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# Stable determination of the nonlinear term in a quasilinear elliptic equation by boundary measurements

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**Abstract.** We establish a Lipschitz stability inequality for the problem of determining the nonlinear term in a quasilinear elliptic equation by boundary measurements. We give a proof based on a linearization procedure together with special solutions constructed from the fundamental solution of the linearized problem.

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## 1. Introduction

### 1.1. *Statement of the problem*

Consider on a bounded Lipschitz domain $\Omega$ of $\mathbb{R}^n$ the quasilinear BVP

$$\begin{cases} \operatorname{div}(a(u)A\nabla u) = 0 & \text{in } \Omega, \\ u_{|\Gamma} = f, \end{cases} \tag{1}$$

where $a$ is a scalar function and $A$ is a matrix with variable coefficients, and $\Gamma$ is the boundary of $\Gamma$. Assume that we can define the map $\Lambda_a : f \mapsto a(u)A\nabla u(f)\cdot \nu$ between two well chosen spaces, where $u(f)$ is the solution of the BVP (1), when it exists, and $\nu$ is the unit exterior normal vector field to $\Gamma$. In the case where $a$ is supposed to be unknown we ask whether $\Lambda_a$ determines uniquely $a$. This problem can be seen as a Calderón type problem for the quasilinear BVP (1). We are mainly interested in establishing a stability inequality for this inverse problem.

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### 1.2. *Assumptions and notations*

Throughout this text, $0<\alpha<1$ and $\Omega$ is a $C^{2,\alpha}$ bounded domain of $\mathbb{R}^n$ $(n\ge 3)$ with boundary $\Gamma$. Fix $A=(a^{ij})\in C^{1,\alpha}(\mathbb{R}^n,\mathbb{R}^{n\times n})$ satisfying $(a^{ij}(x))$ is symmetric for each $x\in\mathbb{R}^n$,

$$\kappa^{-1}|\xi|^2\le A(x)\xi\cdot\xi,\quad x,\xi\in\mathbb{R}^n, \tag{2}$$

and

$$\max_{1\le i,j\le n}\left\|a^{ij}\right\|_{C^{1,\alpha}(\mathbb{R}^n)}\le\kappa, \tag{3}$$

where $\kappa>1$ is a given constant.

Pick $\varkappa>0$. Let $\mu:[0,\infty)\to[\varkappa,\infty)$ and $\gamma:[0,\infty)\to[0,\infty)$ be two nondecreasing functions. If $\varrho=\varrho(n,|\Omega|)$ denotes the constant appearing in (6) then consider the assumptions

(**a1**) $a\in C^1(\mathbb{R})$, $a\ge\varkappa$ and $q_a=a'/a\in C^{0,1}(\mathbb{R})$.
(**a2**) $a(z)\le\mu(\varrho^{-1}|z|)$, $z\in\mathbb{R}$.
(**a3**) $|a'(z)|\le\gamma(\varrho^{-1}|z|)$, $z\in\mathbb{R}$.

Let $\tilde\gamma:[0,\infty)\to[0,\infty)$ be another nondecreasing function. We need also to introduce the following assumption,

($\tilde{\mathbf{a}}$**1**) $a\in C^2(\mathbb{R})$, $a\ge\varkappa$ and $|a''(z)|\le\tilde\gamma(\varrho^{-1}|z|)$, $z\in\mathbb{R}$.

Note that ($\tilde{\mathbf{a}}$**1**) implies (**a1**).

The set of functions $a:\mathbb{R}\to\mathbb{R}$ satisfying (**a1**) (resp. ($\tilde{\mathbf{a}}$**1**), (**a2**) and (**a3**) is denoted hereafter by $\mathscr{A}$ (resp. $\widetilde{\mathscr{A}}$).

In the rest of this text $\Gamma_0$ denotes a nonempty open subset of $\Gamma$. We define $H^{1/2}_{\Gamma_0}(\Gamma)$ as follows

$$H^{1/2}_{\Gamma_0}(\Gamma)=\left\{f\in H^{1/2}(\Gamma);\ \mathrm{supp}(f)\subset\Gamma_0\right\}.$$

We equip $H^{1/2}_{\Gamma_0}(\Gamma)$ with the norm of $H^{1/2}(\Gamma)$.

Also, fix $\Gamma_1\Supset\Gamma_0$ an open subset of $\Gamma$ and $\chi\in C_0^\infty(\Gamma_1)$ so that $\chi=1$ in $\overline{\Gamma}_0$.

For any $t\in\mathbb{R}$, $\mathfrak{f}_t$ will denote the constant function given by $\mathfrak{f}_t(s)=t$, $s\in\Gamma$.

Finally,

$$\begin{aligned}
C_m&=C_m\big(n,\Omega,\kappa,\varkappa,\alpha,\mu(m),\gamma(m)\big)>0,\quad m\ge 0,\\
C_m^0&=C_m^0\big(n,\Omega,\kappa,\varkappa,\alpha,m,\mu(m),\gamma(m)\big)>0,\quad m\ge 0,\\
C_m^1&=C_m^1\big(n,\Omega,\kappa,\varkappa,\alpha,m,\mu(m),\gamma(m),\tilde\gamma(m),\Gamma_0,\Gamma_1\big)>0,\quad m\ge 0,
\end{aligned}$$

will denote generic constants.

### 1.3. *Main result*

We show in Subsection 2.1 that, under assumption (**a1**), for each $f\in C^{2,\alpha}(\Gamma)$ the BVP (1) admits a unique solution $u_a=u_a(f)\in C^{2,\alpha}(\overline{\Omega})$. Furthermore, when $a$ satisfies both (**a1**) and (**a2**) the Dirichlet-to-Neumann map

$$\Lambda_a:C^{2,\alpha}(\Gamma)\to H^{-1/2}(\Gamma):f\mapsto a(u)A\nabla u_a(f)\cdot\nu$$

is well defined, where $\nu$ is the unit normal vector field on $\Gamma$.

Set $C^{2,\alpha}_{\Gamma_0}(\Gamma)=\{f\in C^{2,\alpha}(\Gamma);\ \mathrm{supp}(f)\subset\Gamma_0\}$ and define the family of localized Dirichlet-to-Neumann maps $(\widetilde\Lambda_a^t)_{t\in\mathbb{R}}$ as follows

$$\widetilde\Lambda_a^t:f\in C^{2,\alpha}_{\Gamma_0}(\Gamma)\mapsto\chi\Lambda_a\big(\mathfrak{f}_t+f\big)\in H^{-1/2}(\Gamma),\quad t\in\mathbb{R}.$$

We will prove in Subsection 2.2 that, under the assumption (**a3**), for each $t\in\mathbb{R}$ and $a\in\mathscr{A}$, $\widetilde{\Lambda}_a^t$ is Fréchet differentiable in a neighborhood of the origin. Furthermore, $d\widetilde{\Lambda}_a^t(0)$, the Fréchet differential of $\widetilde{\Lambda}_a^t$ at 0, has an extension, still denoted by $d\widetilde{\Lambda}_a^t(0)$, belonging to $\mathscr{B}(H_{\Gamma_0}^{1/2}(\Gamma), H^{-1/2}(\Gamma))$, and

$$\sup_{|t|\le\tau}\left\|d\widetilde{\Lambda}_a^t(0)\right\|_{\mathrm{op}}<\infty,\quad \tau>0.$$

Here and Henceforth $\|\cdot\|_{\mathrm{op}}$ stands for the norm of $\mathscr{B}(H_{\Gamma_0}^{1/2}(\Gamma), H^{-1/2}(\Gamma))$.

We remark that since $C_{\Gamma_0}^{2,\alpha}(\Gamma)$ is dense in $H_{\Gamma_0}^{1/2}(\Gamma)$ the above extension of $d\widetilde{\Lambda}_a^t(0)$ is entirely determined by $d\widetilde{\Lambda}_a^t(0)$.

