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Compactly supported cohomology of a tower of graphs and generic representations of PGL_n over a local field

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Abstract. Let F be a non-archimedean locally compact field and let G_n be the group $\mathrm{PGL}_n(\mathrm{F})$. In this paper we construct a tower $(\widetilde{\mathrm{X}}_k)_{k\geqslant 0}$ of graphs fibred over the one-skeleton of the Bruhat–Tits building of G_n. We prove that a non-spherical and irreducible generic complex representation of G_n can be realized as a quotient of the compactly supported cohomology of the graph $\widetilde{\mathrm{X}}_k$ for k large enough. Moreover, when the representation is cuspidal then it has a unique realization in a such model.

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1. Introduction

Let F be a non-archimedean locally compact field and let G_n be the locally profinite group $\mathrm{PGL}_n(\mathrm{F})$. In [4], P. Broussous has constructed a projective tower of simplicial complexes fibred over the Bruhat–Tits building of G_n. The idea (due to P. Schneider) consists of constructing simplicial complexes whose structure is very related to that of the Bruhat–Tits building. The goal of a such construction is to try to find geometric interpretation of certain classes of irreducible smooth representations of G_n. Such a geometric interpretation exists for example for the Steinberg representation of G_n which can be realized (see [3, Thm. 3]) as the cohomology with compact support in top dimension of the Bruhat–Tits building. In a second work (see [5]), P. Broussous has constructed in the case $n = 2$ a slightly modified version of his previous construction. More precisely, he construct a tower of directed graphs $(\widetilde{\mathrm{X}}_k)_{k\geqslant 0}$ fibred over the Bruhat–Tits tree of G_2. Based on the existence of new vectors for irreducible generic representations of G_2, he proves that an irreducible generic representation π of G_2 can be realized as a quotient of the compactly supported cohomology space $H^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$, where $c(\pi)$ is an integer related to the conductor of the representation π. He proves moreover that if π is cuspidal then it can be realized as a subrepresentation of the last cohomology space and that a such realization is unique. In a parallel direction, the author has constructed a simplicial complex fibred over the Bruhat–Tits building of G_n whose top compactly supported cohomology realize as subquotient all the irreducible cuspidal level zero representations of G_n, see [9].

In this paper our aim is to generalize the construction of Broussous given in [5] to the case $n \geqslant 3$. More precisely we construct a projective tower $(\widetilde{\mathrm{X}}_k)_{k\geqslant 0}$ of directed graphs fibred over the 1-skeleton of the Bruhat–Tits building of G_n. In our construction, the graphs considered will be defined in terms of combinatorial geodesic paths of the Bruhat–Tits building of G_n.

Let π be an irreducible smooth generic and non-spherical representation of G_n. We prove that there exists an injective intertwining operator

$$\Psi_\pi^\vee : V^\vee \longrightarrow \mathscr{H}_\infty(\widetilde{\mathrm{X}}_{c(\pi)}, \mathbb{C}),$$

where $V^\vee$ is the contragredient representation of π and $\mathscr{H}_\infty(\widetilde{\mathrm{X}}_{c(\pi)}, \mathbb{C})$ is the space of smooth harmonic forms on the graph $\widetilde{\mathrm{X}}_{c(\pi)}$. By applying contragredients to this intertwining operator and then by restriction to $H_c^1(\widetilde{\mathrm{X}}_{c(\pi)}, \mathbb{C})$ we obtain a nonzero intertwining operator

$$\Psi_\pi : H_c^1(\widetilde{\mathrm{X}}_{c(\pi)}, \mathbb{C}) \longrightarrow V.$$

That is the representation π is isomorphic to a quotient of $H_c^1(\widetilde{\mathrm{X}}_{c(\pi)}, \mathbb{C})$. In the case when π is cuspidal, the G_n-equivariant map Ψ_π splits so that π injects in $H_c^1(\widetilde{\mathrm{X}}_{c(\pi)}, \mathbb{C})$. We prove that such an injection is unique, that is :

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{G}_n}(\pi, H_c^1(\widetilde{\mathrm{X}}_{c(\pi)}, \mathbb{C})) = 1.$$

2. Notations and preliminaries

In this article, F will be a non-archimedean locally compact field. We write $\mathfrak{o}_{\mathrm{F}}$ for the ring of integers of F, $\mathfrak{p}_{\mathrm{F}}$ for the maximal ideal of $\mathfrak{o}_{\mathrm{F}}$, $k_{\mathrm{F}} := \mathfrak{o}_{\mathrm{F}}/\mathfrak{p}_{\mathrm{F}}$ for the residue class field of F and q_{F} for the cardinal of k_{F}. We fix a normalized uniformizer ϖ_{F} of $\mathfrak{o}_{\mathrm{F}}$ and we denote by v_{F} the normalized valuation of F.

2.1. *The projective general linear group* $\mathrm{PGL}_n(\mathrm{F})$

For every integer $n \geqslant 2$, the projective general linear group $\mathrm{PGL}_n(\mathrm{F})$ will be denoted by G_n. If $k \geqslant 1$ is an integer, we write $\widetilde{\Gamma}_0(\mathfrak{p}_{\mathrm{F}}^k)$ for the following subgroup of $\mathrm{GL}_n(\mathrm{F})$

$$\widetilde{\Gamma}_0(\mathfrak{p}_{\mathrm{F}}^k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_n(\mathfrak{o}_{\mathrm{F}}) \;\middle|\; a \in \mathrm{GL}_{n-1}(\mathfrak{o}_{\mathrm{F}}),\ d \in \mathfrak{o}_{\mathrm{F}}^\times,\ c \equiv 0 \bmod \mathfrak{p}_{\mathrm{F}}^k \right\} \tag{1}$$

and we write $\Gamma_0(\mathfrak{p}_{\mathrm{F}}^k)$ for its image in G_n. We denote also the image in G_n of the standard maximal compact subgroup of $\mathrm{GL}_n(\mathrm{F})$ by $\Gamma_0(\mathfrak{p}_{\mathrm{F}}^0)$.

2.2. *The Bruhat–Tits building of* G_n

In this section we fix some notations and recall some well-known facts. For more details the reader may refer to [1], [7] or [10]. Recall that a lattice of the vector space F^n is an open compact subgroup of the additive group of F^n. A such lattice is an $\mathfrak{o}_{\mathrm{F}}$-lattice if moreover it is an $\mathfrak{o}_{\mathrm{F}}$-submodule of F^n. Equivalently, an $\mathfrak{o}_{\mathrm{F}}$-lattice of F^n is a free $\mathfrak{o}_{\mathrm{F}}$-submodule L of F^n of rank n. If L is an $\mathfrak{o}_{\mathrm{F}}$-lattice of F^n then $\mathrm{L} = \mathfrak{o}_{\mathrm{F}} f_1 + \cdots + \mathfrak{o}_{\mathrm{F}} f_n$ for some F-basis of F^n. More generally if L and

M are two $\mathfrak{o}_F$-lattices of F^n then there exist an F-basis $(f_1,\dots,f_n)$ of F^n and $(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}^n$, with $\alpha_1\leqslant\cdots\leqslant\alpha_n$, such that

$$L=\mathfrak{o}_F f_1+\cdots+\mathfrak{o}_F f_n \quad \text{and} \quad M=\mathfrak{p}_F^{\alpha_1}f_1+\cdots+\mathfrak{p}_F^{\alpha_n}f_n.$$

For two $\mathfrak{o}_F$-lattices L and M of F^n, we say that L and M are equivalent if $L=\lambda M$ for some $\lambda\in F^\times$, and we denote the class of L by [L]. The Bruhat–Tits building of G_n, denoted by $\mathscr{BT}_n$, can be defined as the simplicial complex whose vertices are the equivalence classes of $\mathfrak{o}_F$-lattices in F^n and in which a collection $\Lambda_0,\Lambda_1,\dots,\Lambda_q$ of pairwise distinct vertices form a q-simplex if we can choose representatives $L_i\in\Lambda_i$, for $i\in\{0,\dots,q\}$, such that

$$\varpi_F L_0 < L_q < L_{q-1} < \cdots < L_0$$

A q-simplex as above define the following flag of the k_F-vector space $L_0/\varpi_F L_0$

$$\{0\} < L_q/\varpi_F L_0 < L_{q-1}/\varpi_F L_0 < \cdots < L_0/\varpi_F L_0$$

The type of a such q-simplex is defined to be the type of the corresponding flag of the k_F-vector space $L_0/\varpi_F L_0\simeq k_F^n$. Note that the maximal dimension of the flag corresponding to a simplex σ of $\mathscr{BT}_n$ is equal to $n-2$. Thus $\mathscr{BT}_n$ is a simplicial complex of dimension $n-1$. The group $GL_n(F)$ acts naturally on $\mathscr{BT}_n$ by simplicial automorphisms and its center $Z(GL_n(F))\simeq F^\times$ acts trivially. So the group G_n acts simplicially on $\mathscr{BT}_n$ and the action is transitive on vertices (resp. chambers, q-simplices of a fixed type). Let's recall that a labelling of $\mathscr{BT}_n$ is a map from the set $\mathscr{BT}_n^0$ of vertices of $\mathscr{BT}_n$ to the set $\{0,\dots,n-1\}$ whose restriction to every chamber is injective. We can construct a labelling $\lambda:\mathscr{BT}_n^0\longrightarrow\{0,\dots,n-1\}$ of $\mathscr{BT}_n$ as follows (see [7, 19.3]). Let L_0 be a fixed $\mathfrak{o}_F$-lattices of F^n. If v is a vertex of $\mathscr{BT}_n$, we can choose a representative L such that $L_0\subset L$. Since $\mathfrak{o}_F$ is a principal ideal domain, the finitely generated torsion $\mathfrak{o}_F$-module L/L_0 is isomorphic to

$$\mathfrak{o}_F/\mathfrak{p}_F^{k_1}\oplus\cdots\oplus\mathfrak{o}_F/\mathfrak{p}_F^{k_n}$$

for some n-tupe of integers $0\leqslant k_1\leqslant k_2\leqslant\cdots\leqslant k_n$. Then

$$\lambda(v)=\sum_{i=0}^{n}k_i \bmod n\,.$$

The simplicial complex $\mathscr{BT}_n$ is the union of a family of subcomplexes, called apartments, defined as follows. A frame is a set $\mathscr{F}=\{d_1,\dots,d_n\}$ of one-dimensional F-vector subspaces of F^n so that $F^n=d_1+\cdots+d_n$. The apartment corresponding to the frame $\mathscr{F}$ is formed by all simplices σ with vertices Λ which are equivalence classes of lattices with representatives $L\in\Lambda$ such that

$$L=L_1+\cdots+L_n,$$

where L_i is a lattice of the F-vector space d_i. If we fix an F-basis $(f_1,\dots,f_n)$ of F^n adapted to the decomposition $F^n=d_1+\cdots+d_n$, then a vertex [L] is in the apartment corresponding to the frame $\mathscr{F}$ if and only if

$$L=\mathfrak{p}_F^{\alpha_1}f_1+\cdots+\mathfrak{p}_F^{\alpha_n}f_n,$$

where $(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}^n$. Note that the set of frames of F^n can be identified with the set of maximal F-split torus of G_n. To a frame $\mathscr{F}=\{d_1,\dots,d_n\}$ we can associate the maximal F-split torus $S\subset G_n$ acting diagonally with respect the decomposition of F^n as direct sum of vectorial lines. Under this identification, for every maximal F-split torus S of G_n, we denote by $\mathscr{A}_S$ the corresponding apartment of $\mathscr{BT}_n$. The apartment corresponding to the diagonal torus T will be called *the standard apartment* of $\mathscr{BT}_n$ and denoted by $\mathscr{A}_0$.

