INSTITUT DE FRANCE
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Comptes Rendus
Mathématique

Martijn Caspers

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Volume 361 (2023), p. 1711-1716

Published online: 21 December 2023

https://doi.org/10.5802/crmath.489

Les Comptes Rendus. Mathématique *sont membres du Centre Mersenne pour l'édition scientifique ouverte*
www.centre-mersenne.org
e-ISSN : 1778-3569

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https://doi.org/10.5802/crmath.489

Functional analysis / *Analyse fonctionnelle*

On the isomorphism class of q-Gaussian W*-algebras for infinite variables

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Abstract. Let $M_q(H_{\mathbb{R}})$ be the q-Gaussian von Neumann algebra associated with a separable infinite dimensional real Hilbert space $H_{\mathbb{R}}$ where $-1 < q < 1$. We show that $M_q(H_{\mathbb{R}}) \not\simeq M_0(H_{\mathbb{R}})$ for $-1 < q \neq 0 < 1$. The C*-algebraic counterpart of this result was obtained recently in [1]. Using ideas of Ozawa we show that this non-isomorphism result also holds on the level of von Neumann algebras.

Keywords. q-Gaussian von Neumann algebras, Akemann–Ostrand property.

2020 Mathematics Subject Classification. 46L35, 46L06.

Funding. Supported by the NWO Vidi grant VI.Vidi.192.018 "Non-commutative harmonic analysis and rigidity of operator algebras".

Manuscript received 23 October 2022, revised 3 March 2023, accepted 6 March 2023.

1. Introduction

Von Neumann algebras of q-Gaussian variables originate from the work of Bożejko and Speicher [3] (see also [2]). To a real Hilbert space $H_{\mathbb{R}}$ and a parameter $-1 < q < 1$ it associates a von Neumann algebra $M_q(H_{\mathbb{R}})$. At parameter $q = 0$ this assignment $H_{\mathbb{R}} \mapsto M_q(H_{\mathbb{R}})$ is known as Voiculescu's free Gaussian functor. The dependence of q of these von Neumann algebras has been an intriguing and very difficult problem. A breakthrough result in this direction was obtained by Guionnet–Shlyakhtenko [8] who showed that for finite dimensional $H_{\mathbb{R}}$ for a range of q close to 0 all von Neumann algebras $M_q(H_{\mathbb{R}})$ are isomorphic. The range for which isomorphism is known decreases as the dimension $H_{\mathbb{R}}$ becomes larger. The Guionnet–Shlyakhtenko approach is based on free analogues of (optimal) transport techniques. Their result also relies on existence and power series estimates of conjugate variables obtained by Dabrowski [5]. In fact the free transport techniques provide even an isomorphism result of underlying q-Gaussian C*-algebras.

In case $H_{\mathbb{R}}$ is infinite dimensional the isomorphism question of q-Gaussian algebras was addressed by Nelson and Zeng [10]. They showed that for *mixed* q-Gaussians for which the array $(q_{ij})_{ij}$ of commutation coefficients decays fast enough to 0 one obtains isomorphism of mixed q-Gaussian C*- and von Neumann algebras. However, the isomorphism question for the original (non-mixed) q-Gaussians remained open, see Questions 1.1 and 1.2 of [10]. In [1] we showed that on the level of C*-algebras there exists a non-isomorphsim result. In the current note we improve on this result: we show that for an infinite dimensional separable real Hilbert space $H_{\mathbb{R}}$ and $-1 < q < 1, q \neq 0$ we have $M_q(H_{\mathbb{R}}) \not\simeq M_0(H_{\mathbb{R}})$. This then fully answers Questions 1.1 and 1.2 of [10] and provides a stark contrast to the results of Guionnet–Shlyakhtenko for finite dimensional $H_{\mathbb{R}}$.

