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Geometry of nondegenerate polynomials: Motivic nearby cycles and Cohomology of contact loci

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Abstract. We study polynomials with complex coefficients which are nondegenerate in two senses, one of Kouchnirenko and the other with respect to its Newton polyhedron, through data on contact loci and motivic nearby cycles. Introducing an explicit description of these quantities we can answer in part the question concerning the motivic nearby cycles of restriction functions in the context of Newton nondegenerate polynomials. Furthermore, in the nondegeneracy in the sense of Kouchnirenko, we give calculations on cohomology groups of the contact loci.

Keywords. arc spaces, contact loci, motivic zeta function, motivic Milnor fiber, motivic nearby cycles, Newton polyhedron, nondegeneracy, sheaf cohomology with compact support.

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1. Introduction

Let \( f \) be a nondegenerate \( \mathbb{C} \)-polynomial in the sense of Kouchnirenko (cf. Section 3.1) vanishing at the origin \( O \) of \( \mathbb{C}^d \). The problem of computing the motivic Milnor fiber \( \mathcal{F}_{f, O} \) in terms of the Newton polyhedron \( \Gamma \) of \( f \) was early mentioned in the works [1] and [6] with materials coming from [4] (see also [7] for a generalization). Recently, Steenbrink and Bultot–Nicaise obtain solutions in terms of toric geometry ([14]), or of log smooth models ([3]). Their formula for \( \mathcal{F}_{f, O} \) together with the additivity of the Hodge spectrum operator allows to reduce the computation of the Hodge spectrum of \( (f, O) \) to that of quasi-homogeneous singularities. In this article, we will show that the formula also provides a way to explore the following problem for Newton nondegenerate polynomials.

**Problem 1.** Let \( f \) be in \( \mathbb{C}^{[x_1, \ldots, x_d]} \) with \( f(O) = 0 \), and let \( H \) be a linear hyperplane in \( \mathbb{C}^d \). What is the relation between \( \mathcal{F}_{f, O} \) and \( \mathcal{F}_{f|_{H}, O} \)?

The question concerns a motivic analogue of a monodromy relation of a complex singularity and its restriction to a generic hyperplane studied early in [9, 10]. For \( n \in \mathbb{N}^* \), the \( n \)-iterated contact locus \( \mathcal{Z}_{n, O}(f) \) (cf. Section 2.3) admits a decomposition as a disjoint union into its \( \mu_n \)-invariant \( \mathbb{C} \)-subvarieties \( \mathcal{Z}_{f, a}^{(m)} \) along \( a \in (\mathbb{N}^*)^d \) and \( J \subseteq [d]: = \{1, \ldots, d\} \). The nondegeneracy of \( f \) allows to describe \( \mathcal{Z}_{f, a}^{(m)} \) via \( \Gamma \), as in Theorem 9, which is the key step to compute the motivic zeta function \( Z_{f, O}(T) \) and the motivic Milnor fiber \( \mathcal{F}_{f, O} \), which yields a proof of Theorem 12. Note that this theorem is well known as mentioned above (see [1, 6, 7]). For every face \( \gamma \) of \( \Gamma \), let \( J_{\gamma} \) be the unique subset of \( [d] \) such that \( \gamma \) is contained in the hyperplanes \( x_j = 0 \) for all \( j \in J_{\gamma} \) and not contained in the other coordinate hyperplanes, and let \( X_{\gamma}(0) \) (resp. \( X_{\gamma}(1) \)) be the \( \mathbb{C} \)-subvariety of \( \mathbb{G}_{m, \mathbb{C}} \) defined by the face function \( f_{\gamma} \) (resp. \( f_{\gamma} - 1 \)).

**Theorem (see Theorem 12).** Let \( f \) be in \( \mathbb{C}^{[x_1, \ldots, x_d]} \) with \( O \in X_0 := f^{-1}(0) \), let \( d_1 \) and \( d_2 \) be in \( \mathbb{N} \) such that \( d = d_1 + d_2 \). The below hold in \( \mathcal{H}_{A_{\mathbb{C}}^{d_1}}^{\geq} \) for (i), in \( \mathcal{H}_{A_{\mathbb{C}}^{d_1}} \) for (ii), and in \( \mathcal{H}_{A_{\mathbb{C}}^{d_1}}^{\geq} \) for (iii).

(i) If \( f \) is Newton nondegenerate, then
\[
\mathcal{F}_f = - \sum_{\gamma \in F \setminus F} \lambda_{\gamma} \left[ X_{\gamma}(1) \to X_0 \right] + \sum_{\gamma \in F} \lambda_{\gamma} \left[ X_{\gamma}(0) \to X_0 \right].
\]

(ii) If \( f \) is Newton nondegenerate and \( t : A_{\mathbb{C}}^{d_1} \equiv A_{\mathbb{C}}^{d_1} \times_{\mathbb{C}} [0]^{d_2} \to X_0 \) is an inclusion, then
\[
t^* \mathcal{F}_f = - \sum_{\gamma \in F \setminus F} \lambda_{\gamma} \left[ X_{\gamma}(1) \times_{X_0} A_{\mathbb{C}}^{d_1} \to A_{\mathbb{C}}^{d_1} \right] + \sum_{\gamma \in F \setminus F} \lambda_{\gamma} \left[ X_{\gamma}(0) \times_{X_0} A_{\mathbb{C}}^{d_1} \to A_{\mathbb{C}}^{d_1} \right].
\]

(iii) If \( f \) is nondegenerate in the sense of Kouchnirenko, then
\[
\mathcal{F}_{f, O} = \sum_{\gamma \in K} (-1)^{|J_{\gamma}| + 1 - \dim(\gamma)} \left( [X_{\gamma}(1)] - [X_{\gamma}(0)] \right).
\]

Here, \( t^* \), \( F \), \( \bar{F} \), \( F(d_1) \), \( K \), and \( \lambda_{\gamma} \) are defined in Sections 2.1, 3.1 and 3.3.

