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Volume 361 (2023), p. 1711-1716

Published online: 21 December 2023

https://doi.org/10.5802/crmath.489

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On the isomorphism class of $q$-Gaussian $W^*$-algebras for infinite variables

Martijn Caspers

Abstract. Let $M_q(H_R)$ be the $q$-Gaussian von Neumann algebra associated with a separable infinite dimensional real Hilbert space $H_R$ where $-1 < q < 1$. We show that $M_q(H_R) \not\cong M_0(H_R)$ for $-1 < q \neq 0 < 1$. The $C^*$-algebraic counterpart of this result was obtained recently in [1]. Using ideas of Ozawa we show that this non-isomorphism result also holds on the level of von Neumann algebras.

Keywords. $q$-Gaussian von Neumann algebras, Akemann–Ostrand property.

2020 Mathematics Subject Classification. 46L35, 46L06.

Funding. Supported by the NWO Vidi grant VI.Vidi.192.018 "Non-commutative harmonic analysis and rigidity of operator algebras".

1. Introduction

Von Neumann algebras of $q$-Gaussian variables originate from the work of Bożejko and Speicher [3] (see also [2]). To a real Hilbert space $H_R$ and a parameter $-1 < q < 1$ it associates a von Neumann algebra $M_q(H_R)$. At parameter $q = 0$ this assignment $H_R \mapsto M_0(H_R)$ is known as Voiculescu’s free Gaussian functor. The dependence of $q$ of these von Neumann algebras has been an intriguing and very difficult problem. A breakthrough result in this direction was obtained by Guionnet–Shlyakhtenko [8] who showed that for finite dimensional $H_R$ for a range of $q$ close to 0 all von Neumann algebras $M_q(H_R)$ are isomorphic. The range for which isomorphism is known decreases as the dimension $H_R$ becomes larger. The Guionnet–Shlyakhtenko approach is based on free analogues of (optimal) transport techniques. Their result also relies on existence and power series estimates of conjugate variables obtained by Dabrowski [5]. In fact the free transport techniques provide even an isomorphism result of underlying $q$-Gaussian $C^*$-algebras.

In case $H_R$ is infinite dimensional the isomorphism question of $q$-Gaussian algebras was addressed by Nelson and Zeng [10]. They showed that for mixed $q$-Gaussians for which the array $(q_{ij})_{ij}$ of commutation coefficients decays fast enough to 0 one obtains isomorphism of mixed $q$-Gaussian $C^*$- and von Neumann algebras. However, the isomorphism question for the original (non-mixed) $q$-Gaussians remained open, see Questions 1.1 and 1.2 of [10]. In [1] we showed...
that on the level of $C^*$-algebras there exists a non-isomorphsim result. In the current note we improve on this result: we show that for an infinite dimensional separable real Hilbert space $H_R$ and $-1 < q < 1$, $q \neq 0$ we have $M_q(H_R) \neq M_0(H_R)$. This then fully answers Questions 1.1 and 1.2 of [10] and provides a stark contrast to the results of Guionnet–Shlyakhtenko for finite dimensional $H_R$.

The distinguishing property of $M_q(H_R)$ and $M_0(H_R)$ is a variation of the Akemann–Ostrand property that was suggested in a note by Ozawa [12] (see also [6]) and which we shall call $W^*_{AO}$. We formally define it in Definition 1. The most important novelty is that we quotient $\mathcal{R}(L^2(M))$ by the $C^*$-algebra $\mathcal{K}_M$ which is much larger than the ideal of compact operators on $L^2(M)$. This larger quotient turns out to provide von Neumann algebraic descriptions of the Akemann–Ostrand property [12]. We use this to distinguish $M_q(H_R)$ and $M_0(H_R)$.

Acknowledgements

The author thanks Mateusz Wasilewski, Changying Ding and Cyril Houdayer for comments on an earlier draft of this paper.

2. Preliminaries

$\mathcal{B}(X, Y)$ denotes the bounded operators between Banach spaces $X \to Y$. $\mathcal{K}(X, Y)$ denotes the compact operators, meaning that they map the unit ball to a relatively compact set. We set $\mathcal{B}(X) := \mathcal{B}(X, X)$ and $\mathcal{K}(X) := \mathcal{K}(X, X)$.