**Theorem 1.** *For any $a_1, a_2\in\widetilde{\mathscr{A}}$ and $\tau>0$ we have*

$$\|a_1-a_2\|_{C([-\tau,\tau])}\le C_\tau^1\sup_{|t|\le\tau}\left\|d\widetilde{\Lambda}_{a_1}^t(0)-d\widetilde{\Lambda}_{a_2}^t(0)\right\|_{\mathrm{op}}.$$

The following uniqueness result is straightforward from the preceding theorem.

**Corollary 2.** *Let $a_j\in C^2(\mathbb{R})$ satisfies $a_j\ge\varkappa$, $j=1,2$. Then $\widetilde{\Lambda}_{a_1}^t=\widetilde{\Lambda}_{a_2}^t$ in a neighborhood of the origin for each $t\in\mathbb{R}$ implies $a_1=a_2$.*

It is worth noticing that if $a_1$ and $a_2$ are as in Corollary 2 then $a_1$ and $a_2$ satisfy also (**a2**), (**a3**) and (**ã1**) with

$$\mu(\tau)=\max_{|z|\le\tau}\max_{j=1,2}a_j(\varrho z),\qquad \gamma(\tau)=\max_{|z|\le\tau}\max_{j=1,2}\left|a_j'(\varrho z)\right|,\quad \tau\ge 0,$$

and

$$\widetilde{\gamma}(\tau)=\max_{|z|\le\tau}\max_{j=1,2}\left|a_j''(\varrho z)\right|,\quad \tau\ge 0.$$

These $\mu$, $\gamma$ and $\widetilde{\gamma}$ depends of course on $a_1$ and $a_2$.

We also emphasize that Theorem 1 and Corollary 2 show that the determination of the nonlinear term in fact can be done only through the knowledge of the Dirichlet-to-Neumann map on an arbitrary subset of the boundary.

### 1.4. *Comments*

There are only very few stability inequalities in the literature devoted to the determination of nonlinear terms in quasilinear and semilinear elliptic equations by boundary measurements. The semilinear case was studied in [5] by using a method based on linearization together with stability inequality for the problem of determining the potential in a Schrödinger equation by boundary measurements. The result in [5] was recently improved in [4]. Both quasilinear and semilinear elliptic inverse problems were considered in [13] where a method exploiting the singularities of fundamental solutions was used to establish stability inequalities. This method was used previously in [3] to obtain a stability inequality at the boundary of the conformal factor in an inverse conductivity problem. We show in the present paper how we can modify the proof of [3, (1.2) of Theorem 1.1] to derive the stability inequality stated in Theorem 1. The localization argument was inspired by that in [13].

There is a recent rich literature dealing with uniqueness issue concerning the determination of nonlinearities in elliptic equations by boundary measurements using the so-called higher order linearization method. We refer to the recent work [2] and references therein for more details. We also quote without being exhaustive the following references [1, 6, 7, 9, 10, 12, 14–21] on semilinear and quasilinear elliptic inverse problems.

## 2. Preliminaries

### 2.1. *Solvability of the BVP and the Dirichlet-to-Neumann map*

Suppose that $a$ satisfies (**a1**) and introduce the divergence form quasilinear operator

$$Q(x,u,\nabla u)=\mathrm{div}(a(u)A\nabla u),\quad x\in\Omega,\ u\in C^2(\Omega).$$

The following observation will be crucial in the sequel : $u\in C^2(\Omega)$ satisfies $Q(x,u,\nabla u)=0$ in $\Omega$ if and only if

$$Q_0(x,u,\nabla u)=\sum_{i,j=1}^n a^{ij}(x)\partial_{ij}^2u(x)+b(x,u,\nabla u)=0,\quad x\in\Omega, \tag{4}$$

where

$$b(x,z,p)=B(x)\cdot p+q_a(z)A(x)p\cdot p,\quad x\in\Omega,\ z\in\mathbb{R},\ p\in\mathbb{R}^n,$$

with

$$B_j(x)=\sum_{i=1}^n\partial_i a^{ij}(x),\quad x\in\Omega,\ 1\le j\le n.$$

As $b(\cdot,\cdot,0)=0$ one can easily check that $Q_0$ satisfies to the assumptions of [8, Theorem 15.12, page 382]. Let $f\in C^{2,\alpha}(\Gamma)$. Il light of the observation above we derive that the quasilinear BVP

$$\begin{cases}\mathrm{div}(a(u)A\nabla u)=0 & \text{in }\Omega,\\ u_{|\Gamma}=f,\end{cases} \tag{5}$$

admits a solution $u_a=u_a(f)\in C^{2,\alpha}(\overline{\Omega})$. The uniqueness of solutions of (1) holds from [8, Theorem 10.7, p. 268] applied to $Q$.

Also, as

$$a(z)p\cdot A(x)p\ge\varkappa\kappa^{-1}|p|^2$$

we infer from [8, Theorem 10.9, p. 272] that

$$\max_{\overline{\Omega}}|u_a(f)|\le\varrho\max_{\Gamma}|f|, \tag{6}$$

where $\varrho=\varrho(n,|\Omega|)$.

On the other hand according to [8, Theorem 6.8, p. 100], for each $f\in C^{2,\alpha}(\Gamma)$, there exists a unique $\mathscr{E}f\in C^{2,\alpha}(\overline{\Omega})$ satisfying

$$\Delta\mathscr{E}f=0\text{ in }\Omega,\quad \mathscr{E}f_{|\Gamma}=f,$$

and from [8, Theorem 6.6, p. 98] we have

$$\|\mathscr{E}f\|_{C^{2,\alpha}(\overline{\Omega})}\le\mathbf{c}\|f\|_{C^{2,\alpha}(\Gamma)}, \tag{7}$$

where $\mathbf{c}=\mathbf{c}(\Omega,\alpha)>0$.

Assume that $a$ satisfies (**a1**) and (**a2**) and let $f\in C^{2,\alpha}(\Gamma)$. Then straightforward computations show that $v=u_a(f)-\mathscr{E}f$ is the solution of the BVP

$$\begin{cases}-\mathrm{div}\big(a(u_a(f))A\nabla v\big)=\mathrm{div}\big(a(u_a(f))A\nabla\mathscr{E}f\big) & \text{in }\Omega,\\ v_{|\Gamma}=0.\end{cases} \tag{8}$$

Multiplying the first equation of (8) by $v$ and integrating over $\Omega$. We then obtain from Green's formula

$$\int_\Omega a\big(u_a(f)\big)A\nabla v\cdot\nabla v\,dx=-\int_\Omega a\big(u_a(f)\big)A\nabla\mathscr{E}f\cdot\nabla v\,dx.$$

Set

$$\mathscr{B}_m=\left\{f\in C^{2,\alpha}(\Gamma);\ \max_{\Gamma}|f|<m\right\}.$$

If $f\in\mathscr{B}_m$ then the last identity together with (6) yield

$$\varkappa\kappa^{-1}\|\nabla v\|_{L^2(\Omega)}^2\le\kappa\mu(m)\|\nabla\mathscr{E}f\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)},$$

which, combined with the fact of $w \in H^1(\Omega) \mapsto \|\nabla w\|_{L^2(\Omega)}$ defines an equivalent norm on $H_0^1(\Omega)$, implies

$$\|v\|_{H^1(\Omega)} \le \aleph\mu(m)\,\|f\|_{C^{2,\alpha}(\Gamma)}.$$

Here and henceforth, $\aleph = \aleph(n,\Omega,\kappa,\varkappa,\alpha) > 0$ is a generic constant.