We choose the hyperplane defined by \( x_d = 0 \) to be \( H \) in Problem 1, and consider for any \( n \geq m \) in \( \mathbb{N}^* \) the so-called \((n, m)\)-iterated contact locus \( \mathcal{X}_{n, m, O}(f, x_d) \) of the pair \( (f, x_d) \). It is a \( \mu_n \)-invariant \( \mathbb{C} \)-subvariety of \( \mathcal{X}_{n, O}(f) \). Then we show in this article that the formal series
\[
Z_{f, x_d, O}^{\Delta}(T) := \sum_{n \geq m \geq 1} \left[ \mathcal{X}_{n, m, O}(f, x_d) \right] L^{-(n+m)d} T^n
\]
is rational and it can be described via data of \( \Gamma \). Here, \( \Delta \) stands for \( \{(n, m) \in (\mathbb{R}_{>0})^2 \mid n \geq m \} \) and the sum runs over \( \Delta \cap (\mathbb{N}^*)^2 \). Put \( Z_{f, x_d, O}^{\Delta} := - \lim_{T \to \infty} Z_{f, x_d, O}^{\Delta}(T) \). Using the description of \( \mathcal{F}_{f, x_d, O} \) together with Theorem 12, a solution to Problem 1 for the nondegeneracy in the sense of Kouchnirenko can be realized as in the following theorem.
Theorem (see Theorem 17). With \( f \) as previous, the identity \( \mathcal{J}_f,0 = \mathcal{J}_{f,1}\mathcal{O} + \mathcal{J}_{f,x,0}^{\Delta} \) holds in the monodromic Grothendieck ring of \( \mathcal{C} \)-varieties with \( \mu \)-action. A similar result also holds for the motivic nearby cycles.

According to [2, Conjecture 1.5], it is expected that the singular cohomology groups with compact support of the \( \mathcal{C} \)-points of the contact loci are nothing but the Floer cohomology groups of the powers of the monodromy of the singularity (cf. [11]). Here, we are interested in a smaller problem on the computation of cohomology groups of \( \mathcal{C}_n,0(f) \) (the reader may compare this with [2, Theorem 1.1]).

Problem 2. Let \( f \) be a polynomial over \( \mathbb{C} \) vanishing at the origin \( O \). Compute the cohomology groups with compact support \( H^n_c(\mathcal{X}_n,0(f), \mathcal{C}) \) for all \( n \in \mathbb{N}^* \) and \( m \in \mathbb{N} \).

We devote Section 4 to study this problem for nondegenerate singularities in the sense of Kouchnirenko not only using sheaf cohomology with compact support but also the Borel–Moore homology \( H_n^{BM} \). Write \( \mathcal{X}_n,0(f) = \bigcup_{(j,a) \in \mathcal{P}_n} \mathcal{X}_j^{(n)} \) as in (2) with \( \mathcal{P}_n \) described in Lemma 8 (ii). Let \( \eta : \mathcal{P}_n \to \mathbb{Z} \) be the function defined by \( \eta(j,a) = \dim_{\mathbb{C}} \mathcal{X}_j^{(n)} \).

Theorem (see Theorems 20, 22). For \( f \) as in Problem 2 and nondegenerate in the sense of Kouchnirenko, for every \( p, q \in \mathbb{N} \), there exist spectral sequences

\[
E_1^{p,q} : \bigoplus_{\eta(j,a) = p} H_{p+q}^{BM}(\mathcal{X}_j^{(n)}) \Rightarrow H_{p+q}^{BM}(\mathcal{X}_n,0(f)),
\]

\[
E_1^{p,q} : \bigoplus_{\eta(j,a) = p} H_{p+q}^{BM}(\mathcal{X}_j^{(n)}, \mathcal{F}) \Rightarrow H_{p+q}^{BM}(\mathcal{X}_n,0(f), \mathcal{F}),
\]

for any sheaf of abelian groups \( \mathcal{F} \) on \( \mathcal{X}_n,0(f) \).

In particular, by applying the second spectral sequence with \( \mathcal{F} \) being a constant sheaf, we obtain a spectral sequence converging to the compact support cohomology groups of contact loci with complex coefficients whose first page is a direct sum of (singular) homology of the spaces defined by the vanishing of the functions \( f_x \) and \( f_y - 1 \) (see Corollary 26).

2. Preliminaries

2.1. Monodromic Grothendieck ring of varieties

Let \( S \) be an algebraic \( \mathbb{C} \)-variety. Let \( \text{Var}_S \) be the category of \( S \)-varieties, with objects being morphisms of algebraic \( \mathbb{C} \)-varieties \( X \to S \) and a morphism in \( \text{Var}_S \) from \( X \to S \) to \( Y \to S \) being a morphism of algebraic \( \mathbb{C} \)-varieties \( X \to Y \) commuting with \( X \to S \) and \( Y \to S \). Denote by \( \mu \) the limit of the projective system \( \mu_m \to \mu_n \) given by \( x \to x^m \), with for any \( n \geq 1 \), \( \mu_n = \text{Spec} \mathbb{C}[\xi]/(\xi^n - 1) \) the group scheme over \( \mathbb{C} \) of \( n \)th roots of unity. Notice that any action of \( \mu \) on a variety \( X \) in the present article is assumed to factorize through an action of \( \mu_n \) for some \( n \in \mathbb{N}^* \).

An action on \( X \) is good if every orbit is contained in an affine open subset of \( X \). By definition, an action of \( \mu \) on an affine Zariski bundle \( X \to B \) is affine if it is a lifting of a good action on \( B \) and its restriction to all fibers is affine.

The Grothendieck group \( K^n_S(\text{Var}_S) \) is defined to be an abelian group generated by symbols \([X \to S]\), \( X \) endowed with a good \( \mu \)-action and \( X \to S \) in \( \text{Var}_S \), such that:

(i) \([X \to S] = [Y \to S]\) if \( X \) and \( Y \) are \( \mu \)-equivariant \( S \)-isomorphic;
(ii) \([X \to S] = [Y \to S] + [X \setminus Y \to S]\) if \( Y \) is a \( \mu \)-invariant closed subvariety in \( X \); and
(iii) \([X \times A^n_C, \sigma] = [X \times A^n_C, \sigma']\) if \( \sigma \) and \( \sigma' \) are liftings of the same \( \mu \)-action on \( X \to X \times A^n_C \).
There is a natural ring structure on $K^\hat\mu_0(\text{Var}_S)$ in which the product is induced by the fiber product over $S$. The unit $1_S$ for the product is the class of the identity morphism $S \to S$ with $S$ endowed with trivial $\hat\mu$-action. Denote by $\mathcal{L}$ (or $\mathcal{L}_S$) the class of the trivial line bundle $S \times \mathbb{A}^1 \to S$, and define the localized ring $\mathcal{M}_S^{\hat\mu}$ to be $K^\hat\mu_0(\text{Var}_S)[[L^{-1}]]$.