The geometric realization $|\mathscr{BT}_n|$ of the building $\mathscr{BT}_n$ is equipped by a metric defined, up to a multiplicative scalar, as follows. The geometric realization of each apartment $|\mathscr{A}|$ can be identified to the euclidian space

$$\mathbb{R}_0^n:=\left\{(x_1,\dots,x_n)\in\mathbb{R}^n\,\middle|\,x_1+\cdots+x_n=0\right\}$$

via the map defined by the following way. We fix an F-basis $(f_1,\dots,f_n)$ of F^n corresponding to the apartment $\mathscr{A}$. The set $\mathscr{A}^0$ of vertices of $\mathscr{A}$ is then embedded in $\mathbb{R}^n_0$ via the map $\varphi : \mathscr{A}^0 \longrightarrow \mathbb{R}^n_0$ defined by

$$\varphi([\mathfrak{p}_{\mathrm{F}}^{x_1} f_1 + \cdots + \mathfrak{p}_{\mathrm{F}}^{x_n} f_n]) = x - \frac{1}{n}\sigma(x)e,$$

where for $x = (x_1,\dots,x_n) \in \mathbb{Z}^n$, $\sigma(x) = x_1 + \cdots + x_n$ and where $e = (1,\dots,1)$. This map extends to a bijection $\varphi : |\mathscr{A}| \longrightarrow \mathbb{R}^n_0$. Via this identification we can then equip $|\mathscr{A}|$ by an euclidian metric. More explicitly, if [L] and [M] are two vertices of $\mathscr{A}$ with

$$\mathrm{L} = \mathfrak{p}_{\mathrm{F}}^{x_1} f_1 + \cdots + \mathfrak{p}_{\mathrm{F}}^{x_n} f_n \quad \text{and} \quad \mathrm{M} = \mathfrak{p}_{\mathrm{F}}^{y_1} f_1 + \cdots + \mathfrak{p}_{\mathrm{F}}^{y_n} f_n,$$

then

$$d_{\mathscr{A}}([\mathrm{L}],[\mathrm{M}]) = \frac{1}{\sqrt{1-\frac{1}{n}}} d_0\left(x - \frac{1}{n}\sigma(x)e, y - \frac{1}{n}\sigma(y)e\right)$$

where d_0 is the euclidian metric of $\mathbb{R}^n_0$. We note that in the above formula the term $1/\sqrt{1-1/n}$ is just used to normalize the metric of the building. The metric d of $|\mathscr{B}\mathscr{T}_n|$ is then defined as follows. If $x, y \in \mathscr{B}\mathscr{T}_n$ then $d(x,y) = d_{\mathscr{A}}(x,y)$ for any apartment $\mathscr{A}$ containing x and y and this is independent of the choice of apartment containing them. Finally we recall that the action of the group G_n on $|\mathscr{B}\mathscr{T}_n|$ is by isometries.

2.3. *Smooth representations of a locally profinite group*

Let G be a locally profinite group. By a representation of G we mean a pair (π, V) formed by a $\mathbb{C}$-vector space V and by a group homomorphism $\pi : G \longrightarrow \mathrm{GL}_{\mathbb{C}}(V)$. A such representation is called smooth if for every $v \in V$ the stabilizer

$$\mathrm{Stab}_G(v) := \left\{g \in \mathscr{G} \,\middle|\, \pi(g).v = v\right\}$$

is an open subgroup of G. *In this paper all the representations will be assumed to be smooth and complex.* A representation (π, V) of G is called admissible if for every compact open subgroup K of G the space $V^K = \{v \in V \mid \forall\, k \in K, \pi(k)v = v\}$ of K-fixed vectors is finite dimensional. If (π, V) is a representation of G, its contragredient $\pi^\vee$ is the representation of G in the subspace $V^\vee$ of the algebraic dual V^* formed by the linear forms whose stabilizers in G is open.

Let H be a closed subgroup of G and (ρ, W) a representation of H. We recall that the induced representation from H to G of (ρ, W), denoted by $\mathrm{Ind}_H^G \rho$, is the representation of G on the space $\mathrm{Ind}_H^G W$ formed by the locally constant functions $f : G \longrightarrow W$ such that $f(hg) = \rho(h).f(h)$ for every $g \in G$ and $h \in H$, where the action of G on $\mathrm{Ind}_H^G \rho$ is by left translation. The compactly induced representation $\text{c-ind}_H^G \rho$ is defined as the subrepresentation of $\mathrm{Ind}_H^G \rho$ formed by the functions $f \in \mathrm{Ind}_H^G W$ whose support is compact modulo H.

2.4. *Locally profinite group acting on directed graphs*

Throughout this paper, we call graph every one dimensional simplicial complex. If Y is a graph, the set of vertices (resp. edges) of Y will be denoted by Y^0 (resp. Y^1). A locally finite graph is a graph Y for which every vertex belongs to a finite number of edges. All graphs in this paper will be assumed to be locally finite. A directed graph is a graph Y with a map $\mathrm{Y}^1 \longrightarrow \mathrm{Y}^0 \times \mathrm{Y}^0$, $a \longmapsto (a^-, a^+)$, such that for every edge a one has $a = \{a^-, a^+\}$, where for any edge a we denote by a^+ and a^- its head and tail respectively. A path in a graph Y is a sequence $(s_0,\dots,s_m)$ of vertices such that two consecutive vertices are linked by an edge. The graph Y is called connected if every pair of vertices are linked by a path. A cover of a graph Y is a family $(Y_\alpha)_{\alpha\in\Delta}$ of subgraphs such that

$$Y = \bigcup_{\alpha\in\Delta} Y_\alpha.$$

The nerve of a such cover, denoted $\mathcal{N}(Y,(Y_\alpha)_{\alpha\in\Delta})$ or just $\mathcal{N}(Y)$ if there is no risk of confusion, is the simplicial complex whose vertex set is Δ and in which a finite number of vertices $\alpha_0,\dots,\alpha_r$ form a simplex if

$$\bigcap_{i=0}^{r} Y_{\alpha_i} \neq \emptyset.$$

In the remainder of this section the notations and definitions are taken from [5]. If Y is a graph, we denote by $C_0(Y,\mathbb{C})$ (resp. $C_1(Y,\mathbb{C})$) the $\mathbb{C}$-vector space with basis Y^0 (resp. Y^1). Let $C_c^i(Y,\mathbb{C})$, $i=1,2$, be the $\mathbb{C}$-vector space of 1-cochains with finite support : $C_c^i(Y,\mathbb{C})$ is the subspace of the algebraic dual of $C_i(Y,\mathbb{C})$ formed of those linear forms whose restrictions to the basis Y^i have finite support. The coboundary map

$$d\,:\,C_c^0(Y,\mathbb{C})\longrightarrow C_c^1(Y,\mathbb{C})$$

is defined by $d(f)(a)=f(a^+)-f(a^-)$. Then the compactly supported cohomology space $H_c^1(Y,\mathbb{C})$ of the graph Y is defined by

$$H_c^1(Y,\mathbb{C})=C_c^1(Y,\mathbb{C})/dC_c^0(Y,\mathbb{C}).$$

Let G be a locally profinite group and Y be a directed graph. We assume that G acts on Y by automorphisms of directed graphs. For all $s\in Y^0$, $a\in Y^1$, the incidence numbers are defined by $[a:a^+]=+1$, $[a:a^-]=-1$, and $[a:s]=0$ if $s\notin\{a^+,a^-\}$. These incidence numbers are equivariant in the sense that $[g.a:g.s]=[a:s]$, for all $g\in G$. The group G acts on $C_i(\mathrm{Y},\mathbb{C})$ and $C_c^i(\mathrm{Y},\mathbb{C})$. If the action of G on Y is proper, that is for every $s\in Y^0$, the stabilizer $\mathrm{Stab}_G(s):=\{g\in G\,|\,g.s=s\}$ is open and compact, then the spaces $C_i(Y,\mathbb{C})$ and $C_c^i(Y,\mathbb{C})$ are smooth G-modules. The coboundary map is G-equivariant so that $H_c^1(Y,\mathbb{C})$ have a structure of a smooth G-module.

The space of harmonic forms of the graph Y is defined as the subspace of $C^1(Y,\mathbb{C})$ formed by the elements $f\in C^1(Y,\mathbb{C})$ verifying the following harmonicity condition (see [5, §(1.3)]):

$$\sum_{a\in Y^1}[a:s]f(a)=0\quad\text{for all } s\in Y^0.$$

This space will be denoted by $\mathscr{H}(Y,\mathbb{C})$. It is naturally provided by a linear action of G. The smooth part of $\mathscr{H}(Y,\mathbb{C})$ under the action of G, i.e. the space of *smooth harmonic forms* is denoted by $\mathscr{H}_\infty(Y,\mathbb{C})$.

Lemma 1 ([5, (1.3.2)]). *The algebraic dual of $H_c^1(Y,\mathbb{C})$ naturally identifies with $\mathscr{H}(Y,\mathbb{C})$. Under this isomorphism, the contragredient representation of $H_c^1(Y,\mathbb{C})$ corresponds to $\mathscr{H}_\infty(Y,\mathbb{C})$.*

3. Combinatorial geodesic paths in $\mathscr{BT}_n$

The aim of this section is to define a class of combinatorial paths in $\mathscr{BT}_n$ and to study the action of the group G_n on this class of paths. The pointwise stabilisers of such paths will be related to the new-vectors subgroups of $\mathrm{GL}_n(\mathrm{F})$ (the subgroups defined in (1)), see [8].

3.1. *Geodesic paths of $\mathscr{BT}_n$ and their prolongations*

Definition 2. *Let $k\geqslant 0$ be an integer. A* geodesic path *of length k in $\mathscr{BT}_n$ (or more simply* geodesic k-path*) is a path $\alpha=(\alpha_0,\alpha_1,\dots,\alpha_k)$ of $\mathscr{BT}_n$ such that for every $i,j\in\{0,\dots,k\}$, $d(\alpha_i,\alpha_j)=|i-j|$. We denote the set of geodesic k-paths of $\mathscr{BT}_n$ by $\mathscr{C}_k(\mathscr{BT}_n)$.*

Remark 3. We notice that when $n\geqslant 4$ the edges of $\mathscr{BT}_n$ are not all of length one, but in the particular cases $n=2$ and $n=3$ all the edges of $\mathscr{BT}_n$ are of length one. We also note that every geodesic k-path of $\mathscr{BT}_n$ lies in a same apartment. In fact if $\alpha\in\mathscr{C}_k(\mathscr{BT}_n)$ is a geodesic k-path as previously, then the geometric realization of any apartment containing the vertices α_0 and α_k contain the segment $[\alpha_0,\alpha_k]$ and then all the vertices of α are contained in the apartment $\mathscr{A}$.

