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Compactly supported cohomology of a tower of graphs and generic representations of $\text{PGL}_n$ over a local field

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Abstract. Let $F$ be a non-archimedean locally compact field and let $G_n$ be the group $\text{PGL}_n(F)$. In this paper we construct a tower $(\mathcal{X}_k)_{k \geq 0}$ of graphs fibred over the one-skeleton of the Bruhat–Tits building of $G_n$. We prove that a non-spherical and irreducible generic complex representation of $G_n$ can be realized as a quotient of the compactly supported cohomology of the graph $\mathcal{X}_k$ for $k$ large enough. Moreover, when the representation is cuspidal then it has a unique realization in such a model.

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1. Introduction

Let $F$ be a non-archimedean locally compact field and let $G_n$ be the locally profinite group $\text{PGL}_n(F)$. In [4], P. Broussous has constructed a projective tower of simplicial complexes fibred over the Bruhat–Tits building of $G_n$. The idea (due to P. Schneider) consists of constructing simplicial complexes whose structure is very related to that of the Bruhat–Tits building. The goal of a such construction is to try to find geometric interpretation of certain classes of irreducible smooth representations of $G_n$. Such a geometric interpretation exists for example for the Steinberg representation of $G_n$ which can be realized (see [3, Thm. 3]) as the cohomology with compact support in top dimension of the Bruhat–Tits building. In a second work (see [5]), P. Broussous has constructed in the case $n = 2$ a slightly modified version of his previous construction. More precisely, he construct a tower of directed graphs $(\mathcal{X}_k)_{k \geq 0}$ fibred over the Bruhat–Tits tree of $G_2$. Based on the existence of new vectors for irreducible generic representations of $G_2$, he proves that an irreducible generic representation $\pi$ of $G_2$ can be realized as a quotient of the compactly supported cohomology space $H^1_c(\mathcal{X}_{c(\pi)}, \mathbb{C})$, where $c(\pi)$ is an integer related to the conductor of
the representation \( \pi \). He proves moreover that if \( \pi \) is cuspidal then it can be realized as a subrepresentation of the last cohomology space and that a such realization is unique. In a parallel direction, the author has constructed a simplicial complex fibred over the Bruhat–Tits building of \( G_n \), whose top compactly supported cohomology realize as subquotient all the irreducible cuspidal level zero representations of \( G_n \), see [9].

In this paper our aim is to generalize the construction of Broussous given in [5] to the case \( n > 3 \). More precisely we construct a projective tower \( (X_k)_{k \geq 0} \) of directed graphs fibred over the 1-skeleton of the Bruhat–Tits building of \( G_n \). In our construction, the graphs considered will be defined in terms of combinatorial geodesic paths of the Bruhat–Tits building of \( G_n \).

Let \( \pi \) be an irreducible smooth generic and non-spherical representation of \( G_n \). We prove that there exists an injective intertwining operator

\[
\Psi^\vee : V^\vee \longrightarrow \mathcal{H}_\infty(\bar{X}_{c(\pi)}, \mathbb{C}),
\]

where \( V^\vee \) is the contragredient representation of \( \pi \) and \( \mathcal{H}_\infty(\bar{X}_{c(\pi)}, \mathbb{C}) \) is the space of smooth harmonic forms on the graph \( \bar{X}_{c(\pi)} \). By applying contragredients to this intertwining operator and then by restriction to \( H^1_c(\bar{X}_{c(\pi)}, \mathbb{C}) \) we obtain a nonzero intertwining operator

\[
\Psi : H^1_c(\bar{X}_{c(\pi)}, \mathbb{C}) \longrightarrow V.
\]

That is the representation \( \pi \) is isomorphic to a quotient of \( H^1_c(\bar{X}_{c(\pi)}, \mathbb{C}) \). In the case when \( \pi \) is cuspidal, the \( G_n \)-equivariant map \( \Psi \) splits so that \( \pi \) injects in \( H^1_c(\bar{X}_{c(\pi)}, \mathbb{C}) \). We prove that such an injection is unique, that is:

\[
dim_{\mathbb{C}} \text{Hom}_{G_n}(\pi, H^1_c(\bar{X}_{c(\pi)}, \mathbb{C})) = 1.
\]

### 2. Notations and preliminaries

In this article, \( F \) will be a non-archimedean locally compact field. We write \( \sigma_F \) for the ring of integers of \( F \), \( p_F \) for the maximal ideal of \( \sigma_F \), \( k_F := \sigma_F / p_F \) for the residue class field of \( F \) and \( q_F \) for the cardinal of \( k_F \). We fix a normalized uniformizer \( \varpi_F \) of \( \sigma_F \) and we denote by \( v_F \) the normalized valuation of \( F \).

#### 2.1. The projective general linear group \( \text{PGL}_n(F) \)

For every integer \( n > 2 \), the projective general linear group \( \text{PGL}_n(F) \) will be denoted by \( G_n \). If \( k \geq 1 \) is an integer, we write \( \Gamma_0(p_F^k) \) for the following subgroup of \( \text{GL}_n(F) \)

\[
\Gamma_0(p_F^k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_n(\sigma_F) \mid a \in \text{GL}_{n-1}(\sigma_F), \; d \in \sigma_F^k, \; c \equiv 0 \mod p_F^k \right\}
\]

and we write \( \Gamma_0(p_F^0) \) for its image in \( G_n \). We denote also the image in \( G_n \) of the standard maximal compact subgroup of \( \text{GL}_n(F) \) by \( \Gamma_0(p_F^0) \).

#### 2.2. The Bruhat–Tits building of \( G_n \)

In this section we fix some notations and recall some well-known facts. For more details the reader may refer to [1], [7] or [10]. Recall that a lattice of the vector space \( F^n \) is an open compact subgroup of the additive group of \( F^n \). A such lattice is an \( \sigma_F \)-lattice if moreover it is an \( \sigma_F \)-submodule of \( F^n \). Equivalently, an \( \sigma_F \)-lattice of \( F^n \) is a free \( \sigma_F \)-submodule \( L \) of \( F^n \) of rank \( n \). If \( L \) is an \( \sigma_F \)-lattice of \( F^n \) then \( L = \sigma_F f_1 + \cdots + \sigma_F f_n \) for some \( F \)-basis of \( F^n \). More generally if \( L \) and
M are two $\sigma_F$-lattices of $F^n$ then there exist an $F$-basis $(f_1, \ldots, f_n)$ of $F^n$ and $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, with $\alpha_1 \leq \cdots \leq \alpha_n$, such that

$$L = \sigma_F f_1 + \cdots + \sigma_F f_n \quad \text{and} \quad M = p_{F}^{\alpha_1} f_1 + \cdots + p_{F}^{\alpha_n} f_n.$$  

For two $\sigma_F$-lattices $L$ and $M$ of $F^n$, we say that $L$ and $M$ are equivalent if $L = \lambda M$ for some $\lambda \in F^\times$, and we denote the class of $L$ by $[L]$. The Bruhat–Tits building of $G_n$, denoted by $\mathcal{B}_n$, can be defined as the simplicial complex whose vertices are the equivalence classes of $\sigma_F$-lattices in $F^n$ and in which a collection $\Lambda_0, \Lambda_1, \ldots, \Lambda_q$ of pairwise distinct vertices form a $q$-simplex if we can choose representatives $L_i \in \Lambda_i$, for $i \in \{0, \ldots, q\}$, such that $\sigma_F L_0 < L_1 < \cdots < L_q$.

A $q$-simplex as above define the following flag of the $k_F$-vector space $L_0/\sigma_F L_0$

$$\{0\} < L_0/\sigma_F L_0 < L_0/\sigma_F L_0 < \cdots < L_0/\sigma_F L_0$$

The type of a such $q$-simplex is defined to be the type of the corresponding flag of the $k_F$-vector space $L_0/\sigma_F L_0 \simeq k_F^n$. Note that the maximal dimension of the flag corresponding to a simplex $\sigma$ of $\mathcal{B}_n$ is equal to $n-2$. Thus $\mathcal{B}_n$ is a simplicial complex of dimension $n-1$. The group $GL_n(F)$ acts naturally on $\mathcal{B}_n$ by simplicial automorphisms and its center $Z(GL_n(F)) \simeq F^\times$ acts trivially.

So the group $G_n$ acts simplicially on $\mathcal{B}_n$ and the action is transitive on vertices (resp. chambers, $q$-simplices of a fixed type). Let's recall that a labelling of $\mathcal{B}_n$ is a map from the set $\mathcal{B}_n^0$ of vertices of $\mathcal{B}_n$ to the set $\{0, \ldots, n-1\}$ whose restriction to every chamber is injective. We can construct a labelling $\lambda : \mathcal{B}_n^0 \rightarrow \{0, \ldots, n-1\}$ of $\mathcal{B}_n$ as follows (see [7, 19.3]). Let $L_0$ be a fixed $\sigma_F$-lattices of $F^n$. If $v$ is a vertex of $\mathcal{B}_n$, we can choose a representative $L$ such that $L_0 \subset L$. Since $\sigma_F$ is a principal ideal domain, the finitely generated torsion $\sigma_F$-module $L/L_0$ is isomorphic to

$$\sigma_F / p_{F}^{k_1} \oplus \cdots \oplus \sigma_F / p_{F}^{k_n}$$

for some $n$-tuple of integers $0 \leq k_1 \leq k_2 \leq \cdots \leq k_n$. Then

$$\lambda(v) = \sum_{i=0}^{n} k_i \mod n.$$  

The simplicial complex $\mathcal{B}_n$ is the union of a family of subcomplexes, called apartments, defined as follows. A frame is a set $\mathcal{F} = \{d_1, \ldots, d_n\}$ of one-dimensional $F$-vector subspaces of $F^n$ so that $F^n = d_1 + \cdots + d_n$. The apartment corresponding to the frame $\mathcal{F}$ is formed by all simplices $\sigma$ with vertices $\Lambda$ which are equivalence classes of lattices with representatives $L \in \Lambda$ such that

$$L = L_1 + \cdots + L_n,$$

where $L_i$ is a lattice of the $F$-vector space $d_i$. If we fix an $F$-basis $(f_1, \ldots, f_n)$ of $F^n$ adapted to the decomposition $F^n = d_1 + \cdots + d_n$, then a vertex $[L]$ is in the apartment corresponding to the frame $\mathcal{F}$ if and only if

$$L = p_{F}^{\alpha_1} f_1 + \cdots + p_{F}^{\alpha_n} f_n,$$

where $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$. Note that the set of frames of $F^n$ can be identified with the set of maximal $F$-split torus of $G_n$. To a frame $\mathcal{F} = \{d_1, \ldots, d_n\}$ we can associate the maximal $F$-split torus $S \subset G_n$ acting diagonally with respect the decomposition of $F^n$ as direct sum of vectorial lines. Under this identification, for every maximal $F$-split torus $S$ of $G_n$, we denote by $\mathcal{A}S$ the corresponding apartment of $\mathcal{B}_n$. The apartment corresponding to the diagonal torus $T$ will be called the standard apartment of $\mathcal{B}_n$ and denoted by $\mathcal{A}T$.

