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Analysis of non scalar control problems for parabolic systems by the block moment method

Analyse de problèmes de contrôle non scalaires pour des systèmes paraboliques par la méthode des moments par blocs

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Abstract. This article deals with abstract linear time invariant controlled systems of parabolic type. In [9], with A. Benabdallah, we introduced the block moment method for scalar control operators. The principal aim of this method is to compute the minimal time needed to drive an initial condition (or a space of initial conditions) to zero, in particular in the case when spectral condensation occurs. The purpose of the present article is to push forward the analysis to deal with any admissible control operator. The considered setting leads to applications to one dimensional parabolic-type equations or coupled systems of such equations.

With such admissible control operator, the characterization of the minimal null control time is obtained thanks to the resolution of an auxiliary vectorial block moment problem (*i.e.* set in the control space) followed by a constrained optimization procedure of the cost of this resolution. This leads to essentially sharp estimates on the resolution of the block moment problems which are uniform with respect to the spectrum of the evolution operator in a certain class. This uniformity allows the study of uniform controllability for various parameter dependent problems. We also deduce estimates on the cost of controllability when the final time goes to the minimal null control time.

We illustrate how the method works on a few examples of such abstract controlled systems and then we deal with actual coupled systems of one dimensional parabolic partial differential equations. Our strategy enables us to tackle controllability issues that seem out of reach by existing techniques.

Résumé. On étudie dans cet article des systèmes de contrôle paraboliques autonomes linéaires abstraits. Dans [9], avec A. Benabdallah, nous avons introduit la méthode des moments par blocs dans le cas d'un opérateur de contrôle scalaire. Le but principal de cette méthode est de permettre de calculer le temps minimal nécessaire pour amener à zéro une donnée initiale fixée (ou un espace de données initiales), en particulier dans le cas où des phénomènes de condensation spectrale sont présents. Le but du présent travail est d'approfondir cette analyse pour prendre en compte n'importe quel opérateur de contrôle admissible. Le cadre proposé permet des applications à des équations ou systèmes paraboliques couplés en dimension un d'espace.

Pour de tels opérateurs de contrôle admissibles, la caractérisation du temps minimal de contrôle est obtenu à l'aide de la résolution de problèmes de moment vectoriels auxiliaires suivie d'une procédure d'optimisation sous contrainte du coût de cette résolution. Cela amène à des estimations essentiellement optimales pour la résolution de ces problèmes de moment par bloc qui, de surcroît, sont uniformes par rapport au spectre de l'opérateur d'évolution à l'intérieur d'une certaine classe. Ce caractère uniforme permet de prouver la contrôlabilité uniforme de divers systèmes dépendant de paramètres. Nous déduisons également des estimations du coût de contrôlabilité quand le temps de contrôle est proche du temps minimal.

Nous illustrons le fonctionnement de cette méthode sur quelques exemples de tels systèmes abstraits mais également sur des exemples plus concrets de systèmes d'équations aux dérivées partielles paraboliques contrôlés en dimension 1. Notre stratégie permet d'étudier des propriétés de contrôlabilité qui semblent hors de portée par les méthodes existantes de la littérature.

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1. Introduction

1.1. *Problem under study and state of the art*

In this paper we study the controllability properties of the following linear control system

$$\begin{cases} y'(t) + \mathscr{A} y(t) = \mathscr{B} u(t), \\ y(0) = y_0. \end{cases} \tag{1}$$

The assumptions on the operator $\mathscr{A}$ (see Section 2.1) will lead to applications to linear parabolic-type equations or coupled systems of such equations mostly in the one dimensional setting. In all this article the Hilbert space of control will be denoted by U and the operator $\mathscr{B}$ will be a general admissible operator.

The question we address is the characterization of the minimal null control time (possibly zero or infinite) from y_0 that is: for a given initial condition y_0, what is the minimal time $T_0(y_0)$ such that, for any $T > T_0(y_0)$, there exists a control $u \in L^2(0,T;U)$ such that the associated solution of (1) satisfies $y(T) = 0$. A more precise definition of the minimal null control time is given in Definition 4 in Section 2.1.1.

For a presentation of null controllability of parabolic control problems as well as the possible existence of a positive minimal null control time for such equations we refer to [4] or [9, Section 1.1] and the references therein. Such a positive minimal null control time is due either to insufficient observation of eigenvectors, or to condensation of eigenvalues or to the geometry of generalized eigenspaces, or even to a combination of all those phenomena. Let us underline that this phenomenon is completely unrelated to the minimal control time arising from constraints on the state or on the control as studied for instance in [31], or to the one arising in hyperbolic problems due to intrinsic finite speed of propagation in the equation.

Under the considered assumptions on $\mathscr{A}$, the problem of characterizing the minimal null control time has been solved for scalar controls ($\dim U = 1$) in [9] where the *block moment method* has been introduced in that purpose. The aim of the present article is to push forward the analysis of [9] to extend it to any admissible control operator. The new difficulties come from the interplay between spectral condensation phenomena and the particular geometry of the control operator.

To present the general ideas, let us assume for simplicity that the operator $\mathscr{A}^*$ has a sequence of real and positive eigenvalues Λ and that the associated eigenvectors ϕ_λ, for $\lambda \in \Lambda$, form a complete family of the state space (the precise functional setting is detailed in Section 2.1). Then, the solution of system (1) satisfies $y(T) = 0$ if and only if the control $u \in L^2(0,T;U)$ solves the following moment problem

$$\int_0^T e^{-\lambda t} \left\langle u(T-t), \mathscr{B}^* \phi_\lambda \right\rangle_U \mathrm{d}t = -e^{-\lambda T} \left\langle y_0, \phi_\lambda \right\rangle, \qquad \forall\, \lambda \in \Lambda. \tag{2}$$

Solving moment problems associated with a scalar control operator. In the scalar case ($U = \mathbb{R}$), provided that $\mathscr{B}^*\phi_\lambda \neq 0$, the moment problem reduces to

$$\int_0^T e^{-\lambda t} u(T-t) \mathrm{d}t = -e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathscr{B}^*\phi_\lambda} \right\rangle, \qquad \forall\, \lambda \in \Lambda. \tag{3}$$

This problem is usually solved by the construction of a biorthogonal family $(q_\lambda)_{\lambda\in\Lambda}$ to the exponentials

$$\left\{ t \in (0,T) \mapsto e^{-\lambda t};\ \lambda \in \Lambda \right\}$$

in $L^2(0,T;U)$, i.e., a family $(q_\lambda)_{\lambda\in\Lambda}$ such that

$$\int_0^T q_\lambda(t) e^{-\mu t} \mathrm{d}t = \delta_{\lambda,\mu}, \qquad \forall\, \lambda, \mu \in \Lambda.$$

From [36], the existence of such biorthogonal family is equivalent to the summability condition

$$\sum_{\lambda\in\Lambda} \frac{1}{\lambda} < +\infty. \tag{4}$$

Remark 1. This condition (which will be assumed in the present article) is the main restriction to apply the moment method. Indeed, due to Weyl's law it imposes on many examples of partial differential equations of parabolic-type a restriction to the one dimensional setting. However, in some particular multi-dimensional geometries, the controllability problem can be transformed into a family of parameter dependent moment problems, each of them satisfying such assumption (see for instance [3, 8, 14] among others).

With such a biorthogonal family, a formal solution of the moment problem (3) is given by

$$u(T-t) = -\sum_{\lambda\in\Lambda} e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathscr{B}^*\phi_\lambda} \right\rangle q_\lambda(t), \quad t \in (0,T).$$

Thus if, for any y_0, the series defining u converges in $L^2(0,T;U)$ one obtains null controllability of system (1) in time T. To do so, it is crucial to prove upper bounds on $\|q_\lambda\|_{L^2(0,T)}$.

Suitable bounds on such biorthogonal families were provided in the pioneering work of Fattorini and Russell [21] in the case where the eigenvalues of $\mathscr{A}^*$ are well separated i.e. satisfy the classical gap condition: $\inf\{|\lambda-\mu|;\ \lambda,\mu\in\Lambda,\ \lambda\neq\mu\} > 0$. When the eigenvalues are allowed to condensate we refer to the work [5] for almost sharp estimates implying the condensation index of the sequence Λ. A discussion on other references providing estimates on biorthogonal families is detailed below. These results have provided an optimal characterization of the minimal null control time when the eigenvectors of $\mathscr{A}^*$ form a Riesz basis of the state space (and thus do not condensate).

However, as analyzed in [9], there are situations in which the eigenvectors also condensate and for which providing estimates on biorthogonal families is not sufficient to characterize the minimal null control time. In [9], it is assumed that the spectrum Λ can be decomposed as a union $\mathscr{G}$ of well separated groups of bounded cardinality. Then, the control u is seeked in the form

$$u(T-t) = \sum_{G\in\mathscr{G}} v_G(t),$$

where, for any $G \in \mathcal{G}$, the function $v_G \in L^2(0,T;U)$ solves the block moment problem

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda t} v_G(t)\mathrm{d}t = e^{-\lambda T}\left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^*\phi_\lambda}\right\rangle, & \forall\, \lambda \in G, \\ \displaystyle\int_0^T e^{-\lambda t} v_G(t)\mathrm{d}t = 0, & \forall\, \lambda \notin G. \end{cases} \tag{5}$$

This enables to deal with the condensation of eigenvectors: the eigenvectors $(\phi_\lambda)_{\lambda\in\Lambda}$ are only assumed to form a complete family of the state space.

Solving moment problems associated with a non scalar control operator. When the control is not scalar there are less available results in the literature. Here again, these results rely on the existence of a biorthogonal family to the exponentials with suitable bounds. For instance, in [6], null controllability in optimal time is proved using a subtle decomposition of the moment problem into two families of moment problems. In a more systematic way, one can take advantage of the biorthogonality in the time variable to seek for a solution u of the moment problem (2) in the form

$$u(T-t) = -\sum_{\lambda\in\Lambda} e^{-\lambda T}\left\langle y_0, \phi_\lambda\right\rangle q_\lambda(t)\frac{\mathcal{B}^*\phi_\lambda}{\left\|\mathcal{B}^*\phi_\lambda\right\|_U^2}.$$

This strategy was introduced by Lagnese in [25] for a one dimensional wave equation and used in the parabolic context for instance in [2, 3, 17, 18].

In the present article we deal with such general admissible control operators. As the eigenvectors will only be assumed to form a complete family, for each initial condition y_0, we study its null control time for system (1) by solving block moment problems of the following form

$$\begin{cases} \displaystyle\int_0^T \left\langle V_G(t), e^{-\lambda t}\mathcal{B}^*\phi_\lambda\right\rangle_U \mathrm{d}t = \left\langle y_0, e^{-\lambda T}\phi_\lambda\right\rangle, & \forall\, \lambda \in G, \\ \displaystyle\int_0^T \left\langle V_G(t), e^{-\lambda t}\mathcal{B}^*\phi_\lambda\right\rangle_U \mathrm{d}t = 0, & \forall\, \lambda \notin G. \end{cases} \tag{6}$$

Let us recall that, for pedagogical purposes, we have restricted this first introductory subsection to the case of real simple eigenvalues. The general form of block moment problems under study in this article is detailed in Section 2.2.

The strategy to solve such block moment problem and estimate its solution is presented on an example in Section 1.3. Let us already notice that the geometry of the finite dimensional space $\mathrm{Span}\{\mathcal{B}^*\phi_\lambda\,;\,\lambda \in G\}$ is crucial.

For instance, if this space is one dimensional, say generated by some $b \in U$, the strategy of Lagnese can be adapted if one seeks for V_G solution of the block moment problem (6) in the form

$$V_G(t) = v_G(t)b,$$

where $v_G \in L^2(0,T;\mathbb{R})$ solves a scalar block moment problem of the same form as (5).

If, instead, the family $(\mathcal{B}^*\phi_\lambda)_{\lambda\in G}$ is composed of linearly independent vectors then it admits a biorthogonal family in U denoted by $(b_\lambda^*)_{\lambda\in G}$. Then, one can for instance seek for V_G solution of the block moment problem (6) in the form

$$V_G(t) = v_G(t)\left(\sum_{\lambda\in G} b_\lambda^*\right).$$

where v_G solves a scalar block moment problem of the form (5). An upper bound of the minimal control time can then be obtained thanks to an estimate of the family $(b_\lambda^*)_{\lambda\in G}$, but without guarantee of optimality.

In the general setting, taking into account the geometry of the observations of eigenvectors to solve block moment problems of the form (6) is a more intricate question that we solve in this article, still under the summability condition (4).

Let us mention that we not only solve block moment problems of the form (6) but we also provide estimates on their solutions to ensure that the series defining the control converges. These estimates will lead to an optimal characterization of the minimal null control time for each given problem.

We pay particular attention to these estimates so that they do not directly depend on the sequence Λ but are uniform for classes of such sequences. This is an important step to tackle uniform controllability for parameter dependent control problems. Estimates of this kind have already proved their efficiency in various contexts such as: numerical analysis of semi-discrete control problems [2], oscillating coefficients [32], analysis of degenerate control problems with respect to the degeneracy parameter [17, 18], analysis of higher dimensional controllability problems by reduction to families of one dimensional control problems [1, 3, 8, 14] or analysis of convergence of Robin-type controls to Dirichlet controls [11].

Another important feature of the estimates we obtain is to track the dependency with respect to the final time T when T goes to the minimal null control time. As presented in Remark 23, this allows applications in higher dimensions (with a cylindrical geometry) or applications to nonlinear control problems.

An overview of some estimates on biorthogonal families.

Finally, let us recall some classical results providing estimates for biorthogonal families to a sequence of exponentials.

Under the classical gap condition, uniform estimates for biorthogonal families were already obtained in [22] and sharp short-time estimates were obtained in [8]. In this setting, bounds with a detailed dependency with respect to parameters were given in [19]. In this work, the obtained bounds take into account the fact that the gap property between eigenvalues may be better in high frequencies. Similar results were also obtained in [26].

Under a weak-gap condition of the form (23), that is when the eigenvalues can be gathered in blocks of bounded cardinality with a gap between blocks (which is the setting of the present article), uniform estimates on biorthogonal sequences follow from the uniform estimates for the resolution of block moment problems proved in [9]. Similar estimates, but where the sharp dependency with respect to T of the different constants is tracked, were obtained in [23]. Using the strategy detailed in [12], the estimates of [9] can also be supplemented with such dependency with respect to T (see Theorem 46). Let us mention that similar results were also obtained in [16] with stronger assumptions, namely with a weak-gap assumption on the square roots of the eigenvalues.

In the absence of any gap-type condition, estimates on biorthogonal families were first proved in [5] involving the condensation index and then later in [3] involving a local measure of the gap.

1.2. *Structure of the article*

To ease the reading, let us give here the detailed outline of this article.

In Section 1.3 we detail, for a simple example, the obtained results as well as our strategy of proof. This allows to explain the contents of this article without introducing too many notations.

In Section 2.1, we detail the framework, assumptions and notations that will be used throughout this article. The main results concerning the resolution of block moment problems with a non scalar control are stated in Section 2.2. The application of these results to the characterization of the minimal null control time is stated in Section 2.3. We provide in Section 2.4 more explicit formulas to compute the minimal null control time. We also deduce from our study some estimates on the cost of controllability that are given in Section 2.5.

The results concerning the resolution of block moment problems are proved in Section 3. The application of these results to the characterization of the minimal null control time and the study of the cost of null controllability are then proved in Section 4. More explicit formulas for the computation of the minimal null control time are proved in Section 5.

Finally we apply these results to different examples. First we deal in Section 6 with academic examples. For these examples the computations are rather simple and this allows to highlight the different phenomena at stake in this minimal null control time study. We end this article with the analysis of null controllability for systems of coupled linear partial differential equations of parabolic type in Section 7.

1.3. *Our analysis on a toy system*

To highlight the ideas we develop in this article (without drowning them in technicalities or notations), let us present our strategy of analysis of null controllability on an abstract simple example.

We consider $X = L^2(0,1;\mathbb{R})^2$ and $\omega \subset (0,1)$ a non empty open set. For a given $a > 0$ we define

$$\Lambda = \left\{\lambda_{k,1} := k^2,\, \lambda_{k,2} := k^2 + e^{-ak^2}\,;\, k \geq 1\right\},$$

and take $(\varphi_k)_{k\geq 1}$ a Hilbert basis of $L^2(0,1;\mathbb{R})$ such that

$$\inf_{k\geq 1} \|\varphi_k\|_{L^2(\omega)} > 0.$$

Let $\phi_{k,1} := \binom{\varphi_k}{\varphi_k}$ and $\phi_{k,2} := \binom{0}{\varphi_k}$. We define the operator $\mathscr{A}^*$ in X by

$$\mathscr{A}^*\phi_{k,1} = \lambda_{k,1}\phi_{k,1}, \qquad \mathscr{A}^*\phi_{k,2} = \lambda_{k,2}\phi_{k,2},$$

with

$$\mathscr{D}(\mathscr{A}^*) = \left\{\sum_{k\geq 1} a_{k,1}\phi_{k,1} + a_{k,2}\phi_{k,2}\,;\, \sum_{k\geq 1} \lambda_{k,1}^2 a_{k,1}^2 + \lambda_{k,2}^2 a_{k,2}^2 < +\infty\right\}.$$

The control operator $\mathscr{B}$ is defined by $U = L^2(0,1;\mathbb{R})$ and

$$\mathscr{B} : u \in U \mapsto \begin{pmatrix} 0 \\ \mathbb{1}_\omega u \end{pmatrix} \in X.$$

The condition $\inf_{k\geq 1} \|\varphi_k\|_{L^2(\omega)} > 0$ yields

$$\mathscr{B}^*\phi_{k,1} = \mathscr{B}^*\phi_{k,2} = \mathbb{1}_\omega \varphi_k \neq 0, \qquad \forall\, k \geq 1. \tag{7}$$

This ensures approximate controllability of system (1).

We insist on the fact that the goal of this article is not to deal with this particular example but to develop a general methodology to analyze the null controllability of system (1). The general assumptions that will be considered in this article are detailed in Section 2.1.

Let $y_0 \in X$. From Proposition 2 and the fact that $\{\phi_{k,1}, \phi_{k,2}\,;\, k \geq 1\}$ forms a complete family of X, system (1) is null controllable from y_0 at time T if and only if there exists $u \in L^2(0,T;U)$ such that for any $k \geq 1$ and any $j \in \{1,2\}$,

$$\int_0^T e^{-\lambda_{k,j} t} \left\langle u(T-t), \mathscr{B}^*\phi_{k,j}\right\rangle_U \mathrm{d}t = -e^{-\lambda_{k,j} T} \left\langle y_0, \phi_{k,j}\right\rangle_X.$$

Following the idea developed in [9], we seek for a control u of the form

$$u(t) = -\sum_{k\geq 1} v_k(T-t) \tag{8}$$

where, for each $k \geq 1$, $v_k \in L^2(0,T;U)$ solves the block moment problem

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,j} t} \left\langle v_k(t), \mathscr{B}^*\phi_{k,j}\right\rangle_U \mathrm{d}t = e^{-\lambda_{k,j} T} \left\langle y_0, \phi_{k,j}\right\rangle_X, \quad \forall\, j \in \{1,2\}, \\ \displaystyle\int_0^T e^{-\lambda_{k',j} t} \left\langle v_k(t), \mathscr{B}^*\phi_{k',j}\right\rangle_U \mathrm{d}t = 0, \quad \forall\, k' \neq k,\, \forall\, j \in \{1,2\}. \end{cases} \tag{9}$$

To solve (9), for a fixed k, we consider the following auxiliary block moment problem in the space U

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,j} t} v_k(t)\mathrm{d}t = \Omega_{k,j}, \quad \forall\, j \in \{1,2\}, \\ \displaystyle\int_0^T e^{-\lambda_{k',j} t} v_k(t)\mathrm{d}t = 0, \quad \forall\, k' \neq k, \forall\, j \in \{1,2\}, \end{cases} \tag{10}$$

where $\Omega_{k,j} \in U$ have to be precised. If we impose that $\Omega_{k,1}$ and $\Omega_{k,2}$ satisfy the constraints

$$\left\langle \Omega_{k,j}, \mathscr{B}^* \phi_{k,j} \right\rangle_U = e^{-\lambda_{k,j} T} \left\langle y_0, \phi_{k,j} \right\rangle_X, \qquad \forall\, j \in \{1,2\}, \tag{11}$$

we obtain that the solutions of (10) also solve (9). The existence of $\Omega_{k,1}$ and $\Omega_{k,2}$ satisfying the constraints (11) is ensured by the approximate controllability condition (7); however there exist infinitely many choices. A crucial point is that, by orthogonal projection, there exists $\Omega_{k,1}$ and $\Omega_{k,2}$ in the space $U_k = \mathrm{Span}\{\mathscr{B}^*\phi_{k,1}, \mathscr{B}^*\phi_{k,2}\}$ satisfying the constraints (11).

Then, for any $\Omega_{k,1}$, $\Omega_{k,2} \in U_k$, since the space U_k is of finite dimension, applying the scalar results of [9] component by component leads to the existence of $v_k \in L^2(0,T;U)$ satisfying (10). It also gives the following estimate

$$\| v_k \|^2_{L^2(0,T;U)} \leq C_{T,\varepsilon} e^{\varepsilon \lambda_{k,1}} F(\Omega_{k,1}, \Omega_{k,2}), \tag{12}$$

with

$$F : \left(\Omega_{k,1}, \Omega_{k,2}\right) \in U^2 \mapsto \|\Omega_{k,1}\|^2_U + \left\| \frac{\Omega_{k,2} - \Omega_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|^2_U.$$

Using (12) and minimizing the function F under the constraints (11) we obtain that there exists $v_k \in L^2(0,T;U)$ solution of the block moment problem (9) such that

$$\| v_k \|^2_{L^2(0,T;U)} \leq C_{T,\varepsilon} e^{\varepsilon \lambda_{k,1}} \inf\left\{ F(\Omega_{k,1}, \Omega_{k,2}) \,;\, \Omega_{k,1}, \Omega_{k,2} \text{ satisfy (11)} \right\}. \tag{13}$$

The corresponding general statements of the resolution of block moment problems are detailed in Section 2.2 (see Theorem 10) and proved in Section 3. Actually using a refined version of the results in [9] (see Theorem 46) we obtain sharper results including dependency with respect to T.

Now that we can solve the block moment problems (9), a way to characterize the minimal null control time is to estimate for which values of T the series (8) defining the control u converges in $L^2(0,T;U)$.

To achieve this goal, we isolate in the estimate (13) the dependency with respect to T. Notice that the function F does not depend on T but that the constraints (11) do.

For any $k \geq 1$ and any $\Omega_{k,1}$, $\Omega_{k,2} \in U_k$ we set

$$\widetilde{\Omega}_{k,j} := e^{\lambda_{k,j} T} \Omega_{k,j}, \qquad \forall\, j \in \{1,2\}.$$

Then, there is equivalence between the constraints (11) and the new constraints

$$\left\langle \widetilde{\Omega}_{k,j}, \mathscr{B}^* \phi_{k,j} \right\rangle_U = \left\langle y_0, \phi_{k,j} \right\rangle_X, \qquad \forall\, j \in \{1,2\}. \tag{14}$$

Now these constraints are independent of the variable T. From the mean value theorem we obtain

$$\begin{aligned} F(\Omega_{k,1}, \Omega_{k,2}) &= \left\| e^{-\lambda_{k,1} T} \widetilde{\Omega}_{k,1} \right\|^2_U + \left\| \frac{e^{-\lambda_{k,2} T} \widetilde{\Omega}_{k,2} - e^{-\lambda_{k,1} T} \widetilde{\Omega}_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|^2_U . \\ &\leq e^{-2\lambda_{k,1} T} \left\| \widetilde{\Omega}_{k,1} \right\|^2_U + 2 e^{-2\lambda_{k,2} T} \left\| \frac{\widetilde{\Omega}_{k,2} - \widetilde{\Omega}_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|^2_U + 2 \left(\frac{e^{-\lambda_{k,2} T} - e^{-\lambda_{k,1} T}}{\lambda_{k,2} - \lambda_{k,1}} \right)^2 \left\| \widetilde{\Omega}_{k,1} \right\|^2_U \\ &\leq 2(1+T^2) e^{-2\lambda_{k,1} T} F(\widetilde{\Omega}_{k,1}, \widetilde{\Omega}_{k,2}). \end{aligned}$$

The general statement of this estimate is given in Lemma 27.

Plugging this estimate into (12) and optimizing the function F under the constraints (14) yields

$$\|v_k\|^2_{L^2(0,T;U)} \leq C_{T,\varepsilon} e^{\varepsilon\lambda_{k,1}} e^{-2\lambda_{k,1}T} \mathscr{C}_k(y_0) \tag{15}$$

where $\mathscr{C}_k(y_0)$ is the quantity, independent of T, given by

$$\mathscr{C}_k(y_0) \quad := \quad \inf\left\{ \|\widetilde{\Omega}_1\|^2_U + \left\| \frac{\widetilde{\Omega}_2 - \widetilde{\Omega}_1}{\lambda_{k,2}-\lambda_{k,1}} \right\|^2_U ; \begin{array}{l} \widetilde{\Omega}_1, \widetilde{\Omega}_2 \in U_k \text{ satisfy} \\ \langle \widetilde{\Omega}_j, \mathscr{B}^*\phi_{k,j}\rangle_U = \langle y_0, \phi_{k,j}\rangle_X, \ \forall\, j \in \{1,2\} \end{array} \right\}. \tag{16}$$

Estimate (15) proves that for any time $T > 0$ such that

$$T > \limsup_{k\to+\infty} \frac{\ln \mathscr{C}_k(y_0)}{2\lambda_{k,1}}$$

the series (8) defining the control u converges in $L^2(0,T;U)$. Thus, null controllability of (1) from y_0 holds for such T.

We also prove that the obtained estimate (15) is sufficiently sharp so that it characterizes the minimal null control time from y_0 as

$$T_0(y_0) = \limsup_{k\to+\infty} \frac{\ln \mathscr{C}_k(y_0)}{2\lambda_{k,1}}. \tag{17}$$

The corresponding general statements regarding the minimal null control time together with bounds on the cost of controllability are detailed in Section 2.2 (see Theorem 11) and proved in Section 4.

At this stage we have characterized the minimal null control time as stated in (17). However to be able to estimate the actual value of $T_0(y_0)$ one should be able to estimate the quantity $\mathscr{C}_k(y_0)$ as defined in (16). This formula is not very explicit and it does not get better in the general setting.

However, we notice that (16) is a finite dimensional optimization problem that we explicitly solve in terms of the eigenelements of $\mathscr{A}^*$ and their observations through $\mathscr{B}^*$.