In the following, if s is a vertex of $\mathscr{BT}_n$ we write $\mathscr{V}(s)$ for its combinatorial neighborhood. That is $\mathscr{V}(s)$ is the set of vertices of $\mathscr{BT}_n$ which are linked to s by an edge.

Definition 4. *Let $\alpha = (\alpha_0, \dots, \alpha_k) \in \mathscr{C}_k(\mathscr{BT}_n)$. A vertex s of $\mathscr{BT}_n$ is called a right (resp. left) prolongation of α if $s \in \mathscr{V}(\alpha_k)$ (resp. $s \in \mathscr{V}(\alpha_0)$) and the sequence $(\alpha_0, \dots, \alpha_k, s)$ (resp. $(s, \alpha_0, \dots, \alpha_k)$) is a geodesic $(k+1)$-path. We denote the set of right and left prolongation of a geodesic k-path α respectively by $\mathscr{P}^+(\alpha)$ and $\mathscr{P}^-(\alpha)$.*

Proposition 5. *Let $k \geqslant 1$ be an integer and let $\alpha = (\alpha_0, \dots, \alpha_k)$ be a geodesic k-path of $\mathscr{BT}_n$. Then for every apartment $\mathscr{A}$ containing α, there exists a unique right (resp. left) prolongation of α in the apartment $\mathscr{A}$.*

Proof. Let $\mathscr{A}$ be an apartment containing the path α. Assume that α have two right prolongations x and y in $\mathscr{A}$, that is $x, y \in \mathscr{V}(\alpha_k)$ and the two sequences $(\alpha_0, \dots, \alpha_k, x)$ and $(\alpha_0, \dots, \alpha_k, y)$ are geodesic $(k+1)$-paths of $\mathscr{A}$. So in the geometric realization $|\mathscr{A}|$ of the apartment $\mathscr{A}$ we have $\alpha_k \in [\alpha_0, x] \cap [\alpha_0, y]$. Therefore we have $\alpha_k = tx + (1-t)\alpha_0$ and $\alpha_k = sy + (1-s)\alpha_0$ for same t and s in $]0,1[$. Moreover the two vertices x and y are of the same distance from α_k, that is $d(x, \alpha_k) = d(y, \alpha_k)$. So we have $\|x - \alpha_k\| = \|y - \alpha_k\|$ (here $\|\cdot\|$ is the euclidian norm of $|\mathscr{A}| \simeq \mathbb{R}_0^n$). From this we obtain $(1-t)\|x - \alpha_0\| = (1-s)\|y - \alpha_0\|$. But $\|x - \alpha_0\| = \|y - \alpha_0\|$ so we get $t = s$ and then $x = y$. □

Let $\alpha = (\alpha_0, \dots, \alpha_k)$ be a geodesic path of $\mathscr{BT}_n$. The inverse of α, denoted by α^{-1}, is defined by $\alpha^{-1} := (\alpha_k, \dots, \alpha_0)$. It is clear that α^{-1} is a geodesic path of $\mathscr{BT}_n$. If $k \geqslant 1$, the tail and the head of α are the two geodesic paths defined respectively by

$$\alpha^- := (\alpha_0, \dots, \alpha_{k-1}) \quad \text{and} \quad \alpha^+ := (\alpha_1, \dots, \alpha_k).$$

We define also the initial and terminal directed edge of α respectively by $e^-(\alpha) := (\alpha_0, \alpha_1)$ and $e^+(\alpha) := (\alpha_{k-1}, \alpha_k)$.

Proposition 6. *Let $k \geqslant 1$ be an integer and let $\alpha, \beta \in \mathscr{C}_k(\mathscr{BT}_n)$. If α and β are contained in a same apartment and if $e^-(\alpha) = e^-(\beta)$ (resp. $e^+(\alpha) = e^+(\beta)$), then $\alpha = \beta$.*

Proof. By induction on k, let $\alpha = (\alpha_0, \dots, \alpha_{k+1})$ and $\beta = (\beta_0, \dots, \beta_{k+1})$ two geodesic $(k+1)$-paths such that $e^-(\alpha) = e^-(\beta)$. Assume that α and β are contained in a same apartment $\mathscr{A}$. Since the two geodesic k-paths α^- and β^- are contained in the same apartment $\mathscr{A}$ and as they have the same initial directed edges then by induction hypothesis we have $\alpha^- = \beta^-$, that is $\alpha_i = \beta_i$ for each $i \in \{0, \dots, k\}$. So the two vertices α_{k+1} and β_{k+1} are two right prolongation of the geodesic k-paths α^- which are contained in the same apartment $\mathscr{A}$. Then by the previous proposition we obtain $\alpha_{k+1} = \beta_{k+1}$ and then $\alpha = \beta$ as required. □

3.2. *Action of* G_n *on the sets* $\mathscr{C}_k(\mathscr{BT}_n)$

The group G_n acts on its building $\mathscr{BT}_n$ by isometries, so G_n acts naturally on the sets $\mathscr{C}_k(\mathscr{BT}_n)$ for each integer $k \geqslant 0$. The action is given by

$$g.(\alpha_0, \dots, \alpha_k) = (g.\alpha_0, \dots, g.\alpha_k)$$

for every $g \in G_n$ and for every $(\alpha_0, \dots, \alpha_k) \in \mathscr{C}_k(\mathscr{BT}_n)$. Note that since the set $\mathscr{C}_0(\mathscr{BT}_n)$ may be identified with the set of vertices of $\mathscr{BT}_n$, then the action of G_n on $\mathscr{C}_0(\mathscr{BT}_n)$ is transitive. In the particular case $n = 2$, the action of G_2 on the sets $\mathscr{C}_k(\mathscr{BT}_2)$ is transitive for every integer $k \geqslant 0$, see [5]. The situation is slightly different when $n \geqslant 3$. We are going to prove that in this last case, the sets $\mathscr{C}_k(\mathscr{BT}_n)$ (for $k \geqslant 1$) have exactly two G_n-orbits. We first define the type of a directed edge of $\mathscr{BT}_n$ and we will prove in the lemma bellow that two geodesic 1-paths are in the same G_n-orbit if and only if they have the same type. Let $e = ([L_0],[L_1])$ be a directed edge of $\mathscr{BT}_n$, where L_0 and L_1 are two $\mathfrak{o}_F$-lattices such that

$$\varpi_F L_0 < L_1 < L_0.$$

The type of the directed edge e, denoted $\xi(e)$, is defined by

$$\xi(e) = \dim_{k_F}\big(L_1/\varpi_F L_0\big).$$

This definition is clearly independent of the choice of representatives. For every directed edge e of $\mathscr{BT}_n$, we write e^{-1} for the inverse of e which is obtained from e by interchanging its vertices.

Lemma 7.

(i) *For every directed edge e of $\mathscr{BT}_n$, $\xi(e^{-1}) = n - \xi(e)$,*

(ii) *For every $e \in \mathscr{C}_1(\mathscr{BT}_n)$, $\xi(e) \in \{1, n-1\}$,*

(iii) *Two elements $e, e^{'} \in \mathscr{C}_1(\mathscr{BT}_n)$ are in the same G_n-orbit if and only if they have the same type.*

Proof. In the proof of the three statements we use the following notations. For each integer $n \geqslant 1$, we write Δ_n for the set of integers $\{1,\dots,n\}$. If $e = ([L_0],[L_1])$ is a directed edge of $\mathscr{BT}_n$ with $\varpi_F L_0 < L_1 < L_0$ and if $(f_1,\dots,f_n)$ is a basis of F^n for which

$$L_0 = \mathfrak{o}_F f_1 + \cdots + \mathfrak{o}_F f_n \quad \text{and} \quad L_1 = \mathfrak{p}_F^{k_1} f_1 + \cdots + \mathfrak{p}_F^{k_n} f_n,$$

where $(k_1,\dots,k_n) \in \mathbb{Z}^n$ and $k_1 \leqslant \cdots \leqslant k_n$, we put $A_0 = \{i \in \Delta_n \,|\, k_i = 0\}$ and $A_1 = \{i \in \Delta_n \,|\, k_i = 1\}$ and we write p and q respectively for their cardinality. The condition $\varpi_F L_0 < L_1 < L_0$ implies that $k_i \in \{0,1\}$ for each $i \in \Delta_n$ and that $p, q \in \{1,\dots,n-1\}$ and $p+q = n$.

(i). Let $e = ([L_0],[L_1])$ be a directed edge with $\varpi_F L_0 < L_1 < L_0$. The inverse of e is then given by $e^{-1} = ([\varpi_F^{-1} L_1],[L_0])$ with $L_1 < L_0 < \varpi_F^{-1} L_1$. Let $(f_1,\dots,f_n)$ be a basis of F^n for which

$$L_0 = \mathfrak{o}_F f_1 + \cdots + \mathfrak{o}_F f_n \quad \text{and} \quad L_1 = \mathfrak{p}_F^{k_1} f_1 + \cdots + \mathfrak{p}_F^{k_n} f_n,$$

where $(k_1,\dots,k_n) \in \mathbb{Z}^n$ with $k_1 \leqslant \cdots \leqslant k_n$. With the previous notations we have the identifications of k_F-vector spaces

$$L_1/\varpi_F L_0 \simeq \bigoplus_{i=1}^{n} \mathfrak{p}_F^{k_i}/\mathfrak{p}_F \simeq \bigoplus_{i \in A_0} \mathfrak{o}_F/\mathfrak{p}_F \oplus \bigoplus_{i \in A_1} \mathfrak{p}_F/\mathfrak{p}_F \simeq k_F^{p} \tag{2}$$

and similarly

$$L_0/L_1 \simeq \bigoplus_{i=1}^{n} \mathfrak{o}_F/\mathfrak{p}_F^{k_i} \simeq \bigoplus_{i \in A_0} \mathfrak{o}_F/\mathfrak{o}_F \oplus \bigoplus_{i \in A_1} \mathfrak{o}_F/\mathfrak{p}_F \simeq k_F^{q}. \tag{3}$$

So we obtain $\dim_{k_F}\big(L_0/L_1\big) = n - \dim_{k_F}\big(L_1/\varpi_F L_0\big)$, and then $\xi(e^{-1}) = n - \xi(e)$.

(ii). Let $e = ([L_0],[L_1])$ be a directed edge of $\mathscr{BT}_n$ with $\varpi_F L_0 < L_1 < L_0$ and let $\mathscr{A}$ be an apartment containing e. To simplify, we can assume that in a some F-basis $(f_1,\dots,f_n)$ of F^n we have $L_0 = \mathfrak{o}_F f_1 + \cdots + \mathfrak{o}_F f_n$ and $L_1 = \mathfrak{p}_F^{x_1} f_1 + \cdots + \mathfrak{p}_F^{x_n} f_n$, where $x = (x_1,\dots,x_n)$ is in $\mathbb{Z}^n$. As previously, the $x_i^{'}$s are in $\{0,1\}$.