The geometric realization $|\mathcal{B}_n|$ of the building $\mathcal{B}_n$ is equipped by a metric defined, up to a multiplicative scalar, as follows. The geometric realization of each apartment $|\mathcal{A}|$ can be identified to the euclidian space

$$\mathbb{R}_{0}^{n} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$$
via the map defined by the following way. We fix an F-basis \((f_1, \ldots, f_n)\) of \(F^n\) corresponding to the apartment \(\mathcal{A}\). The set \(\mathcal{A}^0\) of vertices of \(\mathcal{A}\) is then embedded in \(R^n_0\) via the map \(\varphi: \mathcal{A}^0 \to R^n_0\) defined by

\[
\varphi([p^x_1 f_1 + \cdots + p^x_n f_n]) = x - \frac{1}{n} \sigma(x)e,
\]

where for \(x = (x_1, \ldots, x_n) \in \mathbb{Z}^n\), \(\sigma(x) = x_1 + \cdots + x_n\) and where \(e = (1, \ldots, 1)\). This map extends to a bijection \(\varphi: |\mathcal{A}| \to R^n_0\). Via this identification we can then equip \(|\mathcal{A}|\) by an euclidian metric. More explicitly, if \([L]\) and \([M]\) are two vertices of \(\mathcal{A}\) with

\[
L = p^y_1 f_1 + \cdots + p^y_n f_n \quad \text{and} \quad M = p^y_1 f_1 + \cdots + p^y_n f_n,
\]

then

\[
d_\mathcal{A}([L], [M]) = \frac{1}{\sqrt{1 - \frac{1}{n}}} \left[ x - \frac{1}{n} \sigma(x)e, y - \frac{1}{n} \sigma(y)e \right],
\]

where \(d_0\) is the euclidian metric of \(R^n_0\). We note that in the above formula the term \(1/\sqrt{1 - 1/n}\) is just used to normalize the metric of the building. The metric \(d\) of \(|\mathbb{B}_n|\) is then defined as follows. If \(x, y \in \mathbb{B}_n\) then \(d(x, y) = d_\mathcal{A}(x, y)\) for any apartment \(\mathcal{A}\) containing \(x\) and \(y\) and this is independent of the choice of apartment containing them. Finally we recall that the action of the group \(G_n\) on \(|\mathbb{B}_n|\) is by isometries.

2.3. Smooth representations of a locally profinite group

Let \(G\) be a locally profinite group. By a representation of \(G\) we mean a pair \((\pi, V)\) formed by a \(C\)-vector space \(V\) and by a group homomorphism \(\pi: G \to GL_C(V)\). A such representation is called smooth if for every \(v \in V\) the stabilizer

\[
\text{Stab}_G(v) := \{ g \in G | \pi(g)v = v \}
\]

is an open subgroup of \(G\). In this paper all the representations will be assumed to be smooth and complex. A representation \((\pi, V)\) of \(G\) is called admissible if for every compact open subgroup \(K\) of \(G\) the space \(V^K = \{ v \in V | \forall k \in K, \pi(k)v = v \}\) of \(K\)-fixed vectors is finite dimensional. If \((\pi, V)\) is a representation of \(G\), its contragredient \(\pi^\vee\) is the representation of \(G\) in the subspace \(V^\vee\) of the algebraic dual \(V^*\) formed by the linear forms whose stabilizers in \(G\) is open.

Let \(H\) be a closed subgroup of \(G\) and \((\rho, W)\) a representation of \(H\). We recall that the induced representation from \(H\) to \(G\) of \((\rho, W)\), denoted by \(\text{Ind}_H^G \rho\), is the representation of \(G\) on the space \(\text{Ind}_H^G W\) formed by the locally constant functions \(f: G \to W\) such that \(f(hg) = \rho(h)f(h)\) for every \(g \in G\) and \(h \in H\), where the action of \(G\) on \(\text{Ind}_H^G W\) is by left translation. The compactly induced representation \(c\text{-ind}_H^G \rho\) is defined as the subrepresentation of \(\text{Ind}_H^G \rho\) formed by the functions \(f \in \text{Ind}_H^G W\) whose support is compact modulo \(H\).

2.4. Locally profinite group acting on directed graphs

Throughout this paper, we call graph every one dimensional simplicial complex. If \(Y\) is a graph, the set of vertices (resp. edges) of \(Y\) will be denoted by \(Y^0\) (resp. \(Y^1\)). A locally finite graph is a graph \(Y\) for which every vertex belongs to a finite number of edges. All graphs in this paper will be assumed to be locally finite. A directed graph is a graph \(Y\) with a map \(Y^1 \to Y^0 \times Y^0\), \(a \mapsto (a^-, a^+),\) such that for every edge \(a\) one has \(a = (a^-, a^+)\), where for any edge \(a\) we denote by \(a^+\) and \(a^-\) its head and tail respectively. A path in a graph \(Y\) is a sequence \((s_0, \ldots, s_m)\) of vertices such that two consecutive vertices are linked by an edge. The graph \(Y\) is called connected if every pair of vertices are linked by a path. A cover of a graph \(Y\) is a family \((Y_a)_{a \in \Delta}\) of subgraphs such that

\[
Y = \bigcup_{a \in \Delta} Y_a.
\]
The nerve of a such cover, denoted \( \mathcal{N}(Y,(Y_a)_{a \in \Delta}) \) or just \( \mathcal{N}(Y) \) if there is no risk of confusion, is the simplicial complex whose vertex set is \( \Delta \) and in which a finite number of vertices \( a_0, \ldots, a_r \) form a simplex if
\[
\bigcap_{i=0}^{r} Y_{a_i} = \emptyset.
\]

In the remainder of this section the notations and definitions are taken from [5]. If \( Y \) is a graph, we denote by \( C_0(Y,\mathbb{C}) \) (resp. \( C_1(Y,\mathbb{C}) \)) the \( \mathbb{C} \)-vector space with basis \( Y^0 \) (resp. \( Y^1 \)). Let \( C^1_i(Y,\mathbb{C}), i = 1,2 \), be the \( \mathbb{C} \)-vector space of 1-cochains with finite support : \( C^1_i(Y,\mathbb{C}) \) is the subspace of the algebraic dual of \( C_i(Y,\mathbb{C}) \) formed of those linear forms whose restrictions to the basis \( Y^i \) have finite support. The coboundary map
\[
d : C^0_c(Y,\mathbb{C}) \longrightarrow C^1_c(Y,\mathbb{C})
\]
is defined by \( d(f)(a) = f(a^+) - f(a^-) \). Then the compactly supported cohomology space \( H^1_c(Y,\mathbb{C}) \) of the graph \( Y \) is defined by
\[
H^1_c(Y,\mathbb{C}) = C^1_c(Y,\mathbb{C})/dC^0_c(Y,\mathbb{C}).
\]

Let \( G \) be a locally profinite group and \( Y \) be a directed graph. We assume that \( G \) acts on \( Y \) by automorphisms of directed graphs. For all \( s \in Y^0, a \in Y^1 \), the incidence numbers are defined by \([a: a^+] = +1, [a: a^-] = -1, \) and \([a: s] = 0 \) if \( s \notin \{a^+, a^-\} \). These incidence numbers are equivariant in the sense that \([g.a : g.s] = [a : s] \) for all \( g \in G \). The group \( G \) acts on \( C_i(Y,\mathbb{C}) \) and \( C^1_i(Y,\mathbb{C}) \). If the action of \( G \) on \( Y \) is proper, that is for every \( s \in Y^0 \), the stabilizer \( \text{Stab}_C(s) := \{ g \in G | g.s = s \} \) is open and compact, then the spaces \( C_i(Y,\mathbb{C}) \) and \( C^1_i(Y,\mathbb{C}) \) are smooth \( G \)-modules. The coboundary map is \( G \)-equivariant so that \( H^1_c(Y,\mathbb{C}) \) have a structure of a smooth \( G \)-module.

The space of harmonic forms of the graph \( Y \) is defined as the subspace of \( C^1(Y,\mathbb{C}) \) formed by the elements \( f \in C^1(Y,\mathbb{C}) \) verifying the following harmonicity condition (see [5, §(1.3)]):
\[
\sum_{a \in Y^1} [a : s] f(a) = 0 \quad \text{for all } s \in Y^0.
\]
This space will be denoted by \( \mathcal{H}(Y,\mathbb{C}) \). It is naturally provided by a linear action of \( G \). The smooth part of \( \mathcal{H}(Y,\mathbb{C}) \) under the action of \( G \), i.e. the space of smooth harmonic forms is denoted by \( \mathcal{H}_\infty(Y,\mathbb{C}) \).

**Lemma 1 ([5, (1.3.2)])**. The algebraic dual of \( H^1_c(Y,\mathbb{C}) \) naturally identifies with \( \mathcal{H}(Y,\mathbb{C}) \). Under this isomorphism, the contragredient representation of \( H^1_c(Y,\mathbb{C}) \) corresponds to \( \mathcal{H}_\infty(Y,\mathbb{C}) \).

### 3. Combinatorial geodesic paths in \( \mathcal{B} \mathcal{T}_n \)

The aim of this section is to define a class of combinatorial paths in \( \mathcal{B} \mathcal{T}_n \) and to study the action of the group \( G_n \) on this class of paths. The pointwise stabilisers of such paths will be related to the new-vectors subgroups of \( \text{GL}_n(F) \) (the subgroups defined in (1)), see [8].