Indeed the minimization problem (16) has a unique solution characterized by the existence of multipliers m_1, $m_2 \in \mathbb{R}$ such that for any H_1, $H_2 \in U_k$ we have

$$\langle H_1, \widetilde{\Omega}_1\rangle_U + \left\langle \frac{\widetilde{\Omega}_2 - \widetilde{\Omega}_1}{\lambda_{k,2}-\lambda_{k,1}}, \frac{H_2 - H_1}{\lambda_{k,2}-\lambda_{k,1}} \right\rangle_U = m_1 \langle H_1, \mathscr{B}^*\phi_{k,1}\rangle_U + m_2 \langle H_2, \mathscr{B}^*\phi_{k,2}\rangle_U. \tag{18}$$

Setting $H_1 = H_2 = H$ for any $H \in U_k$ implies

$$\widetilde{\Omega}_1 = m_1 \mathscr{B}^*\phi_{k,1} + m_2 \mathscr{B}^*\phi_{k,2}.$$

Setting $H_1 = 0$ and $H_2 = (\lambda_{k,2}-\lambda_{k,1})H$ for any $H \in U_k$ implies

$$\widetilde{\Omega}_2 = m_1 \mathscr{B}^*\phi_{k,1} + m_2 \mathscr{B}^*\phi_{k,2} + m_2(\lambda_{k,2}-\lambda_{k,1})^2 \mathscr{B}^*\phi_{k,2}.$$

Getting back to the constraints (14) we obtain

$$\begin{pmatrix} \langle y_0, \phi_{k,1}\rangle_X \\ \langle y_0, \phi_{k,2}\rangle_X \end{pmatrix} = M \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \tag{19}$$

where the 2×2 matrix M is defined by

$$M = \mathrm{Gram}_U\left(\mathscr{B}^*\phi_{k,1}, \mathscr{B}^*\phi_{k,2}\right) + \mathrm{Gram}_U\left(0, (\lambda_{k,2}-\lambda_{k,1})\mathscr{B}^*\phi_{k,2}\right).$$

Setting $H_1 = \widetilde{\Omega}_1$ and $H_2 = \widetilde{\Omega}_2$ in (18) and using (19) impliy

$$\begin{aligned} \mathscr{C}_k(y_0) = \|\widetilde{\Omega}_1\|^2_U + \left\| \frac{\widetilde{\Omega}_2 - \widetilde{\Omega}_1}{\lambda_{k,2}-\lambda_{k,1}} \right\|^2_U &= \left\langle \begin{pmatrix} \langle y_0, \phi_{k,1}\rangle_X \\ \langle y_0, \phi_{k,2}\rangle_X \end{pmatrix}, \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} \langle y_0, \phi_{k,1}\rangle_X \\ \langle y_0, \phi_{k,2}\rangle_X \end{pmatrix}, M^{-1} \begin{pmatrix} \langle y_0, \phi_{k,1}\rangle_X \\ \langle y_0, \phi_{k,2}\rangle_X \end{pmatrix} \right\rangle. \end{aligned}$$

Thus, after computations, for the particular example we are considering here, the obtained formula reads

$$\mathscr{C}_k(y_0) = \frac{1}{\|\varphi_k\|^2_{L^2(\omega)}} \left\langle y_0, \begin{pmatrix}\varphi_k\\ \varphi_k\end{pmatrix}\right\rangle_X^2 + \frac{e^{2ak^2}}{\|\varphi_k\|^2_{L^2(\omega)}} \left\langle y_0, \begin{pmatrix}\varphi_k\\ 0\end{pmatrix}\right\rangle_X^2.$$

Then, from (17), it comes that the minimal null control time from X of this example is given by

$$T_0(X) = a.$$

Notice, for instance, that this expression also gives that for a given y_0 if the set

$$\left\{k \in \mathbb{N}^* \,;\, \left\langle y_0, \begin{pmatrix}\varphi_k\\ 0\end{pmatrix}\right\rangle_X \neq 0\right\}$$

is finite, then null controllability from y_0 holds in any positive time, i.e. $T_0(y_0) = 0$.

We obtain different explicit formula depending on the configuration for the multiplicity of the eigenvalues of the considered block. The general statements of an explicit solution of the corresponding optimization problem are detailed in Section 2.4 (see Theorem 14 and Theorem 18) and proved in Section 5.

2. Main results

We state in this section the main results of this article concerning the resolution of block moment problems and the application to the characterization of the minimal null control time. We start by giving the functional setting and assumptions we use.

2.1. *Framework, spectral assumptions and notations*

2.1.1. *Functional setting*

The functional setting for the study of system (1) is the same as in [9]. For the sake of completeness, let us briefly detail it. Unless explicitly stated, all the spaces are assumed to be complex vector spaces.

We consider X a Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle_X$ and $\|\cdot\|_X$ respectively. The space X is identified to its anti-dual through the Riesz theorem. Let $(\mathscr{A}, D(\mathscr{A}))$ be an unbounded operator in X such that $-\mathscr{A}$ generates a C^0−semigroup in X. Its adjoint in X is denoted by $(\mathscr{A}^*, D(\mathscr{A}^*))$. Up to a suitable translation, we can assume that 0 is in the resolvent set of $\mathscr{A}$.

We denote by X_1 (resp. X_1^*) the Hilbert space $D(\mathscr{A})$ (resp. $D(\mathscr{A}^*)$) equipped with the norm $\|x\|_1 := \|\mathscr{A}x\|_X$ (resp. $\|x\|_{1^*} := \|\mathscr{A}^*x\|_X$) and we define X_{-1} as the completion of X with respect to the norm

$$\|y\|_{-1} := \sup_{z\in X_1^*} \frac{\langle y, z\rangle_X}{\|z\|_{1^*}}.$$

Notice that X_{-1} is isometrical to the topological anti-dual of X_1^* using X as a pivot space (see for instance [38, Proposition 2.10.2]). The corresponding duality bracket will be denoted by $\langle\cdot,\cdot\rangle_{-1,1^*}$ and satisfies

$$\langle y, cz\rangle_{-1,1^*} = \bar{c}\langle y, z\rangle_{-1,1^*}, \qquad \forall\, y \in X_{-1}, \forall\, z \in X_1^*, \forall\, c \in \mathbb{C}.$$

The control space U is a Hilbert space (that we will identify to its anti-dual). Its inner product and norm are denoted by $\langle\cdot,\cdot\rangle_U$ and $\|\cdot\|_U$ respectively. Let $\mathscr{B} : U \to X_{-1}$ be a linear continuous control operator and denote by $\mathscr{B}^* : X_1^* \to U$ its adjoint in the duality described above. Let $(X_\diamond^*, \|\cdot\|_{\diamond^*})$ be a Hilbert space such that $X_1^* \subset X_\diamond^* \subset X$ with dense and continuous embeddings. We assume that

$X_\diamond^*$ is stable by the semigroup generated by $-\mathscr{A}^*$. We also define $X_{-\diamond}$ as the subspace of X_{-1} defined by

$$X_{-\diamond} := \left\{ y \in X_{-1} \,;\, \left\| y \right\|_{-\diamond} := \sup_{z \in X_1^*} \frac{\left\langle y, z \right\rangle_{-1,1^*}}{\| z \|_{\diamond^*}} < +\infty \right\},$$

which is also isometrical to the anti-dual of $X_\diamond^*$ with X as a pivot space. The corresponding duality bracket will be denoted by $\langle \cdot, \cdot \rangle_{-\diamond,\diamond}$. Thus, we end up with the following five functional spaces

$$X_1^* \subset X_\diamond^* \subset X \subset X_{-\diamond} \subset X_{-1}.$$

We say that the control operator $\mathscr{B}$ is an admissible control operator for (1) with respect to the space $X_{-\diamond}$ if for any $T > 0$ there exists $C_T > 0$ such that

$$\int_0^T \left\| \mathscr{B}^* e^{-(T-t)\mathscr{A}^*} z \right\|_U^2 \mathrm{d}t \le C_T \| z \|_{\diamond^*}^2, \quad \forall\, z \in X_1^*. \tag{20}$$

Notice that if (20) holds for some $T > 0$ it holds for any $T > 0$. The admissibility condition (20) implies that, by density, we can give a meaning to the map

$$\left(t \mapsto \mathscr{B}^* e^{-(T-t)\mathscr{A}^*} z \right) \in L^2(0, T; U),$$

for any $z \in X_\diamond^*$. Then, we end up with the following well-posedness result (see [9, Proposition 1.2]).

Proposition 2. *Assume that* (20) *holds. Then, for any $T > 0$, any $y_0 \in X_{-\diamond}$, and any $u \in L^2(0, T; U)$, there exists a unique $y \in C^0([0, T]; X_{-\diamond})$ solution to* (1) *in the sense that it satisfies for any $t \in [0, T]$ and any $z_t \in X_\diamond^*$,*

$$\left\langle y(t), z_t \right\rangle_{-\diamond,\diamond} - \left\langle y_0, e^{-t\mathscr{A}^*} z_t \right\rangle_{-\diamond,\diamond} = \int_0^t \left\langle u(s), \mathscr{B}^* e^{-(t-s)\mathscr{A}^*} z_t \right\rangle_U \mathrm{d}s.$$

Moreover there exists $C_T > 0$ such that

$$\sup_{t \in [0,T]} \left\| y(t) \right\|_{-\diamond} \le C_T \left(\left\| y_0 \right\|_{-\diamond} + \| u \|_{L^2(0,T;U)} \right).$$

Remark 3. By analogy with the semigroup notation, when $u = 0$, we set for any $t \in [0, T]$, $e^{-t\mathscr{A}} y_0 := y(t)$. This extends the semigroup $e^{-\cdot\mathscr{A}}$ defined on X to $X_{-\diamond}$ and implies that for any $z \in X_{-\diamond}$,

$$\left\langle e^{-T\mathscr{A}} z, \phi \right\rangle_{-\diamond,\diamond} = \left\langle z, e^{-T\mathscr{A}^*} \phi \right\rangle_{-\diamond,\diamond}, \qquad \forall\, \phi \in X_\diamond^*. \tag{21}$$

With this notion of solution at hand, we finally define the minimal null control time from a subspace of initial conditions Y_0.

Definition 4. *Let Y_0 be a closed subspace of $X_{-\diamond}$ and let $T > 0$. The system* (1) *is said to be null controllable from Y_0 at time T if for any $y_0 \in Y_0$, there exists a control $u \in L^2(0, T; U)$ such that the associated solution of* (1) *satisfies $y(T) = 0$.*

The minimal null control time $T_0(Y_0) \in [0, +\infty]$ is defined by

- *for any $T > T_0(Y_0)$, system* (1) *is null controllable from Y_0 at time T;*
- *for any $T < T_0(Y_0)$, system* (1) *is not null controllable from Y_0 at time T.*

To simplify the notations, for any $y_0 \in X_{-\diamond}$, we define $T_0(y_0) := T_0(\mathrm{Span}\{y_0\})$.

2.1.2. *Spectral assumptions*

In all this article we assume that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy the assumptions of Section 2.1.1. Moreover to solve the control problem we will need some additional spectral assumptions.

Behavior of eigenvalues.

We assume that the spectrum of $\mathscr{A}^*$, denoted by Λ, is only composed of (countably many) eigenvalues. Moreover, we assume that the eigenvalues lie in a suitable sector of the complex plane, i.e., there exists $\tau > 0$ such that

$$\Lambda \subset S_\tau \tag{22}$$

where

$$S_\tau := \Big\{ z \in \mathbb{C}\,;\, \Re z > 0 \text{ and } |\Im z| < (\sinh \tau)\Re z \Big\}.$$

Remark 5. In [9], the assumption on Λ was stronger. Namely, in that article it was assumed that $\Lambda \subset (1,+\infty)$. The fact that $\min \Lambda \geq 1$ was only used in the lower bound on the solution of scalar block moment problems (see estimate (117)). The extension to complex eigenvalues satisfying (22) was done in [12] and is stated in Appendix A.

If necessary, one can replace the operator $\mathscr{A}$ by $\mathscr{A}+\sigma$ without modifying the controllability properties. Then, in the different estimates, the behavior with respect to σ can be carefully tracked if needed.

As in the case of a scalar control (see [9]) we assume that this spectrum satisfies a weak-gap condition. Namely, there exists $p \in \mathbb{N}^*$ and $\varrho > 0$ such that

$$\sharp\Big(\Lambda \cap D(\mu, \varrho/2)\Big) \leq p, \qquad \forall\, \mu \in \mathbb{C}, \tag{23}$$

where $D(\mu,\varrho/2)$ denotes the open disk in the complex plane with center μ and radius $\varrho/2$. This means that the eigenvalues are allowed to condensate by groups but the cardinality of these groups should be bounded. To precise this, let us recall the notion of groupings used in [9, Definition 1.6, Proposition 7.1] and extended to the complex setting in [12, Proposition V.5.28].

Proposition 6. *Let $p \in \mathbb{N}^*$ and $\varrho > 0$. Let $\Lambda \subset \mathbb{C}$ be such that the weak-gap condition* (23) *holds. Then, there exists a countable family $\mathscr{G}$ of disjoint subsets of Λ satisfying*

$$\Lambda = \bigcup_{G \in \mathscr{G}} G \tag{24}$$

and each $G \in \mathscr{G}$ satisfies

$$\operatorname{diam} G \leq \varrho, \tag{25}$$

$$\sharp G \leq p, \tag{26}$$

$$\textit{and} \quad \operatorname{dist}(\operatorname{Conv} G, \Lambda \backslash G) \geq \frac{\varrho}{2 \times 4^{p-1}}. \tag{27}$$

Let us mention that the results do not depend on the particular construction done in [12, Proposition V.5.28] and remain valid for any grouping $\mathscr{G}$ satisfying (24)-(27).

Concerning the asymptotic behavior of the spectrum we will use the counting function associated to Λ defined by

$$N_\Lambda : r > 0 \mapsto \sharp\{\lambda \in \Lambda\,;\, |\lambda| \leq r\}.$$

We assume that there exists $\kappa > 0$ and $\theta \in (0,1)$ such that

$$N_\Lambda(r) \leq \kappa r^\theta, \qquad \forall\, r > 0 \tag{28}$$

and

$$|N_\Lambda(r) - N_\Lambda(s)| \leq \kappa \times \Big(1 + |r-s|^\theta\Big), \qquad \forall\, r,s > 0. \tag{29}$$

Notice that this condition is slightly stronger than the classical summability condition (4) used for instance in [5, 9, 22] and many other works.

Remark 7. Let us underline that if we do not assume (29) to hold all the results of the present article still hold with a slight change in the estimates. To lighten the writing we only detail this change for Theorem 46 concerning the resolution of block moment problems with a scalar control (see Remark 47). However, as proved in Section 7, the assumption (29) holds for many examples.

Notice also that (28), with $r = \inf|\Lambda|$, implies the following lower bound on the bottom of the spectrum

$$\inf|\Lambda| \geq \kappa^{-\theta}.$$

Our goal is not only to study the controllability properties of our system but also to obtain estimates that are uniform in a way to be precised. To do so, we define the following class of sequences: let $p \in \mathbb{N}^*$, $\varrho, \tau, \kappa > 0$, $\theta \in (0,1)$ and consider the class

$$\mathscr{L}_w(p,\varrho,\tau,\theta,\kappa) := \left\{\Lambda \subset S_\tau\,;\, \Lambda \text{ satisfies (23), (28) and (29)}\right\}. \tag{30}$$

Multiplicity of eigenvalues. In our study we allow both algebraic and geometric multiplicities for the eigenvalues. We assume that these multiplicities are finite and that the algebraic multiplicity is globally bounded. More precisely, we assume that

$$\gamma_\lambda := \dim \operatorname{Ker}(\mathscr{A}^* - \lambda) < +\infty, \qquad \forall\, \lambda \in \Lambda, \tag{31}$$

and that there exists $\eta \in \mathbb{N}^*$ such that

$$\operatorname{Ker}(\mathscr{A}^* - \lambda)^\eta = \operatorname{Ker}(\mathscr{A}^* - \lambda)^{\eta+1}, \qquad \forall\, \lambda \in \Lambda. \tag{32}$$

For any $\lambda \in \Lambda$ we denote by α_λ the smallest integer such that

$$\operatorname{Ker}(\mathscr{A}^* - \lambda)^{\alpha_\lambda} = \operatorname{Ker}(\mathscr{A}^* - \lambda)^{\alpha_\lambda+1}$$

and set

$$E_\lambda := \operatorname{Ker}(\mathscr{A}^* - \lambda)^{\alpha_\lambda}.$$

(Generalized) eigenvectors. To study null-controllability, we assume that the Fattorini–Hautus criterion is satisfied

$$\operatorname{Ker}(\mathscr{A}^* - \lambda) \cap \operatorname{Ker}\mathscr{B}^* = \{0\}, \qquad \forall\, \lambda \in \Lambda. \tag{33}$$

It is a necessary condition for approximate controllability. Note that, under additional assumptions on $\mathscr{A}$ and $\mathscr{B}$ it is also a sufficient condition for approximate controllability (see for instance [20, 34]). However, when studying null controllability of system (1) for initial conditions in a closed strict subspace Y_0 of $X_{-\diamond}$ the condition (33) can be too strong, see for instance Sections 7.1.2 and 7.1.3.

We assume that the family of generalized eigenvectors of $\mathscr{A}^*$

$$\Phi = \left\{\phi \in E_\lambda\,;\, \lambda \in \Lambda\right\} = \bigcup_{\lambda\in\Lambda} E_\lambda$$

is complete in $X_\diamond^*$ i.e. for any $y \in X_{-\diamond}$,

$$\left(\left\langle y, \phi\right\rangle_{-\diamond,\diamond} = 0, \quad \forall\, \phi \in \Phi\right) \quad \Longrightarrow \quad y = 0. \tag{34}$$

In the following, to simplify the writing, we gather these assumptions and say that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy (H) if there exists $p \in \mathbb{N}^*$, $\varrho, \tau, \kappa > 0$ and $\theta \in (0,1)$ such that

$$\begin{cases} \mathscr{A} \text{ and } \mathscr{B} \text{ satisfy the assumptions of Section 2.1.1;} \\ \Lambda = \operatorname{Sp}(\mathscr{A}^*) \text{ belongs to } \mathscr{L}_w(p,\varrho,\tau,\theta,\kappa) \text{ and satisfies (31) and (32)}\,; \\ \text{the associated (generalized) eigenvectors satisfy (33) and (34).} \end{cases} \tag{H}$$

2.1.3. *Notation*

We give here some notation that will be used throughout this article.

- For any $a, b \in \mathbb{R}$, we define the following subsets of $\mathbb{N}$:
$$[\![a,b]\!] := [a,b] \cap \mathbb{N}, \qquad [\![a,b[\![:= [a,b) \cap \mathbb{N}.$$
- In all the present paper, $\langle \cdot,\cdot \rangle$ denotes the usual inner product in finite dimension i.e.
$$\langle f, g \rangle = {}^t f \overline{g}.$$
- For any $t \in \mathbb{R}$ we denote by e_t the exponential function
$$\begin{aligned} e_t : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto e^{-tz}. \end{aligned}$$
- We shall denote by $C_{\nu_1,\dots,\nu_l} > 0$ a constant possibly varying from one line to another but depending only on the parameters $\nu_1, \dots, \nu_l$.
- For any non empty subset $\Gamma \subset \Lambda$, we set
$$r_\Gamma := \inf_{\lambda \in \Gamma} \Re \lambda. \tag{35}$$
Notice that assumptions (22) and (23) imply that $r_\Gamma > 0$ for any $\Gamma \subset \Lambda$.
- For any multi-index $\alpha \in \mathbb{N}^n$, we denote its length by $|\alpha| = \sum_{j=1}^n \alpha_j$ and its maximum by $|\alpha|_\infty = \max_{j \in [\![1,n]\!]} \alpha_j$.
  For $\alpha, \mu \in \mathbb{N}^n$, we say that $\mu \le \alpha$ if and only if $\mu_j \le \alpha_j$ for any $j \in [\![1,n]\!]$.
- In all this article the notation $f[\cdots]$ stands for (generalized) divided differences of a set of values (x_j, f_j). Let us recall that, for pairwise distinct $x_1, \dots, x_n \in \mathbb{C}$ and $f_1, \dots, f_n$ in any vector space, the divided differences are defined by
$$f[x_j] = f_j, \qquad f[x_1,\dots,x_j] = \frac{f[x_2,\dots,x_j] - f[x_1,\dots,x_{j-1}]}{x_j - x_1}.$$
The two results that will be the most used in this article concerning divided differences are the Leibniz formula
$$(gf)[x_1,\dots,x_j] = \sum_{k=1}^{j} g[x_1,\dots,x_k] f[x_k,\dots,x_j],$$
and Jensen inequality stating that, when $f_j = f(x_j)$ for an holomorphic function f, we have
$$\left| f[x_1,\dots,x_j] \right| \le \frac{\left| f^{(j-1)}(z) \right|}{(j-1)!},$$
with $z \in \mathrm{Conv}\{x_1,\dots,x_j\}$. For more detailed statements and other useful properties as well as their generalizations when $x_1,\dots,x_n$ are not assumed to be pairwise distinct we refer the reader to [12, Appendix A.2] This generalization is used in the present article whenever there are algebraically multiple eigenvalues.
- For any closed subspace Y of $X_{-\diamond}$ we denote by P_Y the orthogonal projection in $X_{-\diamond}$ onto Y. We denote by $P_Y^* \in L(X_\diamond^*)$ its adjoint in the duality $X_{-\diamond}$, $X_\diamond^*$.

2.2. *Resolution of block moment problems*

Definition of block moment problems. Using the notion of solution given in Proposition 2 and the assumption (34), null controllability from y_0 in time T reduces to the resolution of the following problem: find $u \in L^2(0,T;U)$ such that
$$\int_0^T \left\langle u(t), \mathscr{B}^* e^{-(T-t)\mathscr{A}^*} \phi \right\rangle_U \mathrm{d}t = -\left\langle y_0, e^{-T\mathscr{A}^*} \phi \right\rangle_{-\diamond,\diamond}, \quad \forall\, \phi \in E_\lambda, \forall\, \lambda \in \Lambda. \tag{36}$$

Following the strategy initiated in [9] for scalar controls, we decompose this problem into block moment problems. Hence we look for a control of the form

$$u = -\sum_{G\in\mathscr{G}} v_G(T-\cdot) \tag{37}$$

where $\mathscr{G}$ is a grouping (as stated in Proposition 6) and, for every $G \in \mathscr{G}$, $v_G \in L^2(0,T;U)$ solves the moment problem in the group G i.e.

$$\int_0^T \left\langle v_G(t), \mathscr{B}^* e^{-t\mathscr{A}^*}\phi\right\rangle_U \mathrm{d}t = \left\langle y_0, e^{-T\mathscr{A}^*}\phi\right\rangle_{-\diamond,\diamond}, \qquad \forall\, \phi \in E_\lambda,\ \forall\, \lambda \in G, \tag{38a}$$

$$\int_0^T \left\langle v_G(t), \mathscr{B}^* e^{-t\mathscr{A}^*}\phi\right\rangle_U \mathrm{d}t = 0, \qquad \forall\, \phi \in E_\lambda,\ \forall\, \lambda \in \Lambda\backslash G. \tag{38b}$$

In fact it is sufficient to solve the following block moment problem

$$\int_0^T \left\langle v_G(t), \mathscr{B}^* e^{-t\mathscr{A}^*}\phi\right\rangle_U \mathrm{d}t = \left\langle e^{-T\mathscr{A}} y_0, \phi\right\rangle_{-\diamond,\diamond}, \qquad \forall\, \phi \in E_\lambda,\ \forall\, \lambda \in G, \tag{39a}$$

$$\int_0^T v_G(t)\, t^l e^{-\overline{\lambda}t}\mathrm{d}t = 0, \qquad \forall\, \lambda \in \Lambda\backslash G,\ \forall\, l \in [\![0,\eta[\![\tag{39b}$$

where $e^{-T\mathscr{A}}y_0$ is defined in (21).

Indeed, for any $\phi \in E_\lambda$, from [9, (1.22)], it comes that

$$e^{-t\mathscr{A}^*}\phi = e^{-\lambda t}\sum_{r\geq 0}\frac{(-t)^r}{r!}(\mathscr{A}^*-\lambda)^r\phi = \sum_{r\geq 0} e_t\left[\lambda^{(r+1)}\right](\mathscr{A}^*-\lambda)^r\phi, \tag{40}$$

where the sums are finite (and contains at most the first α_λ terms). Thus, every solution of (39) solves (38). The orthogonality condition (39b) is more restrictive than (38b) but leads to negligible terms in the estimates.

Resolution of block moment problems. In our setting, the block moment problem (39) is proved to be solvable for any $T > 0$. The resolution will follow from the scalar study done in [9] and refined in [12] (see Theorem 46).

Due to (37), the main issue to prove null controllability of (1) is thus to sum those contributions to obtain a solution of (36). This is justified thanks to a precise estimate of the cost of the resolution of (39) for each group G which is the quantity

$$\inf\left\{\|v_G\|_{L^2(0,T;U)}\,;\, v_G \text{ solution of (39)}\right\}.$$

To state this result, we introduce some additional notation.

To solve the moment problem (39) we propose to lift it into a "*vectorial block moment problem*" of the following form (see (59))

$$\begin{cases} \displaystyle\int_0^T v_G(t)\frac{(-t)^l}{l!}e^{-\overline{\lambda}t}\mathrm{d}t = \Omega^l_{\overline{\lambda}}, & \forall\, \lambda \in G,\ \forall\, l \in [\![0,\alpha_\lambda[\![, \\ \displaystyle\int_0^T v_G(t)\, t^l e^{-\overline{\lambda}t}\mathrm{d}t = 0, & \forall\, \lambda \in \Lambda\backslash G,\ \forall\, l \in [\![0,\eta[\![, \end{cases}$$

where $\Omega^l_{\overline{\lambda}}$ belongs to U. Following (40), to recover a solution of (39), we need to impose some constraints on the right-hand side that are given in the following definition.

Definition 8. *For any $\lambda \in \Lambda$ and any $z \in X_{-\diamond}$, we set*

$$\mathscr{O}(\lambda,z) = \left\{(\Omega^0,\ldots,\Omega^{\alpha_\lambda-1}) \in U^{\alpha_\lambda}\,;\, \sum_{l=0}^{\alpha_\lambda-1}\left\langle \Omega^l, \mathscr{B}^*(\mathscr{A}^*-\lambda)^l\phi\right\rangle_U = \langle z,\phi\rangle_{-\diamond,\diamond}, \quad \forall\, \phi \in E_\lambda\right\}. \tag{41}$$

For a given group G, we set

$$\mathscr{O}(G,z) = \prod_{\lambda\in G}\mathscr{O}(\lambda,z) \subset U^{|\alpha|} \tag{42}$$

where α is the multi-index of the algebraic multiplicities of the eigenvalues.

Consider any sequence of multi-indices $(\mu^l)_{l\in[\![0,|\alpha|]\!]}$ such that

$$\begin{cases} \mu^{l-1} \leq \mu^l, & \forall\, l \in [\![1,|\alpha|]\!], \\ \left|\mu^l\right| = l, & \forall\, l \in [\![0,|\alpha|]\!], \\ \mu^{|\alpha|} = \alpha. \end{cases} \tag{43}$$

To measure the cost associated to the group $G = \left\{\lambda_1,\dots,\lambda_g\right\}$ let us define the following functional

$$F : \Omega = \left(\Omega_1^0,\dots,\Omega_1^{\alpha_1-1},\dots,\Omega_g^0,\dots,\Omega_g^{\alpha_g-1}\right) \in U^{|\alpha|} \mapsto \sum_{l=1}^{|\alpha|} \left\| \Omega\left[\overline{\lambda}_{\cdot}^{(\mu^l)}\right]\right\|_U^2 \tag{44}$$

with the convention

$$\Omega\left[\overline{\lambda_j}^{(l+1)}\right] = \Omega_j^l, \quad \forall\, j \in [\![1,g]\!], \forall\, l \in [\![0,\alpha_j[\![.$$

The use of such functional to measure the cost comes from the analysis conducted for scalar controls in [9] (see Proposition 26). It appears in the following lower bound for solutions of block moment problems.