Whence

$$\|u_a(f)\|_{H^1(\Omega)} \le \aleph\mu(m)\,\|f\|_{C^{2,\alpha}(\Gamma)}. \tag{9}$$

We endow $H^{1/2}(\Gamma)$ with the quotient norm

$$\|\varphi\|_{H^{1/2}(\Gamma)} = \min\left\{\|v\|_{H^1(\Omega)};\ v \in \dot{\varphi}\right\},\quad \varphi \in H^{1/2}(\Gamma),$$

where

$$\dot{\varphi} = \left\{v \in H^1(\Omega);\ v_{|\Gamma} = \varphi\right\}.$$

For each $\psi \in H^{-1/2}(\Gamma)$ we define $\chi\psi$ by

$$\left\langle \chi\psi,\varphi\right\rangle_{1/2} = \left\langle \psi,\chi\varphi\right\rangle_{1/2},\quad \varphi \in H^{1/2}(\Gamma),$$

where $\langle\cdot,\cdot\rangle_{1/2}$ is the duality pairing between $H^{1/2}(\Gamma)$ and its dual $H^{-1/2}(\Gamma)$.

It is not difficult to check that $\chi\psi \in H^{-1/2}(\Gamma)$, $\mathrm{supp}(\chi\psi) \subset \Gamma_1$ and the following identity holds

$$\left\langle \chi\psi,\varphi\right\rangle_{1/2} = \langle\psi,\varphi\rangle_{1/2},\quad \varphi \in H^{1/2}_{\Gamma_0}(\Gamma). \tag{10}$$

This identity will be very useful in the sequel.

Let $\varphi \in H^{1/2}(\Gamma)$ and $v \in \dot{\varphi}$. Appying Green's formula, we get

$$\int_\Gamma a(f)A\nabla u_a(f)\cdot \nu\varphi ds = \int_\Omega a\left(u_a(f)\right)A\nabla u_a(f)\cdot\nabla v dx.$$

We recall that $\nu$ is the unit exterior normal vector field on $\Gamma$.

Using that $u_a(f)$ is the solution of the BVP (1), we easily check that the right hand side of the above identity is independent of $v$, $v \in \dot{\varphi}$.

This identity suggests to define the Dirichlet-to-Neumann map

$$\Lambda_a : C^{2,\alpha}(\Gamma) \to H^{-1/2}(\Gamma),$$

associated to $a$, by the formula

$$\left\langle \Lambda_a(f),\varphi\right\rangle_{1/2} = \int_\Omega a\left(u_a(f)\right)A\nabla u_a(f)\cdot\nabla v dx,\quad \varphi \in H^{1/2}(\Gamma),\ v \in \dot{\varphi}.$$

Using (9), we get

$$\left\|\Lambda_a(f)\right\|_{H^{-1/2}(\Gamma)} \le \aleph\mu(m)^2\|f\|_{C^{2,\alpha}(\Gamma)},\quad f \in \mathscr{B}_m. \tag{11}$$

## 2.2. *Differentiability properties*

We need a gradient bound for the solution of the BVP (1). To this end we set

$$\mathscr{B}_m^+ = \left\{f \in \mathscr{B}_m;\ \|f\|_{C^{2,\alpha}(\Gamma)} < \mathbf{c}^{-1}m\right\},\quad m > 0,$$

where $\mathbf{c}$ is the constant in (7).

Fix $f \in \mathscr{B}_m^+$ and $a \in \mathscr{A}$. We apply [8, Theorem 15.9, p. 380] with $\varphi = \mathscr{E}f/\|\mathscr{E}f\|_{C^2(\overline{\Omega})}$ and $u = u_a(f)/\|\mathscr{E}f\|_{C^2(\overline{\Omega})}$ which is the solution of (1) when $a(z)$ is substituted by $a(\|\mathscr{E}f\|_{C^2(\overline{\Omega})}z)$. We obtain

$$\max_{\overline{\Omega}}\left|\nabla u_a(f)\right| \le C_m^0\|f\|_{C^{2,\alpha}(\Gamma)}. \tag{12}$$

Next, we establish that $\Lambda_a$, $a \in \mathscr{A}$, is Fréchet differentiable in a neighborhood of the origin. For $\eta > 0$ define

$$\mathscr{B}_m^\eta = \left\{f \in \mathscr{B}_m^+ \cap C^{2,\alpha}_{\Gamma_0}(\Gamma);\ \|f\|_{C^{2,\alpha}(\Gamma)} < \eta\right\}.$$

**Lemma 3.** *Let $m>0$. There exists $\eta_m=\eta_m(n,\Omega,\kappa,\varkappa,\alpha,m,\mu(m),\gamma(m))>0$ such that for each $a\in\mathscr{A}$ we have*

$$\left\|u_a(f)-u_a(g)\right\|_{H^1(\Omega)}\le C_m\|f-g\|_{C^{2,\alpha}(\Gamma)},\quad f,g\in\mathscr{B}_m^{\eta_m},$$

**Proof.** Let $\eta>0$ to be determined later. Pick $f,g\in\mathscr{B}_m^{\eta}$ and set $h=g-f$. Let $\sigma=a(u_a(g))$ and

$$p(x)=\int_0^1 a'\left(u_a(f)(x)+t\left[u_a(g)(x)-u_a(f)(x)\right]\right)dt,\quad x\in\Omega.$$

It is then straightforward to check that $u=u_a(g)-u_a(f)$ is the solution of the BVP

$$\begin{cases}-\mathrm{div}(\sigma A\nabla u)=\mathrm{div}\left(puA\nabla u_a(f)\right) & \text{in }\Omega,\\ u_{|\Gamma}=h.\end{cases}$$

We split $u$ into two terms $u=\mathscr{E}h+v$, where $v$ is the solution of the BVP

$$\begin{cases}-\mathrm{div}(\sigma A\nabla v)-\mathrm{div}\left(pvA\nabla u_a(f)\right)=\mathrm{div}(F) & \text{in }\Omega,\\ v_{|\Gamma}=0,\end{cases}$$

with

$$F=\sigma A\nabla\mathscr{E}h+q\mathscr{E}hA\nabla u_a(f).$$

Applying Green's formula, we find

$$\int_\Omega\sigma A\nabla v\cdot\nabla v+\int_\Omega qvA\nabla u_a(f)\cdot\nabla v=-\int_\Omega F\cdot\nabla v.$$

From (12) and Poincaré's inequality we derive

$$\varkappa\kappa^{-1}\|\nabla v\|_{L^2(\Omega)}^2-C_m^0\eta\|\nabla v\|_{L^2(\Omega)}^2\le C_m\|\nabla v\|_{L^2(\Omega)}\|h\|_{C^{2,\alpha}(\Gamma)}.$$

If $\eta=\eta_m$ is chosen sufficiently small is such a way that

$$\varkappa\kappa^{-1}-C_m^0\eta\ge\varkappa\kappa^{-1}/2,$$

then we obtain

$$\|\nabla v\|_{L^2(\Omega)}\le C_m\|h\|_{C^{2,\alpha}(\Gamma)},$$

from which the expected inequality follows readily. □

In the sequel $\eta_m$, $m>0$, will denote the constant in Lemma 3.

**Lemma 4.** *Pick $a\in\mathscr{A}$ and $m>0$. Then*

$$\|\Lambda_a(f)-\Lambda_a(g)\|_{H^{-1/2}(\Gamma)}\le C_m\|f-g\|_{C^{2,\alpha}(\Gamma)},\quad f,g\in\mathscr{B}_m^{\eta_m}.$$

**Proof.** Let $f,g\in\mathscr{B}_m^{\eta_m}$. For $\varphi\in H^{1/2}(\Gamma)$ and $v\in\dot{\varphi}$, we have

$$\left\langle\Lambda_a(g)-\Lambda_a(f),\varphi\right\rangle_{1/2}=I_1+I_2,$$

where

$$I_1=\int_\Omega\left[a\left(u_a(g)\right)-a\left(u_a(f)\right)\right]A\nabla u_a(g)\cdot\nabla v\,dx,$$
$$I_2=\int_\Omega a\left(u_a(f)\right)A\left[\nabla u_a(g)-\nabla u_a(f)\right]\cdot\nabla v\,dx.$$

We can proceed similarly to the proof of Lemma 3 to derive that

$$\left|I_j\right|\le C_m\left\|u_a(g)-u_a(f)\right\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)},\quad j=1,2.$$

