \setminus \{0\})$. Moreover, if $\min(H) > 1$, then $d = 1$.*

Corollary 21. *Let $r \geq 2$ be a positive integer and A be a nonempty finite set of $k \geq 7$ nonnegative integers with $0 \in A$ and H be a set of $t \geq 2$ positive integers with $(k-2)r - 1 < \min(H) < \max(H) < (k-1)r$. Let $m_1 = \lfloor \min(H)/r \rfloor$. If*

$$\left|H^{(r)} A\right| = m_1 r(k - m_1) + (\min(H) - m_1 r)(k - 2m_1 - 1) + t,$$

then H is an arithmetic progression with common difference $d \leq r - 1$ and A is an arithmetic progression with common difference $d * \min(A \setminus \{0\})$.

4. Conclusions

In Section 1.1, we have already discussed the relation between generalized H-fold sumset and subsequence sum. Choosing a particular H we get some results of subsequence sum.

Corollary 22 ([7, Theorem 2.1]). *Let k and r be positive integers. Let $\mathbb{A}$ be a finite sequence of nonnegative integers with k distinct terms each with repetitions r.*

If $0 \notin \mathbb{A}$ and $k \geq 3$, then

$$\sum(\mathbb{A}) \geq \frac{rk(k+1)}{2}.$$

If $0 \in \mathbb{A}$ and $k \geq 4$, then

$$\sum(\mathbb{A}) \geq 1 + \frac{rk(k-1)}{2}.$$

The above lower bounds are best possible.

Proof. If $0 \notin \mathbb{A}$, then taking $H = [1, kr]$ in Remark 10 (b), we get $\sum(\mathbb{A}) \geq \frac{rk(k+1)}{2}$. If $0 \in \mathbb{A}$, then taking $H = [1, kr]$ in Remark 12 (a), we get $\sum(\mathbb{A}) \geq 1 + \frac{rk(k-1)}{2}$. □

Corollary 23 ([7, Theorem 2.3]). *Let k and r be positive integers. If $\mathbb{A}$ is a finite sequence of nonnegative integers with k distinct terms each with repetitions r.*

If $0 \notin \mathbb{A}$, $k \geq 6$ and

$$\sum(\mathbb{A}) = \frac{rk(k+1)}{2},$$

then $\mathbb{A} = d * [1, k]_r$ for some positive integer d.

If $0 \in \mathbb{A}$, $k \geq 7$ and

$$\sum(\mathbb{A}) = 1 + \frac{rk(k-1)}{2},$$

then $\mathbb{A} = d * [0, k-1]_r$ for some positive integer d.

Proof. If $0 \notin \mathbb{A}$, then taking $H = [1, kr]$ in Corollary 15, we get $\mathbb{A} = d * [1, k]_r$ for some positive integer d. If $0 \in \mathbb{A}$, then taking $H = [1, kr]$ in Corollary 19, we get $\mathbb{A} = d * [0, k-1]_r$ for some positive integer d. □

Taking $H = [\alpha, kr]$ in Theorem 9 and Remark 10 and taking $H = [\alpha, (k-1)r]$ in Corollary 11 and Remark 12, we get the following result.

Corollary 24 ([4, Corollary 3.2]). *Let $k \geq 4$, $r \geq 1$ and α be integers with $1 \leq \alpha < kr$. Let $m \in [1, k]$ be the integer such that $(m-1)r \leq \alpha < mr$. Let $\mathbb{A}$ be a finite sequence of nonnegative integers with k distinct terms each with repetitions r.*

If $0 \notin \mathbb{A}$, then

$$\sum_{\alpha}(\mathbb{A}) \geq \frac{rk(k+1)}{2} - \frac{rm(m+1)}{2} + m(mr - \alpha) + 1.$$

If $0 \in \mathbb{A}$, then

$$\sum_{\alpha}(\mathbb{A}) \geq \frac{rk(k-1)}{2} - \frac{rm(m-1)}{2} + (m-1)(mr - \alpha) + 1.$$

The above lower bounds are best possible.

Taking $H = [\alpha, kr-2]$, Theorem 14 and Corollaries 15–21 give the following result.

Corollary 25 ([4, Corollary 3.5]). *Let $k \geq 7$, $r \geq 1$ and α be integers with $1 \leq \alpha \leq kr-2$. Let $m \in [1, k]$ be the integer such that $(m-1)r \leq \alpha < mr$. Let $\mathbb{A}$ be a finite sequence of nonnegative integers with k distinct terms each with repetitions r.*

If $0 \notin \mathbb{A}$ and

$$\sum_{\alpha}(\mathbb{A}) = \frac{rk(k+1)}{2} - \frac{rm(m+1)}{2} + m(mr - \alpha) + 1,$$

then $\mathbb{A} = d * [1, k]_r$ for some positive integer d.

If $0 \in \mathbb{A}$ and

$$\sum_{\alpha}(\mathbb{A}) = \frac{rk(k-1)}{2} - \frac{rm(m-1)}{2} + (m-1)(mr - \alpha) + 1,$$

then $\mathbb{A} = d * [0, k-1]_r$ for some positive integer d.

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