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On the generalised André–Pink–Zannier conjecture.

Rodolphe Richard [a] **and Andrei Yafaev** [*, a]

[a] UCL Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK
E-mails: r.richard@ucl.ac.uk, yafaev@ucl.ac.uk

Abstract. We introduce and study the notion of a generalised Hecke orbit in a Shimura variety. We define a height function on such an orbit and study its properties. We obtain lower bounds for the sizes of Galois orbits of points in a generalised Hecke orbit in terms of this height function, assuming the "weakly adelic Mumford–Tate hypothesis" and prove the generalised André–Pink–Zannier conjecture (a special case of the Zilber-Pink conjecture) under this assumption using the Pila–Zannier strategy.

Résumé. On introduit et étudie la notion d'orbite de Hecke généralisée dans une variété de Shimura. On définit une notion de hauteur sur une telle orbite et étudie ses properiétés. On obtient une borne inférieure pour la taille des orbites Galoisiennes de points dans une orbite de Hecke généralisee en termes de cette fonction hauteur en admettant « la conjecture de Mumford–Tate faiblement adélique » et on démontre la conjecture de André–Pink–Zannier généralisée (un cas particulier de la conjecture de Zilber-Pink) en utilisant la stratégie de Pila–Zannier.

Keywords. Shimura varieties, Hecke orbits, Zilber-Pink, Heights, Siegel sets, Mumford–Tate conjecture, Adelic linear groups.

2020 Mathematics Subject Classification. 03C64, 11G18, 11G50, 11F80, 14L30, 20G35, 15A16, 14G35.

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This note describes the results of [21]. We refer to [5] for the language of Shimura varieties, and to [23, §3] for the notion of weakly special subvarieties.

1. Generalised André–Pink–Zannier conjecture and Main results

Let (G,X) be a Shimura datum, let $K \leq G(\mathbb{A}_f)$ be a compact open subgroup, and let $S = Sh_K(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ be the associated Shimura variety. Let $x_0 \in X$ (which we view as a morphism $\mathbb{S} \to G_{\mathbb{R}}$) and let $M \leq G$ be its Mumford–Tate group.

In the following definition $\mathrm{Hom}(M,G)(\mathbb{Q})$ denotes the set of algebraic group morphisms *defined over* $\mathbb{Q}$.

* Corresponding author.

Definition 1. *We define the* Generalised Hecke orbit $\mathscr{H}(x_0)$ *of* x_0 in X *as*

$$\mathscr{H}(x_0) := X \cap \{\phi \circ x_0 : \phi \in \mathrm{Hom}(M,G)(\mathbb{Q})\}$$

and the Generalised Hecke orbit $\mathscr{H}([x_0,g_0])$ *of* $[x_0,g_0]$ in $\mathrm{Sh}_K(G,X)$ *as*

$$\mathscr{H}([x_0,g_0]) := \{[x,g] : x \in \mathscr{H}(x_0), g \in G(\mathbb{A}_f)\}.$$

This notion of Hecke orbit is the most general; we refer to [21, §2.5] for a comparison with some other notions of Hecke orbits. For instance, if A is an abelian variety with Mumford–Tate group $M \le G := GSp(2\dim(A))$, every abelian variety B isogenous to A can be obtained from a morphism $M \to G \in \mathscr{H}(x_0)$. Definition 1 is also readily[1] at least as general than the notion of generalised Hecke orbit defined by [16, §3].

The following conjecture extends questions of [1] of André, [25] of the second author and [16] of Pink. Similar questions have been considered by Zannier [26] in the context of abelian schemes. An argument of Orr ([12]) shows that it is a special case of the Zilber-Pink conjecture for pure Shimura varieties.

Conjecture 2 (Generalised André–Pink–Zannier, [21, Conj. 1.1]). *Let S be a Shimura variety and Σ a subset of a generalised Hecke orbit in S. Then the irreducible components of the Zariski closure of Σ are weakly special subvarieties.*

For a sufficiently large field E of finite type over $\mathbb{Q}$ we have the following (cf. [24]): S and $s_0 = [x_0,1]$ are defined over E and there exists a Galois representation $\rho_{x_0} : \mathrm{Gal}(\overline{E}/E) \to M(\mathbb{A}_f) \cap K$ such that

$$\forall\, \sigma \in \mathrm{Gal}(\overline{E}/E),\ g \in G(\mathbb{A}_f),\ \sigma([x_0,g]) = [x_0, \rho_{x_0}(\sigma)\cdot g].$$

Our main result is the following.