#### 3.1. Geodesic paths of \( \mathcal{B} \mathcal{T}_n \) and their prolongations

**Definition 2.** Let \( k \geq 0 \) be an integer. A geodesic path of length \( k \) in \( \mathcal{B} \mathcal{T}_n \) (or more simply geodesic \( k \)-path) is a path \( \alpha = (a_0, a_1, \ldots, a_k) \) of \( \mathcal{B} \mathcal{T}_n \) such that for every \( i, j \in \{0, \ldots, k\}, \ d(alpha_i, alpha_j) = |i - j| \).

We denote the set of geodesic \( k \)-paths of \( \mathcal{B} \mathcal{T}_n \) by \( \mathcal{C}_k(\mathcal{B} \mathcal{T}_n) \).

**Remark 3.** We notice that when \( n \geq 4 \) the edges of \( \mathcal{B} \mathcal{T}_n \) are not all of length one, but in the particular cases \( n = 2 \) and \( n = 3 \) all the edges of \( \mathcal{B} \mathcal{T}_n \) are of length one. We also note that every geodesic \( k \)-path of \( \mathcal{B} \mathcal{T}_n \) lies in a same apartment. In fact if \( \alpha \in \mathcal{C}_k(\mathcal{B} \mathcal{T}_n) \) is a geodesic \( k \)-path as previously, then the geometric realization of any apartment containing the vertices \( a_0 \) and \( a_k \) contain the segment \( [a_0, a_k] \) and then all the vertices of \( \alpha \) are contained in the apartment \( \mathcal{A} \).
Proposition 5. Let $k \geq 1$ be an integer and let $\alpha = (\alpha_0, \ldots, \alpha_k)$ be a geodesic $k$-path of $\mathcal{BT}_n$. Then for every apartment $\mathcal{A}$ containing $\alpha$, there exists a unique right (resp. left) prolongation of $\alpha$ in the apartment $\mathcal{A}$.

Proof. Let $\mathcal{A}$ be an apartment containing the path $\alpha$. Assume that $\alpha$ have two right prolongations $x$ and $y$ in $\mathcal{A}$, that is $x, y \in V(\alpha_k)$ and the two sequences $(\alpha_0, \ldots, \alpha_k, x)$ and $(\alpha_0, \ldots, \alpha_k, y)$ are geodesic $(k+1)$-paths of $\mathcal{A}$. So in the geometric realization $|\mathcal{A}|$ of the apartment $\mathcal{A}$ we have $\alpha_k \in [\alpha_0, x] \cap [\alpha_0, y]$. Therefore we have $\alpha_k = tx + (1-t)\alpha_0$ and $\alpha_k = sy + (1-s)\alpha_0$ for some $t$ and $s$ in $[0,1]$. Moreover the two vertices $x$ and $y$ are of the same distance from $\alpha_k$, that is $d(x, \alpha_k) = d(y, \alpha_k)$. So we have $\|x - \alpha_k\| = \|y - \alpha_k\|$ (here $\|\|$ is the euclidian norm of $|\mathcal{A}| \simeq \mathbb{R}^n$). From this we obtain $(1-t)\|x - \alpha_0\| = (1-s)\|y - \alpha_0\|$. But $\|x - \alpha_0\| = \|y - \alpha_0\|$ so we get $t = s$ and then $x = y$. \hfill $\square$

Let $\alpha = (\alpha_0, \ldots, \alpha_k)$ be a geodesic path of $\mathcal{BT}_n$. The inverse of $\alpha$, denoted by $\alpha^{-1}$, is defined by $\alpha^{-1} := (\alpha_k, \ldots, \alpha_0)$. It is clear that $\alpha^{-1}$ is a geodesic path of $\mathcal{BT}_n$. If $k \geq 1$, the tail and the head of $\alpha$ are the two geodesic paths defined respectively by

$$\alpha^- := (\alpha_0, \ldots, \alpha_{k-1}) \quad \text{and} \quad \alpha^+ := (\alpha_1, \ldots, \alpha_k).$$

We define also the initial and terminal directed edge of $\alpha$ respectively by $e^- (\alpha) := (\alpha_0, \alpha_1)$ and $e^+ (\alpha) := (\alpha_{k-1}, \alpha_k)$.

Proposition 6. Let $k \geq 1$ be an integer and let $\alpha, \beta \in \mathcal{C}_k(\mathcal{BT}_n)$. If $\alpha$ and $\beta$ are contained in a same apartment and if $e^- (\alpha) = e^- (\beta)$ (resp. $e^+ (\alpha) = e^+ (\beta)$), then $\alpha = \beta$.

Proof. By induction on $k$, let $\alpha = (\alpha_0, \ldots, \alpha_{k+1})$ and $\beta = (\beta_0, \ldots, \beta_{k+1})$ two geodesic $(k+1)$-paths such that $e^- (\alpha) = e^- (\beta)$. Assume that $\alpha$ and $\beta$ are contained in a same apartment $\mathcal{A}$. Since the two geodesic $k$-paths $\alpha^-$ and $\beta^-$ are contained in the same apartment $\mathcal{A}$ and as they have the same initial directed edges then by induction hypothesis we have $\alpha^- = \beta^-$, that is $\alpha_i = \beta_i$ for each $i \in \{0, \ldots, k\}$. So the two vertices $\alpha_{k+1}$ and $\beta_{k+1}$ are two right prolongation of the geodesic $k$-paths $\alpha^-$ which are contained in the same apartment $\mathcal{A}$. Then by the previous proposition we obtain $\alpha_{k+1} = \beta_{k+1}$ and then $\alpha = \beta$ as required. \hfill $\square$

3.2. Action of $G_n$ on the sets $\mathcal{C}_k(\mathcal{BT}_n)$

The group $G_n$ acts on its building $\mathcal{BT}_n$ by isometries, so $G_n$ acts naturally on the sets $\mathcal{C}_k(\mathcal{BT}_n)$ for each integer $k \geq 0$. The action is given by

$$g.(\alpha_0, \ldots, \alpha_k) = (g.\alpha_0, \ldots, g.\alpha_k)$$

for every $g \in G_n$ and for every $(\alpha_0, \ldots, \alpha_k) \in \mathcal{C}_k(\mathcal{BT}_n)$. Note that since the set $\mathcal{C}_0(\mathcal{BT}_n)$ may be identified with the set of vertices of $\mathcal{BT}_n$, then the action of $G_n$ on $\mathcal{C}_0(\mathcal{BT}_n)$ is transitive. In the particular case $n = 2$, the action of $G_2$ on the sets $\mathcal{C}_k(\mathcal{BT}_2)$ is transitive for every integer $k \geq 0$, see [5]. The situation is slightly different when $n \geq 3$. We are going to prove that in this last case, the sets $\mathcal{C}_k(\mathcal{BT}_n)$ (for $k \geq 1$) have exactly two $G_n$-orbits. We first define the type of a directed edge of $\mathcal{BT}_n$ and we will prove in the lemma bellow that two geodesic 1-paths are in the same $G_n$-orbit.
Lemma 7. Let $e = ([L_0], [L_1])$ be a directed edge of $\mathcal{B}T_n$, where $L_0$ and $L_1$ are two $\mathcal{O}_F$-lattices such that
\[
\omega_F L_0 < L_1 < L_0.
\]
The type of the directed edge $e$, denoted $\xi(e)$, is defined by
\[
\xi(e) = \dim_{k_F} (L_1 / \omega_F L_0).
\]
This definition is clearly independent of the choice of representatives. For every directed edge $e$ of $\mathcal{B}T_n$, we write $e^{-1}$ for the inverse of $e$ which is obtained from $e$ by interchanging its vertices.

**Proof.** In the proof of the three statements we use the following notations. For each integer $n \geq 1$, we write $\Delta_n$ for the set of integers $\{1, \ldots, n\}$. If $e = ([L_0], [L_1])$ is a directed edge of $\mathcal{B}T_n$ with $\omega_F L_0 < L_1 < L_0$ and if $(f_1, \ldots, f_n)$ is a basis of $F^n$ for which
\[
L_0 = \mathcal{O}_F f_1 + \cdots + \mathcal{O}_F f_n \quad \text{and} \quad L_1 = p_F^{k_1} f_1 + \cdots + p_F^{k_n} f_n,
\]
where $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $k_1 \leq \cdots \leq k_n$, we put $A_0 = \{i \in \Delta_n | k_i = 0\}$ and $A_1 = \{i \in \Delta_n | k_i = 1\}$ and we write $p$ and $q$ respectively for their cardinality. The condition $\omega_F L_0 < L_1 < L_0$ implies that $k_i \in \{0, 1\}$ for each $i \in \Delta_n$ and that $p, q \in \{1, \ldots, n-1\}$ and $p + q = n$.

(i). Let $e = ([L_0], [L_1])$ be a directed edge with $\omega_F L_0 < L_1 < L_0$. The inverse of $e$ is then given by $e^{-1} = ([\omega_F^{-1} L_1], [L_0])$ with $L_1 < L_0 < \omega_F^{-1} L_1$. Let $(f_1, \ldots, f_n)$ be a basis of $F^n$ for which
\[
L_0 = \mathcal{O}_F f_1 + \cdots + \mathcal{O}_F f_n \quad \text{and} \quad L_1 = p_F^{k_1} f_1 + \cdots + p_F^{k_n} f_n,
\]
where $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ with $k_1 \leq \cdots \leq k_n$. With the previous notations we have the identifications of $k_F$-vector spaces
\[
L_1 / \omega_F L_0 \cong \bigoplus_{i=1}^n p_F^{k_i} / p_F \cong \bigoplus_{i \in A_0} \mathcal{O}_F / p_F \oplus \bigoplus_{i \in A_1} p_F / p_F \cong k_F^n
\]
and similarly
\[
L_0 / L_1 \cong \bigoplus_{i=1}^n \mathcal{O}_F / p_F^{k_i} \cong \bigoplus_{i \in A_0} \mathcal{O}_F / p_F \oplus \bigoplus_{i \in A_1} \mathcal{O}_F / p_F \cong k_F^n.
\]
So we obtain $\dim_{k_F} (L_0 / L_1) = n - \dim_{k_F} (L_1 / \omega_F L_0)$, and then $\xi(e^{-1}) = n - \xi(e)$.