Let $f : S \to S'$ be a morphism of algebraic $\mathbb{C}$-varieties. Then we have two important morphisms associated to $f$, which are the ring homomorphism $f^* : \mathcal{M}^{\hat\mu}_S \to \mathcal{M}^{\hat\mu}_{S'}$ induced from the fiber product (the pullback morphism) and the $\mathcal{M}\mathcal{C}$-linear homomorphism $f_! : \mathcal{M}^{\hat\mu}_S \to \mathcal{M}^{\hat\mu}_{S'}$ defined by the composition with $f$ (the push-forward morphism).

### 2.2. Rational series and limit

Let $\mathcal{A}$ be either $\mathbb{Z}[\mathcal{L}, \mathcal{L}^{-1}]$ or $\mathcal{M}^{\hat\mu}_S$ as a ring. Let $\mathcal{A}[[T]]_{sr}$ be the $\mathcal{A}$-module of elements of the form $\sum_{a} \frac{a}{\mathcal{L}^a}$ with $(a, b)$ in $\mathbb{R} \times \mathbb{N}^*$. Each element of $\mathcal{A}[[T]]_{sr}$ is called a rational series. By [5], there is a unique $\mathcal{A}$-linear morphism $\lim_{T \to \infty} : \mathcal{A}[[T]]_{sr} \to \mathcal{A}$ which sends $\frac{1}{\mathcal{L}^a}$ to $-a$.

For $J$ contained in $[d]$, we denote by $(\mathbb{R}_{\geq 0})^J$ the set of $(a_j)_{j \in J}$ with $a_j$ in $\mathbb{R}_{\geq 0}$ for all $j \in J$, and by $(\mathbb{R}^*)^J$ the subset of $(\mathbb{R}_{\geq 0})^J$ consisting of $(a_j)_{j \in J}$ with $a_j > 0$ for all $j \in J$. Similarly, one can define the sets $(\mathbb{Z}_{\geq 0})^J$, $(\mathbb{Z}^*)^J$, and $(\mathbb{N}^*)^J$. Let $\sigma$ be a rational polyhedral convex cone in $(\mathbb{R}_{\geq 0})^J$ and let $\sigma$ denote its closure in $(\mathbb{R}^*)^J$ with $J$ a finite set. Let $\ell$ and $\ell'$ be two integer linear forms on $\mathbb{Z}^J$ positive on $\sigma \setminus \{0, \ldots, 0\}$. Then the series

$$S_{\sigma, \ell, \ell'}(T) := \sum_{a \in \sigma \cap (\mathbb{N}^*)^J} \frac{\mathcal{L}^{-\ell(a)} \mathcal{T}^{\ell(a)}}{\mathcal{L}^a}$$

is in $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}][[T]]_{sr}$ and $\lim_{T \to \infty} S_{\sigma, \ell, \ell'}(T) = \chi(\sigma)$, the Euler characteristic with compact supports of $\sigma$. If $\sigma$ is relatively open, then $\lim_{T \to \infty} S_{\sigma, \ell, \ell'}(T) = (-1)^{\dim(\sigma)}$ (see [6, Lemma 2.1.5]). We have the following technique lemma.

**Lemma 3.** Let $\sigma$ be a relatively open rational polyhedral convex cone in $(\mathbb{R}_{\geq 0})^J$. Let $K$ and $L$ be disjoint nonempty subsets of $I$. Consider half spaces $H_j$ (with $j \in L$) in $\mathbb{R}^I$ defined by

$$x_j \leq \sum_{i \in K} \alpha_i x_i,$$

where $\alpha_i \geq 0$ for all $i \in K$, such that for any disjoint subsets $L_1, L_2$ of $L$ and any $j \in L \setminus (L_1 \cup L_2)$, the set

$$\sigma \cap \bigcap_{s \in L_1} \left\{ (x_i)_{i \in I} \in \mathbb{R}^I \mid x_s < \sum_{i \in K} \alpha_i x_i \right\} \cap \bigcap_{r \in L_2} \left\{ (x_i)_{i \in I} \in \mathbb{R}^I \mid x_r = \sum_{i \in I} \alpha_i x_i \right\}$$

either is empty or has nonempty intersection with $\mathbb{R}^I \setminus H_j$.

Then the Euler characteristic with compact supports of the set

$$\sigma_L := \sigma \cap \bigcap_{j \in L} H_j$$

is equal to zero.

**Proof.** We prove this lemma by induction on $|L|$. For $L = \{j\}$ a one-point set, we have

$$\sigma = \sigma_L \cup \left\{ (x_i)_{i \in I} \in \sigma \mid x_j > \sum_{i \in K} \alpha_i x_i \right\}.$$

Since the second term on the right hand side and $\sigma$ have the same Euler characteristic with compact supports $(-1)^{|L|}$, we get $\chi(\sigma_L) = 0$. For the case $|L| > 1$, let $j_0 \in L$ and $L' := L \setminus \{j_0\}$. Then $\sigma_L$ is the disjoint union of the following two sets

$$\sigma_{L'} \cap \left\{ (x_i)_{i \in I} \in \mathbb{R}^I \mid x_{j_0} < \sum_{i \in K} \alpha_i x_i \right\}$$

and

$$\sigma_L \cap \left\{ (x_i)_{i \in I} \in \mathbb{R}^I \mid x_{j_0} > \sum_{i \in K} \alpha_i x_i \right\}$$
and

\[ \sigma_L': \cap \left\{ (x_i)_{i \in I} \in \mathbb{R}^I \mid x_{j_0} = \sum_{i \in K} a_i x_i \right\}. \]

By induction, the lemma holds true for \( L' \), thus the Euler characteristic with compact supports of these two sets is zero.

\[ \square \]

2.3. Motivic nearby cycles of regular functions

For any \( \mathbb{C} \)-variety \( X \), let \( \mathcal{L}_n(X) \) be the space of \( n \)-jets on \( X \), and \( \mathcal{L}(X) \) the arc space on \( X \), which is the limit of the projective system of spaces \( \mathcal{L}_n(X) \) and canonical morphisms \( \mathcal{L}_n(X) \to \mathcal{L}_m(X) \) for \( m \geq n \). The group \( \hat{\mu} \) acts on \( \mathcal{L}_n(X) \) via \( \mu_n \) in such a natural way that \( \xi \cdot \varphi(t) = \varphi(\xi t) \) for \( \xi \in \mu_n \).