Proposition 9. *Assume that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy the assumption* (H) *(see Section 2.1.2). Let $T \in (0,+\infty)$, and $G \subset \Lambda$ be a group satisfying* (26)*.*

There exists $C_{p,\eta,r_\Lambda} > 0$ such that, for any $z \in X_{-\diamond}$, any $v_G \in L^2(0,T;U)$ solving

$$\int_0^T \left\langle v_G(t), \mathscr{B}^* e^{-t\mathscr{A}^*}\phi\right\rangle_U \mathrm{d}t = \left\langle z,\phi\right\rangle_{-\diamond,\diamond}, \qquad \forall\, \phi \in E_\lambda, \forall\, \lambda \in G$$

satisfies

$$\|v_G\|^2_{L^2(0,T;U)} \geq C_{p,\eta,r_\Lambda}\mathscr{C}(G,z) \tag{45}$$

where

$$\mathscr{C}(G,z) := \inf\{F(\Omega)\,;\, \Omega \in \mathscr{O}(G,z)\} \tag{46}$$

with F defined in (44) *and $\mathscr{O}(G,z)$ defined in Definition 8.*

The first main result of this article concerns the resolution of block moment problems of the form (39). It roughly states that, up to terms that turns out to be negligible, the lower bound obtained in Proposition 9 is optimal.

Theorem 10. *Assume that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy the assumption* (H) *(see Section 2.1.2). Let $T \in (0,+\infty)$, and $G \subset \Lambda$ be a group satisfying* (25)–(27)*.*

For any $z \in X_{-\diamond}$, there exists $v_G \in L^2(0,T;U)$ solution of

$$\int_0^T \left\langle v_G(t), \mathscr{B}^* e^{-t\mathscr{A}^*}\phi\right\rangle_U \mathrm{d}t = \left\langle z,\phi\right\rangle_{-\diamond,\diamond}, \qquad \forall\, \phi \in E_\lambda, \forall\, \lambda \in G, \tag{47a}$$

$$\int_0^T v_G(t)\, t^l e^{-\overline{\lambda} t}\mathrm{d}t = 0, \qquad \forall\, \lambda \in \Lambda\backslash G, \forall\, l \in [\![0,\eta[\![, \tag{47b}$$

satisfying the following estimate

$$\|v_G\|^2_{L^2(0,T;U)} \leq C\exp\left(\frac{C}{T^{\frac{\theta}{1-\theta}}}\right)\exp\left(Cr_G^\theta\right)\mathscr{C}(G,z). \tag{48}$$

In this estimate, $\mathscr{C}(G,z)$ is defined in (46) *and r_G is defined in* (35)*. The constant $C > 0$ appearing in the estimate only depends on the parameters τ, p, ϱ, η, θ and κ.*

Before giving the application of this resolution of block moment problems to the null controllability of our initial system (1), let us give some comments.

- As it was the case in [9], the considered setting allows for a wide variety of applications. In (34) the generalized eigenvectors are only assumed to form a complete family (and not a Riesz basis as in many previous works) which is the minimal assumption to use a moment method-like strategy. The weak gap condition (23) is also well adapted to study systems of coupled one dimensional parabolic equations (see Section 7).
- The main restriction is the assumption (28). As detailed in Section 1.1, this assumption is common to most of the results based on a moment-like method.

  Though restrictive, let us underline that the moment method is, to the best of our knowledge, the most suitable method to capture very sensitive features such as a minimal null control time for parabolic control problems without constraints.
- The main novelty of this theorem is to ensure solvability of block moment problems coming from control problems with control operators that are only assumed to be admissible. In particular, the space U can be of infinite dimension. Results concerning block moment problems with more general right-hand sides, that is not necessarily coming from a controllability problem, are stated in Appendix C
- The estimate (48) does not explicitly depend on the sequence of eigenvalues Λ but rather on some parameters such as the weak-gap parameters and the asymptotic of the counting function. As presented in Section 1.1, the uniformity of such bounds can be used to deal with parameter dependent problems.
- Let us also underline that the obtained estimate (48) tracks the dependency of the constants with respect to the controllability time T. This will be crucial to estimate the cost of controllability in Proposition 20. We refer to Remark 23 for possible applications of such estimates of the cost of controllability.
- Though quite general and useful for the theoretical characterization of the minimal null control time, the obtained estimate (48) still requires to be able to evaluate quantities of the form $\mathscr{C}(G,z)$, which can be intricate. We provide in Section 2.4 some explicit formulas that makes this estimation possible in many actual examples.

2.3. *Determination of the minimal null control time*

The resolution of block moment problems stated in Theorem 10 allows to obtain the following characterization of the minimal null control time of our abstract control problem from a given initial condition.

Theorem 11. *Assume that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy the assumption* (H) *(see Section 2.1.2) and let $\mathscr{G}$ be an associated grouping as stated in Proposition 6. Then, for any $y_0 \in X_{-\diamond}$, the minimal null control time of* (1) *from y_0 is given by*

$$T_0(y_0) = \limsup_{G\in\mathscr{G}} \frac{\ln^+ \mathscr{C}(G,y_0)}{2r_G} \tag{49}$$

where $\mathscr{C}(G,y_0)$ is defined in (46).

In this statement we have used the notation $\ln^+ s = \max(0,\ln s)$, for any $s \geq 0$.

If one considers a space of initial conditions (instead of a single initial condition), the characterization of the minimal null control time is given in the following corollary.

Corollary 12. *Let Y_0 be a closed subspace of $X_{-\diamond}$. Then, under the assumptions of Theorem 11, the minimal null control time from Y_0 is given by*

$$T_0(Y_0) = \limsup_{G\in\mathscr{G}} \frac{\ln^+ \mathscr{C}(G,Y_0)}{2r_G}$$

with

$$\mathscr{C}(G, Y_0) := \sup_{\substack{y_0 \in Y_0 \\ \|y_0\|_{-\diamond}=1}} \mathscr{C}(G, y_0).$$

2.4. *More explicit formulas*

Assume that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy the assumption (H). Let $G \subset \Lambda$ be such that $\sharp G \leq p$ and $\operatorname{diam} G \leq \varrho$. We have seen in Theorem 11 that the key quantity to compute the minimal null control time from y_0 is

$$\mathscr{C}(G, y_0) = \inf\left\{F(\Omega)\,;\, \Omega \in \mathscr{O}(G, y_0)\right\}.$$

where the function F is defined in (44) and the constraints $\mathscr{O}(G, y_0)$ are defined in (42). Let us give more explicit formulas to compute such costs.

Notice that, for any $z \in X_{-\diamond}$, the quantity $\mathscr{C}(G, z)$ can be expressed as a finite dimensional constrained problem. Indeed, for a given group G we consider the finite dimensional subspace

$$U_G = \mathscr{B}^* \operatorname{Span}\left\{\phi \in E_\lambda\,;\, \lambda \in G\right\} \tag{50}$$

and P_{U_G} the orthogonal projection in U onto U_G. Then, for any $\Omega \in \mathscr{O}(G, z)$ it comes that $P_{U_G}\Omega \in \mathscr{O}(G, z)$ and $F(P_{U_G}\Omega) \leq F(\Omega)$. Thus, the optimization problem defining $\mathscr{C}(G, z)$ reduces to

$$\mathscr{C}(G, z) = \inf\left\{F(\Omega)\,;\, \Omega \in \mathscr{O}(G, z) \cap U_G^{|\alpha|}\right\},$$

which is a finite dimensional optimization problem. From [9, Proposition 7.15], the function F is coercive which implies that the infimum is actually attained:

$$\mathscr{C}(G, z) = \min\left\{F(\Omega)\,;\, \Omega \in \mathscr{O}(G, z) \cap U_G^{|\alpha|}\right\}. \tag{51}$$

In this section, solving the optimization problem (51), we provide more explicit formulas for this cost for some particular configurations for the multiplicities of the eigenvalues in the group G (and only in that particular group).

A group G of geometrically simple eigenvalues. First, assume that the eigenvalues in $G = \{\lambda_1, \dots, \lambda_g\}$ are all geometrically simple i.e. $\gamma_\lambda = 1$ for every $\lambda \in G$ where γ_λ is defined in (31).

For any $j \in [\![1, g]\!]$ we denote by ϕ_j^0 an eigenvector of $\mathscr{A}^*$ associated to the eigenvalue λ_j and by $(\phi_j^l)_{l \in [\![0, \alpha_j[\![}$ an associated Jordan chain i.e.

$$(\mathscr{A}^* - \lambda_j)\phi_j^l = \phi_j^{l-1}, \qquad \forall\, l \in [\![1, \alpha_j[\![.$$

To simplify the writing, we set

$$b_j^l := \mathscr{B}^* \phi_j^l \in U, \qquad \forall\, l \in [\![0, \alpha_j[\![, \ \forall\, j \in [\![1, g]\!].$$

Recall that the sequence of multi-index $(\mu^l)_{l \in [\![0, |\alpha|]\!]}$ satisfy (43) and let

$$M := \sum_{l=1}^{|\alpha|} \Gamma_\mu^l \tag{52}$$

with

$$\Gamma_\mu^l := \operatorname{Gram}_U\left(\underbrace{0, \dots, 0}_{l-1}, b\left[\lambda_{\cdot}^{(\mu^l - \mu^{l-1})}\right], \dots, b\left[\lambda_{\cdot}^{(\mu^{|\alpha|} - \mu^{l-1})}\right]\right)$$

where for every $u_1, \dots, u_n \in U$, $\operatorname{Gram}_U(u_1, \dots, u_n)$ denotes the Gram matrix whose entry on the i-th row and j-th column is $\langle u_j, u_i \rangle_U$. To explicit the cost $\mathscr{C}(G, y_0)$, we will use the inverse of this matrix. Its invertibility is guaranteed by the following proposition which is proved in Section 5.2.

Proposition 13. *Under condition* (33)*, the matrix M defined in* (52) *is invertible.*

The matrix M plays a crucial role in the computation of the cost $\mathscr{C}(G, y_0)$. Let us give some comments. It is a sum of Gram matrices whose construction is summarized in Figure 1 on an example with $G = \{\lambda_1, \lambda_2\}$ with $\alpha_1 = 3$ and $\alpha_2 = 2$. Each of these matrices is of size $|\alpha|$ which is the number of eigenvalues (counted with their algebraic multiplicities) that belong to the group G. Thus, on actual examples (see Section 7), the size of these matrices is usually reasonably small.

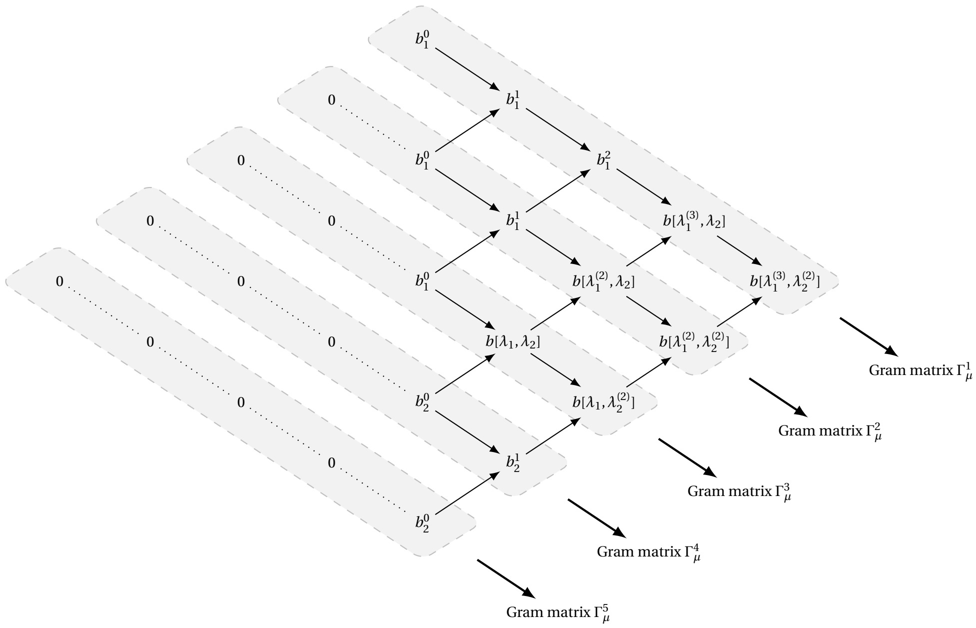

Figure 1. Construction of the Gram matrices Γ_μ^l in the case of a group $G = \{\lambda_1, \lambda_2\}$ with multiplicities $\alpha = (3, 2)$ and the sequence of multi-indices $\mu = \big((0,0), (1,0), (2,0), (3,0), (3,1), (3,2)\big)$

Then, we obtain the following formula for the cost of a group of geometrically simple eigenvalues.

Theorem 14. *Assume that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy the assumption* (H) *(see Section 2.1.2). Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be such that $\sharp G \leq p$ and $\operatorname{diam} G \leq \varrho$ and assume that $\gamma_\lambda = 1$ for every $\lambda \in G$. Then, for any $y_0 \in X_{-\diamond}$, we have*

$$\mathscr{C}(G, y_0) = \left\langle M^{-1}\xi, \xi \right\rangle, \qquad \textit{where } \xi = \begin{pmatrix} \left\langle y_0, \phi\left[\lambda_\cdot^{(\mu^1)}\right]\right\rangle_{-\diamond,\diamond} \\ \vdots \\ \left\langle y_0, \phi\left[\lambda_\cdot^{(\mu^{|\alpha|})}\right]\right\rangle_{-\diamond,\diamond} \end{pmatrix} \in \mathbb{C}^{|\alpha|}$$

and M is defined in (52).

Moreover, if Y_0 is a closed subspace of $X_{-\diamond}$,

$$\mathscr{C}(G, Y_0) = \rho\left(\mathrm{Gram}_{X_\diamond^*}(\psi_1, \dots, \psi_{|\alpha|}) M^{-1}\right) \tag{53}$$

where $\psi_j := P_{Y_0}^* \phi\left[\lambda_\cdot^{(\mu^j)}\right]$ and, for any matrix M, the notation $\rho(\mathsf{M})$ denotes the spectral radius of the matrix M.

Remark 15. Notice that we do not choose any particular eigenvector or Jordan chain. To compute explicitly the cost $\mathscr{C}(G, y_0)$ on actual examples, we will often choose them to satisfy

$$\|b_j^0\|_U = 1, \qquad \left\langle b_j^0, b_j^l \right\rangle_U = 0, \quad \forall\, l \in [\![1, \alpha_j[\![,$$

to simplify the Gram matrices. Obviously, as the quantity $\mathscr{C}(G, y_0)$ is independent of this choice, we can choose any other specific Jordan chains or eigenvectors that are more suitable to each problem.

Remark 16. In the case where the eigenvalues of the considered group G are also algebraically simple, then the expression of M given in (52) reduces to

$$M = \sum_{l=1}^{g} \Gamma^l \quad \text{with} \quad \Gamma^l = \mathrm{Gram}_U\left(\underbrace{0, \dots, 0}_{l-1}, b[\lambda_l], \dots, b[\lambda_l, \dots, \lambda_g]\right) \tag{54}$$

and the expression of ξ reduces to

$$\xi = \begin{pmatrix} \left\langle y_0, \phi[\lambda_1] \right\rangle_{-\diamond,\diamond} \\ \vdots \\ \left\langle y_0, \phi[\lambda_1, \dots, \lambda_g] \right\rangle_{-\diamond,\diamond} \end{pmatrix}.$$

A group G of semi-simple eigenvalues. We now assume that all the eigenvalues in G are semi-simple i.e. for any $\lambda \in G$ we have $\alpha_\lambda = 1$ where α_λ is defined in (32).

For any $j \in [\![1, g]\!]$, we denote by $(\phi_{j,i})_{i \in [\![1, \gamma_j]\!]}$ a basis of $\mathrm{Ker}(\mathscr{A}^* - \lambda_j)$. To simplify the writing, we set

$$b_{j,i} := \mathscr{B}^* \phi_{j,i}, \qquad \forall\, j \in [\![1, g]\!], \forall\, i \in [\![1, \gamma_j]\!]$$

and $\gamma_G := \gamma_1 + \cdots + \gamma_g$.

For any $i \in [\![1, g]\!]$, we set $\delta_1^i := 1$ and

$$\delta_j^i := \prod_{k=1}^{j-1} (\lambda_i - \lambda_k), \qquad \forall\, j \in [\![2, g]\!]. \tag{55}$$

Notice that $\delta_j^i = 0$ as soon as $j > i$.

Let

$$M = \sum_{l=1}^{g} \Gamma^l \text{ with } \Gamma^l = \mathrm{Gram}_U\left(\delta_l^1 b_{1,1}, \dots, \delta_l^1 b_{1,\gamma_1}, \dots, \delta_l^g b_{g,1}, \dots, \delta_l^g b_{g,\gamma_g}\right). \tag{56}$$

Here again, to explicit the cost $\mathscr{C}(G, y_0)$ we will use the inverse of this matrix. Its invertibility is guaranteed by the following proposition which is proved in Section 5.3.

Proposition 17. *Under condition* (33)*, the matrix M defined in* (56) *is invertible.*

Notice that the square matrix Γ^l is of size γ_G and can be seen as a block matrix where the block (i, j) with γ_i rows and γ_j columns is

$$\begin{pmatrix} \left\langle \delta_l^j b_{j,1}, \delta_l^i b_{i,1} \right\rangle_U & \cdots & \left\langle \delta_l^j b_{j,\gamma_j}, \delta_l^i b_{i,1} \right\rangle_U \\ \vdots & & \vdots \\ \left\langle \delta_l^j b_{j,1}, \delta_l^i b_{i,\gamma_i} \right\rangle_U & \cdots & \left\langle \delta_l^j b_{j,\gamma_j}, \delta_l^i b_{i,\gamma_i} \right\rangle_U \end{pmatrix}.$$

Thus, the block (i, j) of Γ^l is identically 0 for $i, j \in [\![1, l[\![$.

Then, we obtain the following formula for the cost of a group made of semi-simple eigenvalues.

Theorem 18. *Assume that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy the assumption* (H) *(see Section 2.1.2). Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be such that $\sharp G \le p$ and $\operatorname{diam} G \le \varrho$ and assume that $\alpha_\lambda = 1$ for every $\lambda \in G$. Then, for any $y_0 \in X_{-\diamond}$, we have*

$$\mathscr{C}(G, y_0) = \left\langle M^{-1}\xi, \xi \right\rangle$$

where

$$\xi = \begin{pmatrix} \left\langle y_0, \phi_{1,1} \right\rangle_{-\diamond,\diamond} \\ \vdots \\ \left\langle y_0, \phi_{1,\gamma_1} \right\rangle_{-\diamond,\diamond} \\ \vdots \\ \left\langle y_0, \phi_{g,1} \right\rangle_{-\diamond,\diamond} \\ \vdots \\ \left\langle y_0, \phi_{g,\gamma_g} \right\rangle_{-\diamond,\diamond} \end{pmatrix}$$

and M is defined in (56).

Moreover, if Y_0 is a closed subspace of $X_{-\diamond}$,

$$\mathscr{C}(G, Y_0) = \rho\left(\operatorname{Gram}_{X_\diamond^*}(\psi_{1,1}, \dots, \psi_{1,\gamma_1}, \dots, \psi_{g,1}, \dots, \psi_{g,\gamma_g}) M^{-1}\right) \tag{57}$$

where $\psi_{j,i} := P_{Y_0}^* \phi_{j,i}$ and, for any matrix M, the notation $\rho(\mathsf{M})$ denotes its spectral radius.

Remark 19. When the eigenvalues of the group G are geometrically and algebraically simple, Theorem 18 gives a characterization of the cost of the block $\mathscr{C}(G, y_0)$ which is different from the one coming from Theorem 14 and detailed in Remark 16. A direct proof of this equivalence (stated in Proposition 55) using algebraic manipulations is given in Appendix D.

Dealing simultaneously with geometric and algebraic multiplicity. Combining Theorems 14 and 18, we can deal with operators $\mathscr{A}^*$ which have both groups of geometrically simple eigenvalues and groups of semi-simple eigenvalues. However, for technical reasons, in the case where both algebraic and geometric multiplicities need to be taken into account into a group G we do obtain a general formula for the cost of this group $\mathscr{C}(G, y_0)$. Nevertheless, if this situation occurs in actual examples, computing this cost is a finite dimensional constrained optimization problem which can be solved "*by hand*". We present in Section 5.4 an example of such resolution for a group G that does not satisfies the assumptions of Theorem 14 nor of Theorem 18.

2.5. *Estimate of the cost of null controllability*

When system (1) is null controllable, we obtain the following bound on the cost of controllability.

Proposition 20. *Assume that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy the assumption* (H) *(see Section 2.1.2) and let $\mathscr{G}$ be an associated grouping as stated in Proposition 6.*

Let $y_0 \in X_{-\diamond}$ and let $T > T_0(y_0)$. There exists a control $u \in L^2(0, T; U)$ such that the associated solution of (1) *initiated from y_0 satisfies $y(T) = 0$ and*

$$\|u\|^2_{L^2(0,T;U)} \le C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{\theta}{1-\theta}}}\right)(1+T)^{2\eta} \sum_{G \in \mathscr{G}} e^{-(T - T_0(y_0)) r_G} e^{-2 r_G T_0(y_0)} \mathscr{C}(G, Y_0).$$

The constant $C > 0$ appearing in the estimate only depends on the parameters τ, p, ϱ, η, θ and κ.

Though quite general the above formula is not very explicit. More importantly, it is proved in [29, Theorem 1.1] that, with a suitable choice of $\mathscr{A}$ and $\mathscr{B}$ satisfying our assumptions, any blow-up of the cost of controllability can occur. We give below a setting (inspired from [29, Theorem 1.2]) in which this upper bound on the cost of controllability is simpler and can have some applications (see Remark 23).

Corollary 21. *Assume that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy the assumption* (H) *(see Section 2.1.2) and let $\mathscr{G}$ be an associated grouping as stated in Proposition 6. Let $\beta > 0$. For any $y_0 \in X_{-\diamond}$ satisfying,*

$$\mathscr{C}(G, y_0) \leq \beta e^{2r_G T_0(y_0)} \|y_0\|_{-\diamond}^2, \qquad \forall\, G \in \mathscr{G}, \tag{58}$$

for any $T > T_0(y_0)$ close enough to $T_0(y_0)$, there exists a control $u \in L^2(0,T;U)$ such that the associated solution of (1) *satisfies $y(T) = 0$ and*

$$\|u\|_{L^2(0,T;U)} \leq C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{\theta}{1-\theta}}}\right) \|y_0\|_{-\diamond},$$

where the constant $C > 0$ only depends on the parameters β, τ, p, ϱ, η, θ and κ.

Remark 22. In the setting of Corollary 21, replacing the assumption (58) by

$$\mathscr{C}(G, y_0) \leq \beta e^{\beta r_G^\sigma} e^{2r_G T_0(y_0)} \|y_0\|_{-\diamond}^2, \qquad \forall\, G \in \mathscr{G},$$

with $\sigma \in (0,1)$ leads to the following estimate

$$\|u\|_{L^2(0,T;U)} \leq C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{\max(\theta,\sigma)}{1-\max(\theta,\sigma)}}}\right) \|y_0\|_{-\diamond}.$$

Remark 23. Giving the best possible estimate on the cost of small time null controllability is a question that has drawn a lot of interest in the past years.

In classical cases, for instance for heat-like equations, null controllability holds in any positive time and the cost of controllability in small time behaves like $\exp\left(\frac{C}{T}\right)$ (see for instance [37]). There are two main applications of such estimate.

- *Controllability in cylindrical domains.* It is proved in [8] that null controllability of parabolic problems in cylindrical geometries (with operators compatible with this geometry) with a boundary control located on the top of the cylinder can be proved thanks to null controllability of the associated problem in the transverse variable together with suitable estimates of the cost of controllability. Their proof relies on an adaptation of the classical strategy of Lebeau and Robbiano [28] and thus uses an estimate of the cost of controllability in small time of the form $\exp\left(\frac{C}{T}\right)$. These ideas were already present in [10] and later generalized in an abstract setting in [1].
- *Nonlinear control problems.* The source term method has been introduced in [30] to prove controllability of a nonlinear fluid-structure system (see also [7, Section 2] for a general presentation of this strategy). Roughly speaking it amounts to prove null controllability with a source term in suitable weighted spaces and then use a fixed point argument. The null controllability with a source term is here proved by an iterative process which strongly uses that the cost of controllability of the linearized system behaves like $\exp\left(\frac{C}{T}\right)$.

Notice that from the upper bound given in Corollary 21, the cost of controllability in small time can explode faster than $\exp\left(\frac{C}{T}\right)$. Yet, as studied in [33] and in [35, Chapter 4], the arguments of the two previous applications can be adapted with an explosion of the cost of the form $\exp\left(\frac{C}{T^{\frac{\theta}{1-\theta}}}\right)$ with $\theta \in (0,1)$.

However, these two applications uses a decomposition of the time interval $[0,T]$ into an infinite number of sub-intervals (which explains the use of the asymptotic of the cost of controllability when the time goes to zero). Thus their extension in the case of a minimal null control time is an open problem.

3. Resolution of block moment problems

In this section we prove Theorem 10 that is we solve the block moment problem (47). To do so, we first consider a vectorial block moment problem (see (59) below) which is proved to be equivalent to the block moment problem (47) in Proposition 24. This equivalence strongly relies on the constraints (41). Then we prove the lower bound for solutions of block moment problems stated in Proposition 9.

Finally, in Section 3.2, we solve the vectorial block moment problem (59) which will conclude the proof of Theorem 10.

3.1. *An auxiliary equivalent vectorial block moment problem*

Let $\Lambda \subset S_\tau$, $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$, $\eta \in \mathbb{N}^*$ and $\alpha = (\alpha_1, \dots, \alpha_g) \in \mathbb{N}^g$ with $|\alpha|_\infty \le \eta$. For any

$$\Omega = \left(\Omega_1^0, \dots, \Omega_1^{\alpha_1 - 1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g - 1}\right) \in U^{|\alpha|},$$

we consider the following auxiliary vectorial block moment problem : find $v_G \in L^2(0,T;U)$ such that

$$\int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\overline{\lambda_j} t} \mathrm{d}t = \Omega_j^l, \qquad \forall\, j \in [\![1,g]\!],\ \forall\, l \in [\![0, \alpha_j[\![, \tag{59a}$$

$$\int_0^T v_G(t) t^l e^{-\overline{\lambda} t} \mathrm{d}t = 0, \qquad \forall\, \lambda \in \Lambda \backslash G,\ \forall\, l \in [\![0, \eta[\![. \tag{59b}$$

This block moment problem is said to be *vectorial*: the right-hand side Ω belongs to $U^{|\alpha|}$ and its solution $v_G(t)$ belongs to the control space U for almost every t. Its resolution with (almost) sharp estimates is given in Proposition 26.

Through (40), when the right-hand side Ω of (59) satisfy the constraints (42), solving this vectorial block moment problem provides a solution of the original block moment problem (47). More precisely we have the following proposition

Proposition 24. *Let $T > 0$ and $z \in X_{-\diamond}$. The following two statements are equivalent:*

(i) *there exists $\Omega \in \mathcal{O}(G,z)$ such that the function $v_G \in L^2(0,T;U)$ solves* (59)*;*

(ii) *the function $v_G \in L^2(0,T;U)$ solves* (47).

Proof. Assume first that there exists $\Omega \in \mathcal{O}(G,z)$ and let $v \in L^2(0,T;U)$ be such that (59) holds.