The expected inequality follows easily by using Lemma 3. □

Let $f \in \mathscr{B}_m^{\eta_m}$. Similarly to the calculations we carried out in the proof of Lemma 3, we show that the bilinear continuous form

$$\mathfrak{b}(v,w) = \int_\Omega \left[a\left(u_a(f)\right)A\nabla v + a'\left(u_a(f)\right)vA\nabla u_a(f)\right]\cdot\nabla w dx, \quad v,w \in H_0^1(\Omega),$$

is coercive. In light of Lemma 12, we obtain that the BVP

$$\begin{cases} \operatorname{div}\left[a\left(u_a(f)\right)A\nabla v + a'\left(u_a(f)\right)v\nabla u_a(f))\right] = 0 & \text{in } \Omega, \\ v_{|\Gamma} = h, \end{cases} \tag{13}$$

admits a unique weak solution $v_a = v_a(f,h) \in H^1(\Omega)$ satisfying

$$\left\|v_a(f,h)\right\|_{H^1(\Omega)} \le C_m\|h\|_{H^{1/2}(\Gamma)} \tag{14}$$

and hence

$$\left\|v_a(f,h)\right\|_{H^1(\Omega)} \le C_m\|h\|_{C^{2,\alpha}(\Gamma)}. \tag{15}$$

We refer to Appendix A for the exact definition of weak solutions.

Next, pick $\epsilon > 0$ such that $f + h \in \mathscr{B}_m^{\eta_m}$ for each $h \in C_{\Gamma_0}^{2,\alpha}(\Gamma)$ satisfying $\|h\|_{C^{2,\alpha}(\Gamma)} < \epsilon$. Set then

$$w = u_a(f+h) - u_a(f) - v_a(f,h).$$

Simple computations show that $w$ is the weak solution of the BVP

$$\begin{cases} \operatorname{div}\left[a\left(u_a(f)\right)A\nabla w + a'\left(u_a(f)\right)w\nabla u_a(f))\right] = \operatorname{div}(F) & \text{in } \Omega, \\ w_{|\Gamma} = 0, \end{cases}$$

with

$$\begin{aligned} F &= a'\left(u_a(f)\right)\left[u_a\left(f+h\right)\right) - u_a(f)\right]\nabla u_a(f) \\ &\qquad\qquad - \left[a\left(u_a(f+h)\right) - a\left(u_a(f)\right)\right]\nabla u_a(f+h) \\ &= \left\{a'\left(u_a(f)\right)\left[u_a(f+h)\right) - u_a(f)\right] - \left[a\left(u_a(f+h)\right) - a\left(u_a(f)\right)\right]\right\}\nabla u_a(f) \\ &\qquad\qquad + \left[a\left(u_a(f+h)\right) - a\left(u_a(f)\right)\right]\left[\nabla u_a(f) - \nabla u_a(f+h)\right]. \end{aligned}$$

In particular, we have

$$\mathfrak{b}(w,w) = \int_\Omega F\cdot\nabla w. \tag{16}$$

Using that

$$\begin{aligned} &a'\left(u_a(f)\right)\left[u_a(f+h)\right) - u_a(f)\right] - \left[a\left(u_a(f+h)\right) - a\left(u_a(f)\right)\right] \\ &\qquad = \left[u_a(f+h)\right) - u_a(f)\right]\int_0^1 \left[a'\left(u_a(f)\right) - a'(u_a(f) + t\left(u_a(f+h) - u_a(f)\right)\right]dt, \end{aligned}$$

and the uniform continuity of $a'$ in $[-\varrho m, \varrho m]$, we obtain

$$\left\|a'\left(u_a(f)\right)\left[u_a(f+h)\right) - u_a(f)\right] - \left[a\left(u_a(f+h)\right) - a\left(u_a(f)\right)\right]\right\|_{L^\infty(\Omega)} = o\left(\|h\|_{C^{2,\alpha}(\Gamma)}\right).$$

On the other hand similar estimates as above give

$$\left\|\left[a\left(u_a(f+h)\right) - a\left(u_a(f)\right)\right]\left[\nabla u_a(f) - \nabla u_a(f+h)\right]\right\|_{H^1(\Omega)} \le C_m\|h\|^2_{C^{2,\alpha}(\Gamma)}.$$

The last two inequalities together with (16) yield

$$\|w\|_{H^1(\Omega)} = o\left(\|h\|_{C^{2,\alpha}(\Gamma)}\right).$$

In other words we proved that $f \in \mathscr{B}_m^{\eta_m} \mapsto u_a(f) \in H^1(\Omega)$ is Fréchet differentiable with

$$du_a(f)(h) = v_a(f,h), \quad f \in \mathscr{B}_m^{\eta_m},\ h \in C_{\Gamma_0}^{2,\alpha}(\Gamma).$$

Using the definition of $\Lambda_a$ we can then state the following result.

**Proposition 5.** *For each $m>0$, the mapping*

$$f \in \mathscr{B}_m^{\eta_m} \mapsto \Lambda_a(f) \in H^{-1/2}(\Gamma)$$

*is Fréchet differentiable with*

$$\langle d\Lambda_a(f)(h),\varphi\rangle_{1/2} = \int_\Omega \left[a\left(u_a(f)\right)A\nabla v_a(f,h) + a'\left(u_a(f)\right)v_a(f,h)A\nabla u_a(f)\right]\cdot\nabla v dx,$$

$f \in \mathscr{B}_m^{\eta_m}$, $h \in C^{2,\alpha}_{\Gamma_0}(\Gamma)$, $\varphi \in H^{1/2}(\Gamma)$ *and* $v \in \dot{\varphi}$.

The fact that $d\Lambda_a(f)$, $f \in \mathscr{B}_m^{\eta_m}$, is extended to a bounded linear map from $H^{1/2}_{\Gamma_0}(\Gamma)$ into $H^{-1/2}(\Gamma)$ is immediate from (14).

**Remark 6.** If the assumption (**a1**) is substituted by the following one

(**a'1**) $a \in C^{1,1}(\mathbb{R})$, $a \geq \varkappa$,

then one can prove that $f \in \mathscr{B}_m^{\eta_m} \mapsto v_a(f,\cdot) \in \mathscr{B}(C^{2,\alpha}_{\Gamma_0}(\Gamma), H^1(\Omega))$ is continuous. In that case $f \in \mathscr{B}_m^{\eta_m} \mapsto \Lambda_a(f) \in H^{-1/2}(\Gamma)$ is continuously Fréchet differentiable.

## 3. Proof of the main result

As we already mentioned we give a proof based on an adaptation of [3, proof of (1.2) of Theorem 1.1] combined with a localization argument borrowed from [13].

### 3.1. *Special solutions*

We construct in a general setting special solutions of a divergence form operator vanishing outside $\Gamma_0$. To this end, let $\mathfrak{A} = (\mathfrak{A}^{ij}) \in C^{1,\alpha}(\mathbb{R}^n, \mathbb{R}^{n\times n})$ satisfying

$$\lambda^{-1}|\xi|^2 \leq \mathfrak{A}(x)\xi\cdot\xi, \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n,$$

and

$$\max_{1\leq i,j\leq n} \left\|\mathfrak{A}^{ij}\right\|_{C^{1,\alpha}(\mathbb{R}^n)} \leq \lambda,$$

for some constant $\lambda > 1$.