The algebraic tensor product (vector space tensor product) is denoted by $\otimes_{alg}$ and $\otimes_{min}$ is the minimal tensor product of $C^*$-algebras. $\otimes$ is used for tensor products of elements.

We refer to [13] as a standard reference on von Neumann algebras. For a von Neumann algebra $M$ we denote by $(M, L^2(M), J, L^2(M)^+)$ its standard form. For $x \in M$ we write $x^{\text{op}} := Jx^* J$ which is the right multiplication with $x$ on the standard space. For a finite von Neumann algebra $M$ with trace $\tau$ we have $M \subseteq L^2(M)$ where $L^2(M)$ is the completion of $M$ with respect to the inner product $\langle x, y \rangle = \tau(y^* x)$. Therefore every $T \in \mathcal{B}(L^2(M))$ determines a map $Q_0(T) \in \mathcal{B}(M, L^2(M))$ given by $x \mapsto T(x)$. Set

$$Q_1 : \mathcal{B}(L^2(M)) \to \mathcal{B}(M, L^2(M))/\mathcal{K}(M, L^2(M)) : T \mapsto Q_0(T) + \mathcal{K}(M, L^2(M)).$$

$Q_1$ is clearly continuous and we define the closed left-ideal $\mathcal{K}_M^L = \ker(Q_1)$ and the hereditary $C^*$-subalgebra $\mathcal{K}_M = (\mathcal{K}_M^L)^* \cap \mathcal{K}_M^L$ of $\mathcal{B}(L^2(M))$ (see also [12]). We let $\mathcal{M}(\mathcal{K}_M) \subseteq \mathcal{B}(L^2(M))$ be the multiplier algebra of $\mathcal{K}_M$; indeed this multiplier algebra is faithfully represented on $L^2(M)$ by [9, Proposition 2.1]. Then $\mathcal{K}_M$ is an ideal in the $C^*$-algebra $\mathcal{M}(\mathcal{K}_M)$. We have $M \subseteq \mathcal{M}(\mathcal{K}_M)$ and $M^{\text{op}} \subseteq \mathcal{M}(\mathcal{K}_M)$.

2.1. A von Neumann version of the Akemann–Ostrand property

Definition 1. Let $M$ be a finite von Neumann algebra. We say that $M$ has $W^*_{AO}$ if the map

$$\theta : M \otimes_{alg} M^{\text{op}} \to \mathcal{M}(\mathcal{K}_M)/\mathcal{K}_M : a \otimes b^{\text{op}} \mapsto ab^{\text{op}} + \mathcal{K}_M.$$  \hspace{1cm} (1)

is continuous with respect to the minimal tensor norm and thus extends to a $\ast$-homomorphism $M \otimes_{min} M^{\text{op}} \to \mathcal{M}(\mathcal{K}_M)/\mathcal{K}_M$.

We recall the following from [12, Section 4]. Let $\Gamma$ be a discrete group and let $\mathcal{L}(\Gamma)$ and $\mathcal{R}(\Gamma)$ be the left and right group von Neumann algebra respectively acting on $\ell^2(\Gamma)$. In this case $L^2(\mathcal{L}(\Gamma)) \approx \ell^2(\Gamma)$ as bimodules with the natural left and right actions of $\mathcal{L}(\Gamma)$ and $\mathcal{R}(\Gamma)$ on $\ell^2(\Gamma)$. We have $J\delta_s = \delta_{s^{-1}}$ which extends to an antilinear isometry on $\ell^2(\Gamma)$. Then $\mathcal{R}(\Gamma) = J\mathcal{L}(\Gamma)J$. 

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Assume $\Gamma$ is icc so that $\mathcal{L}(\Gamma)$ and $\mathcal{R}(\Gamma)$ are factors, i.e. $\mathcal{L}(\Gamma) \cap \mathcal{R}(\Gamma) = \mathbb{C}I$. The map

$$\pi : C^* (\mathcal{L}(\Gamma), \mathcal{R}(\Gamma)) \to \mathcal{B}(\ell^2(\Gamma) \otimes \ell^2(\Gamma)) : ab^{op} \to a \otimes b^{op}, \quad a \in \mathcal{L}(\Gamma), b^{op} \in \mathcal{R}(\Gamma).$$

is a well-defined $*$-homomorphism by Takesaki’s theorem on minimality of the spatial tensor product. In [12, Section 4, Theorem] Ozawa showed the following theorem.