Then, using (40), for any $j \in [\![1,g]\!]$ and any $\phi \in E_{\lambda_j}$ we have

$$\begin{aligned}
\int_0^T \left\langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U \mathrm{d}t &= \int_0^T \left\langle v(t), e^{-\lambda_j t} \sum_{l=0}^{\alpha_j - 1} \frac{(-t)^l}{l!} \mathcal{B}^* (\mathcal{A}^* - \lambda_j)^l \phi \right\rangle_U \mathrm{d}t \\
&= \sum_{l=0}^{\alpha_j - 1} \left\langle \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\overline{\lambda_j} t} \mathrm{d}t, \mathcal{B}^* (\mathcal{A}^* - \lambda_j)^l \phi \right\rangle_U \\
&= \sum_{l=0}^{\alpha_j - 1} \left\langle \Omega_j^l, \mathcal{B}^* (\mathcal{A}^* - \lambda_j)^l \phi \right\rangle_U .
\end{aligned}$$

Since $\left(\Omega_j^0, \dots, \Omega_j^{\alpha_j - 1}\right) \in \mathcal{O}(\lambda_j, z)$ (see (41)), this leads to

$$\int_0^T \left\langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U \mathrm{d}t = \langle z, \phi \rangle_{-\diamond,\diamond}, \qquad \forall\, j \in [\![1,g]\!], \forall\, \phi \in E_{\lambda_j},$$

which proves that v solves (47).

Assume now that $v \in L^2(0,T;U)$ solves (47). Setting

$$\Omega_j^l := \int_0^T v(t)\frac{(-t)^l}{l!}e^{-\overline{\lambda_j}t}\mathrm{d}t$$

we obtain that v solves (59). As in the previous step, the identity (40) implies that $\Omega \in \mathcal{O}(G,z)$. $\square$

Using this vectorial block moment problem allows to prove the lower bound stated in Proposition 9.

Proof of Proposition 9. Let $v_G \in L^2(0,T;U)$ be any solution of (47a). Let

$$\Omega_j^l := \int_0^T v_G(t)\frac{(-t)^l}{l!}e^{-\overline{\lambda_j}t}\mathrm{d}t = \int_0^T v_G(t)e_t\left[\overline{\lambda_j}^{(l+1)}\right]\mathrm{d}t, \quad \forall\, j \in [\![1,g]\!], \forall\, l \in [\![0,\alpha_j[\![.$$

As in the proof of Proposition 24, the use of (40) implies that

$$\Omega = \left(\Omega_1^0,\dots,\Omega_1^{\alpha_1-1},\dots,\Omega_g^0,\dots,\Omega_g^{\alpha_g-1}\right) \in \mathcal{O}(G,z).$$

Thus,

$$\mathscr{C}(G,z) \le F(\Omega) = \sum_{l=1}^{|\alpha|}\left\|\Omega\left[\overline{\lambda}_{\cdot}^{(\mu^l)}\right]\right\|_U^2. \tag{60}$$

Notice that

$$\Omega\left[\overline{\lambda}_{\cdot}^{(\mu^l)}\right] = \int_0^T v_G(t)e_t\left[\overline{\lambda}_{\cdot}^{(\mu^l)}\right]\mathrm{d}t, \quad \forall\, l \in [\![0,|\alpha|]\!].$$

Using Jensen inequality [9, Proposition 6.1] yields,

$$\left|e_t\left[\overline{\lambda}_{\cdot}^{(\mu^l)}\right]\right| = \left|e_t\left[\lambda_{\cdot}^{(\mu^l)}\right]\right| \le \frac{t^{l-1}e^{-r_G t}}{(l-1)!}.$$

Together with Cauchy–Schwarz inequality this implies

$$\left\|\Omega\left[\overline{\lambda}_{\cdot}^{(\mu^l)}\right]\right\|_U \le \left(\int_0^{+\infty}\frac{t^{l-1}e^{-r_G t}}{(l-1)!}\mathrm{d}t\right)^{\frac{1}{2}}\|v_G\|_{L^2(0,T;U)}.$$

Then, as $r_G \ge r_\Lambda$ and $|\alpha| \le p\eta$, estimate (60) ends the proof of Theorem 10. $\square$

3.2. *Solving the original moment problem*

In view of Proposition 24, to solve (47), we prove that there exists at least one Ω satisfying the constraints (41).

Proposition 25. *Let $\lambda \in \Lambda$ and $z \in X_{-\diamond}$. Then, under assumption* (33), *we have*

$$\mathcal{O}(\lambda,z) \neq \emptyset.$$

Proof. Let $T > 0$. The finite dimensional space E_λ is stable by the semigroup $e^{-\cdot\mathscr{A}^*}$ (see for instance (40)). Using the approximate controllability assumption (33) we have that

$$\phi \in E_\lambda \mapsto \left\|\mathscr{B}^* e^{-\cdot\mathscr{A}^*}\phi\right\|_{L^2(0,T,U)}$$

is a norm on E_λ. Then, the equivalence of norms in finite dimension implies that the following HUM-type functional

$$J : \phi \in E_\lambda \mapsto \frac{1}{2}\left\|\mathscr{B}^* e^{-\cdot\mathscr{A}^*}\phi\right\|_{L^2(0,T,U)}^2 - \Re\langle z,\phi\rangle_{-\diamond,\diamond}$$

is coercive. Let $\widetilde{\phi} \in E_\lambda$ be such that

$$J(\widetilde{\phi}) = \inf_{\phi\in E_\lambda} J(\phi)$$

and $v := \mathscr{B}^* e^{-\cdot\mathscr{A}^*}\widetilde{\phi}$. The optimality condition gives (paying attention to the fact that E_λ is a complex vector space)

$$\int_0^T \left\langle v(t), \mathscr{B}^* e^{-t\mathscr{A}^*}\phi \right\rangle_U \mathrm{d}t = \langle z, \phi\rangle_{-\diamond,\diamond}, \quad \forall\, \phi \in E_\lambda. \tag{61}$$

Finally, we set $\Omega := \left(\Omega^0, \dots, \Omega^{\alpha_\lambda - 1}\right)$ with

$$\Omega^l := \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\overline{\lambda} t} \mathrm{d}t, \quad \forall\, l \in [\![0, \alpha_\lambda[\![.$$

Using (61) and following the computations of Proposition 24 we obtain that $\Omega \in \mathscr{O}(\lambda, z)$. □

We now turn to the resolution of the vectorial block moment problem (59).

Proposition 26. *Let $p \in \mathbb{N}^*$, $\varrho, \tau, \kappa > 0$ and $\theta \in (0,1)$. Assume that*

$$\Lambda \in \mathscr{L}_w(p, \varrho, \tau, \theta, \kappa).$$

Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be a group satisfying (25)–(27). *Let $T \in (0, +\infty)$ and $\eta \in \mathbb{N}^*$. For any multi-index $\alpha \in \mathbb{N}^g$ with $|\alpha|_\infty \le \eta$ and any*

$$\Omega = \left(\Omega_1^0, \dots, \Omega_1^{\alpha_1 - 1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g - 1}\right) \in U^{|\alpha|},$$

there exists $v_G \in L^2(0,T;U)$ solution of (59) *such that*

$$\|v_G\|^2_{L^2(0,T;U)} \le C \exp\left(\frac{C}{T^{\frac{\theta}{1-\theta}}}\right) \exp\left(C r_G^\theta\right) F(\Omega),$$

where F is defined in (44) *and r_G is defined in* (35). *The constant $C > 0$ appearing in the estimate only depends on the parameters τ, p, ϱ, η, θ and κ.*

Proof. Let $(e_j)_{j \in [\![1,d]\!]}$ be an orthonormal basis of the finite dimensional subspace of U given by

$$\operatorname{Span}\left\{\Omega_j^l;\ j \in [\![1,g]\!],\ l \in [\![0, \alpha_j[\![\right\}.$$

Then, for any $j \in [\![1,g]\!]$ and $l \in [\![0, \alpha_j[\![$, there exists $\left(a_i\left[\overline{\lambda_j}^{(l+1)}\right]\right)_{i \in [\![1,d]\!]} \in \mathbb{C}^{|\alpha|}$ such that the decomposition of Ω_j^l reads

$$\Omega_j^l = \sum_{i=1}^d a_i\left[\overline{\lambda_j}^{(l+1)}\right] e_i.$$

From Theorem 46, for any $i \in [\![1,d]\!]$, there exists $v_i \in L^2(0,T;\mathbb{C})$ such that

$$\begin{cases} \displaystyle\int_0^T v_i(t) \frac{(-t)^l}{l!} e^{-\overline{\lambda_j} t} \mathrm{d}t = a_i\left[\overline{\lambda_j}^{(l+1)}\right], & \forall\, j \in [\![1,g]\!],\ \forall\, l \in [\![0, \alpha_j[\![, \\ \displaystyle\int_0^T v_i(t) t^l e^{-\overline{\lambda} t} \mathrm{d}t = 0, & \forall\, \lambda \in \Lambda \backslash G,\ \forall\, l \in [\![0, \eta[\![, \end{cases}$$

and

$$\|v_i\|^2_{L^2(0,T;\mathbb{C})} \le C e^{C T^{-\frac{\theta}{1-\theta}}} e^{C r_G^\theta} \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \le \alpha}} \left| a_i\left[\overline{\lambda_1}^{(\mu_1)}, \dots, \overline{\lambda_g}^{(\mu_g)}\right]\right|^2.$$

Setting

$$v := \sum_{i=1}^d v_i e_i,$$

we get that v solves (59) and using [9, Proposition 7.15]

$$\begin{aligned}\|v\|^2_{L^2(0,T;U)} &= \sum_{i=1}^{d} \|v_i\|^2_{L^2(0,T;\mathbb{C})}\\ &\leq Ce^{CT^{-\frac{\theta}{1-\theta}}} e^{Cr_G^\theta} \sum_{i=1}^{d} \max_{\substack{\mu\in\mathbb{N}^g\\ \mu\leq\alpha}} \left| a_i \left[\overline{\lambda_1}^{(\mu_1)},\dots,\overline{\lambda_g}^{(\mu_g)}\right]\right|^2\\ &\leq C_{p,\varrho,\eta} Ce^{CT^{-\frac{\theta}{1-\theta}}} e^{Cr_G^\theta} \sum_{p=1}^{|\alpha|} \left(\sum_{i=1}^{d} \left| a_i \left[\overline{\lambda}_{\cdot}^{(\mu^p)}\right]\right|^2\right)\\ &= Ce^{CT^{-\frac{\theta}{1-\theta}}} e^{Cr_G^\theta} \sum_{p=1}^{|\alpha|} \left\| \Omega\left[\overline{\lambda}_{\cdot}^{(\mu^p)}\right]\right\|^2.\end{aligned}$$

This ends the proof of Proposition 26. □

We now have all the ingredients to prove Theorem 10.

Proof of Theorem 10. From Proposition 25, we have $\mathscr{O}(G,z) \neq \varnothing$. Recall that, from (51), the optimization problem defining $\mathscr{C}(G,z)$ can be reduced to a finite dimensional optimization problem for which the infimum is attained. Thus, let $\Omega \in \mathscr{O}(G,z)$ be such that

$$F(\Omega) = \mathscr{C}(G,z).$$

Let $v_G \in L^2(0,T;U)$ be the solution of (59) given by Proposition 26 with Ω as right-hand side. As $\Omega \in \mathscr{O}(G,z)$, from Proposition 24 we deduce that v_G solves (47). The upper bound (48) on $\|v_G\|_{L^2(0,T;U)}$ is given by Proposition 26. □

4. Application to the determination of the minimal null control time

This section is dedicated to the consequences of Theorem 10 on the null controllability properties of system (1).

From Theorem 10, the resolution of block moment problems (39) associated with null controllability of (1) will involve the quantity $\mathscr{C}(G, e^{-T\mathscr{A}} y_0)$. To formulate the minimal null control time we isolate the dependency with respect to the variable T leading to quantities involving $\mathscr{C}(G, y_0)$. The comparison between these two costs is detailed in Section 4.1.

Then, this leads to the formulation of the minimal null control time stated in Theorem 11. We then prove the estimates on the cost of null controllability stated in Proposition 20 and Corollary 21. This is detailed in Section 4.2.

4.1. *Relating the different costs*

Let us prove that the cost $\mathscr{C}(G, e^{-T\mathscr{A}} z)$ appearing in Theorem 10 roughly behaves like $e^{-2r_G T}\mathscr{C}(G,z)$. More precisely, we have the following estimates.

Lemma 27. *Assume that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy the assumption* (H) *(see Section 2.1.2). There exists $C_{p,\varrho,\eta} > 0$ such that for any $G \subset \Lambda$ with $\sharp G \leq p$ and* $\operatorname{diam} G \leq \varrho$*, for any $T > 0$ and any $z \in X_{-\diamond}$,*

$$\mathscr{C}(G, e^{-T\mathscr{A}} z) \leq C_{p,\varrho,\eta}(1+T)^{2|\alpha|} e^{-2r_G T}\mathscr{C}(G,z) \tag{62}$$

and

$$e^{-2r_G T}\mathscr{C}(G,z) \leq C_{p,\varrho,\eta}(1+T)^{2|\alpha|} e^{2\varrho T}\mathscr{C}(G, e^{-T\mathscr{A}} z). \tag{63}$$

Proof. Recall that from (21) we have

$$\left\langle y_0, e^{-T\mathscr{A}^*}\phi\right\rangle_{-\diamond,\diamond} = \left\langle e^{-T\mathscr{A}} y_0, \phi\right\rangle_{-\diamond,\diamond}, \qquad \forall\, \phi \in X_\diamond^*.$$

We set $G = \{\lambda_1, \dots, \lambda_g\}$.

We start with the proof of (62).

From (51), let $\widetilde{\Omega} \in \mathscr{O}(G,z)$ be such that $F(\widetilde{\Omega}) = \mathscr{C}(G,z)$. We define Ω by

$$\Omega_j^l := (e_T \widetilde{\Omega})\left[\overline{\lambda_j}^{(l+1)}\right], \qquad \forall\, j \in [\![1,g]\!], \forall\, l \in [\![0,\alpha_j[\![,$$

with the convention

$$\widetilde{\Omega}\left[\overline{\lambda_j}^{(l+1)}\right] = \widetilde{\Omega}_j^l, \qquad \forall\, j \in [\![1,g]\!], \forall\, l \in [\![0,\alpha_j[\![.$$

Let us prove that $\Omega \in \mathscr{O}(G, e^{-T\mathscr{A}} z)$. For any $j \in [\![1,g]\!]$ and any $\phi \in E_{\lambda_j}$, using [9, Definition 7.12] we obtain

$$\begin{aligned}
\sum_{l\geq 0}\left\langle \Omega_j^l, \mathscr{B}^*(\mathscr{A}^*-\lambda_j)^l\phi\right\rangle_U &= \sum_{l\geq 0}\sum_{r=0}^{l} e_T\left[\overline{\lambda_j}^{(r+1)}\right]\left\langle \widetilde{\Omega}_j^{l-r}, \mathscr{B}^*(\mathscr{A}^*-\lambda_j)^l\phi\right\rangle_U \\
&= \sum_{r\geq 0} e_T\left[\overline{\lambda_j}^{(r+1)}\right]\sum_{l\geq r}\left\langle \widetilde{\Omega}_j^{l-r}, \mathscr{B}^*(\mathscr{A}^*-\lambda_j)^l\phi\right\rangle_U \\
&= \sum_{r\geq 0} e_T\left[\overline{\lambda_j}^{(r+1)}\right]\sum_{l\geq 0}\left\langle \widetilde{\Omega}_j^{l}, \mathscr{B}^*(\mathscr{A}^*-\lambda_j)^{l+r}\phi\right\rangle_U.
\end{aligned}$$

Since $\widetilde{\Omega} \in \mathscr{O}(G,z)$ and $e_T\left[\overline{\lambda_j}^{(r+1)}\right] = \overline{e_T\left[\lambda_j^{(r+1)}\right]}$ for any $r \geq 0$, using (40) this yields

$$\begin{aligned}
\sum_{l\geq 0}\left\langle \Omega_j^l, \mathscr{B}^*(\mathscr{A}^*-\lambda_j)^l\phi\right\rangle_U &= \sum_{r\geq 0}\overline{e_T\left[\lambda_j^{(r+1)}\right]}\left\langle z, \left((\mathscr{A}^*-\lambda_j)^r\phi\right)\right\rangle_{-\diamond,\diamond} \\
&= \left\langle z, e^{-T\mathscr{A}^*}\phi\right\rangle_{-\diamond,\diamond} = \left\langle e^{-T\mathscr{A}} z, \phi\right\rangle_{-\diamond,\diamond}.
\end{aligned} \tag{64}$$

This proves the claim.

Applying Leibniz formula [9, Proposition 7.13] and Jensen inequality [9, Proposition 6.1] we obtain,

$$\begin{aligned}
\left\|\Omega\left[\overline{\lambda_\cdot}^{(\mu^l)}\right]\right\|_U &= \left\|\sum_{q=1}^{l} e_T\left[\overline{\lambda_\cdot}^{(\mu^l-\mu^{q-1})}\right]\widetilde{\Omega}\left[\overline{\lambda_\cdot}^{(\mu^q)}\right]\right\| \\
&\leq C_{p,\varrho,\eta}(1+T)^{|\alpha|}e^{-r_G T}\left(\sum_{q=1}^{l}\left\|\widetilde{\Omega}\left[\overline{\lambda_\cdot}^{(\mu^q)}\right]\right\|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$F(\Omega) \leq C_{p,\varrho,\eta}(1+T)^{2|\alpha|}e^{-2r_G T}F(\widetilde{\Omega}) = C_{p,\varrho,\eta}(1+T)^{2|\alpha|}e^{-2r_G T}\mathscr{C}(G,z).$$

As $\Omega \in \mathscr{O}(G, e^{-T\mathscr{A}} z)$, this proves (62).

The proof of (63) uses the same ingredients.

From (51), let $\Omega \in \mathscr{O}(G, e^{-T\mathscr{A}} z)$ be such that $F(\Omega) = \mathscr{C}(G, e^{-T\mathscr{A}} z)$. For any $j \in [\![1,g]\!]$ and any $l \in [\![0,\alpha_j[\![$, let

$$\widetilde{\Omega}_j^l := (e_{-T}\Omega)\left[\overline{\lambda_j}^{(l+1)}\right]$$

where

$$\Omega\left[\overline{\lambda_j}^{(l+1)}\right] := \Omega_j^l.$$

As previously, applying Leibniz formula [9, Proposition 7.13] and Jensen inequality [9, Proposition 6.1], since λ_j satisfies $\Re\lambda_j \leq r_G + \varrho$ for any $j \in [\![1,g]\!]$, we obtain

$$\left\|\widetilde{\Omega}\left[\overline{\lambda_\cdot}^{(\mu^l)}\right]\right\|_U \leq C_{p,\varrho,\eta}(1+T)^{2|\alpha|}e^{(r_G+\varrho)T}\left(\sum_{q=1}^{l}\left\|\Omega\left[\overline{\lambda_\cdot}^{(\mu^q)}\right]\right\|^2\right)^{\frac{1}{2}}.$$

The same computations as in (64) give that $\Omega \in \mathscr{O}(G,z)$. Thus

$$\begin{aligned}\mathscr{C}(G,z) \le F(\widetilde{\Omega}) &\le C_{p,\varrho,\eta}(1+T)^{2|\alpha|} e^{2(r_G+\varrho)T} F(\Omega)\\ &= C_{p,\varrho,\eta}(1+T)^{2|\alpha|} e^{2(r_G+\varrho)T} \mathscr{C}(G, e^{-T\mathscr{A}} z)\end{aligned}$$

and (63) is proved. □

4.2. *The minimal null control time*

This section is dedicated to the proof of Theorem 11 and Corollary 12 concerning the minimal null control time. Proposition 20 and Corollary 21 concerning the cost of null controllability will follow from the estimates obtained in the proof of Theorem 11. This is discussed at the end of the current section.

Proof of Theorem 11. We start with the proof of null controllability in time $T > T_0(y_0)$.

We set $\varepsilon = T - T_0(y_0) > 0$. Let $G \in \mathscr{G}$ and let $v_G \in L^2(0,\varepsilon;U)$ be the solution of the block moment problem (47) in time ε associated with $z = e^{-T\mathscr{A}} y_0$ given by Theorem 10 i.e.

$$\begin{aligned}&\int_0^\varepsilon \left\langle v_G(t), \mathscr{B}^* e^{-t\mathscr{A}^*}\phi\right\rangle_U \mathrm{d}t = \left\langle e^{-T\mathscr{A}} y_0, \phi\right\rangle_{-\diamond,\diamond}, && \forall\, \phi \in E_\lambda,\ \forall\, \lambda \in G,\\ &\int_0^\varepsilon v_G(t)\, t^l e^{-\overline{\lambda} t}\mathrm{d}t = 0, && \forall\, \lambda \in \Lambda\backslash G,\ \forall\, l \in [\![0,\eta[\![.\end{aligned}$$

We still denote by $v_G \in L^2(0,T;U)$ the extension of v_G by 0. Thus, v_G satisfies

$$\begin{aligned}&\int_0^T \left\langle v_G(t), \mathscr{B}^* e^{-t\mathscr{A}^*}\phi\right\rangle_U \mathrm{d}t = \left\langle e^{-T\mathscr{A}} y_0, \phi\right\rangle_{-\diamond,\diamond}, && \forall\, \phi \in E_\lambda,\ \forall\, \lambda \in G,\\ &\int_0^T v_G(t)\, t^l e^{-\overline{\lambda} t}\mathrm{d}t = 0, && \forall\, \lambda \in \Lambda\backslash G,\ \forall\, l \in [\![0,\eta[\![.\end{aligned}$$

From (40), this implies that v_G solves (38). Thus, the only point left is to prove that the series (37) defining the control u converges in $L^2(0,T;U)$.

From Theorem 10 we have that

$$\|v_G\|^2_{L^2(0,T;U)} = \|v_G\|^2_{L^2(0,\varepsilon;U)} \le C e^{C\varepsilon^{-\frac{\theta}{1-\theta}}} e^{C r_G^\theta} \mathscr{C}(G, e^{-T\mathscr{A}} y_0).$$

By the Young inequality we get

$$e^{C r_G^\theta} \le e^{C\varepsilon^{-\frac{\theta}{1-\theta}}} e^{\varepsilon r_G}$$

where in the right-hand side the constant $C > 0$ has changed but still depend on the same parameters. Thus,

$$\|v_G\|^2_{L^2(0,T;U)} \le C e^{C\varepsilon^{-\frac{\theta}{1-\theta}}} e^{\varepsilon r_G} \mathscr{C}(G, e^{-T\mathscr{A}} y_0).$$

Using (62) we obtain

$$\|v_G\|^2_{L^2(0,T;U)} \le C e^{C\varepsilon^{-\frac{\theta}{1-\theta}}} (1+T)^{2|\alpha|} e^{-\varepsilon r_G} e^{-2r_G(T-\varepsilon)} \mathscr{C}(G, y_0).$$

Recalling that $\varepsilon = T - T_0(y_0)$ this gives

$$\|v_G\|^2_{L^2(0,T;U)} \le C \exp\left(\frac{C}{\left(T - T_0(y_0)\right)^{\frac{\theta}{1-\theta}}}\right) (1+T)^{2|\alpha|} e^{-(T-T_0(y_0))r_G} e^{-2r_G T_0(y_0)} \mathscr{C}(G, y_0). \tag{65}$$

Recall that in (49) we have defined $T_0(y_0)$ by

$$T_0(y_0) = \limsup_{G\in\mathscr{G}} \frac{\ln^+ \mathscr{C}(G, y_0)}{2r_G}.$$

Thus, when r_G is sufficiently large, we have

$$e^{-2r_G T_0(y_0)}\mathscr{C}(G,y_0) \leq \exp\left(\frac{T-T_0(y_0)}{2} r_G\right).$$

Together with (65) this implies, for r_G sufficiently large,

$$\|v_G\|^2_{L^2(0,T;U)} \leq C\exp\left(\frac{C}{\left(T-T_0(y_0)\right)^{\frac{\theta}{1-\theta}}}\right)(1+T)^{2|\alpha|}\exp\left(-\frac{T-T_0(y_0)}{2} r_G\right)$$

and proves that the series

$$u = \sum_{G\in\mathscr{G}} v_G(T-\cdot) \tag{66}$$

converges in $L^2(0,T;U)$. This proves null controllability of (1) from y_0 in any time $T > T_0(y_0)$.

We now end the proof of Theorem 11 by proving that null controllability does not hold in time $T < T_0(y_0)$. The proof mainly relies on the optimality of the resolution of the block moment problems given in Proposition 9 (see (45)).

Let $T > 0$. Assume that problem (1) is null controllable from y_0 in time T. Thus there exists $u \in L^2(0,T;U)$ such that $y(T) = 0$ and

$$\|u\|_{L^2(0,T;U)} \leq C_T \left\|y_0\right\|_{-\diamond}.$$

Let $v := -u(T-\cdot)$. Then, for any $G \in \mathscr{G}$, v satisfies (47a) with $z = e^{-T\mathscr{A}} y_0$. From (45), this implies

$$C_T^2 \left\|y_0\right\|^2_{-\diamond} \geq \|u\|^2_{L^2(0,T;U)} = \|v\|^2_{L^2(0,T;U)} \geq C_{p,\eta,r_\Lambda}\mathscr{C}(G, e^{-T\mathscr{A}} y_0). \tag{67}$$

Applying (63) we obtain

$$\mathscr{C}(G,y_0) \leq C_{T,p,\varrho,\eta} e^{2r_G T}\mathscr{C}(G, e^{-T\mathscr{A}} y_0).$$

Together with (67) this implies

$$\mathscr{C}(G,y_0) \leq C_{T,p,\varrho,\eta,r_\Lambda} \left\|y_0\right\|^2_{-\diamond} e^{2r_G T}. \tag{68}$$

Getting back to the definition of $T_0(y_0)$ given in (49), this implies that $T \geq T_0(y_0)$ and ends the proof of Theorem 11. □

Remark 28. It is worth noticing that the solution v_G constructed above is only active on the time interval $(0, T-T_0(y_0))$. Thus, whenever $T_0(y_0) > 0$, the series (66) defining the control u proves that it is possible to control y_0 to 0 in any time $T > T_0(y_0)$ with a control that is identically vanishing on the time interval $\left(0, T_0(y_0)\right)$.

We now turn to the proof of Corollary 12.

Proof of Corollary 12. By definition, we have $T_0(Y_0) = \sup_{y_0\in Y_0} T_0(y_0)$. Using the definition of $\mathscr{C}(G,Y_0)$ and Theorem 11, it directly comes that

$$T_0(Y_0) \leq \limsup_{G\in\mathscr{G}} \frac{\ln^+\mathscr{C}(G,Y_0)}{2r_G}.$$

We now focus on the converse inequality. Let $T > 0$ such that

$$T < \limsup_{G\in\mathscr{G}} \frac{\ln^+\mathscr{C}(G,Y_0)}{2r_G}$$

and let us prove that $T \leq T_0(Y_0)$.

There exists $\varepsilon > 0$ and a sequence of groups $(G_k)_{k\in\mathbb{N}} \in \mathscr{G}^{\mathbb{N}}$ such that for any $k \in \mathbb{N}^*$, there exists $y_{0,k} \in Y_0$ with $\left\|y_{0,k}\right\|_{-\diamond} = 1$ satisfying

$$T+\varepsilon < \frac{\ln\mathscr{C}(G_k, y_{0,k})}{2r_{G_k}}. \tag{69}$$

By contradiction, assume that for any $y_0 \in Y_0$, we have $T > T_0(y_0)$. Thus, from (68), there exists $C_{T,p,\varrho,\eta,r_\Lambda} > 0$ such that for any $k \in \mathbb{N}^*$

$$\frac{\ln \mathscr{C}(G_k, y_{0,k})}{2 r_{G_k}} \le \frac{\ln C_{T,p,\varrho,\eta,r_\Lambda}}{2 r_{G_k}} + T.$$

Taking k sufficiently large, this is in contradiction with (69). □

We end this section with the proof of Proposition 20 and Corollary 21 concerning the cost of null controllability.