Now if we assume that $e \in \mathscr{C}_1(\mathscr{BT}_n)$ then $d([L_0],[L_1]) = 1$. We have then

$$d_0\left(0, x - \frac{1}{n}\sigma(x)e\right) = \frac{\sqrt{n-1}}{\sqrt{n}}$$

that is

$$\sum_{i=1}^{n}\left(x_i - \frac{1}{n}\sigma(x)\right)^2 = \frac{n-1}{n}$$

and then

$$\left(\sum_{i=1}^{n} x_i^2\right) - \frac{1}{n}\sigma(x)^2 = \frac{n-1}{n}.$$

But since $x_i \in \{0,1\}$ then $\sigma(x) - \sigma(x)^2/n = (n-1)/n$ which implies that the values of $\sigma(x)$ are 1 or $n-1$. Moreover, from the isomorphisms (2) and (3) we deduce that $\sigma(x) = n - \xi(e)$, so as desired we have $\xi(e) \in \{1, n-1\}$.

(iii). Let $e \in \mathscr{C}_1(\mathscr{BT}_n)$ with $e = ([L_0],[L_1])$ and $\varpi_F L_0 < L_1 < L_0$. Let's prove firstly that if $\xi(e) = 1$ then there exist an F-basis $(f_1,\ldots,f_n)$ of F^n such that $L_0 = \mathfrak{p}_F^{-1} f_1 + \cdots + \mathfrak{p}_F^{-1} f_{n-1} + \mathfrak{o}_F f_n$ and $L_1 = \mathfrak{o}_F f_1 + \cdots + \mathfrak{o}_F f_n$ and if $\xi(e) = n-1$ then there exist an F-basis $(h_1,\ldots,h_n)$ of F^n such that $L_0 = \mathfrak{o}_F h_1 + \cdots + \mathfrak{o}_F h_n$ and $L_1 = \mathfrak{o}_F h_1 + \cdots + \mathfrak{o}_F h_{n-1} + \mathfrak{p}_F h_n$. Assume that $\xi(e) = n-1$ (the proof of the case $\xi(e) = 1$ is similar). For a some F-basis $(h_1,\ldots,h_n)$ of F^n we have $L_0 = \mathfrak{o}_F h_1 + \cdots + \mathfrak{o}_F h_n$ and $L_1 = \mathfrak{p}_F^{k_1} h_1 + \cdots + \mathfrak{p}_F^{k_n} h_n$ where $(k_1,\ldots,k_n) \in \mathbb{Z}^n$ with $k_1 \leqslant \cdots \leqslant k_n$.

As mentioned previously, for each $i \in \Delta_n$ the integer k_i is in $\{0,1\}$. The fact that $k_1 \leqslant \cdots \leqslant k_n$ implies that $(k_1,\ldots,k_n) = (0,\ldots,0,1,\ldots,1)$, where 0 appear p-times and 1 appear q-times.

So we have

$$L_1/\varpi_F L_0 \simeq \bigoplus_{i=1}^{p} \mathfrak{o}_F/\mathfrak{p}_F \oplus \bigoplus_{i=p+1}^{q} \mathfrak{p}_F/\mathfrak{p}_F \simeq k_F^p.$$

But since $\xi(e) = n-1$, that is $\dim_{k_F}(L_1/\varpi_F L_0) = n-1$, then we have $L_1 = \mathfrak{o}_F h_1 + \cdots + \mathfrak{o}_F h_{n-1} + \mathfrak{p}_F h_n$. So as desired we have an F-basis $(h_1,\ldots,h_n)$ of F^n for which $L_0 = \mathfrak{o}_F h_1 + \cdots + \mathfrak{o}_F h_n$ and $L_1 = \mathfrak{o}_F h_1 + \cdots + \mathfrak{o}_F h_{n-1} + \mathfrak{p}_F h_n$. Let's prove now that two elements $e, e' \in \mathscr{C}_1(\mathscr{BT}_n)$ are in the same G_n-orbit if and only if they have the same type. Assume that $e = ([L_0],[L_1])$ (resp. $e' = ([L'_0],[L'_1])$) where L_0 and L_1 (resp. L'_0 and L'_1) are two $\mathfrak{o}_F$-lattices such that $\varpi_F L_0 < L_1 < L_0$ (resp. $\varpi_F L'_0 < L'_1 < L'_0$). If e and e' have the same type, say for example $\xi(e) = \xi(e') = 1$, then by the previous point we can find two F-basis $(f_1,\ldots,f_n)$ and $(f'_1,\ldots,f'_n)$ for which $L_0 = \mathfrak{p}_F^{-1} f_1 + \cdots + \mathfrak{p}_F^{-1} f_{n-1} + \mathfrak{o}_F f_n$ and $L_1 = \mathfrak{o}_F f_1 + \cdots + \mathfrak{o}_F f_n$ and likewise $L'_0 = \mathfrak{p}_F^{-1} f'_1 + \cdots + \mathfrak{p}_F^{-1} f'_{n-1} + \mathfrak{o}_F f'_n$ and $L'_1 = \mathfrak{o}_F f'_1 + \cdots + \mathfrak{o}_F f'_n$. So if $g \in \mathrm{G}_n$ is the unique element sending the F-basis $(f_1,\ldots,f_n)$ on $(f'_1,\ldots,f'_n)$ we have $gL_0 = L'_0$ and $gL_1 = L'_1$, thus $g.e = e'$ and then e and e' are in the same G_n-orbit. The converse is obvious. □

Proposition 8. *Let $n \geqslant 3$ be an integer. For every $k \geqslant 1$, the set $\mathscr{C}_k(\mathscr{BT}_n)$ have two G_n-orbits.*

Proof. Let us prove firstly that two elements α and β of $\mathscr{C}_k(\mathscr{BT}_n)$ are in the same G_n-orbit if and only if their initial directed edges $e^-(\alpha)$ and $e^-(\beta)$ are likewise. If α and β are in the same G_n-orbit then clearly $e^-(\alpha)$ and $e^-(\beta)$ are also in the same G_n-orbit. Conversely, assume that $e^-(\alpha)$ and $e^-(\beta)$ are in the same G_n-orbit, that is for same $g \in \mathrm{G}_n$ one has $e^-(\alpha) = g.e^-(\beta)$. So we have $e^-(\alpha) = e^-(g.\beta)$.

Let $\mathscr{A}$ and $\mathscr{B}$ two apartments containing α and $g.\beta$ respectively. Since the pointwise stabiliser H_0 of the edge $e^-(\alpha)$ acts transitively on the set of apartments containing $e^-(\alpha)$ (see [6, Cor. (7.4.9)]), then there exist $h \in H_0$ such that $h.\mathscr{B} = \mathscr{A}$. So the two geodesic k-paths α and $hg.\beta$ are contained in the same apartment $\mathscr{A}$ and have the same initial directed edge (that is $e^-(\alpha) = e^-(hg.\beta)$). Thus the Proposition 6 implies that $\alpha = hg.\beta$ and then α and β are in the same G_n-orbit. Consequently, two elements α and β of $\mathscr{C}_k(\mathscr{BT}_n)$ are in the same G_n-orbit if and only if $e^-(\alpha)$ and $e^-(\beta)$ are likewise. The result follows then from Lemma 7. □

One can prove that if $\alpha \in \mathscr{C}_k(\mathscr{BT}_n)$ then all the directed edges of α have the same type. So we can define the type of a geodesic k-path α, denoted by $\xi(\alpha)$, as the type of any of its directed edges. The G_n-orbit of $\mathscr{C}_k(\mathscr{BT}_n)$ corresponding to the type $n-1$ (resp. type 1) will be denoted by $\mathscr{C}_k^+(\mathscr{BT}_n)$ (resp. $\mathscr{C}_k^-(\mathscr{BT}_n)$). The Lemma 7 implies that if $\alpha \in \mathscr{C}_k^+(\mathscr{BT}_n)$ then its inverse α^{-1} is in $\mathscr{C}_k^-(\mathscr{BT}_n)$. So for every $\alpha \in \mathscr{C}_k(\mathscr{BT}_n)$ the pair $\{\alpha, \alpha^{-1}\}$ constitute a system of representatives of $\mathscr{C}_k(\mathscr{BT}_n)$ for the action of the group G_n. The path $\gamma = ([L_0],[L_1],\ldots,[L_k])$, where for $i \in \{0,\ldots,k\}$

$$L_i = \mathfrak{o}_F e_1 + \cdots + \mathfrak{o}_F e_{n-1} + \mathfrak{p}_F^i e_n \tag{4}$$

is an element of $\mathscr{C}_k^+(\mathscr{BT}_n)$ contained in the standard apartment of $\mathscr{BT}_n$, this k-path will be called *the standard geodesic k-path.*

Lemma 9. *For every $\alpha \in \mathscr{C}_k(\mathscr{BT}_n)$ the stabilizer $\mathrm{Stab}_{\mathrm{G}_n}(\alpha)$ acts transitively on $\mathscr{P}^+(\alpha)$ and $\mathscr{P}^-(\alpha)$.*

Proof. Let $\alpha = (\alpha_0, \ldots, \alpha_k) \in \mathscr{C}_k(\mathscr{BT}_n)$. We will prove that the action of $\mathrm{Stab}_{\mathrm{G}_n}(\alpha)$ is transitive on $\mathscr{P}^+(\alpha)$. By a similar way we get the same thing for $\mathscr{P}^-(\alpha)$. Let $s, t \in \mathscr{P}^+(\alpha)$, that is $\beta = (\alpha_0, \ldots, \alpha_k, s)$ and $\gamma = (\alpha_0, \ldots, \alpha_k, t)$ are two geodesic $(k+1)$-paths. Since every geodesic path of $\mathscr{BT}_n$ is contained in a some apartment, then there are two apartments $\mathscr{A}$ and $\mathscr{B}$ containing β and γ respectively. The stabilizer $\mathrm{Stab}_{\mathrm{G}_n}(\alpha)$ is also the pointwise stabilizer in G_n of the segment $[\alpha_0, \alpha_k]$. So $\mathrm{Stab}_{\mathrm{G}_n}(\alpha)$ acts transitively on the set of apartments containing α (see [6, Cor. (7.4.9)]). Then there exist $g \in \mathrm{Stab}_{\mathrm{G}_n}(\alpha)$ such that $g.\mathscr{A} = \mathscr{B}$. So $g.s$ is a right prolongation of the geodesic path α contained in the apartment $\mathscr{B}$. Hence, the two vertices t and $g.s$ are two right prolongations of α contained in the apartment $\mathscr{B}$. Then by the Proposition 5, we obtain $g.s = t$ and then as desired the action of $\mathrm{Stab}_{\mathrm{G}_n}(\alpha)$ on $\mathscr{P}^+(\alpha)$ is transitive. □

Corollary 10. *For every $\alpha \in \mathscr{C}_k(\mathscr{BT}_n)$ we have :*

$$\mathscr{P}^+(\alpha) = \mathscr{P}^+(e^+(\alpha)) \quad \textit{and} \quad \mathscr{P}^-(\alpha) = \mathscr{P}^-(e^-(\alpha)),$$

that is the right (resp. left) prolongation of the geodesic path α are exactly the right (resp. left) prolongation of the directed edge $e^+(\alpha)$ (resp. $e^-(\alpha)$).