From now on, we assume that the \( \mathbb{C} \)-variety \( X \) is smooth and of pure dimension \( d \). Consider a regular function \( f : X \to \mathbb{A}^1_{\mathbb{C}} \), with the zero locus \( X_0 \). For \( n \geq 1 \) one defines the \( n \)-iterated contact locus of \( f \) as follows

\[ \mathcal{K}_n(f) = \{ \varphi \in \mathcal{L}_n(X) \mid f(\varphi) = t^n \mod t^{n+1} \}. \]

Clearly, this variety is invariant by the \( \hat{\mu} \)-action on \( \mathcal{L}_n(X) \) and admits a morphism to \( X_0 \) given by \( \varphi(t) \mapsto \varphi(0) \), which defines an element \( [\mathcal{K}_n(f)] := [\mathcal{K}_n(f) \to X_0] \) in \( \mathcal{M}^\hat{\mu}_{X_0} \). We consider Denef–Loeser’s motivic zeta function \( Z_f(T) = \sum_{n \geq 1} [\mathcal{K}_n(f)]L^{-nd}T^n \). They prove in [5] that \( Z_f(T) \) is in \( \mathcal{M}^\hat{\mu}_{X_0}[T]_{\text{st}} \), and call the limit \( \mathcal{Z}_f := -\lim_{T \to -\infty} Z_f(T) \) in \( \mathcal{M}^\hat{\mu}_{X_0} \), the motivic nearby cycles of \( f \). If \( x \) is a closed point of \( X_0 \), the \( \mathbb{C} \)-variety

\[ \mathcal{K}_{n,x}(f) = \{ \varphi \in \mathcal{L}_n(X) \mid f(\varphi) = t^n \mod t^{n+1}, \varphi(0) = x \}, \]

is also invariant by the \( \hat{\mu} \)-action on \( \mathcal{L}_n(X) \), called the \( n \)-iterated contact locus of \( f \) at \( x \). It is also proved that the zeta function \( Z_{f,x}(T) = \sum_{n \geq 1} [\mathcal{K}_{n,x}(f)]L^{-nd}T^n \) is in \( \mathcal{M}^\hat{\mu}_{X_0}(T) \). The limit \( \mathcal{Z}_{f,x} := -\lim_{T \to -\infty} Z_{f,x}(T) \) is called the motivic Milnor fiber of \( f \) at \( x \). Obviously, if \( i \) is the inclusion of \( \{x\} \) in \( X_0 \), then \( \mathcal{Z}_{f,i} = i^* \mathcal{Z}_f \) in \( \mathcal{M}^\hat{\mu}_{\mathbb{C}} \).

We now modify slightly the motivic zeta functions of several functions in [6] and [7]. For a pair of regular functions \( (f,g) \) on \( X \), we denote by \( X_0 := X_0(f,g) \) their common zero locus. For \( n \geq m \) in \( \mathbb{N}^* \), we define

\[ \mathcal{K}_{n,m}(f,g) := \{ \varphi \in \mathcal{L}_n(X) \mid f(\varphi) = t^n \mod t^{n+1}, \text{ord}_1 g(\varphi) = m \}. \]

We can check that \( \mathcal{K}_{n,m}(f,g) \) is invariant under the natural \( \mu_n \)-action on \( \mathcal{L}_n(X) \), and that there is an obvious morphism of \( \mathbb{C} \)-varieties \( \mathcal{K}_{n,m}(f,g) \to X_0 \); from which we obtain the class \([\mathcal{K}_{n,m}(f,g)]\) of that morphism in \( \mathcal{M}^\hat{\mu}_{X_0} \). Consider the series

\[ Z_{f,g}^\Delta(T) := \sum_{n \geq m \geq 1} [\mathcal{K}_{n,m}(f,g)]L^{-nd}T^n \]

in \( \mathcal{M}^\hat{\mu}_{X_0}[T] \). For any closed point \( x \in X_0 \), we can define \( Z_{f,g,x}^\Delta(T) \) in \( \mathcal{M}^\hat{\mu}_{X_0}[T] \) as above with \( \mathcal{K}_{n,m}(f,g) \) replaced by its \( \mu_n \)-invariant subvariety \( \mathcal{K}_{n,m,x}(f,g) := \{ \varphi \in \mathcal{K}_{n,m}(f,g) \mid \varphi(0) = x \} \). The rationality of the series \( Z_{f,g}^\Delta(T) \) and \( Z_{f,g,x}^\Delta(T) \) are stated in [6, Théorème 4.1.2] and [7, Section 2.9], up to the isomorphism of rings \( \mathcal{M}^\hat{\mu}_{X_0} \cong \mathcal{M}^\hat{\mu}_{X_0 \times \mathbb{A}^m} \) (see [8, Proposition 2.6]), where Guibert–Loeser–Merle’s result is done in the framework \( \mathcal{M}^\hat{\mu}_{X_0 \times \mathbb{A}^m} \). Put \( \mathcal{Z}_{f,g}^\Delta := -\lim_{T \to -\infty} Z_{f,g}^\Delta(T) \) and \( \mathcal{Z}_{f,g,x}^\Delta := -\lim_{T \to -\infty} Z_{f,g,x}^\Delta(T) \).
3. Motivic nearby cycles of a nondegenerate polynomial and applications

3.1. Newton polyhedron of a polynomial

Recall that $[d]$ stands for $\{1, \ldots, d\}$, $d \in \mathbb{N}^*$. Let $x = (x_1, \ldots, x_d)$ be a set of $d$ variables, and let $f(x) = \sum_{a \in \mathbb{N}^d} c_a x^a$ be in $\mathbb{C}[x]$ with $f(O) = 0$, with $O$ the origin of $\mathbb{C}^d$. Let $\Gamma$ be the Newton polyhedron of $f$, i.e., the convex hull of the set $\cup_{a \neq 0}(a + (\mathbb{R}_{\geq 0})^d)$ in $(\mathbb{R}_{\geq 0})^d$. For every face $\gamma$ of $\Gamma$ (not necessarily compact, the case $\gamma = \Gamma$ included), define by $f_\gamma(x) = \sum_{a \in \gamma} c_a x^a$ the face function of $f$ with respect to $\gamma$.