A careful inspection of the proof of null controllability in time $T > T_0(y_0)$ detailed in Section 4.2 allows to give a bound on the cost of controllability.

Proof of Proposition 20 and Corollary 21. The proof of Proposition 20 follows directly from (37) and (65).

The proof of Corollary 21 then follows directly from Proposition 20, assumption (58) and the estimate

$$\sum_{G \in \mathscr{G}} e^{-r_G x} \le \frac{C_{\theta,\kappa}}{x^\theta}, \qquad \forall\, x > 0,$$

proved in [12, Proposition A.5.38]. □

5. Computation of the cost of a block

In this section we prove more explicit formulas to estimate the cost $\mathscr{C}(G, y_0)$ of the resolution of a block moment problem depending on the assumptions on the eigenvalues in the group G. More precisely, we prove here Theorems 14 and 18. For pedagogical purpose, we start in Section 5.1 with Theorem 14 for algebraically (and geometrically) simple eigenvalues i.e. when $\alpha_\lambda = \gamma_\lambda = 1$ for any $\lambda \in G$. Then, in Section 5.2, we prove the general statement of Theorem 14 that is when all the eigenvalues in the group are geometrically simple i.e. $\gamma_\lambda = 1$ for any $\lambda \in G$.

The formula for the cost $\mathscr{C}(G, y_0)$ when all the eigenvalues in the group G are semi-simple (i.e. $\alpha_\lambda = 1$ for any $\lambda \in G$) stated in Theorem 18 is then proved in Section 5.3. The extension to spaces of initial conditions (53) and (57) does not depend on the matrix M and follows directly from Lemma 48. Thus, their proofs are not detailed here.

When both algebraic and geometric multiplicities appear in the same group we do not get a general formula but describe the procedure on an example in Section 5.4.

Recall that from (51), computing $\mathscr{C}(G, y_0)$ is a finite dimensional optimization problem given by

$$\mathscr{C}(G, y_0) = \min \left\{ F(\Omega)\,;\, \Omega \in \mathscr{O}(G, y_0) \cap U_G^{|\alpha|} \right\}$$

where the function F is defined in (44), the constraints associated with $\mathscr{O}(G, y_0)$ are defined in (42) and U_G is defined in (50).

5.1. *The case of simple eigenvalues*

In all this section, we consider the simpler case where $\alpha_\lambda = \gamma_\lambda = 1$ for every $\lambda \in G$. Thus, in the rest of this section, we drop the superscript 0 associated to eigenvectors.

We start with the proof of the invertibility of the matrix M stated in Proposition 13.

Proof. Recall that, as $\alpha_\lambda = \gamma_\lambda = 1$, the positive semi-definite matrix M is defined in (54). Let $\tau \in \mathbb{C}^g$ be such that $\langle M\tau, \tau \rangle = 0$. Then, for each $l \in [\![1, g]\!]$, we have

$$\left\langle \Gamma^l \tau, \tau \right\rangle = 0.$$

We prove that $\tau = 0$. By contradiction let

$$l = \max\{j \in [\![1,g]\!]\,;\, \tau_j \neq 0\}.$$

Then from (54) this leads to $\left\langle \Gamma^l \tau, \tau \right\rangle = \|b[\lambda_l]\|_U^2 |\tau_l|^2$. Using (33) implies $\tau_l = 0$. This is in contradiction with the definition of l which proves the invertibility of M. □

We now prove Theorem 14.

Proof. First of all, notice that the function F to minimize reduces to

$$F(\Omega) = \sum_{j=1}^{g} \left\| \Omega[\overline{\lambda_1}, \dots, \overline{\lambda_j}] \right\|^2$$

and, as $\gamma_\lambda = \alpha_\lambda = 1$, the constraints defining the set $\mathcal{O}(\lambda_j, y_0)$ reduce to

$$\left\langle \Omega_j, b_j \right\rangle_U = \left\langle y_0, \phi_j \right\rangle_{-\diamond,\diamond}.$$

Thus, the minimization problem reduces to

$$\mathscr{C}(G, y_0) = \min \left\{ F(\Omega)\,;\, \begin{array}{l} \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \text{ such that} \\ \left\langle \Omega_j, b_j \right\rangle_U = \left\langle y_0, \phi_j \right\rangle_{-\diamond,\diamond},\ \forall\, j \in [\![1,g]\!] \end{array} \right\}. \tag{70}$$

For the sake of generality, let us consider for this proof any $\omega_1, \dots, \omega_g \in \mathbb{C}$ and the more general constraints

$$\left\langle \Omega_j, b_j \right\rangle_U = \omega_j, \qquad \forall\, j \in [\![1,g]\!]. \tag{71}$$

Using the formalism of divided differences, this is equivalent to the family of constraints

$$\langle \Omega, b \rangle_U [\overline{\lambda_1}, \dots, \overline{\lambda_j}] = \omega[\overline{\lambda_1}, \dots, \overline{\lambda_j}], \qquad \forall\, j \in [\![1,g]\!]. \tag{72}$$

We consider the constrained complex minimization problem

$$\min \left\{ F(\Omega)\,;\, \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \text{ such that (72) holds} \right\}.$$

It has a unique solution, which is characterised by the existence of multipliers $(m_j)_{j \in [\![1,g]\!]} \subset \mathbb{C}$ such that

$$\sum_{j=1}^{g} \left\langle H[\overline{\lambda_1}, \dots, \overline{\lambda_j}], \Omega[\overline{\lambda_1}, \dots, \overline{\lambda_j}] \right\rangle_U = \sum_{j=1}^{g} \overline{m_j} \, \langle H, b \rangle_U [\overline{\lambda_1}, \dots, \overline{\lambda_j}], \tag{73}$$

for any $H_1, \dots, H_g \in U_G$.

Then, for a given $q \in [\![1,g]\!]$, using Leibniz formula [12, Proposition A.2.11], the constraints (72) can be rewritten as

$$\omega[\overline{\lambda_1}, \dots, \overline{\lambda_q}] = \langle \Omega, b \rangle_U [\overline{\lambda_1}, \dots, \overline{\lambda_q}] = \sum_{j=1}^{q} \left\langle \Omega[\overline{\lambda_1}, \dots, \overline{\lambda_j}], b[\lambda_j, \dots, \lambda_q] \right\rangle_U \tag{74}$$

To relate (74) and (73), we look for $H_1, \dots, H_g \in U_G$ such that, for a given $q \in [\![1,g]\!]$ we have

$$H[\overline{\lambda_1}, \dots, \overline{\lambda_j}] = \begin{cases} b[\lambda_j, \dots, \lambda_q], & \text{for } j \le q, \\ 0, & \text{for } j > q. \end{cases}$$

This can be done by setting $H_1 = b[\lambda_1, \dots, \lambda_q]$ and, from the interpolation formula [9, Proposition 7.6], by defining H_j by the formula

$$H_j = \sum_{i=1}^{j} \left(\prod_{k=1}^{j-1} (\overline{\lambda_i} - \overline{\lambda_k}) \right) H[\overline{\lambda_1}, \dots, \overline{\lambda_i}], \quad \forall\, j \in [\![2,g]\!].$$

Then, from (74) we obtain

$$\omega[\overline{\lambda_1}, \dots, \overline{\lambda_q}] = \sum_{j=1}^{g} \left\langle \Omega[\overline{\lambda_1}, \dots, \overline{\lambda_j}], H[\overline{\lambda_1}, \dots, \overline{\lambda_j}] \right\rangle_U.$$

Now relation (73) leads, after conjugation, to

$$\overline{\omega[\overline{\lambda_1},\ldots,\overline{\lambda_q}]} = \sum_{j=1}^{g} \overline{m_j}\, \langle H, b\rangle_U\, [\overline{\lambda_1},\ldots,\overline{\lambda_j}].$$

The application of Leibniz formula [12, Proposition A.2.11] yields

$$\begin{aligned}\overline{\omega[\overline{\lambda_1},\ldots,\overline{\lambda_q}]} &= \sum_{j=1}^{g} \overline{m}_j \left(\sum_{l=1}^{j} \left\langle H[\overline{\lambda_1},\ldots,\overline{\lambda_l}], b[\lambda_l,\ldots,\lambda_j]\right\rangle_U\right)\\ &= \sum_{j=1}^{g} \overline{m}_j \left(\sum_{l=1}^{\min(j,q)} \left\langle b[\lambda_l,\ldots,\lambda_q], b[\lambda_l,\ldots,\lambda_j]\right\rangle_U\right).\end{aligned}$$

Conjugating this relation leads to

$$\begin{aligned}\omega[\overline{\lambda_1},\ldots,\overline{\lambda_q}] &= \sum_{j=1}^{g} m_j \left(\sum_{l=1}^{\min(j,q)} \left\langle b[\lambda_l,\ldots,\lambda_j], b[\lambda_l,\ldots,\lambda_q]\right\rangle_U\right)\\ &= \sum_{l=1}^{g}\sum_{j=1}^{g} m_j \Gamma^l_{q,j} = (Mm)_q,\end{aligned}$$

where Γ^l and M are defined in (54).

Let

$$\xi := \begin{pmatrix} \omega[\overline{\lambda_1}] \\ \vdots \\ \omega[\overline{\lambda_1},\ldots,\overline{\lambda_g}] \end{pmatrix} \in \mathbb{C}^g.$$

We have just proved that $m = M^{-1}\xi$. Getting back to (73) with $H = \Omega$ together with the constraints (72), we obtain

$$F(\Omega) = \sum_{j=1}^{g} \overline{m}_j\, \langle \Omega, b\rangle_U\, [\overline{\lambda_1},\ldots,\overline{\lambda_j}] = \left\langle M^{-1}\xi, \xi\right\rangle.$$

With the specific choice, $\omega_j = \left\langle y_0, \phi_j\right\rangle_{-\diamond,\diamond}$, this ends the proof of Theorem 14 with the extra assumption that $\alpha_\lambda = 1$ for all $\lambda \in G$. Indeed, by anti-linearity we have

$$\omega[\overline{\lambda_1},\ldots,\overline{\lambda_j}] = \left\langle y_0, \phi[\lambda_1,\ldots,\lambda_j]\right\rangle_{-\diamond,\diamond}, \qquad \forall\, j \in [\![1,g]\!]. \qquad \square$$

Remark 29. As mentioned in Remark 15, estimate (70) implies that the cost of the block G (i.e. the quantity $\left\langle M^{-1}\xi,\xi\right\rangle$) can be estimated using any eigenvectors: there is no normalization condition.

Remark 30. Rewriting the constraints in the form (72) is not mandatory but, as the function to minimize involves divided differences, it leads to more exploitable formulas and will ease the writing when dealing with algebraic multiplicity of eigenvalues. Dealing directly with (71) would lead to the expression (131) for the cost of the block G as it will appear in the proof of Theorem 18.

5.2. *The case of geometrically simple eigenvalues*

The proof of Proposition 13 and Theorem 14 under the sole assumption $\gamma_\lambda = 1$ for any $\lambda \in G$ follows closely the proof done in Section 5.1. The main difference is the use of generalized divided differences (see [9, Section 7.3]) instead of classical divided differences as detailed below.

Proof of Proposition 13. Due to (43), for any $l \in [\![1,|\alpha|]\!]$ the multi-index $\mu^l - \mu^{l-1}$ is composed of only one 1 and $g-1$ zeros. Thus,

$$b\left[\lambda_{\cdot}^{(\mu^l-\mu^{l-1})}\right] = b_j^0$$

for a certain $j \in [\![1,g]\!]$. From (33) it comes that

$$b\left[\lambda_{\cdot}^{(\mu^l-\mu^{l-1})}\right] \neq 0, \qquad \forall\, l \in [\![1,|\alpha|]\!].$$

The rest of the proof follows as in Section 5.1. □

Proof of Theorem 14. As $\gamma_\lambda = 1$, the constraints defining the set $\mathscr{O}(\lambda_j, y_0)$ reduce to

$$\sum_{r=0}^{l}\left\langle \Omega_j^r, b_j^{l-r}\right\rangle_U = \sum_{r=0}^{l}\left\langle \Omega_j^r, \mathscr{B}^*(\mathscr{A}^*-\lambda_j)^r\phi_j^l\right\rangle_U = \left\langle y_0, \phi_j^l\right\rangle_{-\diamond,\diamond}, \quad \forall\, l \in [\![0,\alpha_j[\![.$$

By definition of $\langle \Omega, b\rangle_U\left[\overline{\lambda_j}^{(l+1)}\right]$, this is equivalent to

$$\langle \Omega, b\rangle_U\left[\overline{\lambda_j}^{(l+1)}\right] = \left\langle y_0, \phi_j^l\right\rangle_{-\diamond,\diamond}, \quad \forall\, l \in [\![0,\alpha_j[\![.$$

Thus,

$$\mathscr{C}(G,y_0) = \min\left\{ F(\Omega)\,;\ \begin{array}{l} \Omega = (\Omega_1^0,\dots,\Omega_1^{\alpha_1-1},\dots,\Omega_g^0,\dots,\Omega_g^{\alpha_g-1}) \in U_G^{|\alpha|} \text{ such that} \\ \langle \Omega, b\rangle_U\left[\overline{\lambda_j}^{(l+1)}\right] = \left\langle y_0, \phi_j^l\right\rangle_{-\diamond,\diamond}, \quad \forall\, j \in [\![1,g]\!], \forall\, l \in [\![0,\alpha_j[\![\end{array}\right\}. \tag{75}$$

For the sake of generality, let us consider for this proof any

$$\left(\omega_1^0,\dots,\omega_1^{\alpha_1-1},\dots,\omega_g^0,\dots,\omega_g^{\alpha_g-1}\right) \in \mathbb{C}^{|\alpha|}$$

and the more general constraints

$$\langle \Omega, b\rangle_U\left[\overline{\lambda_j}^{(l+1)}\right] = \omega_j^l, \quad \forall\, j \in [\![1,g]\!], \forall\, l \in [\![0,\alpha_j[\![.$$

From (43), this is equivalent to the family of constraints

$$\langle \Omega, b\rangle_U\left[\overline{\lambda_{\cdot}}^{(\mu^p)}\right] = \omega\left[\overline{\lambda_{\cdot}}^{(\mu^p)}\right], \quad \forall\, p \in [\![1,|\alpha|]\!],$$

and we proceed as in Section 5.1. The only difference is the use of generalized divided differences. For instance, the equation (73) now reads

$$\sum_{l=1}^{|\alpha|}\left\langle H\left[\overline{\lambda_{\cdot}}^{(\mu^l)}\right], \Omega\left[\overline{\lambda_{\cdot}}^{(\mu^l)}\right]\right\rangle_U = \sum_{l=1}^{|\alpha|}\overline{m}_l\,\langle H, b\rangle_U\left[\overline{\lambda_{\cdot}}^{(\mu^l)}\right], \qquad \forall\, H = \left(H_j^l\right) \in U_G^{|\alpha|}.$$

The rest of the proof remains unchanged. □

Remark 31. As mentioned in Remark 15, estimate (75) implies that the cost of the block G (i.e. the quantity $\langle M^{-1}\xi,\xi\rangle$) can be estimated using any eigenvectors and any associated Jordan chains.

5.3. *The case of semi-simple eigenvalues*

We start with the proof of Proposition 17.

Proof of Proposition 17. Recall that the positive semi-definite matrix M is defined in (56). Let $\tau \in \mathbb{C}^{\gamma_G}$ be such that $\langle M\tau,\tau\rangle = 0$. Then, for any $l \in [\![1,g]\!]$, $\langle \Gamma^l\tau,\tau\rangle = 0$. We prove that $\tau = 0$. By contradiction let

$$\widetilde{l} = \max\{j \in [\![1,\gamma_G]\!]\,;\ \tau_j \neq 0\}$$

and $l \in [\![1,g]\!]$ be such that

$$\gamma_1 + \dots + \gamma_{l-1} < \widetilde{l} \leq \gamma_1 + \dots + \gamma_l$$

with the convention that $l=1$ when $\widetilde{l} \leq \gamma_1$. We denote by $\sigma \in \mathbb{C}^{\gamma_l}$ the l^{th} block of τ i.e.

$$\sigma = \begin{pmatrix} \tau_{\gamma_1+\cdots+\gamma_{l-1}+1} \\ \vdots \\ \tau_{\gamma_1+\cdots+\gamma_l} \end{pmatrix}.$$

From (55) we have $\delta_l^i = 0$ when $i < l$. Thus all the blocks (i,j) of Γ^l are equal to 0 when $i, j \in [\![1, l[\![$. This leads to

$$\left\langle \Gamma^l \tau, \tau \right\rangle = \left|\delta_l^l\right|^2 \left\langle \mathrm{Gram}_U\left(b_{l,1}, \ldots, b_{l,\gamma_l}\right)\sigma, \sigma \right\rangle.$$

As the eigenvalues $\lambda_1, \ldots, \lambda_g$ are distinct it comes that $\delta_l^l \neq 0$ (see (55)) which implies

$$\left\langle \mathrm{Gram}_U\left(b_{l,1}, \ldots, b_{l,\gamma_l}\right)\sigma, \sigma \right\rangle = 0.$$

From (33), we have that $b_{l,1}, \ldots, b_{l,\gamma_l}$ are linearly independent. This proves the invertibility of $\mathrm{Gram}_U\left(b_{l,1}, \ldots, b_{l,\gamma_l}\right)$ and gives $\sigma = 0$. This is in contradiction with the definition of $\widetilde{l}$ which proves the invertibility of M. □

We now turn to the proof of Theorem 18.

Proof of Theorem 18. First of all, notice that the function F to minimize reduces to

$$F(\Omega) = \sum_{j=1}^{g} \left\| \Omega[\overline{\lambda_1}, \ldots, \overline{\lambda_j}] \right\|^2$$

and, as $\alpha_\lambda = 1$, the constraints defining the set $\mathcal{O}(\lambda_j, y_0)$ reduce to

$$\left\langle \Omega_j, \mathscr{B}^* \phi \right\rangle_U = \left\langle y_0, \phi \right\rangle_{-\diamond,\diamond}, \qquad \forall\, \phi \in \mathrm{Ker}(\mathscr{A}^* - \lambda_j).$$

To simplify the writing, let us consider the linear maps

$$B_j := \begin{pmatrix} \left\langle \cdot, \mathscr{B}^* \phi_{j,1} \right\rangle_U \\ \vdots \\ \left\langle \cdot, \mathscr{B}^* \phi_{j,\gamma_j} \right\rangle_U \end{pmatrix} \in \mathscr{L}(U, \mathbb{C}^{\gamma_j}).$$

Then the constraints defining $\mathcal{O}(\lambda_j, y_0)$ can be rewritten as the equality

$$B_j \Omega_j = \begin{pmatrix} \left\langle y_0, \phi_{j,1} \right\rangle_{-\diamond,\diamond} \\ \vdots \\ \left\langle y_0, \phi_{j,\gamma_j} \right\rangle_{-\diamond,\diamond} \end{pmatrix}. \tag{76}$$

Thus,

$$\mathscr{C}(G, y_0) = \min\left\{ F(\Omega)\,;\, \Omega = (\Omega_1, \ldots, \Omega_g) \in U_G^g \text{ such that (76) holds for any } j \in [\![1, g]\!] \right\}. \tag{77}$$

For the sake of generality, let us consider for this proof, for any $j \in [\![1, g]\!]$, any $\omega_j \in \mathbb{C}^{\gamma_j}$ and the more general constraints

$$B_j \Omega_j = \omega_j, \qquad \forall\, j \in [\![1, g]\!]. \tag{78}$$

As the ω_j's have different sizes we avoid in this proof the use of divided differences to rewrite the constraints. This is why we end up with the formula (56) rather than an adaptation of (54) (see also the discussion in Remark 30).

Arguing as before, the solution of our optimisation problem satisfies

$$\sum_{j=1}^{g} \left\langle H[\overline{\lambda_1}, \ldots, \overline{\lambda_j}], \Omega[\overline{\lambda_1}, \ldots, \overline{\lambda_j}] \right\rangle_U = \sum_{j=1}^{g} \left\langle B_j H_j, m_j \right\rangle, \quad \forall\, H_1, \ldots, H_g \in U_G, \tag{79}$$

for some $m_j \in \mathbb{C}^{\gamma_j}$, $j = 1, \ldots, g$.

Recall that in (55) we defined the numbers

$$\delta^i_j = \prod_{k\in[\![1,j[\![} \left(\lambda_i - \lambda_k\right), \qquad \forall\, j \in [\![2,g]\!].$$

Then, from the interpolation formula [9, Proposition 7.6], we obtain that

$$\Omega_i = \sum_{l=1}^{i} \overline{\delta^i_l}\,\Omega[\overline{\lambda_1},\dots,\overline{\lambda_l}]. \tag{80}$$

For any $H \in U_G$ and $i \in [\![1,g]\!]$, let us design $H^{(i)}_1,\dots,H^{(i)}_g \in U_G$ such that

$$H^{(i)}[\overline{\lambda_1},\dots,\overline{\lambda_l}] = \delta^i_l H, \quad \forall\, l \in [\![1,i]\!]. \tag{81}$$

To do so, we set $H^{(i)}_1 = H$ then, using the interpolation formula [9, Proposition 7.6], we define recursively

$$H^{(i)}_j = \sum_{l=1}^{j} \overline{\delta^j_l}\, H^{(i)}[\overline{\lambda_1},\dots,\overline{\lambda_l}] = \left(\sum_{l=1}^{j} \delta^i_l \overline{\delta^j_l}\right) H = a^{(i)}_j H$$

with

$$a^{(i)}_j := \sum_{l=1}^{g} \delta^i_l \overline{\delta^j_l} = \sum_{l=1}^{\min(i,j)} \delta^i_l \overline{\delta^j_l}. \tag{82}$$

This ensures (81). Plugging this set of values $H^{(i)}_j$, $j = 1,\dots,g$ in (79) and taking into account (80), leads to

$$\begin{aligned}
\sum_{j=1}^{g} a^{(i)}_j \langle B_j H, m_j\rangle &= \sum_{j=1}^{g} \left\langle B_j H^{(i)}_j, m_j\right\rangle \\
&= \sum_{j=1}^{g} \delta^i_j \left\langle H, \Omega[\overline{\lambda_1},\dots,\overline{\lambda_j}]\right\rangle_U \\
&= \left\langle H, \sum_{j=1}^{g} \overline{\delta^i_j}\,\Omega[\overline{\lambda_1},\dots,\overline{\lambda_j}]\right\rangle_U \\
&= \langle H, \Omega_i\rangle_U .
\end{aligned}$$

This being true for any $H \in U_G$, we end up with

$$\Omega_i = \sum_{j=1}^{g} \overline{a^{(i)}_j}\, B^*_j m_j. \tag{83}$$

Together with (78), using (82), we obtain that

$$\begin{aligned}
\omega_i &= \sum_{j=1}^{g} \overline{a^{(i)}_j}\, B_i B^*_j m_j \\
&= \sum_{l=1}^{g}\sum_{j=1}^{g} \left(\overline{\delta^i_l} B_i\right)\left(\overline{\delta^j_l} B_j\right)^* m_j \\
&= (Mm)_i
\end{aligned}$$

where M is defined in (56) and $(Mm)_i \in \mathbb{C}^{\gamma_i}$ denotes the i^{th} block of $Mm \in \mathbb{C}^{\gamma_G}$.

Finally, if we set

$$\xi := \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} \in \mathbb{C}^{\gamma_G}, \tag{84}$$

we have proved that the multiplier is given by $m = M^{-1}\xi$. Applying (79) with $H_j = \Omega_j$ and using the constraints (76) leads to

$$F(\Omega) = \sum_{j=1}^{g} \left\| \Omega[\overline{\lambda_1}, \dots, \overline{\lambda_j}] \right\|^2 = \left\langle M^{-1}\xi, \xi \right\rangle,$$

which proves the claim. □

Remark 32. From (83) and the equality $m = M^{-1}\xi$, it comes that

$$\mathscr{C}(G, y_0) = \min \left\{ F(\Omega)\,;\, \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \text{ such that (78) holds for any } j \in [\![1, g]\!] \right\}$$

where the minimum is achieved for

$$\Omega_i = \sum_{j=1}^{g} \left(\sum_{l=1}^{g} \overline{\delta_l^i} \delta_l^j \right) B_j^* (M^{-1}\xi)_j$$

with ξ defined by (84).

Remark 33. As mentioned in Remark 15, estimate (77) implies that the cost of the block G (i.e. the quantity $\left\langle M^{-1}\xi, \xi \right\rangle$) can be estimated using any basis of eigenvectors.

5.4. *Dealing simultaneously with algebraic and geometric multiplicities*

The proof of Theorem 14 strongly relies on the use of divided differences to rewrite the constraints whereas the proof of Theorem 18 is based on the vectorial writing of the constraints through the operators $B_j \in \mathscr{L}(U; \mathbb{C}^{\gamma_j})$. As the target spaces of these operators do not have the same dimension, one cannot directly compute divided differences. Thus, the setting we developed to compute the cost of a given block does not lead to a general formula when both kind of multiplicities need to be taken into account in the same group. However, for actual problems, the computation of this cost is a finite dimensional constrained optimization problem which can be explicitly solved.

Let us give an example of such a group that does not fit into Theorem 14 nor into Theorem 18 but for which we manage to compute the cost *by hand*. To simplify a little the presentation, we give this example in the case of real Hilbert spaces and real eigenvalues.

We consider a group $G = \{\lambda_1, \lambda_2\}$ of two distinct eigenvalues such that $\gamma_{\lambda_1} = \alpha_{\lambda_1} = 2$ and $\gamma_{\lambda_2} = \alpha_{\lambda_2} = 1$. Let $(\phi_{1,1}^0, \phi_{1,2}^0)$ be a basis of $\operatorname{Ker}(\mathscr{A}^* - \lambda_1)$ and $\phi_{2,1}^0$ be an eigenvector of $\mathscr{A}^*$ associated to the eigenvalue λ_2. Assume that the generalized eigenvector $\phi_{1,1}^1$ is such that

$$\left(\mathscr{A}^* - \lambda_1\right) \phi_{1,1}^1 = \phi_{1,1}^0,$$

and that $\{\phi_{1,1}^0, \phi_{1,1}^1, \phi_{1,2}^0\}$ forms a basis of $\operatorname{Ker}(\mathscr{A}^* - \lambda_1)^2$.

For this group, in the same spirit as in Theorems 14 and 18, we obtain the following result.

Proposition 34. *For any $y_0 \in X_{-\diamond}$, we have*

$$\mathscr{C}(G, y_0) = \left\langle M^{-1}\xi, \xi \right\rangle \quad \textit{where} \quad \xi = \begin{pmatrix} \left\langle y_0, \phi_{1,1}^0 \right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \phi_{1,2}^0 \right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \phi_{1,1}^1 \right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \phi_{2,1}^0 \right\rangle_{-\diamond,\diamond} \end{pmatrix}$$

and M is the invertible matrix defined by

$$M = \operatorname{Gram}_U\left(b_{1,1}^0,\, b_{1,2}^0,\, b_{1,1}^1,\, b_{2,1}^0\right) + \operatorname{Gram}_U\left(0, 0, b_{1,1}^0, \delta b_{2,1}^0\right) + \operatorname{Gram}_U\left(0, 0, 0, \delta^2 b_{2,1}^0\right)$$

with $\delta = \lambda_2 - \lambda_1$.