The distinguishing property of $M_q(H_{\mathbb{R}})$ and $M_0(H_{\mathbb{R}})$ is a variation of the Akemann–Ostrand property that was suggested in a note by Ozawa [12] (see also [6]) and which we shall call W*AO. We formally define it in Definition 1. The most important novelty is that we quotient $\mathscr{B}(L^2(M))$ by the C*-algebra $\mathscr{K}_M$ which is much larger than the ideal of compact operators on $L^2(M)$. This larger quotient turns out to provide von Neumann algebraic descriptions of the Akemann–Ostrand property [12]. We use this to distinguish $M_q(H_{\mathbb{R}})$ and $M_0(H_{\mathbb{R}})$.

Acknowledgements

The author thanks Mateusz Wasilewski, Changying Ding and Cyril Houdayer for comments on an earlier draft of this paper.

2. Preliminaries

$\mathscr{B}(X,Y)$ denotes the bounded operators between Banach spaces $X \to Y$. $\mathscr{K}(X,Y)$ denotes the compact operators, meaning that they map the unit ball to a relatively compact set. We set $\mathscr{B}(X) := \mathscr{B}(X,X)$ and $\mathscr{K}(X) := \mathscr{K}(X,X)$.

The algebraic tensor product (vector space tensor product) is denoted by $\otimes_{\mathrm{alg}}$ and $\otimes_{\min}$ is the minimal tensor product of C*-algebras. $\otimes$ is used for tensor products of elements.

We refer to [13] as a standard reference on von Neumann algebras. For a von Neumann algebra M we denote by $(M, L^2(M), J, L^2(M)^+)$ its standard form. For $x \in M$ we write $x^{\mathrm{op}} := Jx^*J$ which is the right multiplication with x on the standard space. For a finite von Neumann algebra M with trace τ we have $M \subseteq L^2(M)$ where $L^2(M)$ is the completion of M with respect to the inner product $\langle x, y\rangle = \tau(y^*x)$. Therefore every $T \in \mathscr{B}(L^2(M))$ determines a map $Q_0(T) \in \mathscr{B}(M, L^2(M))$ given by $x \mapsto T(x)$. Set

$$Q_1 : \mathscr{B}(L^2(M)) \to \mathscr{B}(M, L^2(M))/\mathscr{K}(M, L^2(M)) : T \mapsto Q_0(T) + \mathscr{K}(M, L^2(M)).$$

Q_1 is clearly continuous and we define the closed left-ideal $\mathscr{K}_M^L = \ker(Q_1)$ and the hereditary C*-subalgebra $\mathscr{K}_M = (\mathscr{K}_M^L)^* \cap \mathscr{K}_M^L$ of $\mathscr{B}(L^2(M))$ (see also [12]). We let $\mathscr{M}(\mathscr{K}_M) \subseteq \mathscr{B}(L^2(M))$ be the multiplier algebra of $\mathscr{K}_M$; indeed this multiplier algebra is faithfully represented on $L^2(M)$ by [9, Proposition 2.1]. Then $\mathscr{K}_M$ is an ideal in the C*-algebra $\mathscr{M}(\mathscr{K}_M)$. We have $M \subseteq \mathscr{M}(\mathscr{K}_M)$ and $M^{\mathrm{op}} \subseteq \mathscr{M}(\mathscr{K}_M)$.

2.1. *A von Neumann version of the Akemann–Ostrand property*

Definition 1. *Let M be a finite von Neumann algebra. We say that M has W*AO if the map*

$$\theta : M \otimes_{\mathrm{alg}} M^{\mathrm{op}} \to \mathscr{M}(\mathscr{K}_M)/\mathscr{K}_M : a \otimes b^{\mathrm{op}} \mapsto ab^{\mathrm{op}} + \mathscr{K}_M. \tag{1}$$

is continuous with respect to the minimal tensor norm and thus extends to a $*$-homomorphism $M \otimes_{\min} M^{\mathrm{op}} \to \mathscr{M}(\mathscr{K}_M)/\mathscr{K}_M$.