Theorem 3 ([21, Thm. 1.2]). *In the above situation, let $U = \rho_{x_0}(\mathrm{Gal}(\overline{E}/E))$. We assume:*

$$\exists\, C > 0, \forall\, p, [K \cap M(\mathbb{Q}_p) : U \cap M(\mathbb{Q}_p)] \le C. \tag{1}$$

Then, for any $\Sigma \subseteq \mathscr{H}(x_0)$, every irreducible component of $\overline{\Sigma}^{Zar}$ is weakly special.

We call (1) the weakly adelic Mumford–Tate hypothesis. It is related to a weak form of the conjecture [22, 11.4?] due to Serre. The assumption (1) is known in many cases. This leads to the following unconditional results.

1.1.

Combining Theorem 3 with [4, Thm. A (i)] we have the following, which strictly contains a 2005 result of Pink [16, §7] (and [3, Thm. B]).

Theorem 4 ([21]). *The conjecture 2 is true if S is of abelian type, and Σ contains a point s which satisfies the Mumford–Tate conjecture (at some ℓ, in the sense of [24]).*

The assumptions of Theorem 4 are satisfied in the case where $S = \mathscr{A}_g$ and Σ contains a point $[A]$, where the abelian variety A satisfies the Mumford–Tate conjecture (at some prime ℓ). Examples of such abelian varieties include: $\dim(A) \le 3$; or when $\dim(A)$ is odd and $\mathrm{End}(A) \simeq \mathbb{Z}$. More examples are given in [15], and many examples are mentioned in [10, §2.4].

[1] Or combine [16, Prop. 3.6] and [21, Prop. 2.15].

1.2.

The following is not restricted to Shimura varieties of abelian type.

Theorem 5 ([2, Thm. 1.2]). *We decompose the adjoint datum* $(M^{ad}, X_{M^{ad}})$ *of* $(M, X_M) := (M, M(\mathbb{R}) \cdot x_0)$ *as a product*

$$(p_1, \ldots, p_f) : (M^{ad}, X_{M^{ad}}) \simeq (M_1, X_1) \times \ldots \times (M_f, X_f)$$

with respect to the $\mathbb{Q}$*-simple factors* M_i *of* M^{ad}.

Assume that for some compact open subgroups $K_i \leq M_i(\mathbb{A}_f)$

$$\forall\ i \in \{1;\ldots;f\},\ [p_i \circ ad_M(x_0)] \in Sh_{K_i}(M_i, X_i)(\mathbb{C}) \smallsetminus Sh_{K_i}(M_i, X_i)(\overline{\mathbb{Q}}).$$

Then $U = \rho_{x_0}(\mathrm{Gal}(\overline{E}/E))$ *satisfies* (1).

Theorems 6 and 7 follow from the combination of Theorem 3 with Theorem 5.

Theorem 6 ([21]). *The conjecture 2 is true if* Σ *contains a* $\overline{\mathbb{Q}}$*-Zariski generic point* s *of a special subvariety* $Z \subseteq S$*, namely: for every proper subvariety* $V \subsetneq Z$ *defined over* $\overline{\mathbb{Q}}$*, we have* $s \notin V(\mathbb{C})$.

Theorem 7 ([21]). *The conjecture 2 is true if* M^{ad} *is* $\mathbb{Q}$*-simple and* Σ *contains a point* s *in* $S(\mathbb{C}) \smallsetminus S(\overline{\mathbb{Q}})$.

In the case $M^{ad} = \{1\}$ we recover a result of [6] and [9].

Theorem 8 ([6] and [9]). *The conjecture 2 is true if* Σ *contains a special point.*

1.3. *Previous results towards Conjecture 2*

For the history of the Conjecture and previous results, see the introduction of [21].

The results [16, 17] towards Conjecture 2, based on equidistribution of Hecke points, are limited to the case where s is Hodge generic, and to an assumption similar to (1) ([16, Def. 6.3], [17, §6-7, pp. 57–59]). The case of general S, in the Hodge generic case $Z = S$, can be treated using the results of [17] and an extension of [18, Prop 3.5].

For general Z and S, Conjecture 2 was obtained, under an assumption substantially weaker than (1), but for a much more restrictive notion of Hecke orbits (the "$\mathscr{S}$-Hecke orbit" for a finite set of primes $\mathscr{S}$). See [20] for an approach of based on Ratner's theorems, and see [11] for an approach based on the Pila–Zannier strategy.