Proof. Let

$$\left(\omega^0_{1,1},\omega^0_{1,2},\omega^1_{1,1},\omega^0_{2,1}\right)^t \in \mathbb{R}^4.$$

As in the proofs of Theorems 14 and 18, the goal is to compute the minimum of the function

$$F : \left(\Omega^0_1,\Omega^1_1,\Omega^0_2\right) \in U_G^3 \mapsto \|\Omega^0_1\|^2 + \|\Omega^1_1\|^2 + \|\Omega[\lambda_1^{(2)},\lambda_2]\|^2,$$

under the 4 constraints

$$\begin{aligned}&\left\langle \Omega^0_j, b^0_{j,i}\right\rangle_U = \omega^0_{j,i}, \quad \forall\, i \in [\![1,\gamma_j]\!],\ \forall\, j \in [\![1,2]\!],\\ &\langle \Omega^0_1, b^1_{1,1}\rangle_U + \langle \Omega^1_1, b^0_{1,1}\rangle_U = \omega^1_{1,1}.\end{aligned}$$

Then, the Lagrange multipliers $m^0_{1,1}, m^0_{1,2}, m^1_{1,1}$ and $m^0_{2,1}$ satisfy the equations

$$\begin{aligned}&\langle \Omega^0_1, H^0_1\rangle_U + \langle \Omega^1_1, H^1_1\rangle_U + \left\langle \Omega[\lambda_1^{(2)},\lambda_2], H[\lambda_1^{(2)},\lambda_2]\right\rangle_U\\ &= m^0_{1,1}\langle H^0_1, b^0_{1,1}\rangle_U + m^0_{1,2}\langle H^0_1, b^0_{1,2}\rangle_U + m^1_{1,1}\left(\langle H^0_1, b^1_{1,1}\rangle_U + \langle H^1_1, b^0_{1,1}\rangle_U\right) + m^0_{2,1}\langle H^0_2, b^0_{2,1}\rangle_U, \end{aligned} \quad (85)$$

for every H^0_1, H^1_1, $H^0_2 \in U_G$. Considering successively

$$\begin{aligned}&H^0_1 = b^0_{1,1}, \quad H^1_1 = 0, \quad H^0_2 = b^0_{1,1},\\ &H^0_1 = b^0_{1,2}, \quad H^1_1 = 0, \quad H^0_2 = b^0_{1,2},\\ &H^0_1 = b^1_{1,1}, \quad H^1_1 = b^0_{1,1}, \quad H^0_2 = b^1_{1,1} + (\lambda_2-\lambda_1)b^0_{1,1},\end{aligned}$$

and

$$H^0_1 = b^0_{2,1}, \quad H^1_1 = (\lambda_2-\lambda_1)b^0_{2,1}, \quad H^0_2 = \left(1+(\lambda_2-\lambda_1)^2+(\lambda_2-\lambda_1)^4\right)b^0_{2,1},$$

and plugging it into (85), we obtain that

$$\begin{pmatrix}\omega^0_{1,1}\\ \omega^0_{1,2}\\ \omega^1_{1,1}\\ \omega^0_{2,1}\end{pmatrix} = M \begin{pmatrix} m^0_{1,1}\\ m^0_{1,2}\\ m^1_{1,1}\\ m^0_{2,1}\end{pmatrix}.$$

Then, the same argument as in the proofs of Theorems 14 and 18 ends the proof. □

6. Application to the study of null controllability of academic examples

In this section we provide examples to illustrate how to use the formulas obtained in Theorems 11, 14 and 18 in order to compute the minimal null control time.

We start with academic examples for which computations are simpler. Then, in Section 7, we study coupled systems of actual partial differential equations of parabolic type.

6.1. *Setting and notations*

Let A be the unbounded Sturm–Liouville operator defined in $L^2(0,1;\mathbb{R})$ by

$$D(A) = H^2(0,1;\mathbb{R}) \cap H^1_0(0,1;\mathbb{R}), \qquad A = -\partial_x\left(\gamma\partial_x \cdot\right) + c\,\cdot, \quad (86)$$

with $c \in L^\infty(0,1;\mathbb{R})$ satisfying $c \geq 0$ and $\gamma \in C^1([0,1];\mathbb{R})$ satisfying $\inf_{[0,1]}\gamma > 0$.

The operator A admits an increasing sequence of eigenvalues denoted by $(\nu_k)_{k\in\mathbb{N}^*}$. The associated normalized eigenvectors $(\varphi_k)_{k\in\mathbb{N}^*}$ form a Hilbert basis of $L^2(0,1;\mathbb{R})$.

Remark 35. The assumption $c \geq 0$ ensures that for any $k \geq 1$, the eigenvalues satisfies $\nu_k > 0$. From Remark 5, the controllability results proved in the present article still hold when the function c is bounded from below.

To lighten the notations, for any $I \subset (0,1)$ we set $\|\cdot\|_I = \|\cdot\|_{L^2(I)}$.

Let $f : \mathrm{Sp}(A) \to \mathbb{R}$ be a bounded function. Associated to this function we consider the operator $f(A)$ defined on $D(A)$ by the spectral theorem by

$$f(A) = \sum_{k\geq 1} f(\nu_k) \left\langle \cdot, \varphi_k \right\rangle_{L^2(0,1;\mathbb{R})} \varphi_k. \tag{87}$$

6.2. *Spectral properties of Sturm–Liouville operators*

Let A be the Sturm–Liouville operator defined in (86). All the examples studied in this article are based on this operator. We recall here some spectral properties that will be used in our study.

From [2, Theorem 1.1 and Remark 2.1], there exist $\varrho > 0$ and $C > 0$ such that

$$\varrho < \nu_{k+1} - \nu_k, \qquad \forall\, k \geq 1, \tag{88}$$

$$\frac{1}{C}\sqrt{\nu_k} \leq |\varphi_k'(x)| \leq C\sqrt{\nu_k}, \qquad \forall\, x \in \{0,1\}, \forall\, k \geq 1, \tag{89}$$

and, for any non-empty open set $\omega \subset (0,1)$,

$$\inf_{k\geq 1} \|\varphi_k\|_\omega > 0. \tag{90}$$

Moreover, using [12, Theorem IV.1.3], the associated counting function satisfies

$$N_{(\nu_k)_k}(r) \leq C\sqrt{r}, \quad \forall\, r > 0, \tag{91}$$

and

$$\left|N_{(\nu_k)_k}(r) - N_{(\nu_k)_k}(s)\right| \leq C\left(1 + \sqrt{|r-s|}\right), \quad \forall\, r,s > 0. \tag{92}$$

We also recall the classical Lebeau–Robbiano spectral inequality

$$\left\| \sum_{k\leq K} a_k \varphi_k \right\|_\Omega \leq Ce^{C\sqrt{\nu_K}} \left\| \sum_{k\leq K} a_k\varphi_k \right\|_\omega, \quad \forall\, K \geq 1, \forall\, (a_k)_k \subset \mathbb{R}. \tag{93}$$

Indeed, as detailed for instance in [12, Theorem IV.2.19], the proof of this spectral inequality given in [27] directly extends to the low regularity coefficients considered here.

6.3. *Perturbation of a* 2×2 *Jordan block*

Let $\omega \subset (0,1)$ be a non-empty open set and $U = L^2(\Omega)$. Let A be the Sturm–Liouville operator defined in (86) and $f(A)$ be the operator defined in (87) with $f : \mathrm{Sp}(A) \to \mathbb{R}$ satisfying

$$\left|f(\nu_k)\right| < \frac{\varrho}{2}, \qquad \forall\, k \geq 1.$$

We consider the operator $\mathscr{A}$ on $X = L^2(0,1;\mathbb{R})^2$ defined by

$$\mathscr{A} = \begin{pmatrix} A & I \\ 0 & A + f(A) \end{pmatrix}, \quad D(\mathscr{A}) = D(A) \times D(A), \tag{94}$$

and

$$\mathscr{B} : u \in U \mapsto \begin{pmatrix} 0 \\ \mathbb{1}_\omega u \end{pmatrix}. \tag{95}$$

Then,

$$\mathscr{B}^* : \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in X \mapsto \mathbb{1}_\omega \varphi_2.$$

It is easy to see that $(-\mathscr{A}, D(\mathscr{A}))$ generates a $\mathscr{C}_0$-semigroup on X and that $\mathscr{B} : U \to X$ is bounded. Thus we consider for this example that $X_\diamond^* = X = X_{-\diamond}$.

Proposition 36. *Let us consider the control system* (1) *with* $\mathscr{A}$ *and* $\mathscr{B}$ *given by* (94)–(95). *Then, null-controllability from* $X_{-\diamond}$ *holds in any time i.e.* $T_0(X_{-\diamond}) = 0$.

Proof. The spectrum of $(\mathscr{A}^*, D(\mathscr{A}))$ is given by

$$\Lambda = \{\nu_k\,;\, k \geq 1\} \cup \{\nu_k + f(\nu_k)\,;\, k \geq 1\}.$$

Recall that $(\nu_k)_{k\geq 1}$ satisfies (88), (91) and (92). From [12, Lemma V.4.20] it comes that there exists $\kappa > 0$ such that $\Lambda \in \mathscr{L}_w\left(2, \frac{\varrho}{2}, \frac{1}{2}, \kappa\right)$.

An associated grouping is given by

$$\begin{cases} G_k := \{\lambda_{k,1} := \nu_k,\, \lambda_{k,2} := \nu_k + f(\nu_k)\}, & \text{if } f(\nu_k) \neq 0, \\ G_k := \{\lambda_{k,1} := \nu_k\}, & \text{if } f(\nu_k) = 0. \end{cases}$$

If $f(\nu_k) \neq 0$ the eigenvalues $\lambda_{k,1}$ and $\lambda_{k,2}$ are simple and we consider the associated eigenvectors

$$\phi^0_{k,1} = \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \varphi_k, \qquad \phi^0_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k.$$

If $f(\nu_k) = 0$ the eigenvalue $\lambda_{k,1}$ is algebraically double and we consider the associated Jordan chain

$$\phi^0_{k,1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k, \qquad \phi^1_{k,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k.$$

From (90) it comes that (33) and (34) are satisfied. Thus, from Theorem 11, we obtain that for any $y_0 \in X_{-\diamond}$,

$$T_0(y_0) = \limsup_{k\to+\infty} \frac{\ln^+ \mathscr{C}(G_k, y_0)}{2 \min G_k}.$$

Let us now conclude by estimating $\mathscr{C}(G_k, y_0)$.

- Consider first that $f(\nu_k) \neq 0$. Then, $\phi[\lambda_{k,1}, \lambda_{k,2}] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k$ and

$$b[\lambda_{k,1}, \lambda_{k,2}] = \mathscr{B}^* \phi[\lambda_{k,1}, \lambda_{k,2}] = \frac{\mathbb{1}_\omega \varphi_k - \mathbb{1}_\omega \varphi_k}{f(\nu_k)} = 0.$$

  From Theorem 14 it comes that

$$\mathscr{C}(G_k, y_0) = \left\langle M^{-1}\xi, \xi \right\rangle$$

  with

$$M = \mathrm{Gram}(b[\lambda_{k,1}], b[\lambda_{k,1}, \lambda_{k,2}]) + \mathrm{Gram}(0, b[\lambda_{k,2}]) = \begin{pmatrix} \|\varphi_k\|^2_\omega & 0 \\ 0 & \|\varphi_k\|^2_\omega \end{pmatrix}$$

  and

$$\xi = \begin{pmatrix} \left\langle y_0, \phi[\lambda_{k,1}] \right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}] \right\rangle_{-\diamond,\diamond} \end{pmatrix} = \begin{pmatrix} \left\langle y_0, \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \varphi_k \right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k \right\rangle_{-\diamond,\diamond} \end{pmatrix}.$$

  Thus,

$$\mathscr{C}(G_k, y_0) = \left\langle y_0, \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle^2_{-\diamond,\diamond} + \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle^2_{-\diamond,\diamond}.$$

- Consider now that $f(\nu_k) = 0$. Then, $b[\lambda^{(2)}_{k,1}] = 0$. From Theorem 14 it comes that

$$\mathscr{C}(G_k, y_0) = \left\langle M^{-1}\xi, \xi \right\rangle$$

  with

$$M_k = \mathrm{Gram}(b[\lambda_{k,1}], b[\lambda^{(2)}_{k,1}]) + \mathrm{Gram}(0, b[\lambda_{k,1}]) = \begin{pmatrix} \|\varphi_k\|^2_\omega & 0 \\ 0 & \|\varphi_k\|^2_\omega \end{pmatrix}.$$

and

$$\xi = \begin{pmatrix} \langle y_0, \phi[\lambda_{k,1}] \rangle_{-\diamond,\diamond} \\ \left\langle y_0, \phi[\lambda_{k,1}^{(2)}] \right\rangle_{-\diamond,\diamond} \end{pmatrix} = \begin{pmatrix} \left\langle y_0, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k \right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k \right\rangle_{-\diamond,\diamond} \end{pmatrix}.$$

As previously,

$$\mathscr{C}(G_k, y_0) = \left\langle y_0, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond,\diamond}^2 + \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond,\diamond}^2.$$

Gathering both cases and using estimate (90) we obtain, for any $y_0 \in X_{-\diamond}$,

$$\mathscr{C}(G_k, y_0) \leq C \left\| y_0 \right\|_{-\diamond}^2, \qquad \forall\, k \geq 1.$$

Thus,

$$T_0(y_0) = \limsup_{k\to+\infty} \frac{\ln^+ \mathscr{C}(G_k, y_0)}{2 \min G_k} = 0. \qquad \square$$

6.4. *Competition between different perturbations*

Let $\omega_1, \omega_2 \subset (0,1)$ be two open sets with $\omega_1 \neq \varnothing$ and $U = L^2(\Omega)^2$. Let $B_1, B_2 \in \mathbb{R}^3$. To simplify the computations, we assume that

$$B_i = \begin{pmatrix} 0 \\ B_{i,2} \\ B_{i,3} \end{pmatrix}.$$

Let $\alpha, \beta > 0$ with $\alpha \neq \beta$ and $f, g : \mathrm{Sp}(A) \to \mathbb{R}$ be defined by

$$f(\nu_k) = \frac{\varrho}{2} e^{-\alpha\nu_k}, \qquad g(\nu_k) = \frac{\varrho}{2} e^{-\beta\nu_k}.$$

As previously, we consider the associated operators $f(A)$ and $g(A)$ defined by the spectral theorem and we define the evolution operator $\mathscr{A}$ on $X = L^2(0,1;\mathbb{R})^3$ by

$$\mathscr{A} = \begin{pmatrix} A & I & 0 \\ 0 & A + f(A) & 0 \\ 0 & 0 & A + g(A) \end{pmatrix}, \quad D(\mathscr{A}) = D(A)^3, \tag{96}$$

and the control operator by

$$\mathscr{B} : \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in U \mapsto \mathbb{1}_{\omega_1} u_1 B_1 + \mathbb{1}_{\omega_2} u_2 B_2. \tag{97}$$

Then, the observation operator reads

$$\mathscr{B}^* : \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \in X \mapsto \begin{pmatrix} \mathbb{1}_{\omega_1} \left(B_{1,2}\varphi_2 + B_{1,3}\varphi_3 \right) \\ \mathbb{1}_{\omega_2} \left(B_{2,2}\varphi_2 + B_{2,3}\varphi_3 \right) \end{pmatrix}.$$

Proposition 37. *Let us consider the control system* (1) *with* $\mathscr{A}$ *and* $\mathscr{B}$ *given by* (96)–(97).

(i) *If* $\omega_2 = \varnothing$, *we assume that*

$$B_{1,2}B_{1,3} \neq 0. \tag{98}$$

Then,

$$T_0(X_{-\diamond}) = \beta + \min\{\alpha, \beta\}.$$

(ii) *If* $\omega_2 \neq \varnothing$, *we assume that*

$$\left(B_{1,2}^2 + B_{2,2}^2\right)\left(B_{1,3}^2 + B_{2,3}^2\right) \neq 0. \tag{99}$$

(a) *If* B_1 *and* B_2 *are linearly independent, then,*

$$T_0(X_{-\diamond}) = 0.$$

(b) *If* B_1 *and* B_2 *are not linearly independent, then,*

$$T_0(X_{-\diamond}) = \beta + \min\{\alpha, \beta\}.$$

Proof. It is easy to see that $(-\mathscr{A}, D(\mathscr{A}))$ generates a $\mathscr{C}_0$-semigroup on X and that $\mathscr{B}: U \to X$ is bounded. Thus we consider for this example that $X_\diamond^* = X = X_{-\diamond}$ and $Y_0 = X_{-\diamond}$.

The spectrum of $(\mathscr{A}^*, D(\mathscr{A}))$ is given by $\Lambda = \bigcup_{k\geq 1} G_k$ where

$$G_k := \{\lambda_{k,1} := \nu_k,\ \lambda_{k,2} := \nu_k + f(\nu_k),\ \lambda_{k,3} := \nu_k + g(\nu_k)\}.$$

Again, since $(\nu_k)_{k\geq 1}$ satisfies (88), (91) and (92), the application of [12, Lemma V.4.20] yields the existence of $\kappa > 0$ such that $\Lambda \in \mathscr{L}_w\left(3, \frac{\varrho}{2}, \frac{1}{2}, \kappa\right)$. The sequence $(G_k)_{k\geq 1}$ is an associated grouping.

The eigenvalues are simple and the corresponding eigenvectors are given by

$$\phi_{k,1}^0 = \begin{pmatrix} -f(\nu_k) \\ 1 \\ 0 \end{pmatrix} \varphi_k, \qquad \phi_{k,2}^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \varphi_k, \qquad \phi_{k,3}^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k.$$

Thus, the assumption (34) hold. Moreover,

$$b_1 = b_2 = \begin{pmatrix} \mathbb{1}_{\omega_1}\varphi_k B_{1,2} \\ \mathbb{1}_{\omega_2}\varphi_k B_{2,2} \end{pmatrix}, \quad b_3 = \begin{pmatrix} \mathbb{1}_{\omega_1}\varphi_k B_{1,3} \\ \mathbb{1}_{\omega_2}\varphi_k B_{2,3} \end{pmatrix} \tag{100}$$

From (90) and (98) or (99) (depending on the assumption on ω_2) it comes that (33) is satisfied. Thus, from Theorem 11, it comes that for any $y_0 \in X_{-\diamond}$,

$$T_0(y_0) = \limsup_{k\to+\infty} \frac{\ln^+ \mathscr{C}(G_k, y_0)}{2\min G_k}.$$

Let us now estimate $\mathscr{C}(G_k, y_0)$. From Theorem 14 it comes that

$$\mathscr{C}(G_k, y_0) = \left\langle M^{-1}\xi, \xi\right\rangle$$

with

$$M = \mathrm{Gram}\left(b[\lambda_{k,1}], b[\lambda_{k,1}, \lambda_{k,2}], b[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}]\right) + \mathrm{Gram}\left(0, b[\lambda_{k,2}], b[\lambda_{k,2}, \lambda_{k,3}]\right) + \mathrm{Gram}\left(0, 0, b[\lambda_{k,3}]\right)$$

and

$$\xi = \begin{pmatrix} \left\langle y_0, \phi[\lambda_{k,1}]\right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}]\right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}]\right\rangle_{-\diamond,\diamond} \end{pmatrix}.$$

Explicit computations yield

$$\phi[\lambda_{k,1}] = \begin{pmatrix} -f(\nu_k) \\ 1 \\ 0 \end{pmatrix} \varphi_k, \qquad \phi[\lambda_{k,1}, \lambda_{k,2}] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \varphi_k,$$

and

$$\phi[\lambda_{k,1},\lambda_{k,2},\lambda_{k,3}] = \frac{1}{g(\nu_k)\big(g(\nu_k)-f(\nu_k)\big)}\begin{pmatrix} f(\nu_k)-g(\nu_k)\\ -1\\ 1\end{pmatrix}\varphi_k.$$

(i). Assume that $\omega_2 = \varnothing$.

After the change of variables

$$z = \operatorname{diag}\left(\frac{1}{B_{1,2}},\frac{1}{B_{1,2}},\frac{1}{B_{1,3}}\right)y,$$

the system under study reads

$$\begin{cases} \partial_t z + \begin{pmatrix} A & I & 0\\ 0 & A+f(A) & 0\\ 0 & 0 & A+g(A)\end{pmatrix} z = 1_{\omega_1} u_1(t,x)\begin{pmatrix}0\\1\\1\end{pmatrix},\\ z(t,0)=z(t,1)=0.\end{cases}$$

This leads to

$$b[\lambda_{k,1}] = b[\lambda_{k,2}] = b[\lambda_{k,3}] = \mathbb{1}_{\omega_1}\varphi_k.$$

Thus, $M = \|\varphi_k\|_{\omega_1}^2 I_3$ and

$$\mathscr{C}(G_k,y_0) = \left\langle y_0, \begin{pmatrix}-f(\nu_k)\\ 1\\ 0\end{pmatrix}\frac{\varphi_k}{\|\varphi_k\|_{\omega_1}}\right\rangle_{-\diamond,\diamond}^2 + \left\langle y_0, \begin{pmatrix}1\\0\\0\end{pmatrix}\frac{\varphi_k}{\|\varphi_k\|_{\omega_1}}\right\rangle_{-\diamond,\diamond}^2 \\ + \left(\frac{1}{g(\nu_k)\big(g(\nu_k)-f(\nu_k)\big)}\right)^2\left\langle y_0, \begin{pmatrix} f(\nu_k)-g(\nu_k)\\ -1\\ 1\end{pmatrix}\frac{\varphi_k}{\|\varphi_k\|_{\omega_1}}\right\rangle_{-\diamond,\diamond}^2.$$

From (90), we obtain for any $y_0 \in X_{-\diamond}$,

$$\mathscr{C}(G_k,y_0) \le C\left\|y_0\right\|_{-\diamond}^2\left(1+\left(\frac{1}{g(\nu_k)\big(g(\nu_k)-f(\nu_k)\big)}\right)^2\right).$$

This leads to

$$T_0(X_{-\diamond}) \le \limsup_{k\to+\infty}\frac{-\ln^+\left|g(\nu_k)\big(g(\nu_k)-f(\nu_k)\big)\right|}{\nu_k}.$$

Conversely, with the particular choice

$$y_0 = \sum_{k\ge1}\frac{1}{\nu_k}\begin{pmatrix}0\\0\\1\end{pmatrix}\varphi_k,$$

we have

$$\mathscr{C}(G_k,y_0) = \frac{1}{\nu_k^2\|\varphi_k\|_{\omega_1}^2}\left(\frac{1}{g(\nu_k)\big(g(\nu_k)-f(\nu_k)\big)}\right)^2.$$

Thus, from (90), we obtain

$$T_0(X_{-\diamond}) \ge T_0(y_0) = \limsup_{k\to+\infty}\frac{-\ln\left|g(\nu_k)\big(g(\nu_k)-f(\nu_k)\big)\right|}{\nu_k}$$

which gives

$$T_0(X_{-\diamond}) = \limsup_{k\to+\infty}\frac{-\ln\left|g(\nu_k)\big(g(\nu_k)-f(\nu_k)\big)\right|}{\nu_k}.$$

Then, the same computations as [9, Section 5.1.3] yield

$$T_0(X_{-\diamond}) = \beta + \min\{\alpha,\beta\}.$$

(ii). We now consider the case $\omega_2 \neq \varnothing$.

(a). Assume that B_1 and B_2 are linearly independent. If necessary, we consider smaller control sets so that $\omega_1 \cap \omega_2 = \varnothing$. As we will prove that $T_0(X_{-\diamond}) = 0$, this is not a restrictive assumption.

To ease the reading we drop the index k in what follows. As previously, the vector ξ is not bounded. Let us consider the dilatation $D_\epsilon = \operatorname{diag}(1,1,\epsilon)$ with

$$\epsilon = g(\nu)\big(g(\nu) - f(\nu)\big)$$

and $\widetilde{\xi} = D_\epsilon \xi$. Then, from Section D.1, it comes that

$$\mathscr{C}(G, y_0) = \big\langle \widetilde{M}^{-1}\widetilde{\xi}, \widetilde{\xi} \big\rangle$$

with

$$\widetilde{M} = \operatorname{Gram}\big(b[\lambda_1], b[\lambda_1,\lambda_2], \epsilon b[\lambda_1,\lambda_2,\lambda_3]\big) + \operatorname{Gram}\big(0, b[\lambda_2], \epsilon b[\lambda_2,\lambda_3]\big) + \operatorname{Gram}\big(0,0,\epsilon b[\lambda_3]\big).$$

As $\|\widetilde{\xi}\|$ is bounded, we simply give a lower bound on the smallest eigenvalue of $\widetilde{M}$. Using (100), it comes that

$$b[\lambda_1,\lambda_2] = 0, \quad b[\lambda_2,\lambda_3] = \frac{b_3 - b_1}{g(\nu) - f(\nu)}, \quad b[\lambda_1,\lambda_2,\lambda_3] = \frac{1}{\epsilon}(b_3 - b_1).$$

Thus,

$$\widetilde{M} = \operatorname{Gram}\big(b_1, 0, b_3 - b_1\big) + \operatorname{Gram}\big(0, b_1, g(\nu)(b_3 - b_1)\big) + \operatorname{Gram}\big(0,0,\epsilon b_3\big).$$

This gives that, for any $\tau \in \mathbb{R}^3$, we have

$$\big\langle \widetilde{M}\tau, \tau \big\rangle = \|\tau_1 b_1 + \tau_3(b_3 - b_1)\|_U^2 + \big\|\tau_2 b_1 + g(\nu)\tau_3(b_3 - b_1)\big\|_U^2 + \epsilon^2 \|\tau_3 b_3\|_U^2. \tag{101}$$

To obtain a lower bound on this quantity we use the following lemma.

Lemma 38. *There exists $C > 0$ (independent of k) such that for any $\theta_1, \theta_3 \in \mathbb{R}$,*

$$\|\theta_1 b_1 + \theta_3 b_3\|_U^2 \geq C\big(\theta_1^2 + \theta_3^2\big).$$

Proof of Lemma 38. As $\omega_1 \cap \omega_2 = \varnothing$,

$$\|\theta_1 b_1 + \theta_3 b_3\|_U^2 = \big(B_{1,2}\theta_1 + B_{1,3}\theta_3\big)^2 \|\varphi_k\|_{\omega_1}^2 + \big(B_{2,2}\theta_1 + B_{2,3}\theta_3\big)^2 \|\varphi_k\|_{\omega_2}^2.$$

Using (90) it comes that

$$\begin{aligned}
\|\theta_1 b_1 + \theta_3 b_3\|_U^2 &\geq C\Big(\big(B_{1,2}\theta_1 + B_{1,3}\theta_3\big)^2 + \big(B_{2,2}\theta_1 + B_{2,3}\theta_3\big)^2\Big) \\
&= \left\| \begin{pmatrix} B_{1,2} & B_{1,3} \\ B_{2,2} & B_{2,3} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_3 \end{pmatrix} \right\|^2.
\end{aligned}$$

Since B_1 and B_2 are linearly independent, this ends the proof. □

Applying this lemma twice to (101) yield

$$\begin{aligned}
\big\langle \widetilde{M}\tau, \tau \big\rangle &\geq C\big((\tau_1 - \tau_3)^2 + \tau_3^2 + (\tau_2 - g(\nu)\tau_3)^2 + g(\nu)^2\tau_3^2 + \epsilon^2\tau_3^2\big) \\
&\geq C\big((\tau_1 - \tau_3)^2 + \tau_3^2 + (\tau_2 - g(\nu)\tau_3)^2\big).
\end{aligned}$$

Taking into account that $0 < g(\nu) < \frac{1}{2}$ for ν large enough, the study of this quadratic form in $\mathbb{R}^3$ leads to

$$\big\langle \widetilde{M}\tau, \tau \big\rangle \geq C\big(\tau_1^2 + \tau_2^2 + \tau_3^2\big).$$

Thus the smallest eigenvalue of $\widetilde{M}$ is bounded from below. This leads to the boundedness of $\big\langle \widetilde{M}^{-1}\widetilde{\xi}, \widetilde{\xi} \big\rangle$ which concludes the proof of case (ii)(a).