Recall that the canonical parametrix for the operator $\mathrm{div}(\mathfrak{A}\nabla\cdot)$ is given by

$$H(x,y) = \frac{\left[\mathfrak{A}^{-1}(y)(x-y)\cdot(x-y)\right]^{(2-n)/2}}{(n-2)|\mathbb{S}^{n-1}\left[\det(\mathfrak{A}(y))\right]^{1/2}}, \quad x,y \in \mathbb{R}^n,\ x \neq y.$$

**Theorem 7.** *[ [11, Theorem 3, p. 271]] Pick $\Omega_0 \supset \Omega$. For each $y \in \overline{\Omega}_0$, there exists $u_y \in C^2(\overline{\Omega}_0 \setminus \{y\})$ satisfying* $\mathrm{div}(\mathfrak{A}\nabla u) = 0$ *in* $\Omega_0 \setminus \{y\}$,

$$\left|u_y(x) - H(x,y)\right| \leq C|x-y|^{2-n+\alpha}, \quad x \in \overline{\Omega}_0 \setminus \{y\},$$

$$\left|\nabla u_y(x) - \nabla H(x,y)\right| \leq C|x-y|^{1-n+\alpha}, \quad x \in \overline{\Omega}_0 \setminus \{y\},$$

*where* $C = C(n,\Omega_0,\alpha,\lambda) > 0$.

Pick $x_0 \in \Gamma_0$ and let $r_0 > 0$ sufficiently small in such a way that $B(x_0,r_0)\cap\Gamma \Subset \Gamma_0$. As $B(x_0,r_0)\setminus\overline{\Omega}$ contains a cone with a vertex at $x_0$, we find $\delta_0 > 0$ and a vector $\xi \in \mathbb{S}^{n-1}$ such that, for each $0 < \delta \leq \delta_0$, we have $y_\delta = x_0 + \delta\xi \in B(x_0,r_0)\setminus\overline{\Omega}$ and $\mathrm{dist}(y_\delta,\overline{\Omega}) \geq c\delta$, for some constant $c = c(\Omega) > 0$ (see Figure 1 below).

In the sequel $\Omega_0 = \Omega \cup B(x_0,r_0)$ and $u_\delta = u_{y_\delta}$, $0 < \delta \leq \delta_0$, where $u_{y_\delta}$ is given by Theorem 7. Reducing $\delta_0$ if necessary, we may assume that

$$\mathrm{dist}\left(y_\delta,\partial\Omega_0\right) \geq r_0/2, \quad 0 < \delta \leq \delta_0. \tag{17}$$

On the other hand, using that the continuous bilinear form

$$\mathfrak{b}_0(u,v)=\int_{\Omega_0}\mathfrak{A}\nabla u\cdot\nabla v,\quad u,v\in H_0^1(\Omega_0),$$

is coercive, we derive that the BVP

$$\begin{cases}\mathrm{div}(\mathfrak{A}\nabla v)=0 & \text{in } \Omega_0,\\ v_{|\partial\Omega_0}=u_\delta,\end{cases}\tag{18}$$

admits a unique weak solution $v_\delta\in H^1(\Omega_0)$ satisfying

$$\|v_\delta\|_{H^1(\Omega_0)}\le C\|u_\delta\|_{H^{1/2}(\partial\Omega_0)},\quad 0<\delta\le\delta_0.\tag{19}$$

where $C=C(n,\Omega_0,\lambda)>0$.

**Lemma 8.** *We have*

$$\|v_\delta\|_{H^{1/2}(\Gamma)}\le C,\quad 0<\delta\le\delta_0,\tag{20}$$

*where* $C=C(n,\Omega,\alpha,\lambda,x_0,r_0)>0$.

**Proof.** Let $\widetilde{\Omega}=\{x\in\Omega_0;\ \mathrm{dist}(x,\partial\Omega_0)<r_0/4\}$. By the continuity of the trace operator, we have

$$\|u_\delta\|_{H^{1/2}(\partial\Omega_0)}\le C\|u_\delta\|_{H^1(\tilde{\Omega})},\quad 0<\delta\le\delta_0,\tag{21}$$

for some constant $C=C(\Omega_0,r_0)>0$. Then in light of the inequality

$$\|u_\delta\|_{H^1(\tilde{\Omega})}\le\|u_\delta-H(\cdot,y_\delta)\|_{H^1(\tilde{\Omega})}+\|H(\cdot,y_\delta)\|_{H^1(\tilde{\Omega})},\quad 0<\delta\le\delta_0,$$

(17) and Theorem 7 we obtain

$$\|u_\delta\|_{H^1(\tilde{\Omega})}\le C,\quad 0<\delta\le\delta_0,\tag{22}$$

where $C=C(n,\Omega,\alpha,\lambda,x_0,r_0)>0$.
A combination of (19), (21) and (22) then implies

$$\|v_\delta\|_{H^{1/2}(\Gamma)}\le C,\quad 0<\delta\le\delta_0,$$

where $C$ is as above. □

Let $P\in L^\infty(\mathbb{R}^n,\mathbb{R}^n)$ satisfying $\|P\|_{L^\infty(\mathbb{R}^n)}\le\lambda$ and consider on $H_0^1(\Omega)\times H_0^1(\Omega)$ the continuous bilinear forms

$$\mathfrak{b}(u,v)=\int_\Omega[\mathfrak{A}\nabla u+uP]\cdot\nabla v,\quad u,v\in H_0^1(\Omega).$$

$$\mathfrak{b}^*(u,v)=\int_\Omega[\mathfrak{A}\nabla u\cdot\nabla v-vP\cdot\nabla u],\quad u,v\in H_0^1(\Omega).$$

We assume that $\mathfrak{b}$ and $\mathfrak{b}^*$ are coercive: there exists $c_0>0$ such that

$$\mathfrak{b}(u,u)\ge c_0\|u\|_{H_0^1(\Omega)},\quad \mathfrak{b}^*(u,u)\ge c_0\|u\|_{H_0^1(\Omega)}\quad u\in H_0^1(\Omega).$$

Let $f\in H^{1/2}(\Gamma)$. From Lemma 12 and its proof the BVP

$$\begin{cases}\mathrm{div}(\mathfrak{A}\nabla u+uP)=0 & \text{in } \Omega,\\ u_{|\Gamma}=f,\end{cases}\tag{23}$$

has a unique weak solution $u(f)\in H^1(\Omega)$ satisfying

$$\|u(f)\|_{H^1(\Omega)}\le C\|f\|_{H^{1/2}(\Gamma)},\tag{24}$$

where $C=C(n,\Omega,\lambda,c_0)>0$.
Similarly, the BVP

$$\begin{cases}\mathrm{div}(\mathfrak{A}\nabla u^*)-P\cdot\nabla u^*=0 & \text{in } \Omega,\\ u^*_{|\Gamma}=f,\end{cases}\tag{25}$$

admits unique weak solution $u^*(f)\in H^1(\Omega)$ satisfying

$$\left\|u^*(f)\right\|_{H^1(\Omega)}\le C\|f\|_{H^{1/2}(\Gamma)},\tag{26}$$

where $C$ is as in (24).

Set $f_\delta = (u_\delta - v_\delta)_{|\Gamma} \in H^{1/2}(\Gamma)$, $0 < \delta \le \delta_0$. By construction we have $v_\delta = u_\delta$ on $\partial\Omega_0 \cap \Gamma \supseteq \Gamma \setminus \overline{\Gamma}_0$. Hence $\mathrm{supp}(f_\delta) \subset \Gamma_0$ (see Figure 1 below). That is we have $f_\delta \in H^{1/2}_{\Gamma_0}(\Gamma)$.

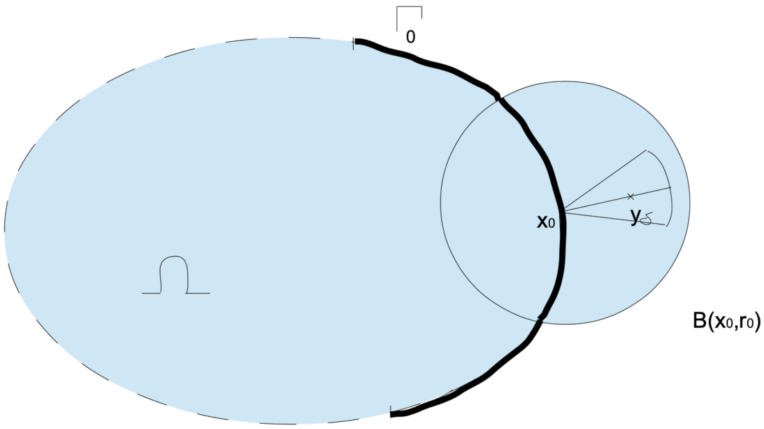


**Figure 1.**

Until the end of this subsection $C = C(n,\Omega,\lambda,\alpha,c_0,x_0,r_0) > 0$ denotes a generic constant.