We recall the following from [12, Section 4]. Let Γ be a discrete group and let $\mathscr{L}(\Gamma)$ and $\mathscr{R}(\Gamma)$ be the left and right group von Neumann algebra respectively acting on $\ell^2(\Gamma)$. In this case $L^2(\mathscr{L}(\Gamma)) \simeq \ell^2(\Gamma)$ as bimodules with the natural left and right actions of $\mathscr{L}(\Gamma)$ and $\mathscr{R}(\Gamma)$ on $\ell^2(\Gamma)$. We have $J\delta_s = \delta_{s^{-1}}$ which extends to an antilinear isometry on $\ell^2(\Gamma)$. Then $\mathscr{R}(\Gamma) = J\mathscr{L}(\Gamma)J$.

Assume Γ is icc so that $\mathscr{L}(\Gamma)$ and $\mathscr{R}(\Gamma)$ are factors, i.e. $\mathscr{L}(\Gamma)\cap\mathscr{R}(\Gamma)=\mathbb{C}1$. The map

$$\pi: C^*(\mathscr{L}(\Gamma),\mathscr{R}(\Gamma)) \to \mathscr{B}(\ell^2(\Gamma)\otimes\ell^2(\Gamma)) : ab^{\mathrm{op}} \mapsto a\otimes b^{\mathrm{op}}, \qquad a\in\mathscr{L}(\Gamma), b^{\mathrm{op}}\in\mathscr{R}(\Gamma).$$

is a well-defined $*$-homomorphism by Takesaki's theorem on minimality of the spatial tensor product. In [12, Section 4, Theorem] Ozawa showed the following theorem.

Theorem 2. *Let Γ be an exact icc group such that the $*$-homomorphism*

$$C_r^*(\Gamma)\otimes_{\mathrm{alg}} C_r^*(\Gamma)^{\mathrm{op}} \to \mathscr{B}(\ell^2(\Gamma))/\mathscr{K}(\ell^2(\Gamma)) : a\otimes b^{\mathrm{op}} \mapsto ab^{\mathrm{op}} + \mathscr{K}(\ell^2(\Gamma)),$$

*is continuous with respect to $\otimes_{\min}$. Then $\mathscr{L}(\Gamma)$ has W*AO.*

Proof. By [12, Section 4, Theorem] we have

$$\ker(\pi) = \mathscr{K}_{\mathscr{L}(\Gamma)} \cap C^*(\mathscr{L}(\Gamma),\mathscr{R}(\Gamma)).$$

Therefore,

$$\mathscr{L}(\Gamma)\otimes_{\min}\mathscr{R}(\Gamma) \xrightarrow{\simeq\pi^{-1}} C^*(\mathscr{L}(\Gamma),\mathscr{R}(\Gamma))/(\mathscr{K}_{\mathscr{L}(\Gamma)}\cap C^*(\mathscr{L}(\Gamma),\mathscr{R}(\Gamma))),$$

which concludes the theorem. □

Remark 3. It follows that if Γ is an icc group that is bi-exact (or said to be in class $\mathscr{S}$, see [4, Section 15]) then $\mathscr{L}(\Gamma)$ has W*AO.

2.2. *q-Gaussians*

Let $-1<q<1$ and let $H_{\mathbb{R}}$ be a real Hilbert space with complexification $H := H_{\mathbb{R}}\oplus iH_{\mathbb{R}}$. Set the symmetrization operator P_q^k on $H^{\otimes k}$,

$$P_q^k(\xi_1\otimes\ldots\otimes\xi_n) = \sum_{\sigma\in S_k} q^{i(\sigma)}\xi_{\sigma(1)}\otimes\ldots\otimes\xi_{\sigma(n)}, \tag{2}$$

where S_k is the symmetric group of permutations of k elements and $i(\sigma) := \#\{(a,b)\mid a<b, \sigma(b)<\sigma(a)\}$ the number of inversions. The operator P_q^k is positive and invertible [3]. Define a new inner product on $H^{\otimes k}$ by