**Theorem 2.** Let $\Gamma$ be an exact icc group such that the $*$-homomorphism

$$C^r_r(\Gamma) \otimes_{alg} C^r_r(\Gamma)^{op} \to \mathcal{B}(\ell^2(\Gamma)) / \mathcal{K}(\ell^2(\Gamma)) : a \otimes b^{op} \to ab^{op} + \mathcal{K}(\ell^2(\Gamma)),$$

is continuous with respect to $\otimes_{\min}$. Then $\mathcal{L}(\Gamma)$ has $W^* \Lambda O$.

**Proof.** By [12, Section 4, Theorem] we have

$$\ker(\pi) = \mathcal{K}_{\mathcal{L}(\Gamma)} \cap C^* (\mathcal{L}(\Gamma), \mathcal{R}(\Gamma)).$$

Therefore,

$$\mathcal{L}(\Gamma) \otimes_{\min} \mathcal{R}(\Gamma) \xrightarrow{\pi} C^* (\mathcal{L}(\Gamma), \mathcal{R}(\Gamma)) / (\mathcal{K}_{\mathcal{L}(\Gamma)} \cap C^* (\mathcal{L}(\Gamma), \mathcal{R}(\Gamma))),$$

which concludes the theorem. \qed

**Remark 3.** It follows that if $\Gamma$ is an icc group that is bi-exact (or said to be in class $\mathcal{S}$, see [4, Section 15]) then $\mathcal{L}(\Gamma)$ has $W^* \Lambda O$.

### 2.2. $q$-Gaussians

Let $-1 < q < 1$ and let $H_{\mathbb{R}}$ be a real Hilbert space with complexification $H := H_{\mathbb{R}} \oplus i H_{\mathbb{R}}$. Set the symmetrization operator $P^k_q$ on $H^\otimes k$,

$$P^k_q(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{\sigma \in S_k} q^{i(\sigma)} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)},$$

where $S_k$ is the symmetric group of permutations of $k$ elements and $i(\sigma) := \# \{ (a, b) \mid a < b, \sigma(b) < \sigma(a) \}$ the number of inversions. The operator $P^k_q$ is positive and invertible [3]. Define a new inner product on $H^\otimes k$ by

$$\langle \xi, \eta \rangle_q := \langle P^k_q \xi, \eta \rangle,$$

and call the new Hilbert space $H^\otimes k_q$. Set the Hilbert space direct sum $F_q(H) := \Omega \oplus (\oplus_{k=1}^\infty H^\otimes k_q)$ where $\Omega$ is a unit vector called the vacuum vector. For $\xi \in H$ let

$$l_q(\xi)(\eta_1 \otimes \cdots \otimes \eta_k) := \xi \otimes \eta_1 \otimes \cdots \otimes \eta_k, \quad l_q(\xi) \Omega = \xi,$$

and then $l_q^*(\xi) = l_q(\xi)^*$. These ‘creation’ and ‘annihilation’ operators are bounded and extend to $F_q(H)$. We define a von Neumann algebra by the double commutant

$$M_q(H_{\mathbb{R}}) := (l_q(\xi) + l_q^*(\xi) \mid \xi \in H_{\mathbb{R}})^\prime \prime.$$

Then $\tau_{\Omega}(x) := \langle x \Omega, \Omega \rangle$ is a faithful tracial state on $M_q(H_{\mathbb{R}})$ which is moreover normal. Now $F_q(H)$ is the standard form Hilbert space of $M_q(H_{\mathbb{R}})$ and $Jx\Omega = x^* \Omega$. For vectors $\xi_1, \ldots, \xi_k \in H$ there exists a unique operator $W_q(\xi_1 \otimes \cdots \otimes \xi_k) \in M_q(H_{\mathbb{R}})$ such that

$$W_q(\xi_1 \otimes \cdots \otimes \xi_k) \Omega = \xi_1 \otimes \cdots \otimes \xi_k.$$

These operators are called Wick operators. It follows that $W^*_q(\xi) \Omega = \xi$.