For $\delta > 0$ define

$$\ell_n(\delta) = \begin{cases} 1, & n = 3, \\ |\ln\delta|^{1/2}, & n = 4, \\ \delta^{2-n/2}, & n \ge 5. \end{cases}$$

**Lemma 9.** *Let $0 < \delta \le \delta_0$ and denote by $w_\delta \in H^1(\Omega)$ the weak solution of the BVP (23) when $f = f_\delta$. Then $w_\delta = H(\cdot, y_\delta) + z_\delta$ with*

$$\|z_\delta\|_{H^1(\Omega)} \le C\ell_n(\delta).$$

**Proof.** We first note that $\tilde{z}_\delta = w_\delta - u_\delta \in H^1(\Omega)$ is the weak solution of the BVP

$$\begin{cases} -\mathrm{div}(\mathfrak{A}\nabla\tilde{z} + \tilde{z}P) = \mathrm{div}(u_\delta P) & \text{in } \Omega, \\ \tilde{z}_{|\Gamma} = -v_\delta, \end{cases}$$

It follows from Lemma 13 that

$$\|\tilde{z}_\delta\|_{H^1(\Omega)} \le C\left(\|v_\delta\|_{H^{1/2}(\Gamma)} + \|u_\delta\|_{L^2(\Omega)}\right).$$

Using that $\mathrm{dist}(y_\delta, \overline{\Omega}) \ge c\delta$ (and hence $\Omega \subset B(R, y_\delta) \setminus B(y_\delta, c\delta/2)$ for some large $R > 0$ independent on $\delta$) we easily derive from Theorem 7

$$\|u_\delta\|_{L^2(\Omega)} \le C\ell_n(\delta).$$

We combine this estimate with (20) in order to obtain

$$\|\tilde{z}_\delta\|_{H^1(\Omega)} \le C\ell_n(\delta). \tag{27}$$

Let $z_\delta = \tilde{z}_\delta + u_\delta - H(\cdot, y_\delta)$. Then we have the decomposition $w_\delta = H(\cdot, y_\delta) + z_\delta$. Using once again Theorem 7 and (27), we obtain

$$\|z_\delta\|_{H^1(\Omega)} \le C\ell_n(\delta)$$

as expected. □

**Lemma 10.** *Assume that $P \in W^{1,\infty}(\Omega)$ with $\|P\|_{W^{1,\infty}(\Omega)} \le \lambda$. Let $0 < \delta \le \delta_0$ and denote by $w_\delta^* \in H^1(\Omega)$ the weak solution of the BVP* (25) *when $f = f_\delta$. Then $w_\delta = H(\cdot, y_\delta) + z_\delta^*$ with*

$$\left\| z_\delta^* \right\|_{H^1(\Omega)} \le C\ell_n(\delta).$$

**Proof.** As $\widetilde{z}_\delta^* = w_\delta^* - u_\delta$ is the solution of the BVP

$$\begin{cases} \mathrm{div}\,(\mathfrak{A}\nabla \widetilde{z}^*) - P\cdot\nabla\widetilde{z}^* = P\cdot\nabla u_\delta & \text{in } \Omega, \\ \widetilde{z}^*_{|\Gamma} = -v_\delta, \end{cases}$$

we obtain from Lemma (14)

$$\left\| \widetilde{z}_\delta^* \right\|_{H^1(\Omega)} \le C\left( \|v_\delta\|_{H^{1/2}(\Gamma)} + \|u_\delta\|_{L^2(\Omega)} \right).$$

The rest of the proof is similar to that of Lemma 9. □

### 3.2. *Stability of determining the conformal factor at the boundary*

Suppose that $\mathfrak{A} = \sigma A$, where $A$ is as in Section 1 and $\sigma \in C^{1,\alpha}(\mathbb{R}^n)$, define

$$\Lambda_\sigma : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

as follows

$$\langle \Lambda_\sigma(f), \varphi\rangle_{1/2} = \int_\Omega \left[\sigma A\nabla u_\sigma(f) + u_\sigma(f)P\right]\cdot\nabla v, \quad \varphi \in H^{1/2}_{\Gamma_0}(\Gamma),\ v \in \dot{\varphi}.$$

where $u_\sigma(f)$ is the solution of (23) when $\mathfrak{A} = \sigma A$. We also consider

$$\widetilde{\Lambda}_\sigma : f \in H^{1/2}_{\Gamma_0}(\Gamma) \mapsto \chi\Lambda_\sigma(f) \in H^{1/2}(\Gamma).$$

Pick $P_j$, $j = 1,2$ satisfying the assumptions of Lemma 10.

Let $u_j = u_{\sigma_j}$ when $P = P_j$, $\Lambda_j = \Lambda_{\sigma_j}$, $j = 1,2$. Set then $\sigma = \sigma_1 - \sigma_2$, $P = P_1 - P_2$ and $u = u_1 - u_2$. With these notations we have

$$\begin{aligned} \langle (\Lambda_1 - \Lambda_2)(f), v_{|\Gamma}\rangle_{1/2} = &\int_\Omega \sigma A\nabla u_1(f)\cdot\nabla v + \int_\Omega \sigma_2 A\nabla u\cdot\nabla v \\ &+ \int_\Omega \left[u_1(f)P - uP_2\right]\cdot\nabla v, \quad v \in H^1(\Omega). \end{aligned}$$

We use this identity with $v = v^*(g)$, $g \in H^{1/2}(\Gamma)$, the weak solution of the BVP

$$\begin{cases} \mathrm{div}\,(\sigma_2 A\nabla v^*) - P_2\cdot\nabla v^* = 0 & \text{in } \Omega, \\ v^*_{|\Gamma} = g. \end{cases}$$

Since

$$\begin{aligned} &\int_\Omega \sigma_2 A\nabla u\cdot\nabla v^*(g)dx - \int_\Omega uP_2\cdot\nabla v^*(g)dx \\ &\qquad = -\int_\Omega u\mathrm{div}\left(\sigma_2\nabla v^*(g)\right)dx - \int_\Omega uP_2\cdot\nabla v^*(g)dx = 0 \end{aligned}$$

we obtain by taking into account (10)

$$\begin{aligned} &\left\langle \left(\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2\right)(f), g\right\rangle_{1/2} \\ &\qquad = \int_\Omega \sigma A\nabla u_1(f)\cdot\nabla v^*(g)dx + \int_\Omega u_1(f)P\cdot\nabla v^*(g)dx, \quad f,g \in H^{1/2}_{\Gamma_0}(\Gamma). \end{aligned}$$

Let $H_j = H$ when $\mathfrak{A} = \sigma_j A$, $j = 1,2$. That is we have

$$H_j(x,y) = \frac{\left[A^{-1}(y)(x-y)\cdot(x-y)\right]^{(2-n)/2}}{(n-2)\left|\mathbb{S}^{n-1}\right|\sigma(y)\left[\det(A(y))\right]^{1/2}}, \quad x,y \in \mathbb{R}^n,\ x \ne y.$$

According to Lemmas 9 and 10, with $f = g = f_\delta$, $\mathfrak{A} = \sigma_j A$ and $P = P_j$, $j = 1$ or $j = 2$, we have

$$u_1(f_\delta) = H_1(\cdot, y_\delta) + z_\delta, \quad 0 < \delta \le \delta_0,$$
$$v^*(f_\delta) = H_2(\cdot, y_\delta) + z_\delta^*, \quad 0 < \delta \le \delta_0,$$

where $z_\delta$ and $z_\delta^*$ satisfies

$$\|z_\delta\|_{H^1(\Omega)} \le C\ell_n(\delta), \quad \left\|z_\delta^*\right\|_{H^1(\Omega)} \le C\ell_n(\delta), \quad 0 < \delta \le \delta_0.$$

In the rest subsection we always need to reduce $\delta_0$. For simplicity convenience we keep the notation $\delta_0$.