Proof. Let's prove the first equality, the proof of the second is similar. It is clear that $\mathscr{P}^+(\alpha) \subset \mathscr{P}^+(e^+(\alpha))$. Since the two sets $\mathscr{P}^+(\alpha)$ and $\mathscr{P}^+(e^+(\alpha))$ are finite it suffice to prove that they have the same cardinality. If Γ_α denoted the subgroup $\mathrm{Stab}_{\mathrm{G}_n}(\alpha)$, then by the previous lemma Γ_α acts transitively on $\mathscr{P}^+(\alpha)$. So for any $s \in \mathscr{P}^+(\alpha)$ we can identify the set $\mathscr{P}^+(\alpha)$ with the quotient set $\Gamma_\alpha / \mathrm{Stab}_{\Gamma_\alpha}(s)$. Similarly, the set $\mathscr{P}^+(e^+(\alpha))$ identifies with the quotient set $\Gamma_{e^+(\alpha)} / \mathrm{Stab}_{\Gamma_{e^+(\alpha)}}(t)$ for any $t \in \mathscr{P}^+(e^+(\alpha))$. Now since the action of G_n on $\mathscr{C}_k(\mathscr{BT}_n)$ have two orbits and since an element $\beta \in \mathscr{C}_k(\mathscr{BT}_n)$ and its inverse β^{-1} have the same stabilizers in G_n then we can assume that α is the standard geodesic k-path defined as previously by $([L_0], [L_1], \ldots, [L_k])$, where $L_i = \mathfrak{o}_\mathrm{F} e_1 + \cdots + \mathfrak{o}_\mathrm{F} e_{n-1} + \mathfrak{p}_\mathrm{F}^i e_n$ for $i \in \{0, \ldots, k\}$. If s is the vertex $[L_{k+1}]$, it is clearly that $s \in \mathscr{P}^+(\alpha)$. By an easy computation we obtain that $\Gamma_\alpha = \Gamma_0(\mathfrak{p}_\mathrm{F}^k)$ and $\mathrm{Stab}_{\Gamma_\alpha}(s) = \Gamma_0(\mathfrak{p}_\mathrm{F}^{k+1})$. Moreover, we can check that $\Gamma_0(\mathfrak{p}_\mathrm{F}^k)/\Gamma_0(\mathfrak{p}_\mathrm{F}^{k+1})$ have cardinality q_F^{n-1}. Similarly, we can check easily that the vertex s whose equivalence class of $\mathfrak{o}_\mathrm{F}$-lattice is represented by L_{k+1} is in $\mathscr{P}^+(e^+(\alpha))$ and that $\Gamma_{e^+(\alpha)} = \Gamma_0(\mathfrak{p}_\mathrm{F})$ and $\mathrm{Stab}_{\Gamma_{e^+(\alpha)}}(s) = \Gamma_0(\mathfrak{p}_\mathrm{F}^2)$. Furthermore, we can check that $\Gamma_0(\mathfrak{p}_\mathrm{F})/\Gamma_0(\mathfrak{p}_\mathrm{F}^2)$ have also cardinality q_F^{n-1}. So as desired we have the equality between the two sets $\mathscr{P}^+(\alpha)$ and $\mathscr{P}^+(e^+(\alpha))$. □

Corollary 11. *For every $\alpha, \beta \in \mathscr{C}_{k+1}(\mathscr{BT}_n)$, if $\alpha^+ = \beta^+$ (resp. $\alpha^- = \beta^-$) then $\mathscr{P}^+(\alpha) = \mathscr{P}^+(\beta)$ (resp. $\mathscr{P}^-(\alpha) = \mathscr{P}^-(\beta)$).*

Proof. If $\alpha^+ = \beta^+$ (resp. $\alpha^- = \beta^-$) then $e^+(\alpha) = e^+(\beta)$ (resp. $e^+(\alpha) = e^+(\beta)$) and then the equality $\mathscr{P}^+(\alpha) = \mathscr{P}^+(\beta)$ (resp. $\mathscr{P}^-(\alpha) = \mathscr{P}^-(\beta)$) follows from the previous corollary. □

Lemma 12. *Let s_0 be a vertex of $\mathscr{BT}_n$. If $L_0 \in s_0$ then for every vertex $x \in \mathscr{V}(s_0)$ there is a unique representative $L \in x$ such that*

$$\varpi_\mathrm{F} L_0 < L < L_0.$$

Proof. Let us fix a representative $L_0 \in s_0$. Let L and L' two representatives of x such that $\varpi_\mathrm{F} L_0 < L < L_0$ and $\varpi_\mathrm{F} L_0 < L' < L_0$. Since L and L' are equivalent then $L' = \lambda L$ for some $\lambda \in \mathrm{F}^\times$. Put $\lambda = \varpi_\mathrm{F}^m u$ for some $m \in \mathbb{Z}$ and $u \in \mathfrak{o}_\mathrm{F}^\times$. We have $\varpi_\mathrm{F} L_0 < L < L_0$ which implies $\varpi_\mathrm{F}^{m+1} L_0 < \lambda L < \varpi_\mathrm{F}^m L_0$, that is $\varpi_\mathrm{F}^{m+1} L_0 < L' < \varpi_\mathrm{F}^m L_0$. The two inclusions $\varpi_\mathrm{F} L_0 < L' < L_0$ and $\varpi_\mathrm{F}^{m+1} L_0 < L' < \varpi_\mathrm{F}^m L_0$ implies then that $m = 0$. Indeed, if we assume to the contrary that $m \neq 0$, say for example $m > 0$, then we have $\varpi_\mathrm{F}^m L_0 \leqslant \varpi_\mathrm{F} L_0$. So from the two inclusions $\varpi_\mathrm{F} L_0 < L' < L_0$ and $\varpi_\mathrm{F}^{m+1} L_0 < L' < \varpi_\mathrm{F}^m L_0$ we obtain $L' < \varpi_\mathrm{F}^m L_0 \leqslant \varpi_\mathrm{F} L_0 < L'$ which is a contradiction. We deduce then that $L' = uL = L$. □

Let s_0 be a vertex of $\mathscr{BT}_n$ and $L_0 \in s_0$ be a fixed representative. By the previous lemma to any vertex $x \in \mathscr{V}(s_0)$ we can associate a non-trivial subspace of the k_F-vector space $\widetilde{V}_{s_0} := L_0/\varpi_\mathrm{F} L_0$.

Indeed, if $x \in \mathscr{V}(s_0)$ and $L_x \in x$ is the unique representative such that $\varpi_F L_0 < L_x < L_0$, then V_x is defined as $L_x/\varpi_F L_0$. For every subspaces X and Y of $\widetilde{V}_{s_0}$, we put

$$\delta(X,Y) = \dim_{k_F}(X+Y) - \dim_{k_F}(X\cap Y).$$

In the following proposition, we give two formulas for the metric of $\mathscr{BT}_n$ on the set of vertices in the neighborhood a fixed vertex s_0 of $\mathscr{BT}_n$ in terms of the corresponding k_F-vector spaces.

Proposition 13. *For every vertex s_0 of $\mathscr{BT}_n$ we have :*

(i) *If $x \in \mathscr{V}(s_0)$, then*

$$d(s_0,x) = \frac{1}{\sqrt{n-1}}\Big(n\dim V_x - (\dim V_x)^2\Big)^{\frac{1}{2}}.$$

(ii) *If $x, y \in \mathscr{V}(s_0)$, then*

$$d(x,y) = \frac{1}{\sqrt{n-1}}\Big(n\delta(V_x,V_y) - (\dim V_x - \dim V_y)^2\Big)^{\frac{1}{2}}.$$

Proof. (i). Let us fix an $\mathfrak{o}_F$-lattice L_0 representing the vertex s_0. Let $x \in \mathscr{V}(s_0)$. We can choose an apartment $\mathscr{A}$ containing s_0 and x. Without loss of generality we can assume that $\mathscr{A}$ is the standard apartment and that $L_0 = \mathfrak{o}_F e_1 + \cdots + \mathfrak{o}_F e_n$, where $(e_1,\ldots,e_n)$ is the standard basis of F^n. Let L_x be the unique representative of the vertex x such that $\varpi_F L_0 < L_x < L_0$. Since the vertex x lies in $\mathscr{A}$ then for some $a = (a_1,\ldots,a_n) \in \mathbb{Z}^n$ we can write $L_x = \mathfrak{p}_F^{a_1} e_1 + \cdots + \mathfrak{p}_F^{a_n} e_n$. As in the proof of Lemma 7, the coordinates $a_i \in \{0,1\}$ and not all the a_i's are zero or one. Moreover, if $A_0 = \{i \in \Delta_n \mid a_i = 0\}$ and $A_1 = \{i \in \Delta_n \mid a_i = 1\}$, then clearly $A_0 \sqcup A_1 = \Delta_n$. So we have

$$L_x = \bigoplus_{i\in A_0} \mathfrak{o}_F \oplus \bigoplus_{i\in A_1} \mathfrak{p}_F$$

and then

$$V_x = L_x/\varpi_F L_0 \simeq \bigoplus_{i\in A_0} \mathfrak{o}_F/\mathfrak{p}_F \oplus \bigoplus_{i\in A_1} \mathfrak{p}_F/\mathfrak{p}_F \simeq k_F^{|A_0|}.$$

Consequently $\dim(V_x) = |A_0|$. We have

$$\begin{aligned}
d(s_0,x) &= \sqrt{\frac{n}{n-1}}\, d_0\Big(0, a - \frac{1}{n}\sigma(a)e\Big) = \sqrt{\frac{n}{n-1}}\left\| a - \frac{\sigma(a)}{n}e\right\| \\
&= \sqrt{\frac{n}{n-1}}\left(\sum_{i=1}^n \left(a_i - \frac{\sigma(a)}{n}\right)^2\right)^{\frac{1}{2}} = \sqrt{\frac{n}{n-1}}\left(\sum_{i=1}^n a_i^2 - \frac{2\sigma(a)}{n}a_i + \frac{\sigma(a)^2}{n^2}\right)^{\frac{1}{2}} \\
&= \sqrt{\frac{n}{n-1}}\left(\sum_{i=1}^n a_i^2 - \frac{2\sigma(a)^2}{n} + \frac{\sigma(a)^2}{n}\right)^{\frac{1}{2}}
\end{aligned}$$

But as $a_i \in \{0,1\}$ for every $i \in \Delta_n$, then

$$d(s_0,x) = \sqrt{\frac{n}{n-1}}\left(\sigma(a) - \frac{\sigma(a)^2}{n}\right)^{\frac{1}{2}}.$$

On the other hand

$$\sigma(a) = \sum_{i=1}^n a_i = \sum_{i\in A_1} 1 = |A_1| = n - \dim V_x.$$

So we get

$$d(s_0,x) = \sqrt{\frac{n}{n-1}}\left(n - \dim V_x - \frac{(n-\dim V_x)^2}{n}\right)^{\frac{1}{2}},$$

and then

$$d(s_0,x) = \frac{1}{\sqrt{n-1}}\Big(n\dim V_x - (\dim V_x)^2\Big)^{\frac{1}{2}}.$$

(ii). The proof of the second formula is obtained by a similar way. □

If x and y are two vertices of $\mathscr{BT}_n$ we write $[x,y]^0$ for the combinatorial segment between x and y. That is $[x,y]^0$ is the set of vertices z of $\mathscr{BT}_n$ such that $d(x,z)+d(z,y)=d(x,y)$.