Note that for every face $\gamma$ of $\Gamma$ (including $\Gamma$ itself), there exists a unique set $J_\gamma \subseteq [d]$ such that $\gamma$ is contained in the hyperplanes $x_j = 0$ for all $j \notin J_\gamma$ and not contained in other coordinate hyperplanes.

**Definition 4.** The polynomial $f$ is called nondegenerate on a face $\gamma$ of $\Gamma$ if the hypersurface $f_\gamma^{-1}(0)$ has no singular point in $\mathbb{G}^J_{m, \mathbb{C}}$. We say that $f$ is nondegenerate in the sense of Kouchnirenko if it is nondegenerate on every compact face $\gamma$. If $f$ is nondegenerate on every face of $\Gamma$ (including noncompact faces, and $\Gamma$ itself), we say that $f$ is nondegenerate in the sense of Newton polyhedron or simply Newton nondegenerate.

Consider the function $\ell = \ell_\Gamma : (\mathbb{R}_{\geq 0})^d \to \mathbb{R}$ which sends $a$ in $(\mathbb{R}_{\geq 0})^d$ to $\min_{b \in \Gamma} \langle a, b \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^d$. For $a$ in $(\mathbb{R}_{\geq 0})^d$, we denote by $\gamma_a$ the maximal face of $\Gamma$ to which the restriction of the function $\langle a, \cdot \rangle$ gets its minimum. Note that $\gamma_a$ is a compact face if and only if $a$ is in $(\mathbb{R}_{\geq 0})^d$ (cf. [4, Property 2.3]). This comes from the fact that $\gamma_a = \{ b \in \Gamma \mid \langle a, b \rangle = \ell(a) \}$. Moreover, $\gamma_a = \Gamma$ when $a = (0, \ldots, 0)$ in $\mathbb{R}^d$, and $\gamma_a$ is a proper face of $\Gamma$ otherwise. For every proper face $\gamma$ of $\Gamma$, we define

$$\sigma_\gamma := \sigma_{[d], \gamma} := \{ a \in (\mathbb{R}_{\geq 0})^d \mid \gamma = \gamma_a \}.$$ 

It is clear that $\sigma_\gamma$ is a rational polyhedral convex cone of dimension $d - \dim(\gamma)$.

For any $J \subseteq [d]$, denote by $f^J$ the polynomial in $\mathbb{C}[(x_j)_{j \in J}]$ obtained from $f(x)$ substituting $x_i$ by 0 for all $i \in [d] \setminus J$. If $f$ is nondegenerate in the sense of Kouchnirenko (resp. Newton nondegenerate) then $f^J$ is also nondegenerate in the sense of Kouchnirenko (resp. Newton polyhedron).

Let $\ell_J$ stand for $\ell_{\Gamma(f^J)}$. For $a \in (\mathbb{R}_{\geq 0})^J$, we define the face $\gamma_a^J$ similarly as above, i.e.

$$\gamma_a^J := \{ b \in \Gamma(f^J) \mid \langle a, b \rangle = \ell_J(a) \}.$$ 

If $\gamma$ is a face of the Newton polyhedron $\Gamma(f^J)$, denote by $\sigma_{J, \gamma}$ the cone $\{ a \in (\mathbb{R}_{\geq 0})^J \mid \gamma = \gamma_a^J \}$ and by $\tilde{\sigma}_{J, \gamma}$ the relative interior of $\sigma_{J, \gamma}$, both of which have dimension $|J| - \dim(\gamma)$. The following lemma is trivial to prove.

**Lemma 5.** There exists a canonical partition of $(\mathbb{R}_{\geq 0})^J$ into rational polyhedral convex cones $\tilde{\sigma}_{J, \gamma}$ with $\gamma$ being all faces of $\Gamma(f^J)$.

A face $\gamma$ of $\Gamma$ is called a coordinate face if $\gamma = \Gamma(f^J)$ for some $J \subseteq [d]$. We have the following description for the coordinate faces.

**Lemma 6.** A face $\gamma$ of $\Gamma$ is a coordinate face if and only if for all $J \supseteq J_\gamma$, the restriction of $\ell_J$ to $\sigma_{J, \gamma}$ is the zero function.

**Proof.** For each $j \in J$, we denote by $e^j$ the vector $(0, \ldots, 0, 1, \ldots, 0) \in \mathbb{R}^J$ with 1 in the $j$-th coordinate. Assume that $\gamma = \Gamma(f^{J_0})$ for some $J_0 \subseteq [d]$. Then, for any $i \in J_0$, $t \geq 0$ and for any $b \in \gamma$, the point $b + te^j$ is also in $\gamma$. For $J \supseteq J_0 = J_\gamma$ and $a = (a_j)_{j \in J} \in \tilde{\sigma}_{J, \gamma}$, we have $\ell_J(a) = \langle a, b + te^j \rangle$ for every $i \in J_0$ and $t \geq 0$. As a consequence, we get $a_j = 0$ for all $j \in J_0$. Hence $\ell_J(a) = \langle a, b \rangle = 0$ for any $b \in \gamma = \Gamma(f^{J_0})$. 
Now, we assume that restriction of $\ell_J$ to $\hat{\sigma}_{J,Y}$ is the zero function for some $J \supseteq J_Y$. Take $a = (a_j)_{j \in J} \in \hat{\sigma}_{J,Y}$, we have $\langle a, b \rangle = \ell_J(a) = 0$ for any $b \in \gamma$. Since $\gamma$ is not contained in any hyperplane $x_k = 0$ for any $k \in J_Y$, we have $a_j = 0$ for all $j \in J_Y$. This together with the description of $\gamma$, namely,
\[
\gamma = \{ b \in \Gamma(f^J) \mid \langle a, b \rangle = 0 \},
\]
implies that $\Gamma(f^J_Y) \subseteq \gamma$. Therefore $\gamma = \Gamma(f^J_Y)$. \hfill \qed

**Notation 7.** In the rest of this article, let $F$ (resp. $K$) denote the set of all the faces (resp. the compact faces) of $\Gamma$, and let $\bar{F}$ denote the set of all the coordinate faces of $\Gamma$.