**Remark 4.** Let $F_\infty$ be the free group with countably infinitely many generators. $F_\infty$ is icc and exact [4] and hence Theorem 2 applies. We conclude that $\mathcal{L}(F_\infty)$ has $W^* \Lambda O$. We have that $\mathcal{L}(F_\infty) \approx \Gamma_0(H_{\mathbb{R}})$ with $H_{\mathbb{R}}$ a separable infinite dimensional real Hilbert space (see [7, Theorem 2.6.2]) and so $\Gamma_0(H_{\mathbb{R}})$ has the $W^* \Lambda O$. 

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3. Non-isomorphism of $q$-Gaussian von Neumann algebras

The following theorem provides a necessary condition for $W^*\text{AO}$.

**Theorem 5.** Let $M$ be a finite von Neumann algebra with finite normal faithful tracial state $\tau$. Suppose there exists a unital von Neumann subalgebra $B \subset M$ and infinitely many subspaces $M_i \subset M$, $i \in \mathbb{N}$ that are left and right $B$-invariant and mutually $\tau$-orthogonal in the sense that $\tau(y^*x) = 0$ for $x \in M_i$, $y \in M_j$, $i \neq j$. Suppose moreover that there exists $\delta > 0$ and finitely many operators $b_j, c_j \in B$, with $\sum_j b_j \otimes c_j$ non-zero, such that for every $i \in \mathbb{N}$ we have

$$\left\| Q_0 \left( \sum_j b_j c_j \right) \right\|_{\mathcal{B}(M_i, L^2(M_i))} \geq (1 + \delta) \left\| \sum_j b_j \otimes c_j \right\|_{B \otimes_{\min} B^\text{op}}.$$ (3)

Then $M$ does not have $W^*\text{AO}$.

**Proof.** Let $X$ be the set of finite rank operators $x \in \mathcal{B}(L^2(M))$ such that there exists $I_x \subset I$ finite with $\ker(x)^0 \subset \oplus_{i \in I_x} L^2(M_i)$. Take $x \in X$ and choose $k \in I \setminus I_x$. Then,

$$\left\| Q_0 \left( \sum_j b_j c_j + x \right) \right\|_{\mathcal{B}(M_k, L^2(M_k))} \geq \left\| Q_0 \left( \sum_j b_j c_j \right) \right\|_{\mathcal{B}(M_k, L^2(M_k))}$$

$$= \left\| Q_0 \left( \sum_j b_j c_j \right) \right\|_{\mathcal{B}(M_i, L^2(M_i))} \geq (1 + \delta) \left\| \sum_j b_j \otimes c_j \right\|_{B \otimes_{\min} B^\text{op}}.$$

The operators in $X$ are norm dense in $\mathcal{K}(L^2(M))$ and by [12, Section 2, Proposition] we have that $Q_0(\mathcal{K}(L^2(M)))$ is dense in $Q_0(\mathcal{K}_M^L)$ in the norm of $\mathcal{B}(M, L^2(M))$. As $Q_0$ is contractive $Q_0(X)$ is dense in $Q_0(\mathcal{K}_M^L)$. It therefore follows that for any $x \in \mathcal{K}_M^L$ we have

$$\left\| Q_0 \left( \sum_j b_j c_j + x \right) \right\|_{\mathcal{B}(M_k, L^2(M_k))} \geq (1 + \delta) \left\| \sum_j b_j \otimes c_j \right\|_{B \otimes_{\min} B^\text{op}}.$$

Since $Q_0$ is contractive for every $x \in \mathcal{K}_M^L$ we have,

$$\left\| \sum_j b_j c_j + x \right\|_{\mathcal{B}(L^2(M))} \geq (1 + \delta) \left\| \sum_j b_j \otimes c_j \right\|_{B \otimes_{\min} B^\text{op}}.$$

Hence, certainly for the Banach space quotient norm we have

$$\left\| \sum_j b_j c_j + \mathcal{K}_M \right\|_{\mathcal{B}(L^2(M)/\mathcal{K}_M)} \geq (1 + \delta) \left\| \sum_j b_j \otimes c_j \right\|_{B \otimes_{\min} B^\text{op}}.$$

As the left hand side norm is the norm of the $C^*$-quotient $\mathcal{M}(\mathcal{K}_M)/\mathcal{K}_M$ this concludes the proof (see [9, Proposition 2.1] as in the preliminaries).

The proof of the following theorem essentially repeats its $C^*$-algebraic counterpart from [1, Theorem 3.3].