Corollary 14. *If $x,y\in\mathscr{V}(s_0)$, then $s_0\in[x,y]^0$ if and only if $V_x\oplus V_y=\widetilde{V}_{s_0}$.*

Proof. Follows from the previous proposition by an easy computation. □

If $\alpha=(\alpha_0,\ldots,\alpha_k)$ is a k-path of $\mathscr{BT}_n$ (where $k\geqslant 1$), the initial (resp. terminal) vertex of α, that is α_0 (resp. α_k), will be denoted by $s^-(\alpha)$ (resp. $s^+(\alpha)$). If α and β are respectively a k-path and an ℓ-path with $s^+(\alpha)=s^-(\beta)$, then their concatenation $\alpha\beta$ is the $(k+\ell)$-path of $\mathscr{BT}_n$ defined by

$$\alpha\beta:=(\alpha_0,\ldots,\alpha_k,\beta_1,\ldots,\beta_\ell).$$

It is not true in general that the concatenation of two geodesic paths of $\mathscr{BT}_n$ is a geodesic path. But we have the following result :

Lemma 15. *Let $\alpha=(\alpha_0,\ldots,\alpha_k)$ and $\beta=(\beta_0,\ldots,\beta_\ell)$ two geodesic paths of $\mathscr{BT}_n$ of length k and ℓ respectively and with $s^+(\alpha)=s^-(\beta)$. Then $\alpha\beta$ is a geodesic $(k+\ell)$-path if and only if $\beta_1\in\mathscr{P}^+(e^+(\alpha))$ (resp. $\alpha_{k-1}\in\mathscr{P}^-(e^-(\beta))$).*

Proof. If $\alpha\beta$ is geodesic then it is clear that $\beta_1\in\mathscr{P}^+(e^+(\alpha))$ (resp. $\alpha_{k-1}\in\mathscr{P}^-(e^-(\beta))$). For the converse, we will prove by induction on $\ell\geqslant 1$ that for every geodesic path $\beta=(\beta_0,\ldots,\beta_\ell)$ of length ℓ such that $s^+(\alpha)=s^-(\beta)$, if $\beta_1\in\mathscr{P}^+(e^+(\alpha))$ (resp. $\alpha_{k-1}\in\mathscr{P}^-(e^-(\beta))$) then the $(k+\ell)$-path $\alpha\beta$ is geodesic. For $\ell=1$ the property follows from Corollary 10. Assume that the property is true for the order ℓ. Let $\beta=(\beta_0,\ldots,\beta_{\ell+1})$ be a geodesic $(\ell+1)$-path of $\mathscr{BT}_n$ such that $s^+(\alpha)=s^-(\beta)$ and with $\beta_1\in\mathscr{P}^+(e^+(\alpha))$ (in the case when $\alpha_{k-1}\in\mathscr{P}^-(e^-(\beta))$ the proof is similar). From the induction hypothesis, the $(k+\ell)$-path $\alpha\beta^-$, that is the path $(\alpha_0,\ldots,\alpha_k,\beta_1,\ldots,\beta_\ell)$, is geodesic. Since moreover the vertex $\beta_{\ell+1}$ is a right prolongation of the directed edge $e^+(\alpha\beta^-)$ then by Corollary 10 the path

$$\alpha\beta=(\alpha_0,\ldots,\alpha_k,\beta_1,\ldots,\beta_\ell,\beta_{\ell+1})$$

is also geodesic. □

Corollary 16. *Let $\alpha\in\mathscr{C}_k(\mathscr{BT}_n)$ and $\beta\in\mathscr{C}_\ell(\mathscr{BT}_n)$, where $k,\ell\geqslant 1$. If α is joined to β by a nontrivial geodesic path, that is there exists an integer $0<m\leqslant\min(k,\ell)$ such that*

$$\alpha_i=\beta_{i-k+m},\ \textit{for every } i\in\{k-m,\ldots,k\},$$

then the sequence $\alpha\cup\beta:=(\alpha_0,\ldots,\alpha_k,\beta_{m+1},\ldots,\beta_\ell)$ is a geodesic path. In particular if $\alpha,\beta\in\mathscr{C}_{k+1}(\mathscr{BT}_n)$ such that $\alpha^+=\beta^-$ (resp. $\alpha^-=\beta^+$) then $\alpha\cup\beta$ is a geodesic $(k+2)$-path.

Proof. The case when $m=\min(k,\ell)$ is obvious since in this case α is a subpath of β or β is a subpath of α. Assume then that $m<\min(k,\ell)$. Since $\widetilde{\alpha}=(\alpha_0,\ldots,\alpha_{k-m})$ is a subpath of α then $\widetilde{\alpha}$ is geodesic. Moreover it is clear that $s^+(\widetilde{\alpha})=s^-(\beta)$ (since from the hypothesis $\alpha_{k-m}=\beta_0$). So the concatenation $\widetilde{\alpha}\beta$ is a path of $\mathscr{BT}_n$. But $\widetilde{\alpha}\beta$ is nothing other than $\alpha\cup\beta$. The vertex β_1 is clearly a right prolongation of the directed edge $e^+(\widetilde{\alpha})$ as $\beta_1=\alpha_{k-m+1}$. So by the previous lemma $\alpha\cup\beta$ is geodesic. □

4. The projective tower of graphs over $\mathscr{BT}_n^{(1)}$

In this section, our purpose is to give the construction of the tower of directed graphs lying equivariantly over the 1-skeleton of the building $\mathscr{BT}_n$ and to give some basic properties of these tower of directed graphs. We note that our construction generalizes the construction of Broussous given in [5] for the case $n=2$. In the sequel, we will be interested then by the case $n\geqslant 3$.

4.1. *The construction*

For every integer $k \geqslant 0$, we define the graph $\widetilde{X}_k$ as the directed graph whose vertex (resp. edges) set is the set $\mathscr{C}_k^+(\mathscr{BT}_n)$ (resp. $\mathscr{C}_{k+1}^+(\mathscr{BT}_n)$). The structure of directed graph of $\widetilde{X}_k$ is given by :

$$a^- = (\alpha_0, \dots, \alpha_k),\ a^+ = (\alpha_1, \dots, \alpha_{k+1}),\ \text{if}\ a = (\alpha_0, \dots, \alpha_{k+1})\ .$$

Let's notice firstly that the graph $\widetilde{X}_0$ is nothing other than the directed graph whose vertices are those of $\mathscr{BT}_n$ and for which the edges set is $\mathscr{C}_1^+(\mathscr{BT}_n)$. The action of G_n on the sets $\mathscr{C}_k^+(\mathscr{BT}_n)$ induce an action on the graph $\widetilde{X}_k$ by automorphisms of directed graphs. Moreover, since the stabilizers of the vertices of $\widetilde{X}_k$ are open and compact then the action is proper. From the previous section, the action of G_n on the graph $\widetilde{X}_k$ is transitive on vertices and edges. For every vertex s (resp. edge a) of $\widetilde{X}_k$, we write Γ_s (resp. Γ_a) for the stabilizer in G_n of s (resp. a). The stabilizer in G_n of the standard vertex (resp. edge) of $\widetilde{X}_k$, that is the standard geodesic k-path (resp. $(k+1)$-path) given in (4), is the subgroup $\Gamma_0(\mathfrak{p}_F^k)$ (resp. $\Gamma_0(\mathfrak{p}_F^{k+1})$).

Proposition 17. *For every vertex s of $\widetilde{X}_k$ the stabilizer Γ_s acts transitively on the two sets of neighborhoods :*

$$\mathscr{V}^-(s) = \left\{a \in \widetilde{X}_k^1 \,\middle|\, a^- = s\right\} \quad \textit{and} \quad \mathscr{V}^+(s) = \left\{a \in \widetilde{X}_k^1 \,\middle|\, a^+ = s\right\}$$

Proof. Follows immediately from Lemma 9. □

Recall that the 1-skeleton of the building $\mathscr{BT}_n$, denoted by $\mathscr{BT}_n^{(1)}$, is the subcomplex of $\mathscr{BT}_n$ formed by the faces of dimension at most one. When $k = 2m$ is even, there is a natural simplicial projection $p_k : \widetilde{X}_k \longrightarrow \mathscr{BT}_n^{(1)}$ defined on vertices by

$$p_k(s_{-m}, \dots, s_0, \dots, s_m) = s_0.$$

Similarly, When $k = 2m+1$ is odd, there is a natural simplicial projection $p_k : \widetilde{X}_k^{sd} \longrightarrow \widetilde{\mathscr{BT}}{}_n^{(1)}$, where $\widetilde{X}_k^{sd}$ and $\widetilde{\mathscr{BT}}{}_n^{(1)}$ are respectively the barycentric subdivision of the graphs $\widetilde{X}_k$ and $\mathscr{BT}_n^{(1)}$. The family of graphs $(\widetilde{X}_k)_{k\geqslant 0}$ constitute a tower of graphs over the graph $\mathscr{BT}_n^{(1)}$ in the sense that we have the following diagram of simplicial maps

$$\cdots \longrightarrow \widetilde{X}_{k+1} \xrightarrow{\varphi_k^\varepsilon} \widetilde{X}_k \longrightarrow \cdots \longrightarrow \widetilde{X}_0 \xrightarrow{p_0} \mathscr{BT}_n^{(1)}$$

where for $\varepsilon = \pm$ and for $k \geqslant 0$, the map $\varphi_k^\varepsilon : \widetilde{X}_{k+1} \longrightarrow \widetilde{X}_k$ is the simplicial map defined on vertices by $\varphi_k^\varepsilon(s) = s^\varepsilon$.

4.2. *Connectivity of the graphs*

The aim of this section is the study of the connectivity of the graphs $\widetilde{X}_k$. We begin by defining a cover of $\widetilde{X}_{k+1}$ by finite subgraphs whose nerve is a graph isomorphic to $\widetilde{X}_k$. Assume that $k \geqslant 0$ is an integer. For every vertex s of $\widetilde{X}_k$ we define the subgraph $\widetilde{X}_{k+1}(s)$ of the graph $\widetilde{X}_{k+1}$ as the subgraph whose edges are the geodesic $(k+2)$-paths $\alpha \in \mathscr{C}_{k+2}^+(\mathscr{BT}_n)$ of the form $\alpha = (x, s_0, \dots, s_k, y)$, where x (resp. y) is a left (resp. right) prolongation of the path s. The vertices of $\widetilde{X}_{k+1}(s)$ are exactly those $v \in \widetilde{X}_{k+1}^0$ such that $v^- = s$ or $v^+ = s$. Obviously the subgraphs $\widetilde{X}_{k+1}(s)$, when s range over the set of vertices of $\widetilde{X}_k$, form a cover the graph $\widetilde{X}_{k+1}$. That is

$$\widetilde{X}_{k+1} = \bigcup_{s \in \widetilde{X}_k^0} \widetilde{X}_{k+1}(s). \tag{5}$$

For every vertex s_0 of $\widetilde{X}_0$ (considered as a vertex of $\mathscr{BT}_n$) the subgraph $\widetilde{X}_1(s_0)$ of $\widetilde{X}_1$ has two types of vertices : the directed edges $(x, s_0) \in \mathscr{C}_1^+(\mathscr{BT}_n)$ and the directed edges $(s_0, y) \in \mathscr{C}_1^+(\mathscr{BT}_n)$. Let us denote the k_F-vector space k_F^n by $\overline{V}$. The Lemma 7 implies that the vertex set of $\widetilde{X}_1(s_0)$ may be identified with the set $\mathbb{P}^1(\overline{V}) \sqcup \mathbb{P}^1(\overline{V}^*)$, where $\mathbb{P}^1(\overline{V})$ is the set of one dimensional subspaces and

$\mathbb{P}^1(\overline{V}^*)$ is the set of one codimensional subspaces of $\overline{V}$. By the Corollary 14 we deduce that the graph $\widetilde{X}_1(s_0)$ is isomorphic to the graph $\Delta(\overline{V})$ whose vertex set is $\mathbb{P}^1(\overline{V}) \sqcup \mathbb{P}^1(\overline{V}^*)$ and in which a vertex $D \in \mathbb{P}^1(\overline{V})$ is linked to a vertex $H \in \mathbb{P}^1(\overline{V}^*)$ if and only if $D \oplus H = \overline{V}$ and there is no edges between two distinct vertices of $\mathbb{P}^1(\overline{V})$ (resp. $\mathbb{P}^1(\overline{V}^*)$). One can prove easily that $\Delta(\overline{V})$ is a connected bipartite graph so that $\widetilde{X}_1(s_0)$ is connected and bipartite for every vertex s_0 of $\widetilde{X}_0$.