$$\langle\xi,\eta\rangle_q := \langle P_q^k\xi,\eta\rangle,$$

and call the new Hilbert space $H_q^{\otimes k}$. Set the Hilbert space direct sum $F_q(H) := \mathbb{C}\Omega\oplus(\bigoplus_{k=1}^\infty H_q^{\otimes k})$ where Ω is a unit vector called the vacuum vector. For $\xi\in H$ let

$$l_q(\xi)(\eta_1\otimes\ldots\otimes\eta_k) := \xi\otimes\eta_1\otimes\ldots\otimes\eta_k, \qquad l_q(\xi)\Omega = \xi,$$

and then $l_q^*(\xi) = l_q(\xi)^*$. These 'creation' and 'annihilation' operators are bounded and extend to $F_q(H)$. We define a von Neumann algebra by the double commutant

$$M_q(H_{\mathbb{R}}) := \{l_q(\xi)+l_q^*(\xi)\mid \xi\in H_{\mathbb{R}}\}''.$$

Then $\tau_\Omega(x) := \langle x\Omega,\Omega\rangle$ is a faithful tracial state on $M_q(H_{\mathbb{R}})$ which is moreover normal. Now $F_q(H)$ is the standard form Hilbert space of $M_q(H_{\mathbb{R}})$ and $Jx\Omega = x^*\Omega$. For vectors $\xi_1,\ldots,\xi_k\in H$ there exists a unique operator $W_q(\xi_1\otimes\ldots\otimes\xi_k)\in M_q(H_{\mathbb{R}})$ such that

$$W_q(\xi_1\otimes\ldots\otimes\xi_k)\Omega = \xi_1\otimes\ldots\otimes\xi_k.$$

These operators are called Wick operators. It follows that $W_q(\xi)^{\mathrm{op}}\Omega = \xi$.

Remark 4. Let $\mathbb{F}_\infty$ be the free group with countably infinitely many generators. $\mathbb{F}_\infty$ is icc and exact [4] and hence Theorem 2 applies. We conclude that $\mathscr{L}(\mathbb{F}_\infty)$ has W*AO. We have that $\mathscr{L}(F_\infty)\simeq\Gamma_0(H_{\mathbb{R}})$ with $H_{\mathbb{R}}$ a separable infinite dimensional real Hilbert space (see [7, Theorem 2.6.2]) and so $\Gamma_0(H_{\mathbb{R}})$ has the W*AO.

3. Non-isomorphism of q-Gaussian von Neumann algebras

The following theorem provides a necessary condition for W*AO.

Theorem 5. *Let M be a finite von Neumann algebra with finite normal faithful tracial state τ. Suppose there exists a unital von Neumann subalgebra $B \subseteq M$ and infinitely many subspaces $M_i \subseteq M, i \in \mathbb{N}$ that are left and right B-invariant and mutually τ-orthogonal in the sense that $\tau(y^*x) = 0$ for $x \in M_i, y \in M_j, i \neq j$. Suppose moreover that there exists $\delta > 0$ and finitely many operators $b_j, c_j \in B$, with $\sum_j b_j \otimes c_j^{\mathrm{op}}$ non-zero, such that for every $i \in \mathbb{N}$ we have*

$$\left\| Q_0\left(\sum_j b_j c_j^{op}\right)\right\|_{\mathscr{B}(M_i, L^2(M_i))} \geq (1+\delta)\left\|\sum_j b_j \otimes c_j^{\mathrm{op}}\right\|_{B\otimes_{\min} B^{\mathrm{op}}}. \tag{3}$$