(ii). Let $e = ([L_0], [L_1])$ be a directed edge of $\mathcal{B}T_n$ with $\omega_F L_0 < L_1 < L_0$ and let $\mathcal{A}$ be an apartment containing $e$. To simplify, we can assume that in a some $F$-basis $(f_1, \ldots, f_n)$ of $F^n$ we have $L_0 = \mathcal{O}_F f_1 + \cdots + \mathcal{O}_F f_n$ and $L_1 = p_F^{x_1} f_1 + \cdots + p_F^{x_n} f_n$, where $x = (x_1, \ldots, x_n)$ is in $\mathbb{Z}^n$. As previously, the $x_i$'s are in $\{0, 1\}$.

Now if we assume that $e \in \mathcal{C}_1(\mathcal{B}T_n)$ then $d([L_0], [L_1]) = 1$. We have then
\[
d_0 \left(0, x - \frac{1}{n} \sigma(x) e \right) = \frac{\sqrt{n-1}}{\sqrt{n}}
\]
that is
\[
\sum_{i=1}^n \left( x_i - \frac{1}{n} \sigma(x) \right)^2 = \frac{n-1}{n}
\]
and then
\[
\sum_{i=1}^n x_i^2 - \frac{1}{n} \sigma(x)^2 = \frac{n-1}{n}.
\]
Let $n \in \mathbb{N}$.

**Proof.** Let us prove firstly that two elements $e$ and $f$ are in the same $G_n$-orbit if and only if they have the same type. Assume that $e$ and $f$ have the same type. Then there exist an $F$-basis $(f_1, \ldots, f_n)$ of $F^n$ for which $L_0 = \sigma F f_1 + \cdots + \sigma F f_n$ and $L_1 = \sigma F f_1 + \cdots + \sigma F f_n$ and if $\xi(e) = \xi(f) = n - 1$ then there exist an $F$-basis $(h_1, \ldots, h_n)$ of $F^n$ such that $L_0 = \sigma F h_1 + \cdots + \sigma F h_n$ and $L_1 = \sigma F h_1 + \cdots + \sigma F h_n + \sigma F h_n$. Assume that $\xi(e) = \xi(f) = n - 1$ (the proof of the case $\xi(e) = 1$ is similar).

As mentioned previously, for each $i \in \Delta_n$ the integer $k_i$ is in $\{0, 1\}$. The fact that $k_1 \leq \cdots \leq k_n$ implies that $(k_1, \ldots, k_n) = (0, \ldots, 0, 1, \ldots, 1)$, where 0 appear $p$-times and 1 appear $q$-times.

So we have

$$L_i / \sigma F L_0 = \bigoplus_{i=1}^{p} \sigma F / \sigma F + \bigoplus_{i=p+1}^{q} \sigma F / \sigma F \cong F^p.$$

But since $\xi(e) = n - 1$, that is $\dim_{k_i} (L_i / \sigma F L_0) = n - 1$, then we have $L_i = \sigma F h_1 + \cdots + \sigma F h_n + \sigma F h_n$. So as desired we have an $F$-basis $(h_1, \ldots, h_n)$ of $F^n$ for which $L_0 = \sigma F h_1 + \cdots + \sigma F h_n$ and $L_1 = \sigma F h_1 + \cdots + \sigma F h_n + \sigma F h_n$. Let's prove now that two elements $e, e' \in \mathcal{C}_k(\mathcal{R}\mathcal{T}_n)$ are in the same $G_n$-orbit if and only if they have the same type. Assume that $e = ([L_0], [L_1])$ (resp. $e' = ([L'_0], [L'_1])$) where $L_0$ and $L_1$ (resp. $L'_0$ and $L'_1$) are two $\sigma F$-lattices such that $\sigma F L_0 < L_1 < L_0$ (resp. $\sigma F L'_0 < L'_1 < L'_0$).

If $e$ and $e'$ have the same type, say for example $\xi(e) = \xi(e') = 1$, then by the previous point we can find two $F$-basis $(f_1, \ldots, f_n)$ and $(f'_1, \ldots, f'_n)$ for which $L_0 = \sigma F f_1 + \cdots + \sigma F f_n$ and $L_1 = \sigma F f_1 + \cdots + \sigma F f_n$ and likewise $L'_0 = \sigma F f'_1 + \cdots + \sigma F f'_n$ and $L'_1 = \sigma F f'_1 + \cdots + \sigma F f'_n$. So if $g \in G_n$ is the unique element sending the $F$-basis $(f_1, \ldots, f_n)$ on $(f'_1, \ldots, f'_n)$ we have $g L_0 = L'_0$ and $g L_1 = L'_1$, thus $g e = e'$ and then $e$ and $e'$ are in the same $G_n$-orbit. The converse is obvious.

**Proposition 8.** Let $n \geq 3$ be an integer. For every $k \geq 1$, the set $\mathcal{C}_k(\mathcal{R}\mathcal{T}_n)$ have two $G_n$-orbits.

**Proof.** Let us prove firstly that two elements $\alpha$ and $\beta$ of $\mathcal{C}_k(\mathcal{R}\mathcal{T}_n)$ are in the same $G_n$-orbit if and only if their initial directed edges $e^-(\alpha)$ and $e^-(\beta)$ are likewise. If $\alpha$ and $\beta$ are in the same $G_n$-orbit then clearly $e^- (\alpha)$ and $e^- (\beta)$ are also in the same $G_n$-orbit. Conversely, assume that $e^- (\alpha)$ and $e^- (\beta)$ are in the same $G_n$-orbit, that is for same $g \in G_n$ one has $e^- (\alpha) = g e^- (\beta)$. So we have $e^- (\alpha) = e^- (g, \beta)$.

Let $\mathcal{A}$ and $\mathcal{B}$ two apartments containing $\alpha$ and $g, \beta$ respectively. Since the pointwise stabiliser $H_0$ of the edge $e^-(\alpha)$ acts transitively on the set of apartments containing $e^-(\alpha)$ (see [6, Cor. (7.4.9)]), then there exist $h \in H_0$ such that $h.\mathcal{A} = \mathcal{A}$. So the two geodesic $k$-paths $\alpha$ and $h g, \beta$ are contained in the same apartment $\mathcal{A}$ and have the same initial directed edge (that is $e^- (\alpha) = e^- (h g, \beta)$). Thus the Proposition 6 implies that $\alpha = h g, \beta$ and then $\alpha$ and $\beta$ are in the same $G_n$-orbit. Consequently, two elements $\alpha$ and $\beta$ of $\mathcal{C}_k(\mathcal{R}\mathcal{T}_n)$ are in the same $G_n$-orbit if and only if $e^- (\alpha)$ and $e^- (\beta)$ are likewise. The result follows then from Lemma 7.

One can prove that if $\alpha \in \mathcal{C}_k(\mathcal{R}\mathcal{T}_n)$ then all the directed edges of $\alpha$ have the same type. So we can define the type of a geodesic $k$-path $\alpha$, denoted by $\xi(\alpha)$, as the type of any of its directed edges. The $G_n$-orbit of $\mathcal{C}_k(\mathcal{R}\mathcal{T}_n)$ corresponding to the type $n - 1$ (resp. type 1) will be denoted by $\mathcal{C}^+ (\mathcal{R}\mathcal{T}_n)$ (resp. $\mathcal{C}^- (\mathcal{R}\mathcal{T}_n)$). The Lemma 7 implies that if $\alpha \in \mathcal{C}^+ (\mathcal{R}\mathcal{T}_n)$ then its inverse $\alpha^{-1}$ is in $\mathcal{C}^- (\mathcal{R}\mathcal{T}_n)$. So for every $\alpha \in \mathcal{C}_k(\mathcal{R}\mathcal{T}_n)$ the pair $(\alpha, \alpha^{-1})$ constitute a system of representatives of $\mathcal{C}_k(\mathcal{R}\mathcal{T}_n)$ for the action of the group $G_n$. The path $\gamma = ([L_0], [L_1], \ldots, [L_k])$, where for $i \in \{0, \ldots, k\}$

$$L_i = \sigma F e_i + \cdots + \sigma F e_{i-1} + \sigma F e_n$$

is an element of $\mathcal{C}^+_k (\mathcal{R}\mathcal{T}_n)$ contained in the standard apartment of $\mathcal{R}\mathcal{T}_n$, this $k$-path will be called the standard geodesic $k$-path.
Lemma 9. For every \( \alpha \in \mathcal{C}_k(\mathcal{B}_n) \) the stabilizer \( \text{Stab}_{G_n}(\alpha) \) acts transitively on \( \mathcal{P}^+(\alpha) \) and \( \mathcal{P}^-(\alpha) \).

Proof. Let \( \alpha = (\alpha_0, \ldots, \alpha_k) \in \mathcal{C}_k(\mathcal{B}_n) \). We will prove that the action of \( \text{Stab}_{G_n}(\alpha) \) is transitive on \( \mathcal{P}^+(\alpha) \). By a similar way we get the same thing for \( \mathcal{P}^-(\alpha) \). Let \( s, t \in \mathcal{P}^+(\alpha) \), that is \( \beta = (\alpha_0, \ldots, \alpha_k, s) \) and \( \gamma = (\alpha_0, \ldots, \alpha_k, t) \) are two geodesic \((k + 1)\)-paths. Since every geodesic path of \( \mathcal{B}_n \) is contained in a some apartment, then there are two apartments \( \mathcal{A} \) and \( \mathcal{B} \) containing \( \beta \) and \( \gamma \) respectively. The stabilizer \( \text{Stab}_{G_n}(\alpha) \) is also the pointwise stabilizer in \( G_n \) of the segment \([\alpha_0, \alpha_k]\). So \( \text{Stab}_{G_n}(\alpha) \) acts transitively on the set of apartments containing \( \alpha \) (see [6, Cor. (7.4.9)]). Then there exist \( g \in \text{Stab}_{G_n}(\alpha) \) such that \( g.\mathcal{A} = \mathcal{B} \). So \( g.\mathcal{s} = t \) is a right prolongation of the geodesic path \( \alpha \) contained in the apartment \( \mathcal{B} \). Hence, the two vertices \( t \) and \( g.\mathcal{s} \) are two right prolongations of \( \alpha \) contained in the apartment \( \mathcal{B} \). Then by the Proposition 5, we obtain \( g.\mathcal{s} = t \) and then as desired the action of \( \text{Stab}_{G_n}(\alpha) \) on \( \mathcal{P}^+(\alpha) \) is transitive. \( \square \)

Corollary 10. For every \( \alpha \in \mathcal{C}_k(\mathcal{B}_n) \) we have:

\[
\mathcal{P}^+(\alpha) = \mathcal{P}^+(e^+(\alpha)) \quad \text{and} \quad \mathcal{P}^-(\alpha) = \mathcal{P}^-(e^-(\alpha)),
\]

that is the right (resp. left) prolongation of the geodesic path \( \alpha \) are exactly the right (resp. left) prolongation of the directed edge \( e^+(\alpha) \) (resp. \( e^-(\alpha) \)).