Lemma 18. *Let $k \geqslant 1$ be an integer. Then we have :*

(i) *For every $s \in \widetilde{X}_k^0$, the graph $\widetilde{X}_{k+1}(s)$ is a complete bipartite graph and hence connected,*
(ii) *The nerve $\mathcal{N}(\widetilde{X}_{k+1})$ of the cover of $\widetilde{X}_{k+1}$ given in* (5) *is isomorphic to the graph $\widetilde{X}_k$.*

Proof. (i). Let $s \in \widetilde{X}_k^0$. The set of vertices of $\widetilde{X}_{k+1}(s)$ is clearly partitioned into two subsets. The set $\mathcal{U}$ of vertices $v \in \widetilde{X}_{k+1}^0$ such that $v^- = s$ and the set $\mathcal{V}$ of vertices $v \in \widetilde{X}_{k+1}^0$ such that $v^+ = s$. By Corollary 16 we deduce that every vertex in $\mathcal{U}$ is linked to every vertex in $\mathcal{V}$. So as desired the graph $\widetilde{X}_{k+1}(s)$ is a complete bipartite graph and then connected.

(ii). Let s and t two distinct vertices of $\widetilde{X}_k$. If s and t are linked by an edge then by Corollary 16 the two subgraphs $\widetilde{X}_{k+1}(s)$ and $\widetilde{X}_{k+1}(t)$ have at least a common vertex, namely the vertex $s \cup t$. Conversely, if the two subgraphs $\widetilde{X}_{k+1}(s)$ and $\widetilde{X}_{k+1}(t)$ have at least a common vertex, say v, then we have $v^- = s$ or $v^+ = s$ and $v^- = t$ or $v^+ = t$. As s and t are distinct then we deduce that $v^- = s$ and $v^+ = t$ or $v^- = t$ and $v^+ = s$. The Corollary 16 implies then that s and t are linked by an edge. So the nerve of the cover of $\widetilde{X}_{k+1}$ by the subgraphs $\widetilde{X}_{k+1}(s)$, for $s \in \widetilde{X}_k^0$, is the graph $\widetilde{X}_k$. □

Theorem 19. *For every integer $k \geqslant 0$, the geometric realization of $\widetilde{X}_k$ is connected and locally compact.*

Proof. The locally compactness of $|\widetilde{X}_k|$ follows from the fact that the graphs $\widetilde{X}_k$ are locally finite. For the connectedness, we will prove firstly that $\widetilde{X}_0$ is connected. Let $s = [L]$ and $t = [M]$ be two distinct vertices of $\widetilde{X}_0$, where L and M are two $\mathfrak{o}_F$-lattices. Let us choose an F-basis $(v_1, \dots, v_n)$ of F^n for which $L = \mathfrak{o}_F v_1 + \cdots + \mathfrak{o}_F v_n$ and $M = \mathfrak{p}_F^{k_1} v_1 + \cdots + \mathfrak{p}_F^{k_n} v_n$, where $(k_1, \dots, k_n) \in \mathbb{Z}^n$ with $k_1 \leqslant \cdots \leqslant k_n$. By changing the representative $M \in [M]$ we can assume that $0 < k_1$. Now let us consider the sequence $(L_0, \dots, L_m)$ of $\mathfrak{o}_F$-lattices, where $m = k_1 + \cdots + k_n$, defined as follows. For every integer $0 \leqslant i \leqslant m$, if $k_1 + \cdots + k_{j-1} + 1 \leqslant i \leqslant k_1 + \cdots + k_j$, where $1 \leqslant j \leqslant n$, then

$$L_i = \bigoplus_{\ell=1}^{j-1} \mathfrak{p}_F^{k_\ell} v_\ell \oplus \mathfrak{p}_F^{i-(k_1+\cdots+k_{j-1})} v_j \oplus \bigoplus_{\ell=j+1}^{n} \mathfrak{o}_F v_\ell.$$

By a straightforward computation, we can check easily that the sequence $([L_0], \dots, [L_m])$ is a path of the graph $\widetilde{X}_0$ linking the vertex s to the vertex t. So as desired $\widetilde{X}_0$ is connected. Now we will prove by induction that the graphs $\widetilde{X}_k$ are connected for every non-negative integer k. Let $k \geqslant 0$ be an integer. Assume that the graph $\widetilde{X}_k$ is connected and let's prove that $\widetilde{X}_{k+1}$ is also connected. Let u and v be two distinct vertices of $\widetilde{X}_{k+1}$. Since $\widetilde{X}_{k+1}$ is covered by the subgraph $\widetilde{X}_{k+1}(s)$, when s range over the set of vertices of $\widetilde{X}_k$, then there exist two vertices $s, t \in \widetilde{X}_k^0$ such that $u \in \widetilde{X}_{k+1}^0(s)$ and $v \in \widetilde{X}_{k+1}^0(t)$. As $\widetilde{X}_k$ is connected then there exist a path $p = (p_0, \dots, p_m)$ in $\widetilde{X}_k$ linking the two vertices s and t (say $p_0 = s$ and $p_m = t$). For every integer $i \in \{1, \dots, m\}$, let v_i be any vertex of the non-empty graph $\widetilde{X}_{k+1}(p_{i-1}) \cap \widetilde{X}_{k+1}(p_i)$. Let's also put $v_0 = u$ and $v_{\ell+1} = v$. By the previous lemma the graphs $\widetilde{X}_{k+1}(p_i)$ are connected. So for $i \in \{0, \dots, \ell\}$, since p_i and p_{i+1} are two vertices of the graph $\widetilde{X}_{k+1}(p_i)$ then there exist a path in $\widetilde{X}_{k+1}$ from p_i to p_{i+1}. Consequently there exist a path in $\widetilde{X}_{k+1}$ connecting the two vertices u and v and then the graph $\widetilde{X}_{k+1}$ is connected. We have then the connectedness of the graphs $\widetilde{X}_k$ for every integer $k \geqslant 0$ which implies the connectedness of their geometric realization. □

5. Realization of the generic representations of G_n in the cohomology of the tower of graphs

5.1. *Generic representations of* G_n

Let us firstly recall some basic facts and introduce some notations. Let ψ be a fixed additive smooth character of F trivial on $\mathfrak{p}_\mathrm{F}$ and nontrivial on $\mathfrak{o}_\mathrm{F}$. We define a character θ_ψ of the group U_n of upper unipotent matrices as follows

$$\theta_\psi\left(\begin{pmatrix} 1 & u_{1,2} & \dots & \dots & u_{1,n} \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & u_{n-1,n} \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}\right) = \psi(u_{1,2}+\dots+u_{n-1,n}).$$

Let (π,V) be an irreducible admissible representation of G_n considered as an irreducible admissible representation of $\mathrm{GL}_n(\mathrm{F})$ with trivial central character. The representation (π,V) is called generic if

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathrm{F})}(\pi, \mathrm{Ind}_{\mathrm{U}_n}^{\mathrm{GL}_n(\mathrm{F})}\theta_\psi) \neq 0.$$

By Frobenius reciprocity, this is equivalent to the existence of a nonzero linear form $\ell : \mathrm{V} \longrightarrow \mathbb{C}$ such that $\ell(\pi(u).v) = \theta_\psi(u)\ell(v)$ for every $v \in \mathrm{V}$ and $u \in \mathrm{U}_n$. Thus a generic representation (π, V) of G_n can be realized on a same space of functions f with the property $f(ug) = \theta_\psi(u)f(g)$ for every $u \in \mathrm{U}_n$ and $g \in \mathrm{GL}_n(\mathrm{F})$ and for which the action of $\mathrm{GL}_n(\mathrm{F})$ on the space of π is by right translation. Such a realization is called the Whittaker model of π. The following theorem, due to Bernstein and Zelevinski, shows that generic representations have a unique Whittaker model.

Theorem 20 ([2, V.16]). *Let (π, V) be an irreducible admissible representation of G_n. Then the dimension of the space $\mathrm{Hom}_{\mathrm{GL}_n(\mathrm{F})}(\pi, \mathrm{Ind}_{\mathrm{U}_n}^{\mathrm{GL}_n(\mathrm{F})}\theta_\psi)$ is at most one, that is*

$$\dim_\mathbb{C} \mathrm{Hom}_{\mathrm{GL}_n(\mathrm{F})}\left(\pi, \mathrm{Ind}_{\mathrm{U}_n}^{\mathrm{GL}_n(\mathrm{F})}\theta_\psi\right) \leqslant 1.$$

In particular, if π is generic then π has a unique Whittaker model.

We have the following result which is due to H. Jacquet, J. L. Piatetski-Shapiro and J. Shalika, see [8, Thm. (5.1)]:

Theorem 21. *Let (π,V) be an irreducible generic representation of G_n.*

- (i) *For k large enough, the space of fixed vectors $V^{\Gamma_0(\mathfrak{p}_\mathrm{F}^k)}$ is non-zero.*
- (ii) *Let $c(\pi)$ the smallest integer such that $V^{\Gamma_0(\mathfrak{p}_\mathrm{F}^{c(\pi)+1})} \neq 0$, then for every integer $k \geqslant c(\pi)$, we have :*

$$\dim_\mathbb{C} V^{\Gamma_0(\mathfrak{p}_\mathrm{F}^{k+1})} = k - c(\pi) + 1.$$

5.2. *Realization of the Generic representations of* G_n

In this section, we fix an irreducible generic representation (π, V) of G_n and we make the following assumption:

Assumption 22. π is non-spherical, that is the space of $\Gamma_0(\mathfrak{p}_\mathrm{F}^0)$-fixed vectors

$$V^{\Gamma_0(\mathfrak{p}_\mathrm{F}^0)} := \left\{v \in V \,\middle|\, \forall\, g \in \Gamma_0(\mathfrak{p}_\mathrm{F}^0),\ \pi(g)v = v\right\}$$

is zero.