**Theorem 6.** Assume $\dim(H_\mathbb{R}) = \infty$ and $-1 < q < 1, q \neq 0$. Then the von Neumann algebra $M_q(H_\mathbb{R})$ does not have $W^*\text{AO}$.

**Proof.** Let $d \geq 2$ be such that $q^2 d > 1$. Let

$$M := M_q(\mathbb{R}^d \oplus H_\mathbb{R}), \quad B := M_q(\mathbb{R}^d \oplus 0).$$
Let \( \{f_i\}_i \) be an infinite set of orthogonal vectors in \( 0 \oplus H_\mathbb{R} \) such that \( \|W_q(f_i)\| = 1 \). Let \( M_{q,i} := BW_q(f_i)B \) which is a \( B' \)-invariant subset of \( M \). Then \( M_{q,i} \) and \( M_{q,j} \) are \( \tau_\Omega \)-orthogonal if \( i \neq j \).

For \( k \in \mathbb{N} \) let

\[
\mathcal{B}(k) = \{ W_q(\xi) \mid \xi \in (\mathbb{R}^d \oplus 0)^{\otimes k} \}.
\]

It is proved in [1, Equation (3.2)] that for \( b, c \in \mathcal{B}(k) \) we have

\[
\langle bW_q(f_j)\xi, f_i \rangle_q = \langle b\xi^{op}, f_i \rangle_q = q^k\langle b\xi^{op}, \Omega \rangle_q.
\]

Then for finitely many \( b_j, c_j \in \mathcal{B}(k) \) we have

\[
\left\| Q_0 \left( \sum_j b_j c_j^{op} \right) \right\|_{\mathcal{B}(M_{q,i}, L^2(M_{q,i})))} \geq \left\| \sum_j b_j W_q(f_j) c_j \right\|_{L^2(M_{q,i})} \geq \left| \sum_j \langle b_j W_q(f_j) c_j \Omega, f_i \rangle_q \right| = \left| \sum_j q^k \langle b_j \xi^{op}, \Omega \rangle_q \right|.
\]

Now (4) yields that for all \( k \geq 1 \) and all \( i \),

\[
\left\| Q_0 \left( \sum_{j \in J_k} W_q(\xi_j)^* W_q(\xi_j)^{op} \right) \right\|_{\mathcal{B}(M_{q,i}, L^2(M_{q,i})))} \geq \sum_{j \in J_k} q^k \langle W_q(\xi_j)^* \Omega W_q(\xi_j), \Omega \rangle_q = \sum_{j \in J_k} q^k \langle \Omega W_q(\xi_j), W_q(\xi_j) \Omega \rangle_q = \sum_{j \in J_k} q^k \langle \xi_j, \xi_j \rangle_q = q^k d^k.
\]

From [11, Proof of Theorem 2] (or see [1, Proof of Theorem 3.3]) we find,

\[
\left\| \sum_{j \in J_k} W_q(\xi_j)^* \otimes W_q(\xi_j)^{op} \right\|_{B_{\min}B^{op}} \leq \left( \prod_{i=1}^{\infty} (1 - q^{-i})^{-1} \right)^{3} (k + 1)^2 d^{k/2}.
\]

Therefore, as \( q^2 d > 1 \) there exists \( \delta > 0 \) such that for \( k \) large enough we have for every \( i \),

\[
\left\| Q_0 \left( \sum_{j \in J_k} W_q(\xi_j)^* W_q(\xi_j)^{op} \right) \right\|_{\mathcal{B}(M_{q,i}, L^2(M_{q,i})))} \geq (1 + \delta) \left\| \sum_{j \in J_k} W_q(\xi_j)^* \otimes W_q(\xi_j)^{op} \right\|_{B_{\min}B^{op}}.
\]

Hence the assumptions of Theorem 5 are witnessed which shows that \( W^{*}A^{o} \) does not hold. \( \square \)

**Corollary 7.** Let \( H_\mathbb{R} \) be an infinite dimensional real separable Hilbert space. The von Neumann algebras \( \Gamma_0(H_\mathbb{R}) \) and \( \Gamma_q(H_\mathbb{R}) \) with \( -1 < q < 1, q \neq 0 \) are non-isomorphic.

**Proof.** This is a consequence of Theorem 6 and Remark 4 as \( W^{*}A^{o} \) is preserved under isomorphism. \( \square \)

**References**