2. Some useful results

The proof of Theorem 3 in Section 3 relies on the following results, which are of independent interest.

2.1. *Geometric Hecke orbits*

We define $W = G{\cdot}\phi_0 \subseteq \mathrm{Hom}(M, G)$ the conjugacy class of the injection $\phi_0 : M \hookrightarrow G$, as an algebraic variety. We define the geometric Hecke orbit as

$$\mathscr{H}^g(x_0) = X \cap \{\phi \circ x_0 : \phi \in W(\mathbb{Q})\} \subset \mathscr{H}(x_0).$$

The following is an essential tool in reduction steps in the proof of Theorem 3.

Theorem 9 ([21, §2.4]). *The generalised Hecke orbit* $\mathscr{H}(x_0)$ *is a union of finitely many geometric Hecke orbits.*

2.2. *Height functions and estimates*

Let $\mathfrak{m}$ and $\mathfrak{g}$ be Lie algebras over $\mathbb{Q}$, and choose $\mathbb{Z}$-structures on $\mathfrak{m}$ and $\mathfrak{g}$. We denote by $\mathfrak{m}_{\widehat{\mathbb{Z}}} \leq \mathfrak{m}\otimes\mathbb{A}_f$ and $\mathfrak{g}_{\widehat{\mathbb{Z}}} \leq \mathfrak{g}\otimes\mathbb{A}_f$ the corresponding $\widehat{\mathbb{Z}}$-structures and define

$$H_{\mathbb{A}_f} : \mathrm{Hom}(\mathfrak{m},\mathfrak{g})\otimes\mathbb{A}_f \to \mathbb{Z}_{\geq 1} \quad \text{by} \quad \Phi \mapsto \min\{n \in \mathbb{Z}_{\geq 1} : n\cdot\Phi(\mathfrak{m}_{\widehat{\mathbb{Z}}}) \subseteq \mathfrak{g}_{\widehat{\mathbb{Z}}}\}.$$

2.2.1.

We choose $\mathbb{Z}$-structures in such a way that $\mathfrak{g}_{\widehat{\mathbb{Z}}}$ is invariant under the adjoint action of K and $\mathfrak{m}_{\widehat{\mathbb{Z}}} = \mathfrak{m}\otimes\mathbb{A}_f \cap \mathfrak{g}_{\widehat{\mathbb{Z}}}$.

Proposition 10 ([21, §4.3]). *There is a well defined function $H_{[x_0,1]} : \mathcal{H}([x_0,1]) \to \mathbb{Z}_{\geq 1}$ given by*

$$[\phi\circ x_0, g] \mapsto H_{\mathbb{A}_f}(d(g^{-1}\cdot\phi\cdot g)).$$

This function is constant on the orbits of $\mathrm{Gal}(\overline{E}/E)$ *in* $\mathcal{H}([x_0,1])$.

For a set A and two functions $f, g : A \to \mathbb{R}_{\geq 0}$, we write $f \preccurlyeq g$, resp. $f \approx g$ when

$$\exists\, b, c, d \in \mathbb{R}_{>0}, \forall\, a \in A, f(a) \leq d + b\cdot g(a)^c, \quad \text{resp. } f \preccurlyeq g \text{ and } g \preccurlyeq f.$$

The following estimate requires the assumption (1).

Theorem 11 ([21, Thm. 6.4 and Prop. 3.6]). *As $s = [\phi\circ x_0, g]$ ranges through $\mathcal{H}([x_0,1])$, we have*

$$|\mathrm{Gal}(\overline{E}/E)\cdot s| \approx H_{\mathbb{A}_f}(d(g^{-1}\cdot\phi\cdot g)). \tag{2}$$

2.2.2.

Only the lower bound $H_{\mathbb{A}_f}(d(g^{-1}\cdot\phi\cdot g)) \preccurlyeq |\mathrm{Gal}(\overline{E}/E)\cdot s|$ is needed for the proof of Theorem 3. This lower bound is derived from the following.