In the following, our aim is to prove that the representation π can be realized as a quotient of the cohomology space $H_c^1(\widetilde{\mathrm{X}}_{c(\pi)}, \mathbb{C})$ and if moreover π is cuspidal then in fact it can be realized as a subrepresentation of this cohomology space. Furthermore, as in Theorem (5.3.2) of [5], we obtain a multiplicity one result for cuspidals but in a more simpler way. The proofs of the results below are similar to those given in [5, §(3.2)]. Let us recall that for every vertex s (resp. edge a) of $\widetilde{\mathrm{X}}_{c(\pi)}$, Γ_s (resp. Γ_a) denotes the stabilizer in G_n of s (resp. a). We recall that

$$\Gamma_{s_0} = \Gamma_0(\mathfrak{p}_{\mathrm{F}}^{c(\pi)}) \quad \text{and} \quad \Gamma_{a_0} = \Gamma_0(\mathfrak{p}_{\mathrm{F}}^{c(\pi)+1}),$$

where s_0 (resp. a_0) is the standard vertex (resp. edge) of $\widetilde{\mathrm{X}}_{c(\pi)}$.

Lemma 23.

(i) *For every edge a of $\widetilde{\mathrm{X}}_{c(\pi)}$, V^{Γ_a} is of dimension one.*

(ii) *Let a be an edge of $\widetilde{\mathrm{X}}_{c(\pi)}$ and s be a vertex of a. Then for every $v \in V^{\Gamma_a}$ we have*

$$\sum_{g \in \Gamma_s/\Gamma_a} \pi(g)v = 0.$$

Proof. (i). Since G_n acts transitively on the set of edges of $\widetilde{\mathrm{X}}_{c(\pi)}$ then the subgroup Γ_a is conjugate to Γ_{a_0} which gives the result.

(ii). Clearly the vector

$$v_0 := \sum_{g \in \Gamma_s/\Gamma_a} \pi(g)v$$

is fixed by Γ_s. But by transitivity of the action of G_n on the set of vertices of $\widetilde{\mathrm{X}}_{c(\pi)}$, the subgroup Γ_s is conjugate to Γ_{s_0}. So Theorem 21 implies that $v_0 = 0$. □

We define a map

$$\Psi_\pi^\vee : V^\vee \longrightarrow C^1\left(\widetilde{\mathrm{X}}_{c(\pi)}, \mathbb{C}\right)$$

as follows. Let us fix a non-zero vector $v_0 \in V^{\Gamma_{a_0}}$. For every edge a of $\widetilde{\mathrm{X}}_{c(\pi)}$, we put

$$v_a = \pi(g).v_0, \quad \text{where } a = g.a_0 \tag{6}$$

This definition is well defined since G_n acts transitively on $\widetilde{\mathrm{X}}^1_{c(\pi)}$ and it does not depend on the choice of $g \in \mathrm{G}_n$ such that $v_a = g.v_0$ as v_0 is fixed by Γ_{a_0}. The map $\Psi^\vee$ is then defined by

$$\Psi^\vee(\varphi)(a) = \varphi(v_a)$$

for every $\varphi \in V^\vee$ and $a \in \widetilde{\mathrm{X}}^1_{c(\pi)}$. From (6) the map $\Psi^\vee$ is G_n-equivariant.

Lemma 24. *The map $\Psi^\vee$ is injective and its image is contained in $\mathscr{H}_\infty(\widetilde{\mathrm{X}}_{c(\pi)}, \mathbb{C})$.*

Proof. The G_n-equivariant map $\Psi^\vee$ is injective as it is nonzero and as the representation π is irreducible. Let $\varphi \in V^\vee$. Let us prove that for every vertex s of $\widetilde{\mathrm{X}}^1_{c(\pi)}$,

$$\sum_{a \in \widetilde{\mathrm{X}}^1_{c(\pi)}} [a:s]\varphi(v_a) = 0.$$

Let s be a vertex of $\widetilde{\mathrm{X}}^1_{c(\pi)}$. By Proposition 17, the stabilizer Γ_s acts transitively on the two sets

$$\mathcal{V}^-(s) = \left\{a \in \widetilde{\mathrm{X}}^1_{c(\pi)} \,\middle|\, a^- = s\right\} \quad \text{and} \quad \mathcal{V}^+(s) = \left\{a \in \widetilde{\mathrm{X}}^1_{c(\pi)} \,\middle|\, a^+ = s\right\}.$$

Let us fix $a_s^+ \in \mathcal{V}^+(s)$ and $a_s^- \in \mathcal{V}^-(s)$. We have then

$$\begin{aligned}\sum_{a \in \widetilde{\mathrm{X}}^1_{c(\pi)}} [a:s]\varphi(v_a) &= \varphi\left(\sum_{a \in \mathcal{V}^+(s)} v_a - \sum_{a \in \mathcal{V}^-(s)} v_a\right) \\ &= \varphi\left(\sum_{g \in \Gamma_s/\Gamma_{a_s^+}} \pi(g).v_{a_s^+} - \sum_{g \in \Gamma_s/\Gamma_{a_s^-}} \pi(g).v_{a_s^-}\right) = 0\end{aligned}$$

by Lemma 23. Consequently, $\mathrm{Im}(\Psi^\vee)$ is contained in $\mathscr{H}(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$ which implies that it is contained in $\mathscr{H}_\infty(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$. □

By Lemma 1 we have the isomorphism of smooth G_n-module

$$\mathscr{H}_\infty\left(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C}\right) \simeq H^1_c\left(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C}\right)^\vee.$$

So applying contragredients to the operator $\Psi^\vee_\pi : V^\vee \longrightarrow \mathscr{H}_\infty(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$ we obtain an intertwining operator

$$\Psi^{\vee\vee}_\pi : H^1_c\left(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C}\right)^{\vee\vee} \longrightarrow V^{\vee\vee}.$$

It is well known that a smooth G_n-module W have a canonical injection in the contragredient of its contragredient $W^{\vee\vee}$. So the smooth G_n-module $H^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$ canonically injects in $H^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})^{\vee\vee}$. Moreover the representation π is irreducible and hence admissible then V and $V^{\vee\vee}$ are canonically isomorphic. In the following, if $\omega \in C^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$ we write $\overline{\omega}$ for its image in $H^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$.

Theorem 25. *The restriction of $\Psi^{\vee\vee}_\pi$ to the space $H^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$ define a nonzero intertwining operator*

$$\Psi_\pi : H^1_c\left(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C}\right) \longrightarrow V$$

given by

$$\Psi_\pi(\overline{\omega}) = \sum_{a\in\tilde{\mathrm{X}}^1_{c(\pi)}} \omega(a)\, v_a$$

In particular, (π,V) is isomorphic to a quotient of $H^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$. Moreover, if (π,V) is cuspidal then it is isomorphic to a subrepresentation of $H^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$.

Proof. The fact that the restriction of the map $\Psi^{\vee\vee}_\pi$ to the space $H^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$ is given exactly by the map Ψ_π follows by a straightforward computation. Let $\omega_0 \in C^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$ defined on the basis $\widetilde{\mathrm{X}}^1_{c(\pi)}$ of $C_1(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$ as follows : for every edge a of $\widetilde{\mathrm{X}}_{c(\pi)}$, $\omega_0(a) = 1$ if $a = a_0$ and $\omega_0(a) = 0$ otherwise. We have

$$\Psi_\pi(\overline{\omega}_0) = \sum_{a\in\tilde{\mathrm{X}}^1_{c(\pi)}} \omega_0(a)\, v_a = v_0 \neq 0.$$

So the map Ψ_π is nonzero. Hence by irreducibility of π the map Ψ_π is surjective and then as desired (π,V) is isomorphic to a quotient of $H^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$. If the representation (π,V) is cuspidal, so in particular generic, then it is isomorphic to a quotient of $H^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$. But (π,V) is cuspidal and then it is projective in the category of smooth complex representation of G_n. So we have in fact an embedding of (π,V) in $H^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$. □

Theorem 26. *If the representation (π,V) is cuspidal then it have a unique realization in the cohomology space $H^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$, that is*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{G}_n}\left(\pi, H^1_c\left(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C}\right)\right) = 1.$$

Proof. Since G_n acts transitively on the set of vertices and edges of $\widetilde{\mathrm{X}}_{c(\pi)}$ then the two G_n-modules $\mathrm{C}^0_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$ and $\mathrm{C}^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$ are respectively isomorphic to the following compactly induced representation

$$\text{c-ind}^{\mathrm{G}_n}_{\Gamma_0(\mathfrak{p}_{\mathrm{F}}^{c(\pi)})} 1 \quad \text{and} \quad \text{c-ind}^{\mathrm{G}_n}_{\Gamma_0(\mathfrak{p}_{\mathrm{F}}^{c(\pi)+1})} 1$$

(where 1 denotes the trivial character). The space $H^1_c(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$ is by definition the cokernel of the coboundary map

$$C^0_c\left(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C}\right) \xrightarrow{\ d\ } C^1_c\left(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C}\right)$$

Then we have a surjective map

$$\varphi : \text{c-ind}^{\mathrm{G}_n}_{\Gamma_0(\mathfrak{p}_{\mathrm{F}}^{c(\pi)+1})} 1 \longrightarrow \mathrm{H}^1_c\left(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C}\right)$$

and so we obtain an injective map

$$\widetilde{\varphi}: \mathrm{Hom}_{\mathrm{G}_n}\left(H_c^1\left(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C}\right),\pi\right) \longrightarrow \mathrm{Hom}_{\mathrm{G}_n}\left(\text{c-ind}_{\Gamma_0(\mathfrak{p}_\mathrm{F}^{c(\pi)+1})}^{\mathrm{G}_n} 1,\pi\right)$$

On the other hand, by Frobenius reciprocity we have

$$\mathrm{Hom}_{\mathrm{G}_n}\left(\text{c-ind}_{\Gamma_0(\mathfrak{p}_\mathrm{F}^{c(\pi)+1})} 1,\pi\right) \simeq V^{\Gamma_n(\mathfrak{p}_\mathrm{F}^{c(\pi)+1})}$$

But by the Theorem 21, the space of fixed vectors $V^{\Gamma_n(\mathfrak{p}_\mathrm{F}^{c(\pi)+1})}$ is of dimension one. Thus we obtain

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{G}_n}\left(H_c^1\left(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C}\right),\pi\right) \leqslant 1.$$

On the other hand, since the representation (π, V) is cuspidal then it is a projective object of the category of smooth representations of G_n. So the two spaces $\mathrm{Hom}_{\mathrm{G}_n}(H_c^1(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C}),\pi)$ and $\mathrm{Hom}_{\mathrm{G}_n}(\pi, H_c^1(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C})$ are in fact isomorphic. But by the previous theorem $\mathrm{Hom}_{\mathrm{G}_n}(H_c^1(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C}),\pi)$ is nonzero. So as desired the space $\mathrm{Hom}_{\mathrm{G}_n}(\pi, H_c^1(\widetilde{\mathrm{X}}_{c(\pi)},\mathbb{C}))$ is one dimensional. □

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