Proof. Let's prove the first equality, the proof of the second is similar. It is clear that \( \mathcal{P}^+(\alpha) \subset \mathcal{P}^+(e^+(\alpha)) \). Since the two sets \( \mathcal{P}^+(\alpha) \) and \( \mathcal{P}^+(e^+(\alpha)) \) are finite it suffice to prove that they have the same cardinality. If \( \Gamma_\alpha \) denoted the subgroup \( \text{Stab}_{G_n}(\alpha) \), then by the previous lemma \( \Gamma_\alpha \) acts transitively on \( \mathcal{P}^+(\alpha) \). So for any \( s \in \mathcal{P}^+(\alpha) \) we can identify the set \( \mathcal{P}^+(\alpha) \) with the quotient set \( \Gamma_\alpha / \text{Stab}_{\Gamma_\alpha}(s) \). Similarly, the set \( \mathcal{P}^+(e^+(\alpha)) \) identifies with the quotient set \( \Gamma_{e^+(\alpha)}/ \text{Stab}_{\Gamma_{e^+(\alpha)}}(t) \) for any \( t \in \mathcal{P}^+(e^+(\alpha)) \). Now since the action of \( G_n \) on \( \mathcal{C}_k(\mathcal{B}_n) \) have two orbits and since an element \( \beta \in \mathcal{C}_k(\mathcal{B}_n) \) and its inverse \( \beta^{-1} \) have the same stabilizers in \( G_n \) then we can assume that \( \alpha \) is the standard geodesic \( k \)-path defined as previously by \( (L_0, [L_1], [L_2], \ldots, [L_k]) \), where \( L_i = \alpha_0 e_1 + \cdots + \alpha_i e_{n_i} \) for \( i \in [0, k] \). If \( s \) is the vertex \( [L_{k+1}] \), it is clearly that \( s \in \mathcal{P}^+(\alpha) \). By an easy computation we obtain that \( \Gamma_\alpha = \Gamma_0(p_f^k) \) and \( \text{Stab}_{\Gamma_\alpha}(s) = \Gamma_0(p_f^k) \). Moreover, we can check that \( \Gamma_0(p_f^k)/\Gamma_0(p_f^{k+1}) \) have cardinality \( q_f^{n-1} \). Similarly, we can check easily that the vertex \( s \) whose equivalence class of \( \alpha_0 \)-lattice is represented by \( L_{k+1} \) is in \( \mathcal{P}^+(e^+(\alpha)) \) and that \( \Gamma_{e^+(\alpha)} = \Gamma_0(p_f) \) and \( \text{Stab}_{\Gamma_{e^+(\alpha)}}(s) = \Gamma_0(p_f^2) \). Furthermore, we can check that \( \Gamma_0(p_f)/\Gamma_0(p_f^2) \) have also cardinality \( q_f^{n-1} \). So as desired we have the equality between the two sets \( \mathcal{P}^+(\alpha) \) and \( \mathcal{P}^+(e^+(\alpha)) \). \( \square \)

Corollary 11. For every \( \alpha, \beta \in \mathcal{C}_{k+1}(\mathcal{B}_n) \), if \( \alpha^+ = \beta^+ \) (resp. \( \alpha^- = \beta^- \)) then \( \mathcal{P}^+(\alpha) = \mathcal{P}^+(\beta) \) (resp. \( \mathcal{P}^-(\alpha) = \mathcal{P}^-(\beta) \)).

Proof. If \( \alpha^+ = \beta^+ \) (resp. \( \alpha^- = \beta^- \)) then \( e^+(\alpha) = e^+(\beta) \) (resp. \( e^-(\alpha) = e^-(\beta) \)) and then the equality \( \mathcal{P}^+(\alpha) = \mathcal{P}^+(\beta) \) (resp. \( \mathcal{P}^-(\alpha) = \mathcal{P}^-(\beta) \)) follows from the previous corollary. \( \square \)

Lemma 12. Let \( s_0 \) be a vertex of \( \mathcal{B}_n \). If \( L_0 \in s_0 \) then for every vertex \( x \in \mathcal{V}(s_0) \) there is a unique representative \( L \in x \) such that

\[
\omega_{\mathcal{F}} L_0 < L < L_0.
\]

Proof. Let us fix a representative \( L_0 \in s_0 \). Let \( L \) and \( L' \) two representatives of \( x \) such that \( \omega_{\mathcal{F}} L_0 < L < L_0 \) and \( \omega_{\mathcal{F}} L_0 < L' < L_0 \). Since \( L \) and \( L' \) are equivalent then \( L' = \lambda L \) for some \( \lambda \in F^* \). Put \( \lambda = \omega_{\mathcal{F}}^{m+1} u \) for some \( m \in Z \) and \( u \in \alpha_0^* \). We have \( \omega_{\mathcal{F}} L_0 < L < L_0 \) which implies \( \omega_{\mathcal{F}}^{m+1} L_0 < \lambda L < \omega_{\mathcal{F}}^m L_0 \), that is \( \omega_{\mathcal{F}}^{m+1} L_0 < L' < \omega_{\mathcal{F}}^m L_0 \). The two inclusions \( \omega_{\mathcal{F}} L_0 < L' < L_0 \) and \( \omega_{\mathcal{F}}^{m+1} L_0 < L' < \omega_{\mathcal{F}}^m L_0 \) implies then that \( m = 0 \). Indeed, if we assume to the contrary that \( m \neq 0 \), say for example \( m > 0 \), then we have \( \omega_{\mathcal{F}}^m L_0 < \omega_{\mathcal{F}} L_0 \). So from the two inclusions \( \omega_{\mathcal{F}} L_0 < L' < L_0 \) and \( \omega_{\mathcal{F}}^{m+1} L_0 < L' < \omega_{\mathcal{F}}^m L_0 \) we obtain \( L' < \omega_{\mathcal{F}}^m L_0 < \omega_{\mathcal{F}} L_0 < L' \) which is a contradiction. We deduce then that \( L' = u L = L \). \( \square \)

Let \( s_0 \) be a vertex of \( \mathcal{B}_n \) and \( L_0 \in s_0 \) be a fixed representative. By the previous lemma to any vertex \( x \in \mathcal{V}(s_0) \) we can associate a non-trivial subspace of the \( k_{\mathcal{F}} \)-vector space \( \mathcal{V}_s := L_0/\omega_{\mathcal{F}} L_0 \).
Indeed, if \( x \in \mathcal{V}(s_0) \) and \( L_x \in x \) is the unique representative such that \( \partial_F L_0 < L_x < L_0 \), then \( V_x \) is defined as \( L_x/\partial_F L_0 \). For every subspaces \( X \) and \( Y \) of \( \mathcal{V}_s \), we put
\[
\delta(X, Y) = \dim k \cdot (X + Y) - \dim k \cdot (X \cap Y).
\]
In the following proposition, we give two formulas for the metric of \( \mathcal{B}_F \mathcal{N} \) on the set of vertices in the neighborhood a fixed vertex \( s_0 \) of \( \mathcal{B}_F \mathcal{N} \) in terms of the corresponding \( k_F \)-vector spaces.

**Proposition 13.** For every vertex \( s_0 \) of \( \mathcal{B}_F \mathcal{N} \) we have:

(i) If \( x \in \mathcal{V}(s_0) \), then
\[
d(s_0, x) = \frac{1}{\sqrt{n-1}} \left( n \dim V_x - (\dim V_x)^2 \right)^{\frac{1}{2}}.
\]

(ii) If \( x, y \in \mathcal{V}(s_0) \), then
\[
d(x, y) = \frac{1}{\sqrt{n-1}} \left( n \delta(V_x, V_y) - (\dim V_x - \dim V_y)^2 \right)^{\frac{1}{2}}.
\]

**Proof.** (i). Let us fix an \( \sigma_F \)-lattice \( L_0 \) representing the vertex \( s_0 \). Let \( x \in \mathcal{V}(s_0) \). We can choose an apartment \( \mathcal{A} \) containing \( s_0 \) and \( x \). Without loss of generality we can assume that \( \mathcal{A} \) is the standard apartment and that \( L_0 = \sigma_F e_1 + \cdots + \sigma_F e_n \), where \( (e_1,\ldots,e_n) \) is the standard basis of \( F^n \).