(b). Assume now that B_1 and B_2 are not linearly independent. Then, there exist x_1, $x_2 \in \mathbb{R}$ such that

$$\begin{cases} x_1 B_{1,2} + x_2 B_{1,3} = 0 \\ x_1 B_{2,2} + x_2 B_{2,3} = 0. \end{cases}$$

Up to a change of normalization of the eigenvectors (independent of k) we obtain

$$b_1 = b_2 = b_3 = \begin{pmatrix} \mathbb{1}_{\omega_1} \varphi_k x_1 B_{1,2} \\ \mathbb{1}_{\omega_2} \varphi_k x_1 B_{2,2} \end{pmatrix}$$

and this amounts to case (i). □

7. Analysis of controllability for systems of partial differential equations

We now turn to the analysis of null controllability of actual partial differential equations. We consider here coupled systems of two linear one dimensional parabolic equations.

7.1. *Coupled heat equations with different diffusion coefficients*

In this application, we consider the Sturm–Liouville operator A defined in (86) and we define in $X = L^2(0,1;\mathbb{R})^2$ the operator

$$\mathscr{A} = \begin{pmatrix} A & I \\ 0 & dA \end{pmatrix}, \qquad D(\mathscr{A}) = D(A)^2,$$

with $d > 0$. We will assume $d \neq 1$, since the case $d = 1$ is much simpler and already studied in the literature: see the computations of Section 6.3 in the case $f = 0$ or, for instance, [24] for a more general study based on Carleman estimates.

We will consider two cases : the case where two boundary controls are applied to the system, and the case where we consider the same distributed control in the two equations of the system.

7.1.1. *Spectrum of $\mathscr{A}^*$*

Let $\Lambda_1 := \mathrm{Sp}(A) = \{\nu_k\,;\, k \geq 1\}$ and $\Lambda_2 := d\Lambda_1$.

The spectrum of $\mathscr{A}^*$ is given by $\Lambda = \Lambda_1 \cup \Lambda_2$ which belongs to $\mathscr{L}_w\left(2, \varrho, \frac{1}{2}, \kappa\right)$ for some $\varrho, \kappa > 0$ (see [12, Lemma V.4.20]).

For any $\lambda \in \Lambda$, there are two non mutually exclusive cases:

- If $\lambda = \nu_k \in \Lambda_1$, then we can associate an eigenvector given by

$$\phi_{\lambda,1} = \begin{pmatrix} 1 \\ \varepsilon_k \end{pmatrix} \varphi_k,$$

  with $\varepsilon_k = \frac{1}{\nu_k(1-d)}$. Note that ε_k tends to zero when k goes to infinity.
- If $\lambda = d\nu_l \in \Lambda_2$, then we can associate an eigenvector given by

$$\phi_{\lambda,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_l.$$

It clearly appears that the elements in $\Lambda_1 \cap \Lambda_2$ (if this set is not empty) are geometrically double eigenvalues of $\mathscr{A}^*$, since in that case $\phi_{\lambda,1}$ and $\phi_{\lambda,2}$ are linearly independent.

Note that (34) holds for the choices of $X_\diamond^*$ that we will make in the sequel, since $(\varphi_k)_{k\geq 1}$ is a Hilbert basis of $L^2(0,1;\mathbb{R})$.

7.1.2. *Two boundary controls*

In this section, we study the following boundary control system

$$\begin{cases} \partial_t y + \mathscr{A} y = 0, & t \in (0,T), \\ y(t,0) = B_0 u_0(t), \quad y(t,1) = B_1 u_1(t), & t \in (0,T), \end{cases} \tag{102}$$

with

$$B_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{103}$$

The control operator $\mathscr{B}$ is defined in a weak sense as in [38]. The expression of its adjoint is given by

$$\mathscr{B}^* : \begin{pmatrix} f \\ g \end{pmatrix} \in X_1^* \mapsto \begin{pmatrix} -B_0^* \begin{pmatrix} f'(0) \\ g'(0) \end{pmatrix} \\ B_1^* \begin{pmatrix} f'(1) \\ g'(1) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -(f'(0)+g'(0)) \\ g'(1) \end{pmatrix}.$$

Considering $X_\diamond^* = H_0^1(0,1;\mathbb{R})^2$, we obtain that $\mathscr{B}$ is admissible with respect to $X_{-\diamond} = H^{-1}(0,1;\mathbb{R})^2$.

Proposition 39. *For any $d \neq 1$, there exists Y_0 a closed subspace of $H^{-1}(0,1;\mathbb{R})^2$ of finite codimension such that*

- *for any $y_0 \notin Y_0$, system* (102) *is not approximately controllable;*
- *for any $y_0 \in Y_0$, system* (102) *is null controllable in any time $T > 0$.*

Remark 40. The situation with a single control is quite different. Indeed, considering $B_0 = \binom{0}{1}$ and $B_1 = 0$, it is proved in [5] that, when A is the Dirichlet Laplace operator, approximate controllability holds if and only $\sqrt{d} \notin \mathbb{Q}$ and in this case that

$$T_0(X_{-\diamond}) = \limsup_{\substack{\lambda\to\infty \\ \lambda\in\Lambda}} \frac{\ln^+\left(\dfrac{1}{\operatorname{dist}\left(\lambda, \Lambda\backslash\{\lambda\}\right)}\right)}{\lambda}.$$

With this formula the authors prove that, for any $\tau \in [0,+\infty]$, there exists a diffusion ratio $d > 0$ such that the minimal null control time of system (102) satisfies $T_0(X_{-\diamond}) = \tau$.

Remark 41. From the definition of Y_0 in the following proof, we directly obtain that in the case where $\mathscr{A}$ is the Dirichlet Laplace operator on the interval $(0,1)$, then $Y_0 = H^{-1}(0,1;\mathbb{R})^2$.

Remark 42. The particular choice of B_0 and B_1 is done to simplify the computations. Notice that with this choice, it is not possible to steer to zero the second equation and then control the first equation. This would be the case with the simpler choice

$$B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Proof. Let us compute the observations associated to the eigenvectors of $\mathscr{A}^*$.

For any $k \geq 1$, we define $s_k \in \mathbb{R}$ be such that $\varphi_k'(1) = s_k \varphi_k'(0)$. From (89), there exists $C > 0$ such that

$$\frac{1}{C} \leq |s_k| \leq C, \qquad \forall\, k \geq 1. \tag{104}$$

- For any $\lambda = \nu_k \in \Lambda_1$, we have

$$\mathscr{B}^* \phi_{\lambda,1} = -\varphi_k'(0) \begin{pmatrix} 1+\varepsilon_k \\ -s_k \varepsilon_k \end{pmatrix}. \tag{105}$$

- For any $\lambda = d\nu_l \in \Lambda_2$, we have

$$\mathscr{B}^* \phi_{\lambda,2} = -\varphi_l'(0) \begin{pmatrix} 1 \\ -s_l \end{pmatrix}. \tag{106}$$

Due to (89) and (104), it comes that (33) holds for any simple eigenvalue $\lambda \in (\Lambda_1 \setminus \Lambda_2) \cup (\Lambda_2 \setminus \Lambda_1)$.

However, for a geometrically double eigenvalue $\lambda \in \Lambda_1 \cap \Lambda_2$, there can be non-observable modes. Indeed, let k and l such that $\lambda = \nu_k = d\nu_l$. Then, the condition

$$\operatorname{Ker}(\mathscr{A}^* - \lambda) \cap \operatorname{Ker} \mathscr{B}^* \neq \{0\}$$

is equivalent to the fact that $\mathscr{B}^* \phi_{\lambda,1}$ and $\mathscr{B}^* \phi_{\lambda,2}$ given by (105)–(106) are linearly independent, which is itself equivalent to the condition

$$s_k \varepsilon_k = s_l (1 + \varepsilon_k). \tag{107}$$

Due to the asymptotics $\varepsilon_k \underset{k \to +\infty}{\longrightarrow} 0$ it turns out that the set

$$\Theta := \{\lambda = \nu_k = d\nu_l \in \Lambda_1 \cap \Lambda_2 \,;\, (107) \text{ holds}\},$$

is finite.

For any $\lambda \in \Theta$, we can find $\psi_\lambda \in \operatorname{Span}(\phi_{\lambda,1}, \phi_{\lambda_2})$ such that $\mathscr{B}^* \psi_\lambda = 0$ and $\psi_\lambda \neq 0$, that is a non observable mode.

Finally, we introduce the set

$$Y_0 := \left\{ y_0 \in X_{-\diamond} \,;\, \langle y_0, \psi_\lambda \rangle_{-\diamond,\diamond} = 0, \ \forall\, \lambda \in \Theta \right\}$$

which is, by construction, of finite codimension. For $y_0 \in Y_0$, the associated moment problem reduces to the one where the geometrically double eigenvalues $\lambda \in \Theta$ are now considered as simple eigenvalues with associated eigenvector $\phi_{\lambda,2}$, since the moment equation is automatically satisfied for the other eigenvector ψ_λ.

We consider now a grouping $\mathscr{G}$ as given by Proposition 6, with $p = 2$ and $\varrho > 0$ small enough such that for $i \in \{1, 2\}$ we have

$$|\lambda - \mu| > \varrho, \quad \forall\, \lambda, \mu \in \Lambda_i, \lambda \neq \mu. \tag{108}$$

Hence, Theorem 11 gives the formula

$$T_0(y_0) = \limsup_{G \in \mathscr{G}} \frac{\ln^+ \mathscr{C}(G, y_0)}{2 r_G}.$$

We will prove in the sequel, analyzing the different possible blocks, that

$$\sup_{G \in \mathscr{G}} \mathscr{C}(G, y_0) < +\infty, \tag{109}$$

which will let us conclude the claim, that is $T_0(y_0) = 0$.

Blocks of a simple eigenvalue. We immediately obtain

$$\mathscr{C}(G, y_0) = \begin{cases} \dfrac{\left| \langle y_0, \phi_{\lambda,1} \rangle_{-\diamond,\diamond} \right|^2}{\left((1 + \varepsilon_k)^2 + s_k^2 \varepsilon_k^2 \right) |\varphi_k'(0)|^2}, & \text{if } \lambda = \nu_k, \\[2ex] \dfrac{\left| \langle y_0, \phi_{\lambda,2} \rangle_{-\diamond,\diamond} \right|^2}{\left(1 + s_l^2 \right) |\varphi_l'(0)|^2}, & \text{if } \lambda = d\nu_l. \end{cases}$$

Using again (89) the estimate (104) and the fact that $(\varepsilon_k)_k$ goes to 0 as k goes to infinity, we observe that the blocks consisting of a single simple eigenvalue do not contribute to the minimal time: the quantity $\mathscr{C}(G, y_0)$ is bounded independently of G.

Moreover, by the discussion above, the blocks consisting of a single double eigenvalue belonging to Θ do not contribute either.

Blocks of two simple eigenvalues: $G = \{\lambda_1 := \nu_k\} \cup \{\lambda_2 := d\nu_l\}$**.** From Theorem 18 we obtain

$$\mathscr{C}(G, y_0) = \left\langle M^{-1}\xi, \xi \right\rangle$$

with

$$M = \mathrm{Gram}\left(b[\lambda_1], b[\lambda_2]\right) + \mathrm{Gram}\left(0, (\lambda_2 - \lambda_1) b[\lambda_2]\right)$$

and

$$\xi = \begin{pmatrix} \left\langle y_0, \phi_{\lambda_1,1} \right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \phi_{\lambda_2,2} \right\rangle_{-\diamond,\diamond} \end{pmatrix}.$$

To ease the reading, we use the following change of normalization for the eigenvectors

$$\widetilde{\phi}_{\lambda_1} := \frac{\phi_{\lambda_1,1}}{-\varphi_k'(0)}, \qquad \widetilde{\phi}_{\lambda_2} := \frac{\phi_{\lambda_2,2}}{-\varphi_l'(0)},$$

and we denote by $\widetilde{M}$ and $\widetilde{\xi}$ the associated quantities. Notice that, due to (89), the quantity $\|\widetilde{\xi}\|$ is bounded. Thus, to estimate $\mathscr{C}(G, y_0)$ we give a lower bound on the smallest eigenvalue of $\widetilde{M}$. We have

$$\begin{aligned} \widetilde{M} &= \mathrm{Gram}\left(\widetilde{b}[\lambda_1], \widetilde{b}[\lambda_2]\right) + \mathrm{Gram}\left(0, (\lambda_2 - \lambda_1)\widetilde{b}[\lambda_2]\right) \\ &= \underbrace{\begin{pmatrix} \epsilon_k^2 s_k^2 + (1+\varepsilon_k)^2 & 1 + \varepsilon_k + \varepsilon_k s_k s_l \\ 1 + \varepsilon_k + \varepsilon_k s_k s_l & 1 + s_l^2 \end{pmatrix}}_{=\Gamma^1} + \begin{pmatrix} 0 & 0 \\ 0 & (\lambda_2 - \lambda_1)^2 (1 + s_l^2) \end{pmatrix}. \end{aligned}$$

For any $\tau \in \mathbb{R}^2$, $\left\langle \widetilde{M}\tau, \tau \right\rangle \geq \left\langle \Gamma^1 \tau, \tau \right\rangle$. Then,

$$\min \mathrm{Sp}(\Gamma^1) \geq \frac{\det(\Gamma^1)}{\mathrm{tr}(\Gamma^1)} = \frac{((1+\varepsilon_k)s_l - \varepsilon_k s_k)^2}{1 + (1+\epsilon_k)^2 + \varepsilon_k^2 s_k^2 + s_l^2}$$

From (104), it comes that, for k large enough, $\min \mathrm{Sp}(\Gamma^1)$ is bounded from below by a positive constant independent of G.

Blocks made of a geometrically double eigenvalue which does not belong to Θ. Consider $G = \{\lambda\}$ with $\lambda = \nu_k = d\nu_l \in \Lambda_1 \cap \Lambda_2$. With the same notations as previously, Theorem 18 implies that

$$\mathscr{C}(G, y_0) = \left\langle \widetilde{M}^{-1} \widetilde{\xi}, \widetilde{\xi} \right\rangle$$

where

$$\widetilde{\xi} = \begin{pmatrix} \left\langle y_0, \frac{\phi_{\lambda,1}}{-\varphi_k'(0)} \right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \frac{\phi_{\lambda,2}}{-\varphi_l'(0)} \right\rangle_{-\diamond,\diamond} \end{pmatrix}$$

and

$$\widetilde{M} = \mathrm{Gram}\left(\frac{\mathscr{B}^* \phi_{\lambda,1}}{-\varphi_k'(0)}, \frac{\mathscr{B}^* \phi_{\lambda,2}}{-\varphi_l'(0)} \right) = \Gamma^1.$$

Notice that since $\lambda \notin \Theta$, we have $\det(\Gamma^1) = ((1+\varepsilon_k)s_l - \varepsilon_k s_k)^2 > 0$.

Thus, the study of the previous item proves that, for λ large enough, $\min \mathrm{Sp}(\Gamma^1)$ is bounded from below by a positive constant independent of λ.

Gathering all cases, we deduce (109) and the proof is complete. □

7.1.3. *Simultaneous distributed control*

Let us now consider the following control problem

$$\begin{cases} \partial_t y + \mathscr{A} y = \mathbb{1}_\omega \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) u(t,x), & t \in (0,T), \\ y(t,0) = y(t,1) = 0, & t \in (0,T). \end{cases} \tag{110}$$

In that case, the observation operator $\mathscr{B}^*$ is given by

$$\mathscr{B}^* : \begin{pmatrix} f \\ g \end{pmatrix} \in X_1^* \mapsto \mathbb{1}_\omega (f+g),$$

and is clearly admissible with respect to the pivot space X. Our result concerning this example is very similar to Proposition 39 and reads as follows.

Proposition 43. *For any $d \neq 1$, there exists Y_0 a closed subspace of $H^{-1}(0,1;\mathbb{R})^2$ of codimension less or equal than 1 such that*

- *for any $y_0 \notin Y_0$, system (110) is not approximately controllable;*
- *for any $y_0 \in Y_0$, system (110) is null controllable in any time $T > 0$.*

Remark 44. During the proof it will appear that there exists a countable set $D \subset (1,+\infty)$ such that for any $d \notin D \cup \{1\}$, we have $Y_0 = H^{-1}(0,1;\mathbb{R})^2$, which means that our system is null-controllable at any time $T > 0$ for any initial data. In particular, it is noticeable that this property holds for any $d < 1$, that is in the case where the diffusion coefficient is lower in the second equation (the one which does not contain coupling terms).

Proof. We start by computing the observations related to the eigenelements of $\mathscr{A}^*$

- For any $\lambda = \nu_k \in \Lambda_1$, we have
$$\mathscr{B}^* \phi_{\lambda,1} = (1+\varepsilon_k)\varphi_k \mathbb{1}_\omega. \tag{111}$$
- For any $\lambda = d\nu_l \in \Lambda_2$, we have
$$\mathscr{B}^* \phi_{\lambda,2} = \varphi_l \mathbb{1}_\omega. \tag{112}$$

If for some k we have $1+\varepsilon_k = 0$, then we clearly get chat (33) does not hold. We can thus introduce the set

$$\Theta := \{\lambda = \nu_k\, ;\, 1+\varepsilon_k = 0\},$$

which is of cardinal less or equal than 1 (by definition of the sequence $(\varepsilon_k)_k$, see Section 7.1.1). Note also that for $d < 1$, we always have $\varepsilon_k > 0$, so that $\Theta = \varnothing$, see Remark 44.

We notice however that, for any $\lambda = d\nu_l$, we have $\mathscr{B}^*\phi_{\lambda,2} \neq 0$ and that if $\lambda = \nu_k = d\nu_l \in \Lambda_1 \cap \Lambda_2$, with $\lambda \notin \Theta$, then $\mathscr{B}^*\phi_{\lambda,1}$ and $\mathscr{B}^*\phi_{\lambda,2}$ are linearly independent.

Let us introduce

$$Y_0 := \left\{ y_0 \in X;\ \text{s.t. } \left\langle y_0, \phi_{\lambda,1} \right\rangle_X = 0, \forall\, \lambda \in \Theta \right\}.$$

By definition of this set, for any initial data in Y_0, the moment equation (2) related to the eigenvector $\phi_{\lambda,1}$ for $\lambda \in \Theta$ is automatically satisfied for any control since both members are equal to zero.

As in the proof of Proposition 39, we consider a grouping $\mathscr{G}$ as given by Proposition 6, with $p = 2$ and $\varrho > 0$ small enough such that for $i \in \{1,2\}$ we have

$$|\lambda - \mu| > \varrho, \quad \forall\, \lambda, \mu \in \Lambda_i, \lambda \neq \mu.$$

Hence, Theorem 11 gives the formula

$$T_0(y_0) = \limsup_{G \in \mathscr{G}} \frac{\ln^+ \mathscr{C}(G, y_0)}{2 r_G}.$$

Let us now evaluate the quantities $\mathscr{C}(G, y_0)$ for every possible block.

Blocks made of a simple eigenvalue that does not belong to Θ. We immediately obtain

$$\mathscr{C}(G,y_0)=\begin{cases}\dfrac{\left|\left\langle y_0,\phi_{\lambda,1}\right\rangle_X\right|^2}{(1+\varepsilon_k)^2\|\varphi_k\|_\omega^2}, & \text{if } \lambda=\nu_k,\\ \dfrac{\left|\left\langle y_0,\phi_{\lambda,2}\right\rangle_X\right|^2}{\|\varphi_l\|_\omega^2}, & \text{if } \lambda=d\nu_l,\end{cases}$$

which is a bounded quantity thanks to (90) and the fact that $(\varepsilon_k)_k$ tends to zero at infinity.

Blocks made of two eigenvalues: $G=\{\lambda_1:=\nu_k\}\cup\{\lambda_2:=d\nu_l\}$**.** Note that the proof below works exactly the same in the case where $\lambda_1\neq\lambda_2$, that is if the two eigenvalues are simple, or in the case where $\lambda_1=\lambda_2$, that is if there is only a geometrically double eigenvalue.

By the discussion above, we can assume that λ_1 does not belong to Θ (if not, this block has to be considered as a block containing only the simple eigenvalue λ_2).

Thanks to Theorem 18 we have $\mathscr{C}(G,y_0)\leq\langle\widetilde{M}^{-1}\xi,\xi\rangle$ where

$$\widetilde{M}=\operatorname{Gram}(\mathbb{1}_\omega\varphi_k,\mathbb{1}_\omega\varphi_l),$$
$$\xi=\begin{pmatrix}\dfrac{\left\langle y_0,\phi_{\lambda_1,1}\right\rangle_X}{1+\varepsilon_k}\\ \left\langle y_0,\phi_{\lambda_2,2}\right\rangle_X\end{pmatrix}.$$

By using the Lebeau–Robbiano inequality (93), and the fact that $|\lambda_1-\lambda_2|\leq\varrho$, we have that

$$\langle\widetilde{M}^{-1}\xi,\xi\rangle\leq C_1e^{C_1\sqrt{r_G}}\|\xi\|^2\leq C_2e^{C_1\sqrt{r_G}}\left\|y_0\right\|_X^2,$$

where C_1,C_2 only depends on ϱ, ω and on the operator $\mathscr{A}$.

All in all, we have obtained that

$$\ln^+\mathscr{C}(G,y_0)\leq C\left(1+\sqrt{r_G}\right).$$

Gathering all cases, we conclude that $T_0(y_0)=0$. □

7.2. *Other applications*

Let us consider the following control system

$$\begin{cases}\partial_t y+\begin{pmatrix}-\partial_{xx}+c_1(x) & 1\\ 0 & -\partial_{xx}+c_2(x)\end{pmatrix}y=\begin{pmatrix}0\\ \mathbb{1}_\omega u(t,x)\end{pmatrix}, & (t,x)\in(0,T)\times(0,1),\\ y(t,0)=y(t,1)=0, & t\in(0,T),\\ y(0,x)=y_0(x),\end{cases}\tag{113}$$

where $c_1,c_2\in L^2(0,1;\mathbb{R})$.

With the technics developed in this article, one can prove the following controllability result.

Proposition 45. *For any non-negative potentials c_1, c_2, system* (113) *is null controllable in any time $T>0$ from $L^2(0,1;\mathbb{R})^2$.*

The proof follows closely the computations done for the same system with a boundary control in [9, Section 5.2.1]. The only difference is that the contributions of terms of the form $\|\mathscr{B}^*\cdot\|_U=\|\cdot\|_\omega$ are estimated using (90).

As the result stated in Proposition 45 is already known (it is for instance an application of [24] with a proof based on Carleman estimates), we do not detail the proof here to lighten this article.

With the technics developed in this article we can also analyze null controllability for the following control system

$$\begin{cases} \partial_t y + \begin{pmatrix} A & q(x) \\ 0 & A \end{pmatrix} y = \begin{pmatrix} 0 \\ \mathbb{1}_\omega u(t,x) \end{pmatrix}, & (t,x) \in (0,T)\times(0,1), \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), \end{cases} \tag{114}$$

where the coupling function q belongs to $L^\infty(0,1;\mathbb{R})$ and $\omega \subset (0,1)$ is a non empty open set. We manage to characterize the value of the minimal null-control time without any other assumption on q and ω.

This analysis extends previous results of [15] where approximate controllability was studied and those of [6] where null controllability was studied in the particular case where $\mathscr{A}$ is the Dirichlet Laplace operator and ω is an interval disjoint of $\operatorname{Supp} q$. Our formalism also allows us to recover null controllability in any time when q has a strict sign on a subdomain of ω as proved in [24] by means of Carleman estimates.

Since the analysis of this example makes use of refined spectral properties of the underlying operator whose proofs are rather intricate, we will develop it in the forthcoming paper [13].

Appendix A. Some refinements in the case of scalar controls

In [9], the block moment method was introduced to solve null controllability problems with scalar controls ($U = \mathbb{R}$). With respect to block moment problems, the main result of this paper is [9, Theorem 4.1]. In this work there were no assumptions on the counting function. The spectrum Λ was only assumed to satisfy $\Lambda \subset [1,+\infty)$ and

$$\sum_{\lambda\in\Lambda} \frac{1}{\lambda} < +\infty.$$

Using the slightly more restrictive condition (28) on the asymptotics of the counting function we allow the eigenvalues to be complex valued and we obtain sharper estimates together with the explicit dependency of the constants with respect to the final time T (see Remark 23 for possible applications of such estimates). This improved resolution of scalar block moment problems reads as follow and is proved in [12, Theorem V.4.26].

Theorem 46. *Let $p \in \mathbb{N}^*$, $\varrho, \tau, \kappa > 0$ and $\theta \in (0,1)$. Assume that*

$$\Lambda \in \mathscr{L}_w(p,\varrho,\tau,\theta,\kappa).$$

Let $G = \{\lambda_1,\dots,\lambda_g\} \subset \Lambda$ be a group satisfying (25)–(27). Let $T \in (0,+\infty)$ and $\eta \in \mathbb{N}^*$. For any multi-index $\alpha \in \mathbb{N}^g$ with $|\alpha|_\infty \le \eta$ and any

$$\omega = \left(\omega_1^0,\dots,\omega_1^{\alpha_1-1},\dots,\omega_g^0,\dots,\omega_g^{\alpha_g-1}\right) \in \mathbb{C}^{|\alpha|},$$

there exists $v_G \in L^2(0,T;\mathbb{C})$ satisfying

$$\int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\overline{\lambda_j} t} \mathrm{d}t = \omega_j^l, \qquad \forall\, j \in [\![1,g]\!],\ \forall\, l \in [\![0,\alpha_j[\![, \tag{115a}$$

$$\int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\overline{\lambda} t} \mathrm{d}t = 0, \qquad \forall\, \lambda \in \Lambda\backslash G,\ \forall\, l \in [\![0,\eta[\![. \tag{115b}$$

The solution v_G satisfies the following estimate

$$\|v_G\|_{L^2(0,T;U)} \le C \exp\left(\frac{C}{T^{\frac{\theta}{1-\theta}}}\right) \exp\left(C r_G^\theta\right) \max_{\substack{\mu\in\mathbb{N}^g \\ \mu\le\alpha}} \left|\omega\left[\overline{\lambda}_\cdot^{(\mu)}\right]\right|, \tag{116}$$

where r_G *is defined in* (35) *and with the convention*

$$\omega\left[\overline{\lambda_j}^{(l+1)}\right] = \omega_j^l, \qquad \forall\, j \in [\![1,g]\!],\ \forall\, l \in [\![0,\alpha_j[\![.$$

The constant $C>0$ *appearing in the estimate only depends on the parameters* τ, p, ϱ, η, θ *and* κ.