*Then M does not have W*AO.*

Proof. Let X be the set of finite rank operators $x \in \mathscr{B}(L^2(M))$ such that there exists $I_x \subseteq I$ finite with $\ker(x)^\perp \subseteq \oplus_{i\in I_x} L^2(M_i)$. Take $x \in X$ and choose $k \in I \backslash I_x$. Then,

$$\begin{aligned}\left\| Q_0\left(\sum_j b_j c_j^{op} + x\right)\right\|_{\mathscr{B}(M, L_2(M))} &\geq \left\| Q_0\left(\sum_j b_j c_j^{op} + x\right)\right\|_{\mathscr{B}(M_k, L^2(M_k))} \\ &= \left\| Q_0\left(\sum_j b_j c_j^{op}\right)\right\|_{\mathscr{B}(M_k, L^2(M_k))} \\ &\geq (1+\delta)\left\|\sum_j b_j \otimes c_j^{\mathrm{op}}\right\|_{B\otimes_{\min} B^{\mathrm{op}}}.\end{aligned}$$

The operators in X are norm dense in $\mathscr{K}(L^2(M))$ and by [12, Section 2, Proposition] we have that $Q_0(\mathscr{K}(L^2(M)))$ is dense in $Q_0(\mathscr{K}_M^L)$ in the norm of $\mathscr{B}(M, L^2(M))$. As Q_0 is contractive $Q_0(X)$ is dense in $Q_0(\mathscr{K}_M^L)$. It therefore follows that for any $x \in \mathscr{K}_M^L$ we have

$$\left\| Q_0\left(\sum_j b_j c_j^{op} + x\right)\right\|_{\mathscr{B}(M, L_2(M))} \geq (1+\delta)\left\|\sum_j b_j \otimes c_j^{\mathrm{op}}\right\|_{B\otimes_{\min} B^{\mathrm{op}}}.$$

Since Q_0 is contractive for every $x \in \mathscr{K}_M^L$ we have,

$$\left\|\sum_j b_j c_j^{op} + x\right\|_{\mathscr{B}(L^2(M))} \geq (1+\delta)\left\|\sum_j b_j \otimes c_j^{\mathrm{op}}\right\|_{B\otimes_{\min} B^{\mathrm{op}}}.$$

Hence, certainly for the Banach space quotient norm we have

$$\left\|\sum_j b_j c_j^{op} + \mathscr{K}_M\right\|_{\mathscr{B}(L^2(M))/\mathscr{K}_M} \geq (1+\delta)\left\|\sum_j b_j \otimes c_j^{\mathrm{op}}\right\|_{B\otimes_{\min} B^{\mathrm{op}}}.$$

As the left hand side norm is the norm of the C*-quotient $\mathscr{M}(\mathscr{K}_M)/\mathscr{K}_M$ this concludes the proof (see [9, Proposition 2.1] as in the preliminaries). □

The proof of the following theorem essentially repeats its C*-algebraic counterpart from [1, Theorem 3.3].

Theorem 6. *Assume $\dim(H_\mathbb{R}) = \infty$ and $-1 < q < 1, q \neq 0$. Then the von Neumann algebra $M_q(H_\mathbb{R})$ does not have W*AO.*

Proof. Let $d \geq 2$ be such that $q^2 d > 1$. Let

$$M := M_q(\mathbb{R}^d \oplus H_\mathbb{R}), \qquad B := M_q(\mathbb{R}^d \oplus 0).$$

Let $\{f_i\}_i$ be an infinite set of orthogonal vectors in $0 \oplus H_{\mathbb{R}}$ such that $\|W_q(f_i)\| = 1$. Let $M_{q,i} := BW_q(f_i)B$ which is a B-B invariant subset of M. Then $M_{q,i}$ and $M_{q,j}$ are τ_Ω-orthogonal if $i \neq j$. For $k \in \mathbb{N}$ let

$$\mathscr{B}(k) = \{W_q(\xi) \mid \xi \in (\mathbb{R}^d \oplus 0)^{\otimes k}\}.$$