Let \( L_x \) be the unique representative of the vertex \( x \) such that \( \partial_F L_0 < L_x < L_0 \). Since the vertex \( x \) lies in \( \mathcal{A} \) then for some \( a = (a_1,\ldots,a_n) \in \mathbb{Z}^n \) we can write \( L_x = p_F^{a_1} e_1 + \cdots + p_F^{a_n} e_n \). As in the proof of Lemma 7, the coordinates \( a_i \in \{0,1\} \) and not all the \( a_i \)'s are zero or one. Moreover, if \( A_0 = \{i \in \Delta_n | a_i = 0\} \) and \( A_1 = \{i \in \Delta_n | a_i = 1\} \), then clearly \( A_0 \cup A_1 = \Delta_n \). So we have
\[
L_x = \bigoplus_{i \in A_0} \sigma_F \oplus \bigoplus_{i \in A_1} p_F
\]
and then
\[
V_x = L_x/\partial_F L_0 = \bigoplus_{i \in A_0} \sigma_F / \sigma_F \oplus \bigoplus_{i \in A_1} p_F / p_F \cong k_F^{\lfloor A_0 \rfloor}.
\]
Consequently \( \dim(V_x) = |A_0| \). We have
\[
d(s_0, x) = \sqrt{\frac{n}{n-1} d_0(0, a - \frac{1}{n} \sigma(a) e)} = \sqrt{\frac{n}{n-1} \left\| a - \frac{\sigma(a)}{n} e \right\|}
\]
\[
= \sqrt{\frac{n}{n-1} \sum_{i=1}^{n} \left( a_i - \frac{\sigma(a)}{n} \right)^2} = \sqrt{\frac{n}{n-1} \sum_{i=1}^{n} a_i^2 - \frac{2 \sigma(a)}{n} \sum_{i=1}^{n} a_i + \frac{\sigma(a)^2}{n^2} \sum_{i=1}^{n} a_i^2}
\]
\[
= \sqrt{\frac{n}{n-1} \sum_{i=1}^{n} a_i^2 + \frac{2 \sigma(a)^2}{n} - \frac{\sigma(a)^2}{n} \sum_{i=1}^{n} a_i^2}
\]
But as \( a_i \in \{0,1\} \) for every \( i \in \Delta_n \), then
\[
d(s_0, x) = \sqrt{\frac{n}{n-1} \left( \sigma(a) - \frac{\sigma(a)^2}{n} \right)^2}.
\]
On the other hand
\[
\sigma(a) = \sum_{i=1}^{n} a_i = \sum_{i \in A_1} 1 = |A_1| = n - \dim V_x.
\]
So we get
\[
d(s_0, x) = \sqrt{\frac{n}{n-1} \left( n - \dim V_x - \frac{(n - \dim V_x)^2}{n} \right)^2}
\]
and then
\[
d(s_0, x) = \frac{1}{\sqrt{n-1}} \left( n \dim V_x - (\dim V_x)^2 \right)^{\frac{1}{2}}.
\]
(ii). The proof of the second formula is obtained by a similar way. \( \square \)
Lemma 15. But we have the following result:

$$\alpha$$ is a geodesic path, that is there exists an integer $$\ell$$ if 

$$(x, y) \in V(s_0),$$ then $$s_0 \in (x, y)$$ if and only if $$V_x \oplus V_y = V_{s_0}.$$ 

Proof. Follows from the previous proposition by an easy computation. 

4. The projective tower of graphs over $$\mathcal{B}T_n$$

In this section, our purpose is to give the construction of the tower of directed graphs lying equivariantly over the 1-skeleton of the building $$\mathcal{B}T_n$$ and to give some basic properties of these tower of directed graphs. We note that our construction generalizes the construction of Broussous given in [5] for the case $$n = 2.$$ In the sequel, we will be interested then by the case $$n \geq 3.$$
4.1. The construction

For every integer \( k \geq 0 \), we define the graph \( \tilde{X}_k \) as the directed graph whose vertex (resp. edges) set is the set \( \mathcal{E}^+_k(\mathcal{B} \mathcal{T}_n) \) (resp. \( \mathcal{E}^+_{k+1}(\mathcal{B} \mathcal{T}_n) \)). The structure of directed graph of \( \tilde{X}_k \) is given by:

\[
a^-(a) = (a_0, \ldots, a_k), \quad a^+(a) = (a_1, \ldots, a_{k+1}), \quad \text{if} \quad a = (a_0, \ldots, a_{k+1}).
\]

Let’s notice firstly that the graph \( \tilde{X}_0 \) is nothing other than the directed graph whose vertices are those of \( \mathcal{B} \mathcal{T}_n \) and for which the edges set is \( \mathcal{E}^+_1(\mathcal{B} \mathcal{T}_n) \). The action of \( G_n \) on the sets \( \mathcal{E}^+_k(\mathcal{B} \mathcal{T}_n) \) induce an action on the graph \( \tilde{X}_k \) by automorphisms of directed graphs. Moreover, since the stabilizers of the vertices of \( \tilde{X}_k \) are open and compact then the action is proper. From the previous section, the action of \( G_n \) on the graph \( \tilde{X}_k \) is transitive on vertices and edges. For every vertex \( s \) (resp. edge \( a \)) of \( \tilde{X}_k \), we write \( \Gamma_s \) (resp. \( \Gamma_a \)) for the stabilizer in \( G_n \) of \( s \) (resp. \( a \)). The stabilizer in \( G_n \) of the standard vertex (resp. edge) of \( \tilde{X}_k \) that is the standard geodesic \( k \)-path (resp. \( (k + 1) \)-path) given in (4), is the subgroup \( \Gamma_0(p^k_1) \) (resp. \( \Gamma_0(p^{k+1}_1) \)).

**Proposition 17.** For every vertex \( s \) of \( \tilde{X}_k \) the stabilizer \( \Gamma_s \) acts transitively on the two sets of neighborhoods:

\[
\nu^-(s) = \{ a \in \tilde{X}_k^1 \mid a^- = s \} \quad \text{and} \quad \nu^+(s) = \{ a \in \tilde{X}_k^1 \mid a^+ = s \}
\]

**Proof.** Follows immediately from Lemma 9. \( \square \)

Recall that the 1-skeleton of the building \( \mathcal{B} \mathcal{T}_n \), denoted by \( \mathcal{B} \mathcal{T}_n^{(1)} \), is the subcomplex of \( \mathcal{B} \mathcal{T}_n \) formed by the faces of dimension at most one. When \( k = 2m \) is even, there is a natural simplicial projection \( p_k : \tilde{X}_k \to \mathcal{B} \mathcal{T}_n^{(1)} \) defined on vertices by

\[
p_k(s_m, \ldots, s_0, \ldots, s_m) = s_0.
\]

Similarly, When \( k = 2m + 1 \) is odd, there is a natural simplicial projection \( p_k : \tilde{X}_k^{id} \to \mathcal{B} \mathcal{T}_n^{(1)} \), where \( \tilde{X}_k^{id} \) and \( \mathcal{B} \mathcal{T}_n^{(1)} \) are respectively the barycentric subdivision of the graphs \( \tilde{X}_k \) and \( \mathcal{B} \mathcal{T}_n^{(1)} \). The family of graphs \( (\tilde{X}_k)_{k \geq 0} \) constitute a tower of graphs over the graph \( \mathcal{B} \mathcal{T}_n^{(1)} \) in the sense that we have the following diagram of simplicial maps

\[
\cdots \longrightarrow \tilde{X}_{k+1} \xrightarrow{\varphi_k^\varepsilon} \tilde{X}_k \longrightarrow \cdots \longrightarrow \tilde{X}_0 \xrightarrow{p_0} \mathcal{B} \mathcal{T}_n^{(1)}
\]

where for \( \varepsilon = \pm \) and for \( k \geq 0 \), the map \( \varphi_k^\varepsilon : \tilde{X}_{k+1} \longrightarrow \tilde{X}_k \) is the simplicial map defined on vertices by \( \varphi_k^\varepsilon(s) = s^\varepsilon \).

4.2. Connectivity of the graphs

The aim of this section is the study of the connectivity of the graphs \( \tilde{X}_k \). We begin by defining a cover of \( \tilde{X}_{k+1} \) by finite subgraphs whose nerve is a graph isomorphic to \( \tilde{X}_k \). Assume that \( k \geq 0 \) is an integer. For every vertex \( s \) of \( \tilde{X}_k \) we define the subgraph \( \tilde{X}_{k+1}(s) \) of the graph \( \tilde{X}_{k+1} \) as the subgraph whose edges are the geodesic \( (k + 2) \)-paths \( \alpha \in \mathcal{E}^+_{k+2}(\mathcal{B} \mathcal{T}_n) \) of the form \( \alpha = (x, s_0, \ldots, s_k, y) \), where \( x \) (resp. \( y \)) is a left (resp. right) prolongation of the path \( s \). The vertices of \( \tilde{X}_{k+1}(s) \) are exactly those \( v \in \tilde{X}_{k+1}^0 \) such that \( v^- = s \) or \( v^+ = s \). Obviously the subgraphs \( \tilde{X}_{k+1}(s) \), when \( s \) range over the set of vertices of \( \tilde{X}_k \), form a cover the graph \( \tilde{X}_{k+1} \). That is

\[
\tilde{X}_{k+1} = \bigcup_{s \in \tilde{X}_k^0} \tilde{X}_{k+1}(s). \tag{5}
\]

For every vertex \( s_0 \) of \( \tilde{X}_0 \) (considered as a vertex of \( \mathcal{B} \mathcal{T}_n \)) the subgraph \( \tilde{X}_1(s_0) \) of \( \tilde{X}_1 \) has two types of vertices: the directed edges \( (x, s_0) \in \mathcal{E}^+_1(\mathcal{B} \mathcal{T}_n) \) and the directed edges \( (s_0, y) \in \mathcal{E}^+_1(\mathcal{B} \mathcal{T}_n) \). Let us denote the \( k \)-vector space \( k^0_\varepsilon \) by \( \tilde{V} \). We use the vector space \( \tilde{V} \) to identify the set \( \mathbb{P}^1(\tilde{V}) \). The Lemma 7 implies that the vertex set of \( \tilde{X}_1(s_0) \) may be identified with the set \( \tilde{V} \).
$p^1(V^\ast)$ is the set of one codimensional subspaces of $V$. By the Corollary 14 we deduce that the graph $\tilde{X}_1(s_0)$ is isomorphic to the graph $\Delta(V)$ whose vertex set is $p^1(V) \cup p^1(V^\ast)$ and in which a vertex $D \in p^1(V)$ is linked to a vertex $H \in p^1(V^\ast)$ if and only if $D \cap H = V$ and there is no edges between two distinct vertices of $p^1(V)$ (resp. $p^1(V^\ast)$). One can prove easily that $\Delta(V)$ is a connected bipartite graph so that $\tilde{X}_1(s_0)$ is connected and bipartite for every vertex $s_0$ of $\tilde{X}_0$.

**Lemma 18.** Let $k \geq 1$ be an integer. Then we have:

(i) For every $s \in \tilde{X}_k^0$, the graph $\tilde{X}_{k+1}(s)$ is a complete bipartite graph and hence connected,

(ii) The nerve $\mathcal{N}(\tilde{X}_{k+1})$ of the cover of $\tilde{X}_{k+1}$ given in (5) is isomorphic to the graph $\tilde{X}_k$.