Fix $\Upsilon$ a nonempty closed subset of $\Gamma_0$ and assume that $\|\sigma\|_{C(\Upsilon)} = \sigma(x_0)$. Proceeding as in the proof [3, (2.8)], we get

$$C\|\sigma\|_{C(\Upsilon)} \le \delta^{n-2}\int_\Omega \sigma A\nabla u_1(f_\delta)\cdot\nabla v^*(f_\delta)dx + \delta^\alpha$$

We also prove in a similar manner

$$\left|\int_\Omega u_1(f)P\cdot\nabla v^*(f_\delta)dx\right| \le C\ell_n(\delta)\delta^{1-n/2},$$
$$\left|\left\langle\left(\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2\right)(f_\delta), v^*(f_\delta)\right\rangle_{1/2}\right| \le C'\left\|\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2\right\|_{\mathrm{op}}\delta^{2-n}.$$

Here and in the sequel $C' = C'(n, \Omega, \lambda, \alpha, \Gamma_0, \Gamma_1, x_0, r_0) > 0$ denotes a generic constant. Hence

$$C'\|\sigma\|_{C(\Upsilon)} \le \left\|\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2\right\|_{\mathrm{op}} + \max\left(\delta^{n/2-1}\ell_n(\delta), \delta^\alpha\right), \quad 0 < \delta \le \delta_0,$$

from which we derive

$$\|\sigma\|_{C(\Upsilon)} \le C'\left\|\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2\right\|_{\mathrm{op}}. \tag{28}$$

### 3.3. *Proof of Theorem 1*

Let $a_1, a_2 \in \widetilde{\mathscr{A}}$. We apply the preceding result with

$$\sigma_j = a_j\left(u_{a_j}(f)\right), \quad P_j = a'\left(u_{a_j}(f)\right)\nabla u_{a_j}(f), \quad f \in B_m^{\eta_m},\ j = 1, 2,\ t \in \mathbb{R}.$$

Note that the coercivity of $\mathfrak{b}$ and $\mathfrak{b}^*$ when $P = P_j$, $j = 1, 2$, was already demonstrated in the previous section, with coercivity constant independent on $f \in B_m^{\eta_m}$.

By taking $f = 0$ we easily get from (28)

$$|a_1(0) - a_2(0)| \le C_0^1\left\|d\widetilde{\Lambda}_{a_1}^0(0) - d\widetilde{\Lambda}_{a_2}^0(0)\right\|_{\mathrm{op}}. \tag{29}$$

For each $a \in \mathscr{A}$ and $t \in \mathbb{R}$, we obtain by straightforward computations that

$$u_{a^t}(f) = u_a\left(f + \mathfrak{f}_t\right) - \mathfrak{f}_t,$$

where $a^t(z) = a(z+t)$, $z \in \mathbb{R}$. This identity yields

$$\Lambda_a\left(f + \mathfrak{f}_t\right) = \Lambda_{a^t}(f), \quad f \in C_{\Gamma_0}^{2,\alpha}(\Gamma). \tag{30}$$

The following assumptions hold for the family $(a^t)_{t\in\mathbb{R}}$ : for any $\tau > 0$, we have

$$a^t(z) \le \mu_\tau\left(\varrho^{-1}|z|\right) = \mu\left(\varrho^{-1}(|z|+\tau)\right), \quad a \in \mathbb{R},\ |t| \le \tau,$$
$$\left|(a^t)'(z)\right| \le \gamma_\tau\left(\varrho^{-1}|z|\right) = \gamma\left(\varrho^{-1}(|z|+\tau)\right), \quad a \in \mathbb{R},\ |t| \le \tau,$$
$$\left|(a^t)''(z)\right| \le \widetilde{\gamma}_\tau\left(\varrho^{-1}|z|\right) = \widetilde{\gamma}\left(\varrho^{-1}(|z|+\tau)\right), \quad a \in \mathbb{R},\ |t| \le \tau.$$

Identity (30) shows that $\widetilde{\Lambda}_a^t$ is Fréchet differentiable in a neighborhood of the origin with

$$d\widetilde{\Lambda}_a^t(0) = d\widetilde{\Lambda}_{a^t}(0).$$

Furthermore, (14) and Proposition (5) with $a = a^t$ gives

$$\sup_{|t|\le\tau} \left\| d\widetilde{\Lambda}_a^t(0)\right\|_{\mathrm{op}} \le C_\tau^0. \tag{31}$$

Now, we easily get by applying (29), with $a_1$ and $a_2$ substituted by $a_1^t$ and $a_2^t$,

$$|a_1(t) - a_2(t)| \le C_\tau^1 \left\| d\widetilde{\Lambda}_{a_1}^t(0) - d\widetilde{\Lambda}_{a_2}^t(0)\right\|_{\mathrm{op}}, \quad |t|\le\tau.$$

That is we have

$$\|a_1 - a_2\|_{C([-\eta,\eta])} \le C_\tau^1 \sup_{|t|\le\eta} \left\| d\widetilde{\Lambda}_{a_1}^t(0) - d\widetilde{\Lambda}_{a_2}^t(0)\right\|_{\mathrm{op}},$$

which is the expected inequality.

## Appendix A. Technical elementary lemmas

Let $\mathfrak{A} = (\mathfrak{A}^{ij}) \in L^\infty(\mathbb{R}^n, \mathbb{R}^{n\times n})$ satisfying

$$2\beta|\xi|^2 \le \mathfrak{A}(x)\xi\cdot\xi, \quad x,\xi\in\mathbb{R}^n,$$

for some $\beta > 0$, and $P \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

Pick a bounded domain $\Omega$ of $\mathbb{R}^n$ and consider on $H_0^1(\Omega)\times H_0^1(\Omega)$ the continuous bilinear forms

$$\mathfrak{b}(u,v) = \int_\Omega [\mathfrak{A}\nabla u + uP]\cdot\nabla v, \quad u,v\in H_0^1(\Omega).$$

$$\mathfrak{b}^*(u,v) = \int_\Omega [\mathfrak{A}\nabla u\cdot\nabla v - vP\cdot\nabla u], \quad u,v\in H_0^1(\Omega).$$

Denote by $\mu_\Omega$ the Poincaré's constant of $\Omega$:

$$\|u\|_{L^2(\Omega)} \le \mu_\Omega \|\nabla u\|_{L^2(\Omega)}, \quad u\in H_0^1(\Omega).$$

**Lemma 11.** *Under the assumption $\|P\|_{L^\infty(\mathbb{R}^n)} \le \mu_\Omega^{-1}\beta$, we have*

$$\mathfrak{b}(u,u) \ge \beta\|\nabla u\|_{L^2(\Omega)}^2, \quad u\in H_0^1(\Omega), \tag{32}$$

$$\mathfrak{b}^*(u,u) \ge \beta\|\nabla u\|_{L^2(\Omega)}^2, \quad u\in H_0^1(\Omega). \tag{33}$$

**Proof.** Let $u\in H_0^1(\Omega)$. As

$$\int_\Omega uP\cdot\nabla u \le \|P\|_{L^\infty(\mathbb{R}^n)}\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} \le \mu_\Omega\|P\|_{L^\infty(\mathbb{R}^n)}\|\nabla u\|_{L^2(\Omega)}^2,$$

we get

$$\mathfrak{b}(u,u) \ge \left(2\beta - \mu_\Omega\|P\|_{L^\infty(\mathbb{R}^n)}\right)\|\nabla u\|_{L^2(\Omega)}^2.$$

Therefore (32) follows. The proof of (33) is similar. □

We introduce a definition. Let $f\in H^{1/2}(\Gamma)$. We say that $u\in H^1(\Omega)$ is a weak solution of the BVP

$$\begin{cases} \operatorname{div}(\mathfrak{A}\nabla u + uP) = 0 & \text{in } \Omega, \\ u_{|\Gamma} = f \end{cases} \tag{34}$$

if $u_{|\Gamma} = f$ (in the trace sense) and

$$\mathfrak{b}(u,v) = 0, \quad v\in H_0^1(\Omega).$$

Note that this last condition implies that the first equation in (34) holds in $H^{-1}(\Omega)$.