It is proved in [1, Equation (3.2)] that for $b, c \in \mathscr{B}(k)$ we have

$$\langle bW_q(f_i)c\Omega, f_i\rangle_q = \langle bc^{\mathrm{op}} f_i, f_i\rangle_q = q^k \langle bc^{\mathrm{op}}\Omega, \Omega\rangle_q.$$

Then for finitely many $b_j, c_j \in \mathscr{B}(k)$ we have

$$\begin{aligned}\left\| Q_0\left(\sum_j b_j c_j^{\mathrm{op}}\right)\right\|_{\mathscr{B}(M_{q,i}, L^2(M_{q,i}))} &\geq \left\|\sum_j b_j W_q(f_i) c_j\right\|_{L^2(M_{q,i})} \\ &\geq \left|\left\langle \sum_j b_j W_q(f_i) c_j \Omega, f_i\right\rangle_q\right| = \left|\sum_j q^k \left\langle b_j \Omega c_j, \Omega\right\rangle_q\right|. \quad (4)\end{aligned}$$

Now let $\{e_1, \ldots, e_d\}$ be an orthonormal basis of $\mathbb{R}^d \oplus 0$ and for $j = (j_1, \ldots, j_k) \in \{1, \ldots, d\}^k$ let $e_j = e_{j_1} \otimes \ldots \otimes e_{j_k}$. Let J_k be the set of all such multi-indices of length k. So $\# J_k = d^k$. Set $\xi_j = (P_q^k)^{-\frac{1}{2}} e_j$ so that $\langle \xi_j, \xi_j\rangle_q = \langle P_q^k \xi_j, \xi_j\rangle = 1$.

Now (4) yields that for all $k \geq 1$ and all i,

$$\begin{aligned}\left\| Q_0\left(\sum_{j\in J_k} W_q(\xi_j)^* W_q(\xi_j)^{\mathrm{op}}\right)\right\|_{\mathscr{B}(M_{q,i}, L^2(M_{q,i}))} &\geq \sum_{j\in J_k} q^k \langle W_q(\xi_j)^* \Omega W_q(\xi_j), \Omega\rangle_q \\ &= \sum_{j\in J_k} q^k \langle \Omega W_q(\xi_j), W_q(\xi_j)\Omega\rangle_q \\ &= \sum_{j\in J_k} q^k \langle \xi_j, \xi_j\rangle_q = q^k d^k.\end{aligned}$$

From [11, Proof of Theorem 2] (or see [1, Proof of Theorem 3.3]) we find,

$$\left\|\sum_{j\in J_k} W_q(\xi_j)^* \otimes W_q(\xi_j)^{\mathrm{op}}\right\|_{B\otimes_{\min} B^{\mathrm{op}}} \leq \left(\prod_{i=1}^{\infty} (1-q^i)^{-1}\right)^3 (k+1)^2 d^{k/2}.$$

Therefore, as $q^2 d > 1$ there exists $\delta > 0$ such that for k large enough we have for every i,

$$\left\| Q_0\left(\sum_{j\in J_k} W_q(\xi_j)^* W_q(\xi_j)^{\mathrm{op}}\right)\right\|_{\mathscr{B}(M_{q,i}, L^2(M_{q,i}))} \geq (1+\delta)\left\|\sum_{j\in J_k} W_q(\xi_j)^* \otimes W_q(\xi_j)^{\mathrm{op}}\right\|_{B\otimes_{\min} B^{\mathrm{op}}}.$$

Hence the assumptions of Theorem 5 are witnessed which shows that W*AO does not hold. ☐

Corollary 7. *Let $H_{\mathbb{R}}$ be an infinite dimensional real separable Hilbert space. The von Neumann algebras $\Gamma_0(H_{\mathbb{R}})$ and $\Gamma_q(H_{\mathbb{R}})$ with $-1 < q < 1, q \neq 0$ are non-isomorphic.*

Proof. This is a consequence of Theorem 6 and Remark 4 as W*AO is preserved under isomorphism. ☐

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