Moreover, there exists a constant $C_{p,\eta,r_\Lambda}>0$ *such that any* $v_G \in L^2(0,T;U)$ *solution of* (115a) *satisfy*

$$\|v_G\|_{L^2(0,T;\mathbb{C})} \geq C_{p,\eta,r_\Lambda} \max_{\substack{\mu\in\mathbb{N}^g\\ \mu\leq\alpha}} \left|\omega\left[\overline{\lambda_{\cdot}}^{(\mu)}\right]\right|. \tag{117}$$

Remark 47. If every assumption hold except (29) in the definition of the class $\mathscr{L}_w(p,\varrho,\tau,\theta,\kappa)$, Theorem 46 remains valid replacing θ in estimate (116) by any $\theta' \in (\theta,1)$ (see [12, Theorem V.4.26]).

Since every estimate on the resolution of block moment problems proved in this paper follows from (116), this remark holds in the whole current paper. Notably it applies to Theorem 10 and to the estimates of the cost of controllability stated in Proposition 20 and Corollary 21.

Appendix B. An auxiliary optimization argument

Lemma 48. *Let* Y *be a closed subspace of* $X_{-\diamond}$. *Let* $g\in\mathbb{N}^*$ *and* $\psi_1,\dots,\psi_g \in P_Y^* X_\diamond^*$. *For any* $y\in Y$, *let*

$$\xi_y = \begin{pmatrix} \langle y,\psi_1\rangle_{-\diamond,\diamond} \\ \vdots \\ \langle y,\psi_g\rangle_{-\diamond,\diamond}\end{pmatrix}.$$

Then, for any positive semi-definite hermitian square matrix $M\in\mathscr{M}_g(\mathbb{C})$, *we have*

$$\sup_{\substack{y\in Y\\ \|y\|_{-\diamond}=1}} \langle M\xi_y,\xi_y\rangle = \rho(\mathsf{G}_\psi M) \tag{118}$$

with $\mathsf{G}_\psi = \mathrm{Gram}_{X_\diamond^*}(\psi_1,\dots,\psi_g)$.

In the course of the proof we will use that there exists an isometric linear bijection $I : X_{-\diamond} \mapsto X_\diamond^*$ such that

$$\langle y,\varphi\rangle_{-\diamond,\diamond} = (Iy,\varphi)_{\diamond^*}, \quad \forall\, y\in X_{-\diamond}, \forall\,\varphi\in X_\diamond^*.$$

Note that it satisfies

$$(Iy,\varphi)_{\diamond^*} = (y, I^{-1}\varphi)_{-\diamond}, \quad \forall\, y\in X_{-\diamond}, \forall\,\varphi\in X_\diamond^*.$$

Proof. Let S be the value of the supremum in the left-hand side of (118). By assumption on the $(\psi_i)_i$, we first observe that the supremum can be taken on the whole space $X_{-\diamond}$ instead of Y without changing its value. Then, for any $1\leq i\leq g$, we have

$$\langle y,\psi_i\rangle_{-\diamond,\diamond} = (y, I^{-1}\psi_i)_{-\diamond},$$

and therefore the value of S does not change if we take the supremum over the set

$$\widetilde{\Psi} = \mathrm{Span}(\widetilde{\psi}_1,\dots,\widetilde{\psi}_g) \subset X_{-\diamond},$$

with

$$\widetilde{\psi}_i = I^{-1}\psi_i. \tag{119}$$

We write any element $y \in \widetilde{\Psi}$ as follows $y = \sum_{i=1}^{g} x_i \widetilde{\psi}_i$, with $x = (x_j)_{j\in[\![1,g]\!]} \in \mathbb{C}^g$ so that we can compute

$$\left(y, \widetilde{\psi}_i\right)_{-\diamond} = \sum_{j=1}^{g} x_j \left(\widetilde{\psi_j}, \widetilde{\psi_i}\right)_{-\diamond} = (\mathsf{G}_{\widetilde{\psi}} x)_i, \qquad \forall\, i \in [\![1,g]\!],$$

$$\left(y, y\right)_{-\diamond} = \sum_{i=1}^{g} \sum_{j=1}^{g} \bar{x}_i x_j \left(\widetilde{\psi_j}, \widetilde{\psi_i}\right)_{-\diamond} = \left\langle \mathsf{G}_{\widetilde{\psi}} x, x\right\rangle,$$

where $\mathsf{G}_{\widetilde{\psi}}$ is the Gram matrix in $X_{-\diamond}$ of the family $\{\widetilde{\psi}_1,\dots,\widetilde{\psi}_g\}$. Using that I is an isometry from $X_{-\diamond}$ onto $X_\diamond^*$ it actually appears that

$$\mathsf{G}_{\widetilde{\psi}} = \mathsf{G}_{\psi}.$$

Finally, we have proved that

$$\xi_y = \mathsf{G}_\psi x, \quad \text{and} \quad \|y\|_{-\diamond}^2 = \left\langle \mathsf{G}_\psi x, x\right\rangle.$$

The supremum we are looking for thus reads

$$S = \sup_{\substack{x\in\mathbb{C}^g \\ \langle \mathsf{G}_\psi x, x\rangle = 1}} \left\langle M\mathsf{G}_\psi x, \mathsf{G}_\psi x\right\rangle.$$

- By compactness, we know that this supremum is actually achieved at some point $x_0 \in \mathbb{C}^g$, that is

  $$\left\langle M\mathsf{G}_\psi x_0, \mathsf{G}_\psi x_0\right\rangle = S, \quad \text{and} \quad \left\langle \mathsf{G}_\psi x_0, x_0\right\rangle = 1.$$

  The Lagrange multiplier theorem gives that there exists $\lambda \in \mathbb{C}$ such that

  $$\left\langle M\mathsf{G}_\psi x_0, \mathsf{G}_\psi h\right\rangle = \lambda \left\langle \mathsf{G}_\psi x_0, h\right\rangle, \quad \forall\, h \in \mathbb{C}^g. \tag{120}$$

  Taking $h = x_0$ in this equation, we get

  $$\left\langle M\mathsf{G}_\psi x_0, \mathsf{G}_\psi x_0\right\rangle = \lambda \left\langle \mathsf{G}_\psi x_0, x_0\right\rangle = \lambda,$$

  and thus $\lambda = S$, in particular λ is a non negative real number.

  From (120), we deduce

  $$\mathsf{G}_\psi M \mathsf{G}_\psi x_0 = \lambda \mathsf{G}_\psi x_0.$$

  and since $\mathsf{G}_\psi x_0 \neq 0$ (we recall that $\left\langle \mathsf{G}_\psi x_0, x_0\right\rangle = 1$), we conclude that λ is an eigenvalue of $\mathsf{G}_\psi M$ and therefore

  $$S = \lambda \le \rho(\mathsf{G}_\psi M).$$

  We have thus proved that

  $$S \le \rho(\mathsf{G}_\psi M).$$

- If $\rho(\mathsf{G}_\psi M) = 0$, the claim is proved. If not, we set

  $$\lambda = \rho(\mathsf{G}_\psi M) = \rho(M\mathsf{G}_\psi) = \rho\left(\mathsf{G}_\psi^{\frac{1}{2}} M \mathsf{G}_\psi^{\frac{1}{2}}\right),$$

  which is a positive number which is an eigenvalue of the three matrices above. In particular, there exists $x_0 \in \mathbb{C}^g \setminus \{0\}$ such that

  $$M\mathsf{G}_\psi x_0 = \lambda x_0.$$

  Taking the inner product with $\mathsf{G}_\psi x_0$ we obtain

  $$\left\langle M\mathsf{G}_\psi x_0, \mathsf{G}_\psi x_0\right\rangle = \lambda \left\langle x_0, \mathsf{G}_\psi x_0\right\rangle,$$

  and since $\left\langle x_0, \mathsf{G}_\psi x_0\right\rangle = \left\|\mathsf{G}_\psi^{\frac{1}{2}} x_0\right\|^2$ cannot be equal to zero, we deduce that

  $$\lambda \le S,$$

  and the proof is complete. □

Appendix C. Solving general block moment problems

As this paper is oriented towards control theory we do not deal with the most general block moment problems. Indeed, in Theorem 10, the considered block moment problems have a specific right-hand side which is a linear form. This formalism is chosen in order to avoid exhibiting a particular basis of the generalized eigenspaces. The price to pay is this restriction on the considered right-hand sides. However the proofs detailed in Sections 3 and 5 directly lead to the following more general results.

The study with a group composed of geometrically simple eigenvalues (see Sections 5.1 and 5.2) leads to the following theorem.

Theorem 49. *Let* $p \in \mathbb{N}^*$, $\varrho, \tau, \kappa > 0$ *and* $\theta \in (0,1)$. *Assume that*

$$\Lambda \in \mathscr{L}_w(p, \varrho, \tau, \theta, \kappa).$$

Recall that this class of sequences is defined in (30). *Let* $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ *be a group satisfying* (25)–(27). *Let* $T \in (0, +\infty)$ *and* $\eta \in \mathbb{N}^*$. *For any multi-index* $\alpha \in \mathbb{N}^g$ *with* $|\alpha|_\infty \leq \eta$, *any*

$$\omega = \left(\omega_1^0, \dots, \omega_1^{\alpha_1 - 1}, \dots, \omega_g^0, \dots, \omega_g^{\alpha_g - 1}\right) \in \mathbb{C}^{|\alpha|},$$

and any $b \in U^{|\alpha|}$ *with*

$$b_j^0 \neq 0, \qquad \forall\, j \in [\![1, g]\!],$$

there exists $v_G \in L^2(0, T; U)$ *satisfying*

$$\int_0^T \left\langle v_G(t), (e_t b)\left[\overline{\lambda_j}^{(l+1)}\right]\right\rangle_U \mathrm{d}t = \omega_j^l, \qquad \forall\, j \in [\![1, g]\!], \forall\, l \in [\![0, \alpha_j[\![, \tag{121a}$$

$$\int_0^T v_G(t)\, t^l e^{-\overline{\lambda} t}\, \mathrm{d}t = 0, \qquad \forall\, \lambda \in \Lambda \setminus G, \forall\, l \in [\![0, \eta[\![. \tag{121b}$$

The solution v_G *satisfies the following estimate*

$$\|v_G\|^2_{L^2(0,T;U)} \leq C \exp\left(\frac{C}{T^{\frac{\theta}{1-\theta}}}\right) \exp\left(C r_G^\theta\right) \left\langle M^{-1} \xi, \xi \right\rangle,$$

where

$$\xi := \begin{pmatrix} \omega\left[\overline{\lambda}_\cdot^{(\mu^1)}\right] \\ \vdots \\ \omega\left[\overline{\lambda}_\cdot^{(\mu^{|\alpha|})}\right] \end{pmatrix},$$

the sequence $(\mu^p)_{p \in [\![0, |\alpha|]\!]}$ *is defined in* (43), *the associated matrix* M *is defined in* (52), r_G *is defined in* (35) *and with the convention*

$$\omega\left[\overline{\lambda_j}^{(l+1)}\right] = \omega_j^l, \qquad \forall\, j \in [\![1, g]\!], \forall\, l \in [\![0, \alpha_j[\![.$$

The constant $C > 0$ *appearing in the estimate only depends on the parameters* τ, p, ϱ, η, θ *and* κ.

Moreover, there exists a constant $C_{p,\eta,r_\Lambda} > 0$ *such that any* $v_G \in L^2(0, T; U)$ *solution of* (121a) *satisfy*

$$\|v_G\|^2_{L^2(0,T;U)} \geq C_{p,\eta,r_\Lambda} \left\langle M^{-1} \xi, \xi \right\rangle.$$

Remark 50. As detailed in Remark 16, when the eigenvalues in G are also algebraically simple, i.e. $\alpha_\lambda = \gamma_\lambda = 1$ for any $\lambda \in G$, the expression of ξ reduces to

$$\xi := \begin{pmatrix} \omega\left[\overline{\lambda_1}\right] \\ \vdots \\ \omega\left[\overline{\lambda_1}, \dots, \overline{\lambda_g}\right] \end{pmatrix},$$

and the expression of M reduces to the one given in (54).

The study with a group composed of semi-simple eigenvalues (see Section 5.3) leads to the following theorem.

Theorem 51. *Let $p \in \mathbb{N}^*$, $\varrho, \tau, \kappa > 0$ and $\theta \in (0,1)$. Assume that*

$$\Lambda \in \mathscr{L}_w(p, \varrho, \tau, \theta, \kappa).$$

Recall that this class of sequences is defined in (30). *Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be a group satisfying* (25)–(27). *Let $\gamma_1, \dots, \gamma_g \in \mathbb{N}^*$ and $\gamma_G = \gamma_1 + \dots + \gamma_g$. Let $\eta \in \mathbb{N}^*$ and $T \in (0, +\infty)$.*

For any $\left(\omega_{j,i}\right)_{j \in [\![1,g]\!], i \in [\![1,\gamma_j]\!]} \in \mathbb{C}^{\gamma_G}$ and any $\left(b_{j,i}\right)_{j \in [\![1,g]\!], i \in [\![1,\gamma_j]\!]} \in U^{\gamma_G}$ such that $b_{j,1}, \dots, b_{j,\gamma_j}$ are linearly independent for every $j \in [\![1,g]\!]$, there exists $v_G \in L^2(0,T;U)$ satisfying

$$\int_0^T \left\langle v_G(t), e^{-\overline{\lambda_j} t} b_{j,i} \right\rangle_U \mathrm{d}t = \omega_{j,i}, \qquad \forall\, j \in [\![1,g]\!],\ \forall\, i \in [\![1,\gamma_j]\!], \tag{122a}$$

$$\int_0^T v_G(t) t^l e^{-\overline{\lambda} t} \mathrm{d}t = 0, \qquad \forall\, \lambda \in \Lambda \backslash G,\ \forall\, l \in [\![0, \eta[\![. \tag{122b}$$

The solution v_G satisfies the following estimate

$$\| v_G \|^2_{L^2(0,T;U)} \leq C \exp\left(\frac{C}{T^{\frac{\theta}{1-\theta}}} \right) \exp\left(C r_G^{\theta} \right) \left\langle M^{-1} \xi, \xi \right\rangle,$$

where $\xi \in \mathbb{C}^{\gamma_G}$ is defined by blocks with

$$\xi_j := \begin{pmatrix} \omega_{j,1} \\ \vdots \\ \omega_{j,g} \end{pmatrix},$$

the associated matrix M is defined in (56) *and r_G is defined in* (35). *The constant $C > 0$ appearing in the estimate only depends on the parameters τ, p, ϱ, η, θ and κ.*

Moreover, there exists a constant $C_{p,\eta,r_\Lambda} > 0$ such that any $v_G \in L^2(0,T;U)$ solution of (122a) *satisfy*

$$\| v_G \|^2_{L^2(0,T;U)} \geq C_{p,\eta,r_\Lambda} \left\langle M^{-1} \xi, \xi \right\rangle.$$

Appendix D. Post-processing formulas

The minimal null control time given in Theorem 11, together with the computation of the contribution of each group given in Theorems 14 and 18, allow to answer the question of minimal null control time for a wide variety of one dimensional parabolic control problems. However, for a given problem, the precise estimate of the quantity of interest $\left\langle M^{-1}\xi, \xi \right\rangle$ can remain a tricky question.

There is no normalization condition on the eigenvectors and no uniqueness of the considered Jordan chains. Thus, it happens that there are choices for which the quantity of interest $\left\langle M^{-1}\xi, \xi \right\rangle$ is easier to compute (see for instance Remark 15). We gather here some results that are used in Sections 6 and 7 to estimate such quantities.

We will make an intensive use of the following reformulation. Let $n \in \mathbb{N}^*$ and let $T, M \in \mathrm{GL}_n(\mathbb{C})$. For any $\xi \in \mathbb{C}^n$, let $\widetilde{\xi} := T\xi$. Then,

$$\left\langle M^{-1}\xi, \xi \right\rangle = \left\langle M^{-1} T^{-1} \widetilde{\xi}, T^{-1} \widetilde{\xi} \right\rangle = \left\langle \widetilde{M}^{-1} \widetilde{\xi}, \widetilde{\xi} \right\rangle \tag{123}$$

where

$$\widetilde{M} := T M T^*. \tag{124}$$

As the matrix M is a sum of Gram matrices we will also use the following lemma.

Lemma 52. *Let X be a Hilbert space. Let $n \in \mathbb{N}^*$ and $e = (e_1, \dots, e_n) \in X^n$. Let $T \in \mathcal{M}_n(\mathbb{C})$. Then,*

$$T \operatorname{Gram}_X(e_1, \dots, e_n) T^* = \operatorname{Gram}_X\left((\overline{T}e)_1, \dots, (\overline{T}e)_n\right)$$

where, for any $i \in [\![1, n]\!]$, $(\overline{T}e)_i$ is defined by

$$(\overline{T}e)_i := \sum_{j=1}^{n} \overline{T_{i,j}} e_j.$$

Proof. For any $\omega \in \mathbb{C}^n$, it comes that

$$\left\langle T \operatorname{Gram}_X(e_1, \dots, e_n) T^* \omega, \omega \right\rangle = \left\langle \operatorname{Gram}_X(e_1, \dots, e_n) \left(T^* \omega\right), \left(T^* \omega\right) \right\rangle \tag{125}$$

$$= \left\| \sum_{i=1}^{n} (T^* \omega)_i e_i \right\|^2 \tag{126}$$

$$= \left\| \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{T_{j,i}} \omega_j e_i \right\|^2 \tag{127}$$

$$= \left\| \sum_{j=1}^{n} \omega_j (\overline{T}e)_j \right\|^2 \tag{128}$$

$$= \left\langle \operatorname{Gram}_X\left((\overline{T}e)_1, \dots, (\overline{T}e)_n\right) \omega, \omega \right\rangle. \tag{129}$$

□

Depending on the phenomenon at stake on actual examples, with a suitable choice of $\widetilde{\xi}$ (i.e. of T), the quantity $\langle \widetilde{M}^{-1} \widetilde{\xi}, \widetilde{\xi} \rangle$ can be easier to estimate than $\langle M^{-1} \xi, \xi \rangle$.

D.1. *Dilatations*

Notice that

$$\langle \widetilde{M}^{-1} \widetilde{\xi}, \widetilde{\xi} \rangle \leq \|\widetilde{M}^{-1}\| \, \|\widetilde{\xi}\|^2.$$

When the minimal null control time can be estimated with rough estimates (this can only characterize the minimal time when $T_0 = 0$), it can simplify the computations to have a bounded $\|\widetilde{\xi}\|$. To do so, it is convenient to consider dilatations of ξ.

Let X be a Hilbert space. Let $n \in \mathbb{N}^*$ and $e_1, \dots, e_n \in X$. Let $\xi \in \mathbb{C}^n$ and $\beta \in \mathbb{C}^n$ with non-zero entries. Let

$$T = D_\beta := \operatorname{diag}(\beta) \in \operatorname{GL}_n(\mathbb{C}), \quad \text{and} \quad \widetilde{\xi} = T\xi.$$

Then, from Lemma 52, it comes that

$$T \operatorname{Gram}_X(e_1, \dots, e_n) T^* = \operatorname{Gram}_X\left(\overline{\beta_1} e_1, \dots, \overline{\beta_n} e_n\right).$$

D.2. *Invariance by scale change*

In our assumptions there is no normalization condition on the eigenvectors (see Remark 15). This allows to have simpler expressions for these eigenvectors. Actually, the computation of $\langle M^{-1} \xi, \xi \rangle$ can be done with a different scale change on every generalized eigenvector as detailed in the following proposition.

Proposition 53. *Let M and ξ be as defined in Theorem 14. Let $\beta \in \mathbb{C}^{|\alpha|}$ be such that $\beta_j^0 \neq 0$ for all $j \in [\![1, g]\!]$. Set*

$$\widetilde{\xi} = \begin{pmatrix} \left\langle y_0, (\beta\phi)\left[\lambda^{(\mu^1)}\right] \right\rangle_{-\diamond,\diamond} \\ \vdots \\ \left\langle y_0, (\beta\phi)\left[\lambda^{(\mu^{|\alpha|})}\right] \right\rangle_{-\diamond,\diamond} \end{pmatrix}.$$

Then,

$$\langle M^{-1}\xi,\xi\rangle = \langle \widetilde{M}^{-1}\widetilde{\xi},\widetilde{\xi}\rangle$$

where

$$\widetilde{M} := \sum_{l=1}^{|\alpha|} \mathrm{Gram}_U\left(\underbrace{0,\dots,0}_{l-1},(\beta b)\left[\lambda^{(\mu^l-\mu^{l-1})}\right],\dots,(\beta b)\left[\lambda^{(\mu^{|\alpha_k|}-\mu^{l-1})}\right]\right). \tag{130}$$

Proof. From Leibniz formula [9, Proposition 7.13], it comes that for any $p\in[\![1,|\alpha|]\!]$,

$$(\beta\phi)\left[\lambda^{(\mu^p)}\right] = \sum_{q=1}^{|\mu^p|}\beta\left[\lambda^{(\mu^p-\mu^{q-1})}\right]\phi\left[\lambda^{(\mu^q)}\right].$$

Thus, $\widetilde{\xi} = T\xi$ where T is the following lower triangular matrix

$$T = \left(\mathbb{1}_{q\le p}\overline{\beta\left[\lambda^{(\mu^p-\mu^{q-1})}\right]}\right)_{p,q\in[\![1,|\alpha|]\!]}.$$

The diagonal entries of this lower triangular matrix are $\overline{\beta_j^0}$ and thus $T\in \mathrm{GL}_{|\alpha|}(\mathbb{C})$. From (124), the associated matrix is

$$\widetilde{M} := \sum_{l=1}^{|\alpha|} T\,\mathrm{Gram}_U\left(\underbrace{0,\dots,0}_{l-1},b\left[\lambda^{(\mu^l-\mu^{l-1})}\right],\dots,b\left[\lambda^{(\mu^{|\alpha|}-\mu^{l-1})}\right]\right)T^*.$$

Let $l\in[\![1,|\alpha|]\!]$ and

$$e_1=\dots=e_{l-1}=0,$$
$$e_p = b\left[\lambda^{(\mu^p-\mu^{l-1})}\right],\quad \forall\, p\in[\![l,|\alpha|]\!].$$

Then, for any $p\in[\![1,|\alpha|]\!]$,

$$(\overline{T}e)_p = \sum_{q=1}^{|\alpha|}\mathbb{1}_{q\le p}\beta\left[\lambda^{(\mu^p-\mu^{q-1})}\right]e_q.$$

Thus, $(\overline{T}e)_1=\dots=(\overline{T}e)_{l-1}=0$ and, for any $p\in[\![l,|\alpha|]\!]$,

$$(\overline{T}e)_p = \sum_{q=1}^{|\alpha|}\mathbb{1}_{q\le p}\beta\left[\lambda^{(\mu^p-\mu^{q-1})}\right]e_q = \sum_{q=l}^{p}\beta\left[\lambda^{(\mu^p-\mu^{q-1})}\right]b\left[\lambda^{(\mu^q-\mu^{l-1})}\right].$$

Then, using again Leibniz formula [9, Proposition 7.13], we obtain

$$(\overline{T}e)_p = (\beta b)\left[\lambda^{(\mu^p-\mu^{l-1})}\right].$$

Finally, applying (123) and Lemma 52 ends the proof of Proposition 53. □

Remark 54. As there is no normalization condition on the eigenvectors a similar statement automatically holds with M and ξ defined in Theorem 18.

D.3. *An equivalent formula for simple eigenvalues*

In this section, we consider the case of a group of simple eigenvalues i.e. $\alpha_\lambda=\gamma_\lambda=1$ for every $\lambda\in G$. In that case, the cost of the group G can be computed either using the formula of Theorem 14 for geometrically simple eigenvalues or the formula of Theorem 18 for semi-simple eigenvalues. Even though these theorems imply that those two formulas coincide (as they are both the cost of the group) we give a direct proof of this statement.

Proposition 55. *Let M and ξ be the matrix and the vector given in Theorem 14 i.e.*

$$M := \sum_{l=1}^{g} \mathrm{Gram}_U \left(\underbrace{0,\dots,0}_{l-1}, b[\lambda_l], \dots, b[\lambda_l,\dots,\lambda_g] \right)$$

and

$$\xi = \begin{pmatrix} \langle y_0, \phi[\lambda_1] \rangle_{-\diamond,\diamond} \\ \vdots \\ \langle y_0, \phi[\lambda_1,\dots,\lambda_g] \rangle_{-\diamond,\diamond} \end{pmatrix}.$$

Let $\widetilde{M}$ and $\widetilde{\xi}$ be the matrix and the vector given in Theorem 18 i.e.

$$\widetilde{M} := \sum_{l=1}^{g} \mathrm{Gram}_U \left(\delta_l^1 b[\lambda_1], \dots, \delta_l^g b[\lambda_g] \right) \quad \text{and} \quad \widetilde{\xi} := \begin{pmatrix} \langle y_0, \phi[\lambda_1] \rangle_{-\diamond,\diamond} \\ \vdots \\ \langle y_0, \phi[\lambda_g] \rangle_{-\diamond,\diamond} \end{pmatrix}. \tag{131}$$

Then,

$$\langle M^{-1}\xi, \xi \rangle = \langle \widetilde{M}^{-1}\widetilde{\xi}, \widetilde{\xi} \rangle$$

Proof. The usual interpolation formula [9, Proposition 7.6] gives

$$\phi[\lambda_i] = \sum_{j=1}^{i} \left(\prod_{k=1}^{j-1} (\lambda_i - \lambda_k) \right) \phi[\lambda_1,\dots,\lambda_j]. \tag{132}$$

Recall that the notation δ_j^i has been introduced in (55). With these notations, $\widetilde{\xi} = T\xi$ where T is the following lower triangular matrix

$$T = \left(\overline{\delta_j^i} \right)_{i,j \in [\![1,g]\!]} \in \mathrm{GL}_g(\mathbb{C}).$$

From (124), we define

$$\widehat{M} := \sum_{l=1}^{g} T\, \mathrm{Gram}_U \left(\underbrace{0,\dots,0}_{l-1}, b[\lambda_l], \dots, b\left[\lambda_l,\dots,\lambda_g\right] \right) T^*,$$

so that we have $\langle M^{-1}\xi, \xi \rangle = \langle \widehat{M}^{-1}\widetilde{\xi}, \widetilde{\xi} \rangle$. We will now prove that $\widehat{M} = \widetilde{M}$.

Let $l \in [\![1,g]\!]$ and

$$e_1 = \dots = e_{l-1} = 0,$$
$$e_j = b[\lambda_l,\dots,\lambda_j], \quad \forall\, j \in [\![l,g]\!].$$

Then, $(\overline{T}e)_1 = \dots = (\overline{T}e)_{l-1} = 0$ and for $i \in [\![l,g]\!]$, using again the interpolation property [9, Proposition 7.6], we obtain

$$\begin{aligned} (\overline{T}e)_i &= \sum_{j=l}^{g} \delta_j^i b[\lambda_l,\dots,\lambda_j] \\ &= \sum_{j=l}^{i} \delta_j^i b[\lambda_l,\dots,\lambda_j] \\ &= \delta_l^i \sum_{j=l}^{i} \left(\prod_{k=l}^{j-1} (\lambda_i - \lambda_k) \right) b[\lambda_l,\dots,\lambda_j] \\ &= \delta_l^i b[\lambda_i]. \end{aligned}$$

Recalling that $\delta_l^1 = \dots = \delta_l^{l-1} = 0$, we thus obtain

$$(\overline{T}e)_i = \delta_l^i b[\lambda_i], \qquad \forall\, i \in [\![1,g]\!].$$

Finally, from Lemma 52, we deduce that $\widehat{M} = \widetilde{M}$ which ends the proof of Proposition 55. □

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