Also, we say that $u^*\in H^1(\Omega)$ is a weak solution of the BVP

$$\begin{cases} \operatorname{div}(\mathfrak{A}\nabla u^*) - P\cdot\nabla u^* = 0 & \text{in } \Omega, \\ u_{|\Gamma}^* = f, \end{cases} \tag{35}$$

if $u_{|\Gamma}^* = f$ and

$$\mathfrak{b}^*\left(u^*, v\right) = 0, \quad v\in H_0^1(\Omega).$$

Let us assume that $\mathfrak{A}$ satisfies in addition

$$\max_{1\le i,j\le n}\left\|\mathfrak{A}^{ij}\right\|_{L^\infty(\mathbb{R}^n)}\le\widetilde{\beta},$$

for some $\widetilde{\beta}>0$.

Let $\mathscr{E}f$ be the unique element of $\dot{f}$ so that $\|\mathscr{E}f\|_{H^1(\Omega)}=\|f\|_{H^{1/2}(\Gamma)}$ ($\mathscr{E}f$ is nothing but the orthogonal projection of $0\in H^1(\Omega)$ on the closed convex set $\dot{f}$). Then

$$\left\|\mathrm{div}\left(\mathfrak{A}\nabla\mathscr{E}f+\mathscr{E}fP\right)\right\|_{H^{-1}(\Omega)}\le\left(\widetilde{\beta}+\|P\|_{L^\infty(\mathbb{R}^n)}\right)\|f\|_{H^{1/2}(\Gamma)}. \tag{36}$$

Furthermore, we have

$$\left\langle\mathrm{div}\left(\mathfrak{A}\nabla\mathscr{E}f+\mathscr{E}fP\right),v\right\rangle_{-1}=\mathfrak{b}(\mathscr{E}f,v),\quad v\in H_0^1(\Omega), \tag{37}$$

where $\langle\cdot,\cdot\rangle_{-1}$ is the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

**Lemma 12.** *Suppose that $\|P\|_{L^\infty(\mathbb{R}^n)}\le\mu_\Omega^{-1}\beta$. Then we have*

(i) *the BVP* (34) *admits a unique weak solution $u\in H^1(\Omega)$ satisfying*

$$\|u\|_{H^1(\Omega)}\le C\|f\|_{H^{1/2}(\Gamma)}, \tag{38}$$

*and*

(ii) *the BVP* (35) *has a unique weak solution $u^*\in H^1(\Omega)$ satisfying*

$$\left\|u^*\right\|_{H^1(\Omega)}\le C\|f\|_{H^{1/2}(\Gamma)}, \tag{39}$$

*where $C=C(n,\Omega,\beta,\widetilde{\beta})>0$.*

**Proof.** We provide the proof of (i) and omit that of (ii) which is quite similar to that of (i).

Since $\mathfrak{b}$ is coercive by Lemma 11, in light of (37) we get by applying Lax–Milgram's lemma that there exists a unique $u_0\in H_0^1(\Omega)$ satisfying

$$\mathfrak{b}(u_0,v)=-\mathfrak{b}\left(\mathscr{E}f,v\right),\quad v\in H_0^1(\Omega).$$

In particular we have

$$\mathfrak{b}(u_0,u_0)=-\mathfrak{b}\left(\mathscr{E}f,u_0\right),\quad v\in H_0^1(\Omega).$$

Using (32), (36) and (37), we derive

$$\beta\|\nabla u_0\|_{L^2(\Omega)}\le\left(\widetilde{\beta}+\|P\|_{L^\infty(\mathbb{R}^n)}\right)\|f\|_{H^{1/2}(\Gamma)}. \tag{40}$$

Clearly, $u=u_0+\mathscr{E}f$ satisfies

$$\mathfrak{b}(u,v)=0,\quad v\in H_0^1(\Omega).$$

and $u_{|\Gamma}=f$. In other words $u$ is a weak solution of (34) and, as a consequence of (40), $u$ satisfies (38).

We complete the proof by noting that the uniqueness of solutions of (34) is a straightforward consequence of the coercivity of $\mathfrak{b}$. □

Consider now the BVP

$$\begin{cases}\mathrm{div}\,(\mathfrak{A}\nabla u+uP)=\mathrm{div}(F)\quad\text{in }\Omega,\\ u_{|\Gamma}=f,\end{cases} \tag{41}$$

where $f\in H^{1/2}(\Gamma)$ and $F\in L^2(\Omega)^n$.

Assume first that $f=0$. In that case the variational problem associated to (41) has the form

$$\mathfrak{b}(u,v)=\langle\mathrm{div}F,v\rangle_{-1},\quad v\in H_0^1(\Omega), \tag{42}$$

As in the preceding proof we show, with the help of Lax-Milgram's lemma, that the variational problem (42) admits a unique solution $u(F)\in H_0^1(\Omega)$ satisfying

$$\|u(F)\|_{H^1(\Omega)}\le C\|F\|_{L^2(\Omega)^n},$$

where $C=C(n,\Omega,\beta,\widetilde{\beta})>0$.

By linearity, the solution of (41) is the sum of the solution of (41) with $f = 0$ and the solution of (41) with $F = 0$ (corresponding to (34)).

We derive from Lemma 12 the following result.

**Lemma 13.** *Suppose that* $\|P\|_{L^\infty(\mathbb{R}^n)} \le \mu_\Omega^{-1}\beta$. *Let* $f \in H^{1/2}(\Gamma)$ *and* $F \in L^2(\Omega)^n$. *Then the BVP* (41) *admits a unique weak solution* $u \in H^1(\Omega)$ *satisfying*

$$\|u\|_{H^1(\Omega)} \le C\left(\|f\|_{H^{1/2}(\Gamma)} + \|F\|_{L^2(\Omega)^n}\right), \tag{43}$$

*where* $C = C(n, \Omega, \beta, \widetilde{\beta}) > 0$.

Next, consider the BVP

$$\begin{cases} \operatorname{div}(\mathfrak{A}\nabla u^*) - P\cdot\nabla u^* = R\cdot\nabla g & \text{in}\,\Omega, \\ u^*_{|\Gamma} = f, \end{cases} \tag{44}$$

Assume that $R \in W^{1,\infty}(\Omega)$ with $\|R\|_{W^{1,\infty}(\Omega)} \le \varrho$, for some $\varrho > 0$, and $g \in H^1(\Omega)$. A simple integration by parts enables us to show that

$$\|R\cdot\nabla g\|_{H^{-1}(\Omega)} \le c\|g\|_{L^2(\Omega)},$$

where $c = c(n, \Omega, \varrho)$.

With the help of this inequality we can proceed as above in order to derive the following lemma.

**Lemma 14.** *Suppose that* $\|P\|_{L^\infty(\mathbb{R}^n)} \le \mu_\Omega^{-1}\beta$. *Let* $g \in H^1(\Omega)$ *and* $R \in W^{1,\infty}(\Omega)$ *satisfying* $\|R\|_{W^{1,\infty}(\Omega)} \le \varrho$, *for some* $\varrho > 0$. *Then the BVP* (44) *has a unique weak solution* $u^* \in H^1(\Omega)$ *satisfying*

$$\left\|u^*\right\|_{H^1(\Omega)} \le C\left(\|f\|_{H^{1/2}(\Gamma)} + \|g\|_{L^2(\Omega)}\right), \tag{45}$$

*where* $C = C(n, \Omega, \beta, \widetilde{\beta}, \varrho) > 0$.

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