Theorem 12 ([21, Thm. B.1]). *Let $M \leq GL(N)$ be a linear algebraic subgroup defined over $\mathbb{Q}$, denote by $\phi_0 : M \to GL(N)$ the identity morphism and W the $GL(N)$-conjugacy class of ϕ_0. We choose a basis of* $\mathfrak{m}$ *such that* $\mathfrak{m}_{\widehat{\mathbb{Z}}} = \mathfrak{m}\otimes\mathbb{A}_f \cap \mathfrak{gl}(N,\widehat{\mathbb{Z}})$*, and define $M(\widehat{\mathbb{Z}}) = M(\mathbb{A}_f)\cap GL(N,\widehat{\mathbb{Z}})$.*

There exists $c = c(\phi_0) \in \mathbb{R}_{>0}$ such that, as ϕ ranges through $W(\mathbb{A}_f)$, we have

$$[\phi(M(\widehat{\mathbb{Z}})) : \phi(M(\widehat{\mathbb{Z}}))\cap GL(N,\widehat{\mathbb{Z}})] \geq \frac{1}{c^{\omega(H_{\mathbb{A}_f}(d\phi))}}\cdot H_{\mathbb{A}_f}(d\phi). \tag{3}$$

(Where $\omega(n)$ is the number of prime factors of n.)

A main tool in proving (3) is Theorem 13. We establish Theorem 13 using estimates [21, Prop. A.1] on p-adic norms of exponentials of a matrix $X \in M_d(\mathbb{Q}_p)$.

Theorem 13 (Lemma of the exponentials, [21, Thm. A.3]). *Let $X \in M_d(\mathbb{Q}_p)$ be such that* $\exp(X)$ *converges and denote by* $\exp(X)^{\mathbb{Z}}$ *the subgroup generated by* $\exp(X)$ *in $GL_d(\mathbb{Q}_p)$. We define $H_p(X) = \max\{1; \|X\|\}$. Then we have*

$$[\exp(X)^{\mathbb{Z}} : \exp(X)^{\mathbb{Z}} \cap GL_d(\mathbb{Z}_p)] \geq H_p(X)/d.$$

If $p > d$, we have $[\exp(X)^{\mathbb{Z}} : \exp(X)^{\mathbb{Z}} \cap GL_d(\mathbb{Z}_p)] \geq H_p(X)$.

2.3. *Comparison with the global height*

Let $\mathfrak{S} \subseteq G(\mathbb{R})^+$ be a finite union of Siegel sets, and denote by $\mathfrak{S}\cdot\phi_0 \subseteq W(\mathbb{R})$ its image.

There exists a closed affine embedding $W \to \mathbb{A}^n$ defined over $\mathbb{Q}$, say $\iota : \phi \mapsto d\phi : W \to \mathrm{Hom}(\mathfrak{m},\mathfrak{g})$. For $(q_1,\ldots,q_n) \in \mathbb{Q}^n$, we denote by $H_{\mathbb{Q}}(q_1,\ldots,q_n) \in \mathbb{Z}_{\geq 1}$ the usual height of $(q_1,\ldots,q_n)$ and we denote by $H_f(q_1,\ldots,q_n) \in \mathbb{Z}_{\geq 1}$ the lowest common multiple of the denominators of the q_i. We recover $H_{\mathbb{A}_f}(d\phi) = H_f(\iota(\phi))$ on $W(\mathbb{Q})$.

Theorem 14 ([21, Thm. 5.16]). *As ϕ ranges through $\mathfrak{S}\cdot\phi_0 \cap W(\mathbb{Q})$, we have*

$$H_f(\iota(\phi)) \approx H_{\mathbb{Q}}(\iota(\phi)). \tag{4}$$

Theorem 14 was [13, Thm. 1.1] obtained by M. Orr, *in the case $W = G$*. This is insufficient for us. We replaced the "block version of Pila–Wilkie" (used by Orr) by Theorem 15.

The above (4) is used to relate the height used in (2) with the height used in Theorem 15. We deduce Theorem 15 from [14, Thm. 1.7]. The referee informed us that Theorem 15 can also be deduced from [7, Cor. 7.2]. Theorem 15 and its proof in [21] are both much simpler.

Theorem 15 ([21, Thm. 7.1]). *Let $W \subseteq \mathbb{A}^N$ be a closed subvariety defined over $\mathbb{Q}$, let X be a semialgebraic set, and let $p : W(\mathbb{R}) \to X$ be a semialgebraic map.*

Let $Z \subseteq X$ be a definable subset, and denote Z^{alg} be the union of the semialgebraic subsets of X which are contained in Z and of non-zero dimension.