**Proof.** (i). Let $s \in \tilde{X}_k^0$. The set of vertices of $\tilde{X}_{k+1}(s)$ is clearly partitioned into two subsets. The set $\mathcal{E}$ of vertices $v \in \tilde{X}_{k+1}^0$ such that $v^{-} = s$ and the set $\mathcal{V}$ of vertices $v \in \tilde{X}_{k+1}^0$ such that $v^{+} = s$. By Corollary 16 we deduce that every vertex in $\mathcal{E}$ is linked to every vertex in $\mathcal{V}$. So as desired the graph $\tilde{X}_{k+1}(s)$ is a complete bipartite graph and then connected.

(ii). Let $s$ and $t$ two distinct vertices of $\tilde{X}_k$. If $s$ and $t$ are linked by an edge then by Corollary 16 the two subgraphs $\tilde{X}_{k+1}(s)$ and $\tilde{X}_{k+1}(t)$ have at least a common vertex, namely the vertex $s \cup t$. Conversely, if the two subgraphs $\tilde{X}_{k+1}(s)$ and $\tilde{X}_{k+1}(t)$ have at least a common vertex, say $v$, then we have $v^{-} = s$ or $v^{+} = s$ and $v^{-} = t$ or $v^{+} = t$. As $s$ and $t$ are distinct then we deduce that $v^{-} = s$ and $v^{+} = t$ or $v^{-} = t$ and $v^{+} = s$. The Corollary 16 implies then that $s$ and $t$ are linked by an edge. So the nerve of the cover of $\tilde{X}_{k+1}(s)$ by the subgraphs $\tilde{X}_{k+1}(s)$, for $s \in \tilde{X}_k^0$, is the graph $\tilde{X}_k$. 

**Theorem 19.** For every integer $k \geq 0$, the geometric realization of $\tilde{X}_k$ is connected and locally compact.

**Proof.** The locally compactness of $|\tilde{X}_k|$ follows from the fact that the graphs $\tilde{X}_k$ are locally finite. For the connectedness, we will prove firstly that $\tilde{X}_0$ is connected. Let $s = [L]$ and $t = [M]$ be two distinct vertices of $\tilde{X}_0$, where $L$ and $M$ are two $\mathcal{F}$-lattices. Let us choose an $\mathcal{F}$-basis $(v_1, \ldots, v_n)$ of $\mathcal{F}^n$ for which $L = \mathcal{F}v_1 + \cdots + \mathcal{F}v_n$ and $M = \mathcal{F}v_1 + \cdots + \mathcal{F}v_n$, where $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ with $k_1 \leq \cdots \leq k_n$. By changing the representative $M \in [\mathcal{F}]$ we can assume that $0 < k_1$. Now let us consider the sequence $(L_0, \ldots, L_m)$ of $\mathcal{F}$-lattices, where $m = k_1 + \cdots + k_n$, defined as follows. For every integer $0 \leq i \leq m$, if $k_1 + \cdots + k_{j-1} + 1 \leq i \leq k_1 + \cdots + k_j$, where $1 \leq j \leq n$, then

$$L_i = \bigoplus_{t=1}^{j-1} p_{k+t}^{i-(k_1+\cdots+k_{j-1})} v_{t} \oplus p_{k}^{i-(k_1+\cdots+k_{j-1})} v_{j} \oplus \bigoplus_{t=j+1}^{n} \mathcal{F}v_{t}.$$  

By a straightforward computation, we can check easily that the sequence $(|L_0|, \ldots, |L_m|)$ is a path of the graph $\tilde{X}_0$ linking the vertex $s$ to the vertex $t$. So as desired $\tilde{X}_0$ is connected. Now we will prove by induction that the graphs $\tilde{X}_k$ are connected for every non-negative integer $k$. Let $k \geq 0$ be an integer. Assume that the graph $\tilde{X}_k$ is connected and let’s prove that $\tilde{X}_{k+1}$ is also connected. Let $u$ and $v$ be two distinct vertices of $\tilde{X}_{k+1}$. Since $\tilde{X}_{k+1}$ is covered by the subgraph $\tilde{X}_{k+1}(s)$, when $s$ range over the set of vertices of $\tilde{X}_k$, then there exist two vertices $s, t \in \tilde{X}_k^0$ such that $u \in \tilde{X}_{k+1}^0(s)$ and $v \in \tilde{X}_{k+1}^0(t)$. As $\tilde{X}_k$ is connected then there exist a path $p = (p_0, \ldots, p_m)$ in $\tilde{X}_k$ linking the two vertices $s$ and $t$ (say $p_0 = s$ and $p_m = t$). For every integer $i \in \{1, \ldots, m\}$, let $v_i$ be any vertex of the non-empty graph $\tilde{X}_{k+1}(p_{i-1}) \cap \tilde{X}_{k+1}(p_i)$. Let’s also put $v_0 = u$ and $v_{m+1} = v$. By the previous lemma the graphs $\tilde{X}_{k+1}(p_i)$ are connected. So for $i \in \{0, \ldots, \ell\}$, since $p_i$ and $p_{i+1}$ are two vertices of the graph $\tilde{X}_{k+1}(p_i)$ then there exist a path in $\tilde{X}_{k+1}$ from $p_i$ to $p_{i+1}$. Consequently there exist a path in $\tilde{X}_{k+1}$ connecting the two vertices $u$ and $v$ and then the graph $\tilde{X}_{k+1}$ is connected. We have then the connectedness of the graphs $\tilde{X}_k$ for every integer $k \geq 0$ which implies the connectedness of their geometric realization. 

\[\square\]
5. Realization of the generic representations of $G_n$ in the cohomology of the tower of graphs

5.1. Generic representations of $G_n$

Let us firstly recall some basic facts and introduce some notations. Let $\psi$ be a fixed additive smooth character of $F$ trivial on $p_F$ and nontrivial on $o_F$. We define a character $\theta_\psi$ of the group $U_n$ of upper unipotent matrices as follows

$$\theta_\psi \left( \begin{pmatrix} 1 & u_{1,2} & \cdots & u_{1,n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & u_{n-1,n} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right) = \psi(u_{1,2} + \cdots + u_{n-1,n}).$$

Let $(\pi, V)$ be an irreducible admissible representation of $G_n$ considered as an irreducible admissible representation of $GL_n(F)$ with trivial central character. The representation $(\pi, V)$ is called generic if

$$\text{Hom}_{GL_n(F)}(\pi, \text{Ind}_{U_n}^{GL_n(F)} \theta_\psi) \neq 0.$$ 

By Frobenius reciprocity, this is equivalent to the existence of a nonzero linear form $\ell : V \rightarrow C$ such that $\ell(\pi(u) v) = \theta_\psi(u) \ell(v)$ for every $v \in V$ and $u \in U_n$. Thus a generic representation $(\pi, V)$ of $G_n$ can be realized on a same space of functions $f$ with the property $f(ug) = \theta_\psi(u)f(g)$ for every $u \in U_n$ and $g \in GL_n(F)$ and for which the action of $GL_n(F)$ on the space of $\pi$ is by right translation. Such a realization is called the Whittaker model of $\pi$. The following theorem, due to Bernstein and Zelevinski, shows that generic representations have a unique Whittaker model.

**Theorem 20 ([2, V.16]).** Let $(\pi, V)$ be an irreducible admissible representation of $G_n$. Then the dimension of the space $\text{Hom}_{GL_n(F)}(\pi, \text{Ind}_{U_n}^{GL_n(F)} \theta_\psi)$ is at most one, that is

$$\dim_C \text{Hom}_{GL_n(F)}(\pi, \text{Ind}_{U_n}^{GL_n(F)} \theta_\psi) \leq 1.$$ 

In particular, if $\pi$ is generic then $\pi$ has a unique Whittaker model.

We have the following result which is due to H. Jacquet, J. L. Piatetski-Shapiro and J. Shalika, see [8, Thm. (5.1)]:

**Theorem 21.** Let $(\pi, V)$ be an irreducible generic representation of $G_n$.

(i) For $k$ large enough, the space of fixed vectors $V^\Gamma_0(p_F^k)$ is non-zero.

(ii) Let $c(\pi)$ the smallest integer such that $V^\Gamma_0(p_F^{c(\pi)+1}) \neq 0$, then for every integer $k \geq c(\pi)$, we have:

$$\dim_C V^\Gamma_0(p_F^{k+1}) = k - c(\pi) + 1.$$

5.2. Realization of the Generic representations of $G_n$

In this section, we fix an irreducible generic representation $(\pi, V)$ of $G_n$ and we make the following assumption:

**Assumption 22.** $\pi$ is non-spherical, that is the space of $\Gamma_0(p_F^0)$-fixed vectors

$$V^\Gamma_0(p_F^0) := \{ v \in V \mid \forall g \in \Gamma_0(p_F^0), \pi(g)v = v \}$$

is zero.
In the following, our aim is to prove that the representation \( \pi \) can be realized as a quotient of the cohomology space \( H^1_c(\mathcal{X}_{c(\pi)}, \mathcal{C}) \) and if moreover \( \pi \) is cuspidal then in fact it can be realized as a subrepresentation of this cohomology space. Furthermore, as in Theorem (5.3.2) of \([5]\), we obtain a multiplicity one result for cuspals but in a more simpler way. The proofs of the results below are similar to those given in \([5, \S(3.2)]\). Let us recall that for every vertex \( s \) (resp. edge \( a \)) of \( \mathcal{X}_{c(\pi)} \), \( \Gamma_s \) (resp. \( \Gamma_a \)) denotes the stabilizer in \( G_n \) of \( s \) (resp. \( a \)). We recall that

\[
\Gamma_{s_0} = \Gamma_0(p^{c(\pi)}) \quad \text{and} \quad \Gamma_{a_0} = \Gamma_0(p^{c(\pi)+1}),
\]

where \( s_0 \) (resp. \( a_0 \)) is the standard vertex (resp. edge) of \( \mathcal{X}_{c(\pi)} \).

**Lemma 23.**

(i) For every edge \( a \) of \( \mathcal{X}_{c(\pi)} \), \( V^T_a \) is of dimension one.

(ii) Let \( s \) be a vertex of \( \mathcal{X}_{c(\pi)} \) and \( a \) be an edge of \( \mathcal{X}_{c(\pi)} \). Then for every \( \nu \in V^T_a \) we have

\[
\sum_{g \in G_s/\Gamma_a} \pi(g) \nu = 0.
\]

**Proof.** (i). Since \( G_n \) acts transitively on the set of edges of \( \mathcal{X}_{c(\pi)} \) then the subgroup \( \Gamma_a \) is conjugate to \( \Gamma_{a_0} \) which gives the result.