### 3.2. Contact loci

Let $(x_1, \ldots, x_d)$ be the standard coordinates of $\mathbb{A}^d_{\mathbb{C}}$ and let $f(x_1, \ldots, x_d)$ be as above. For $n \in \mathbb{N}^*$, $k \in \mathbb{N}$ and $J \subseteq [d]$, denote by $\Delta_f^{(n,k)}(J)$ the set of $a \in \{0, \ldots, n\}^J$ (resp. $a \in \{n\}^J$) such that $\ell_J(a) + k = n$. Clearly, $\Delta_f^{(n,k)} \subseteq \Delta_f^{(n,k)}$. For $a \in \Delta_f^{(n,k)}$, put
\[
\mathcal{A}^{(n)}_{f,a} := \{ \varphi \in \mathcal{D}_n(f) \mid \text{ord}_x x_j(\varphi) = a_j \ \forall \ j \in J, \ x_j(\varphi) = 0 \ \forall \ i \notin J \}.
\]
This subvariety of $\mathcal{D}_n(f)$ is invariant by the $\mu_n$-action given by $\xi \cdot \varphi(t) = \varphi(\xi \cdot t)$, and it defines an element $[\mathcal{A}^{(n)}_{f,a}] \in [\mathcal{D}_n^{(n)} \to X_0]$ in $\mathbb{K}_0(\text{Var}_{X_0})$, where the structure map is given by $\varphi \to \varphi(0)$. Let $\mathcal{P}_n$ and $\mathcal{Q}_n$ be the index sets consisting of all such pairs $(J, a)$ such that
\[
\mathcal{D}_n(f) = \bigcup_{(J, a) \in \mathcal{P}_n} \mathcal{A}^{(n)}_{f,a}
\]
and
\[
\mathcal{D}_n,0(f) = \bigcup_{(J, a) \in \mathcal{Q}_n} \mathcal{A}^{(n)}_{f,a}.
\]

**Lemma 8.**

(i) $\mathcal{P}_n$ is the set of all the pairs $(J, a)$ such that $J \supseteq J_Y$, $a \in \bigcup_{k \in \mathbb{N}} (\hat{\sigma}_{J,Y} \cap \Delta_f^{(n,k)})$ and $\gamma \in F$.

(ii) $\mathcal{Q}_n$ is the set of all the pairs $(J, a)$ such that $J \supseteq J_Y$, $a \in \bigcup_{k \in \mathbb{N}} (\hat{\sigma}_{J,Y} \cap \Delta_f^{(n,k)})$ and $\gamma \in K$.

**Proof.** For any $\varphi$ in $\mathcal{D}_n(f)$, there exists a unique subset $J$ of $\{1, \ldots, d\}$ such that $x_i(\varphi) \equiv 0$ for all $i \notin J$ and that $x_j(\varphi) \neq 0$ for all $j \in J$. Put $a := (\text{ord}_x x_j(\varphi))_{j \in J}$ and put $\gamma := \gamma_a$. Then we have $J_Y \subseteq J$ and
\[
f(\varphi) = f_J(\bar{\varphi}(0)) + \text{higher terms},
\]
where $\bar{\varphi} := (t^{-a_j} x_j(\varphi))_{j \in J}$, thus $\ell_J(a) \leq n$ and $\varphi \in \mathcal{D}_n^{(n)}$. The proof for (i) is completed by using Lemma 5. Similar arguments work for (ii). \hfill \qed

For $\gamma \in \bar{F}$, if $a \in \hat{\sigma}_{J,Y} \cap \Delta_f^{(n,k)}$ and $k \neq n$, then $\mathcal{A}^{(n)}_{f,a} = \emptyset$.

For every face $\gamma \in F$ of $\Gamma(f^J)$, let us consider the $\mathbb{C}$-varieties
\[
X_{J,Y}(1) := \{ x \in \mathbb{C}^J_{m,\mathbb{C}} \mid f_J(x) = 1 \}, \quad X_{J,Y}(0) := \{ x \in \mathbb{C}^J_{m,\mathbb{C}} \mid f_J(x) = 0 \}.
\]
When $J = J_Y$ we write simply $X_Y(e)$ instead of $X_{J,Y}(e)$, for $e = 0, 1$. We always consider the trivial action of $\mu$ on the variety $X_{J,Y}(0)$. Let $a$ be in $\hat{\sigma}_{J,Y}$. Then the variety $X_{J,Y}(1)$ admits a natural $\mu_{\ell_J(a)}$-action as follows
\[
e^{2\pi ir/\ell_J(a)}(x_j)_{j \in J} := (e^{2\pi ir a_j/\ell_J(a)} x_j)_{j \in J},
\]
for $r \in [\ell_J(a)]$. Note that the class $[X_{J,Y}(1)]$ in $\mathbb{A}^{(n)}_{\mathbb{C}}$ does not depend on $a$ provided $a$ is in $\hat{\sigma}_{J,Y}$ and $\ell_J(a) = n$, which follows from the construction of the Grothendieck ring (see [13, Proposition 3.13]).
The result and proof ideas of the following theorem are well known due to \([1, 6, 7]\). In the present article, we are going to contribute a detailed explanation for every step of proof. Denote \(|a| := \sum_{j \in J} a_j\) for \(a = (a_j)_{j \in J} \in \mathbb{R}^J\).