Then, for every $\epsilon \in \mathbb{R}_{>0}$, there exists $C(\epsilon, Z) \in \mathbb{R}_{>0}$, such that

$$\forall\, T \gg 0, \quad |(Z \smallsetminus Z^{\mathrm{alg}}) \cap p(\{w \in W(\mathbb{Q}) : H_{\mathbb{Q}}(w) \le T\})| \le C(\epsilon, Z)\cdot T^{\epsilon}.$$

3. Outline of the proof of Theorem 3

We reduce the Conjecture 2 to the case where $V := \overline{\Sigma} = \overline{\{s_0; s_1; \ldots\}}$ is irreducible, G is adjoint and V is Hodge generic in S. We rely on functoriality properties of geometric and generalised Hecke orbits.[2] Theorem 9 allows us to use geometric and generalised Hecke orbits interchangeably. We also prove, cf. [21, §6.3], functoriality properties of the assumption (1).

The final objective of the proof is to apply the geometric part of the André–Oort conjecture [23] (or [19]), and use induction on the number of simple factors of M^{ad}. For every n large enough, we construct a weakly special subvariety $Z_n \subseteq V$ of non-zero dimension such that $s_n \in Z_n$. Then [19, 23] describes $\overline{\bigcup Z_n}$, and we deduce Conjecture 2.

In order to construct the non-zero dimensional Z_n, we use the Pila–Zannier strategy. The generalised Hecke orbit is naturally related to $W(\mathbb{Q})$ where $W = G\cdot\phi_0 \simeq G/Z_G(M)$ (cf. [21, Lem. 2.5].)

The goal is to apply the variant Theorem 15 of the Pila–Wilkie theorem, after constructing many rational points of small height in some set definable in an o-minimal structure. This definable set is

$$\widetilde{V} = \left(\overset{-1}{\pi}(V) \cap \mathfrak{S}\right) / Z_{G(\mathbb{R})}(M) \subseteq W(\mathbb{R})$$

where $\pi : G(\mathbb{R}) \to X \to S$ is the uniformisation map, and $\mathfrak{S} \subseteq G(\mathbb{R})$ is a finite union of Siegel sets such that $S = \pi(\mathfrak{S})$.

Let E be field of definition of V. Then V contains the Galois orbits $\mathrm{Gal}(\overline{E}/E)\cdot s_n$. Each point $s' \in \mathrm{Gal}(\overline{E}/E)\cdot s_n$ lifts to a rational point $\widetilde{s'} \in \widetilde{V} \cap W(\mathbb{Q})$.

By Proposition 10, the value of $H_{[x_0,1]}$ is constant as s' ranges through $\mathrm{Gal}(\overline{E}/E)\cdot s_n$. It follows from (2) that there are $\#\mathrm{Gal}(\overline{E}/E)\cdot s_n \approx H_{[x_0,1]}(s_n)$ such points.[3] By (4), we have $H_{[x_0,1]}(s_n) \approx H_{\mathbb{Q}}(\widetilde{s_n})$. We introduce

$$Q_n := \{\phi \in \mathfrak{S}\cdot\phi_0 \cap W(\mathbb{Q}) : [\phi \circ x_0 : 1] \in \mathrm{Gal}(\overline{E}/E)\cdot s_n\} \subseteq \widetilde{V}.$$

For $\phi \in Q_n$, we have $H_{\mathbb{A}_f}(d\phi) = H_{[x_0,1]}([\phi \circ x_0, 1]) = H_{[x_0,1]}(s_n) \approx H_{\mathbb{Q}}(\widetilde{s_n})$.

Denote by p the map $G(\mathbb{R})\cdot\phi_0 \to X$ with $G(\mathbb{R})\cdot\phi_0 \subseteq W(\mathbb{R})$. We have surjections $Q_n \to p(Q_n) \to \mathrm{Gal}(\overline{E}/E)\cdot s_n$. Thus $\#Q_n \ge \#\mathrm{Gal}(\overline{E}/E)\cdot s_n \approx H_{\mathbb{Q}}(\widetilde{s_n})$.

Thus $\widetilde{V}$ contains $\#Q_n \approx H_{\mathbb{Q}}(\widetilde{s_n})$ points of height $\approx H_{\mathbb{Q}}(\widetilde{s_n})$.

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[2] This avoids one difficulty in the approach [12] of Orr.

[3] This is where the assumption (1) is needed.

By Theorem 15, for sufficiently large n, there exist ϕ_n in Q_n such that $p(\phi_n) \in Z^{\mathrm{alg}}$, with $Z = p(\widetilde{V})$. By Ax–Lindemann–Weierstrass theorem [8], it follows that $s'_n = [\phi_n, 1] \in Z_n \subseteq V$, for a non-zero dimensional weakly special subvariety Z_n. Using Galois action, we may assume $s'_n = s_n$.

This concludes the proof of Theorem 3.

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