(ii). Clearly the vector

\[
\nu_0 := \sum_{g \in G_s/\Gamma_a} \pi(g) \nu
\]

is fixed by \( \Gamma_s \). But by transitivity of the action of \( G_n \) on the set of vertices of \( \mathcal{X}_{c(\pi)} \), the subgroup \( \Gamma_s \) is conjugate to \( \Gamma_{s_0} \). So Theorem 21 implies that \( \nu_0 = 0 \).

We define a map

\[
\Psi^\nu_\pi : V^\nu \rightarrow C^1(\mathcal{X}_{c(\pi)}, \mathcal{C})
\]

as follows. Let us fix a non-zero vector \( \nu_0 \in V^{T_{a_0}} \). For every edge \( a \) of \( \mathcal{X}_{c(\pi)} \), we put

\[
\nu_a = \pi(g).\nu_0, \quad \text{where} \quad a = g.a_0 \tag{6}
\]

This definition is well defined since \( G_n \) acts transitively on \( \mathcal{X}_{c(\pi)} \) and it does not depend on the choice of \( g \in G_n \) such that \( \nu_a = g.\nu_0 \) as \( \nu_0 \) is fixed by \( \Gamma_{a_0} \). The map \( \Psi^\nu \) is then defined by

\[
\Psi^\nu(\varphi)(a) = \varphi(\nu_a)
\]

for every \( \varphi \in V^\nu \) and \( a \in \mathcal{X}_{c(\pi)} \). From (6) the map \( \Psi^\nu \) is \( G_n \)-equivariant.

**Lemma 24.** The map \( \Psi^\nu \) is injective and its image is contained in \( \mathcal{H}_\infty(\mathcal{X}_{c(\pi)}, \mathcal{C}) \).

**Proof.** The \( G_n \)-equivariant map \( \Psi^\nu \) is injective as it is nonzero and as the representation \( \pi \) is irreducible. Let \( \varphi \in V^\nu \). Let us prove that for every vertex \( s \) of \( \mathcal{X}_{c(\pi)} \),

\[
\sum_{a \in \mathcal{X}_{c(\pi)}} [a : s] \varphi(\nu_a) = 0.
\]

Let \( s \) be a vertex of \( \mathcal{X}_{c(\pi)} \). By Proposition 17, the stabilizer \( \Gamma_s \) acts transitively on the two sets

\[
V^-(s) = \{ a \in \mathcal{X}_{c(\pi)} \mid a^- = s \} \quad \text{and} \quad V^+(s) = \{ a \in \mathcal{X}_{c(\pi)} \mid a^+ = s \}.
\]

Let us fix \( a_s^+ \in V^+(s) \) and \( a_s^- \in V^-(s) \). We have then

\[
\sum_{a \in \mathcal{X}_{c(\pi)}} [a : s] \varphi(\nu_a) = \varphi \left( \sum_{a \in V^+(s)} \nu_a - \sum_{a \in V^-(s)} \nu_a \right)
\]

\[
= \varphi \left( \sum_{g \in G_s/\Gamma_{a_s^+}} \pi(g).\nu_{a_s^+} - \sum_{g \in G_s/\Gamma_{a_s^-}} \pi(g).\nu_{a_s^-} \right) = 0
\]
by Lemma 23. Consequently, $\text{Im}(\Psi^V)$ is contained in $\mathcal{H}(\tilde{X}_c(\pi), \mathbb{C})$ which implies that it is contained in $\mathcal{H}_\infty(\tilde{X}_c(\pi), \mathbb{C})$. □

By Lemma 1 we have the isomorphism of smooth $G_n$-module
\[ \mathcal{H}_\infty(\tilde{X}_c(\pi), \mathbb{C}) \cong H^1_c(\tilde{X}_c(\pi), \mathbb{C})^\vee. \]
So applying contragredients to the operator $\Psi^V: V^\vee \to \mathcal{H}_\infty(\tilde{X}_c(\pi), \mathbb{C})$ we obtain an intertwining operator
\[ \Psi^V: H^1_c(\tilde{X}_c(\pi), \mathbb{C})^\vee \to V^\vee. \]
It is well known that a smooth $G_n$-module $W$ has a canonical injection in the contragredient of its contragredient $W^{\vee\vee}$. So the smooth $G_n$-module $H^1_c(\tilde{X}_c(\pi), \mathbb{C})$ canonically injects in $H^1_c(\tilde{X}_c(\pi), \mathbb{C})^{\vee\vee}$. Moreover the representation $\pi$ is irreducible and hence admissible then $V$ and $V^{\vee\vee}$ are canonically isomorphic. In the following, if $\omega \in C^1_c(\tilde{X}_c(\pi), \mathbb{C})$ we write $\tilde{\omega}$ for its image in $H^1_c(\tilde{X}_c(\pi), \mathbb{C})$.

**Theorem 25.** The restriction of $\Psi^V$ to the space $H^1_c(\tilde{X}_c(\pi), \mathbb{C})$ define a nonzero intertwining operator
\[ \Psi_\pi: H^1_c(\tilde{X}_c(\pi), \mathbb{C}) \to V \]
given by
\[ \Psi_\pi(\tilde{\omega}) = \sum_{a \in \tilde{X}_c(\pi)} \omega(a) v_a. \]
In particular, $(\pi, V)$ is isomorphic to a quotient of $H^1_c(\tilde{X}_c(\pi), \mathbb{C})$. Moreover, if $(\pi, V)$ is cuspidal then it is isomorphic to a subrepresentation of $H^1_c(\tilde{X}_c(\pi), \mathbb{C})$.

**Proof.** The fact that the restriction of the map $\Psi^V$ to the space $H^1_c(\tilde{X}_c(\pi), \mathbb{C})$ is given exactly by the map $\Psi_\pi$ follows by a straightforward computation. Let $\omega_0 \in C^1_c(\tilde{X}_c(\pi), \mathbb{C})$ defined on the basis $\tilde{X}^1_c(\pi)$ of $C^1_c(\tilde{X}_c(\pi), \mathbb{C})$ as follows: for every edge $a$ of $\tilde{X}_c(\pi)$, $\omega_0(a) = 1$ if $a = a_0$ and $\omega_0(a) = 0$ otherwise. We have
\[ \Psi_\pi(\tilde{\omega}_0) = \sum_{a \in \tilde{X}_c(\pi)} \omega_0(a) v_a = v_0 \neq 0. \]
So the map $\Psi_\pi$ is nonzero. Hence by irreducibility of $\pi$ the map $\Psi_\pi$ is surjective and then as desired $(\pi, V)$ is isomorphic to a quotient of $H^1_c(\tilde{X}_c(\pi), \mathbb{C})$. If the representation $(\pi, V)$ is cuspidal, so in particular generic, then it is isomorphic to a quotient of $H^1_c(\tilde{X}_c(\pi), \mathbb{C})$. But $(\pi, V)$ is cuspidal and then it is projective in the category of smooth complex representation of $G_n$. So we have in fact an embedding of $(\pi, V)$ in $H^1_c(\tilde{X}_c(\pi), \mathbb{C})$.

**Theorem 26.** If the representation $(\pi, V)$ is cuspidal then it have a unique realization in the cohomology space $H^1_c(\tilde{X}_c(\pi), \mathbb{C})$, that is
\[ \dim \text{Hom}_{G_n}(\pi, H^1_c(\tilde{X}_c(\pi), \mathbb{C})) = 1. \]

**Proof.** Since $G_n$ acts transitively on the set of vertices and edges of $\tilde{X}_c(\pi)$ then the two $G_n$-modules $C^0_c(\tilde{X}_c(\pi), \mathbb{C})$ and $C^1_c(\tilde{X}_c(\pi), \mathbb{C})$ are respectively isomorphic to the following compactly induced representation
\[ c\text{-ind}^G_n_{\Gamma_0(p^{(n)}_F)} 1 \quad \text{and} \quad c\text{-ind}^G_n_{\Gamma_0(p^{(n)}_F)+1} 1 \]
(where 1 denotes the trivial character). The space $H^1_c(\tilde{X}_c(\pi), \mathbb{C})$ is by definition the cokernel of the coboundary map
\[ C^0_c(\tilde{X}_c(\pi), \mathbb{C}) \xrightarrow{d} C^1_c(\tilde{X}_c(\pi), \mathbb{C}). \]
Then we have a surjective map
\[ \varphi: c\text{-ind}^G_n_{\Gamma_0(p^{(n)}_F)+1} H^1_c(\tilde{X}_c(\pi), \mathbb{C}) \]

and so we obtain an injective map

$$\varphi : \text{Hom}_{G_n}(H^1_c(\mathcal{X}_c(\pi), \mathbb{C}), \pi) \rightarrow \text{Hom}_{G_n}(c\text{-ind}_{\Gamma_0(p^{(n)}_F + 1)} G_n, 1, \pi)$$

On the other hand, by Frobenius reciprocity we have

$$\text{Hom}_{G_n}(c\text{-ind}_{\Gamma_0(p^{(n)}_F + 1)} G_n, 1, \pi) \simeq \text{V}^\Gamma_{n}(p^{(n)}_F + 1)$$

But by the Theorem 21, the space of fixed vectors $V^{\Gamma_{n}}(p^{(n)}_F + 1)$ is of dimension one. Thus we obtain

$$\dim_{\mathbb{C}} \text{Hom}_{G_n}(H^1_c(\mathcal{X}_c(\pi), \mathbb{C}), \pi) \leq 1.$$ 

On the other hand, since the representation $(\pi, V)$ is cuspidal then it is a projective object of the category of smooth representations of $G_n$. So the two spaces $\text{Hom}_{G_n}(H^1_c(\mathcal{X}_c(\pi), \mathbb{C}), \pi)$ and $\text{Hom}_{G_n}(\pi, H^1_c(\mathcal{X}_c(\pi), \mathbb{C}))$ are in fact isomorphic. But by the previous theorem $\text{Hom}_{G_n}(H^1_c(\mathcal{X}_c(\pi), \mathbb{C}), \pi)$ is nonzero. So as desired the space $\text{Hom}_{G_n}(\pi, H^1_c(\mathcal{X}_c(\pi), \mathbb{C}))$ is one dimensional. □

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