**Theorem 9** (cf. \([1, 6, 7]\)). Let \(f \in \mathbb{C}[x_1, \ldots, x_d]\) such that \(f(0) = 0\). Assume that \(f\) is nondegenerate on a face \(\gamma \in F\). Let \(J \subseteq [d]\) containing \(f\). If \(a \in \partial J, \gamma \cap \Delta^{(n,0)}\) and \(J^{(n,0)}\) is nonempty, then there is a naturally \(\mu_n\)-equivariant isomorphism of \(\mathbb{C}\)-varieties

\[
\tau : \mathcal{X}^{(n)}_{J,a} \to X_{f,\gamma}(1) \times_{\mathbb{C}} \mathbb{A}^{1\{f(a)-|a|\}}_{\mathbb{C}}.
\]

If \(k \in \mathbb{N}^*\), \(a \in \partial J, \gamma \cap \Delta^{(n,k)}\), and if \(J^{(n,0)}\) is nonempty, there is a Zariski locally trivial fibration

\[
\pi : \mathcal{X}^{(n)}_{J,a} \to X_{f,\gamma}(0)
\]

with fiber \(\mathbb{A}^{1\{f(a)-|a|\}-k}_{\mathbb{C}}\).

**Proof.** It suffices to prove the theorem for \(J = [d]\). Let \(a = (a_1, \ldots, a_d)\) be in \(\partial \gamma \cap \Delta^{(n,0)}\), hence \(n = \ell(a)\) and \(\gamma = \gamma_a\). Every element \(\varphi\) in \(\mathcal{X}^{(n)}_{J,a}\) has the form

\[
(\ell(a) b_{1i} t^j, \ldots, b_{di} t^j)
\]

with \(b_{ia} \neq 0\) for \(1 \leq i \leq d\). The coefficient of \(t^{\ell(a)}\) in \(f(\varphi(t))\) is nothing but \(f_{\gamma a}(b_{1a}, \ldots, b_{da})\), thus \((b_{1a}, \ldots, b_{da})\) is in \(X_{[d], \gamma_a}(1)\). We deduce that \(\mathcal{X}^{(\ell(a))}_{J,a}\) is \(\mu_{\ell(a)}\)-equivariant isomorphic to \(X_{[d], \gamma_a}(1) \times_{\mathbb{C}} \mathbb{A}^{\ell(a)-|a|}_{\mathbb{C}}\) (where \(\mu_{\ell(a)}\) acts trivially on \(\mathbb{A}^{\ell(a)-|a|}_{\mathbb{C}}\)) via the map

\[
\tau : \varphi(t) \to ((b_{1a}), 1 \leq i \leq d, (b_{ij}) 1 \leq i \leq d, a_j < j \leq \ell(a)).
\]

Indeed, for every \(\xi \in \mu_{\ell(a)}\), the element \(\varphi(\xi t)\) is sent to

\[
((\xi^{a_i} b_{ia}), 1 \leq i \leq d, (b_{ij}) 1 \leq i \leq d, a_j < j \leq \ell(a)) = \xi \cdot ((b_{ia}), 1 \leq i \leq d, (b_{ij}) 1 \leq i \leq d, a_j < j \leq \ell(a)).
\]

Thus \(\tau\) is a \(\mu_{\ell(a)}\)-equivariant isomorphism.

Now we prove the second statement. Let \(a\) be in \(\partial \gamma \cap \Delta^{(n,k)}\) for \(k \in \mathbb{N}^*\), hence \(n = \ell(a) + k\) and \(\gamma = \gamma_a\). For \(\varphi\) in \(\mathcal{X}^{(n)}_{J,a}\), putting

\[
\tilde{\varphi} := (t^{-a_1} x_1(\varphi), \ldots, t^{-a_d} x_d(\varphi)),
\]

we get

\[
f(\varphi) = t^{\ell(a)} f_{\gamma a}(\tilde{\varphi}) + \sum_{k \geq 1} \ell(a) + k \sum_{\langle a, a \rangle = \ell(a) + k} c_a \tilde{\varphi}^a.
\]

Defining

\[
\tilde{f}(\tilde{\varphi}, t) := f_{\gamma a}(\tilde{\varphi}) + \sum_{k \geq 1} t^k \sum_{\langle a, a \rangle = \ell(a) + k} c_a \tilde{\varphi}^a,
\]

we obtain a function

\[
\tilde{f} : \mathcal{L}_{\ell(a) + k + 1 - a_1}(\mathbb{A}^1_{\mathbb{C}}) \times_{\mathbb{C}} \cdots \times_{\mathbb{C}} \mathcal{L}_{\ell(a) + k + 1 - a_d}(\mathbb{A}^1_{\mathbb{C}}) \times_{\mathbb{C}} \mathbb{A}^{1\{f(a)-|a|\}}_{\mathbb{C}} \to \mathbb{A}^{1\{f(a)-|a|\}}_{\mathbb{C}}
\]

given by

\[
\tilde{f}(\tilde{\varphi}, t_0) := \tilde{f}(\tilde{\varphi}(t_0), t_0).
\]

It thus follows from (5) that \(\varphi\) is in \(\mathcal{X}^{(\ell(a))}_{J,a}\) if and only if \(\tilde{f}(\tilde{\varphi}, t) = t^k\) mod \(t^{k+1}\). Putting \(\tilde{\varphi}(t) = \sum_{j=0}^{\ell(a)-a+1} b_{ij} t^j\) for \(1 \leq i \leq d\), the latter means that

\[
\begin{align*}
(f_{\gamma a}(b_{10}, \ldots, b_{d0}) = 0 & \quad \text{with } b_{i0} \neq 0 \text{ for } 1 \leq i \leq d, \\
q_j(b_{1j}, \ldots, b_{dj}) + p_j((b_{i' j'})_{i' j'}) = 0 & \quad \text{for } 1 \leq j \leq k-1, \\
q_k(b_{1k}, \ldots, b_{dk}) + p_k((b_{i' j'})_{i' j'}) = 1, & \quad \text{for } 1 \leq i' \leq d, 1 \leq j' \leq d.
\end{align*}
\]
where $p_j$, for $1 \leq j \leq k$, are polynomials in variables $b_{i,j'}$ with $i' \leq d$ and $j' < j$, and

$$q_j(b_{1j},...,b_{dj}) = \sum_{i=1}^{d} \frac{\partial f_{i,a}}{\partial x_i} (b_{10},...,b_{di}) b_{ij}.$$

We consider the morphism

$$\pi : \mathcal{X}^{(\ell(a)+k)}_{d[a,a]} \rightarrow X_{d[i,a]}(0)$$

which sends the $\phi$ described previously to $(b_{10},...,b_{di})$. Since $\mu$ acts trivially on $X_{d[i,a]}(0)$, we only need to prove that $\pi$ is a locally trivial fibration with fiber $\mathbb{A}^d_{\mathbb{C}}(\ell(a)+k)$. For every $1 \leq i \leq d$, we put

$$U_i := \{(x_1,\ldots,x_d) \in X_{d[i,a]}(0) | \frac{\partial f_{i,a}}{\partial x_i} (x_1,\ldots,x_d) \neq 0 \}.$$ (6)

The nondegeneracy of $f$ on the face $\gamma = \gamma_a$ gives us an open covering $\{U_1,\ldots,U_d\}$ of $X_{d[i,a]}(0)$. We construct trivializations of $\pi$ as follows

$$\pi^{-1}(U_i) \xrightarrow{\Phi_{U_i}} U_i \times \mathbb{A}^e_{\mathbb{C}}$$

where $e = \sum_{i=1}^{d} (\ell(a) - a_i + k) - k$ and we identify $\mathbb{A}^e_{\mathbb{C}}$ with the subvariety of $\mathbb{A}^{\sum_{i=1}^{d} (\ell(a) - a_i + k)}_{\mathbb{C}}$ defined by the equations $\tilde{b}_{ij} = 0$ for $1 \leq j \leq k - 1$ and $\tilde{b}_{ik} = 1$ in the coordinate system $(\tilde{b}_{ij})$, and for $\phi$ as previous,

$$\Phi_{U_i}(\phi) = (\tilde{b}_{10},\ldots,\tilde{b}_{di},(\tilde{b}_{ij})_{1 \leq l \leq d, 1 \leq j \leq \ell(a) - a_i + k}),$$

with $\tilde{b}_{ij} = 0$ if $1 \leq j \leq k - 1$, $\tilde{b}_{ik} = 1$, and $\tilde{b}_{ij} = b_{ij}$ otherwise. Furthermore, the inverse map $\Phi_{U_i}^{-1}$ of $\Phi_{U_i}$ is also a regular morphism given explicitly as follows

$$\Phi_{U_i}^{-1}(\tilde{b}_{ij}) = \left(\sum_{j=0}^{\ell(a) - a_i + k} b_{ij}^{l+a_i},\ldots,\sum_{j=0}^{\ell(a) - a_i + k} b_{dj}^{l+a_i} \right),$$

where $\tilde{b}_{ij} = b_{ij}$ for either that $l \neq i$ or that $l = i$ and $k < j \leq \ell(a) - a_i + k$, and

$$b_{ij} = -\frac{p_j((b_{ij})_{l \leq d, j' < j}) - \sum_{l \leq d, i \neq j} (\frac{\partial f_{i,a}}{\partial x_i})(\tilde{b}_{10},\ldots,\tilde{b}_{di}) \tilde{b}_{ij}}{\sum_{l \leq d, i \neq j} (\frac{\partial f_{i,a}}{\partial x_i})(\tilde{b}_{10},\ldots,\tilde{b}_{di}) \tilde{b}_{ij}},$$

for $1 \leq j \leq k - 1$, and

$$b_{ik} = \frac{1 - p_k((b_{ij})_{l \leq d, j' < k}) - \sum_{l \leq d, i \neq j} (\frac{\partial f_{i,a}}{\partial x_i})(\tilde{b}_{10},\ldots,\tilde{b}_{di}) \tilde{b}_{ik}}{\sum_{l \leq d, i \neq j} (\frac{\partial f_{i,a}}{\partial x_i})(\tilde{b}_{10},\ldots,\tilde{b}_{di}) \tilde{b}_{ik}}.$$

This proves that $\pi$ is a (Zariski) locally trivial fibration with fiber $\mathbb{A}^e_{\mathbb{C}}$. \hfill $\Box$

### 3.3. Motivic nearby cycles

For every $\gamma = \gamma_a \in F$ with $a \in \partial f_{i,y} \cap \Delta^{0\ldots0}_f$ we consider the morphism $\Phi_a : X_{f,y}(1) \rightarrow X_0$ which sends $(x_i)_{i \in J}$ to $(\tilde{x}_1,\ldots,\tilde{x}_d)$, where $\tilde{x}_l = 0$ if either $i \in |d| \setminus J$ or $a_i \geq 1$, and $\tilde{x}_l = x_l$ if $a_i = 0$. With this morphism it follows from Theorem 9 the below commutative diagram

$$\begin{array}{ccc}
\mathcal{X}^{(n)}_{f,a} & \xrightarrow{\cong} & X_{f,y}(1) \times \mathbb{A}^{d_1} \setminus \{y \} \\
\phi(t)^{\phi(0)} & \downarrow & \Phi_{\phi(0)} \\
X_0 & \xrightarrow{\Phi_{\phi(t)}} & X_0
\end{array}$$ (7)
Lemma 10. If $a, b \in \delta_{J, Y} \cap \Delta_f^{(n, 0)}$, then $[\Phi_a : X_{J, Y}(1) \to X_0] = [\Phi_b : X_{J, Y}(1) \to X_0]$ in $\mathcal{M}_X^H$.

Proof. Suggested from [13, Section 3.4.3], we stratify $(\mathbb{R}_{\geq 0})^I$ into the cones

$$C_\delta := \{(k_j)_I \in (\mathbb{R}_{\geq 0})^I \mid k_i > 0 \text{ if } \delta_i = 1, k_i = 0 \text{ if } \delta_i = 0\}$$

with $\delta \in (0, 1)^I$. We have a stratification of $\sigma_{J, Y} \cap \Delta_f^{(n, 0)}$ into the strata $C_\delta \cap \sigma_{J, Y} \cap \Delta_f^{(n, 0)}$. It is a fact that $\Phi_a = \Phi_b$ if and only if $a, b$ belong the same stratum. Now, to deduce the lemma, we use the same arguments as in the proof of [13, Proposition 3.13].

In particular, when $\gamma_j^J = \Gamma$ we have $a = (0, \ldots, 0), J = [d]$, and $f_{|a}(x) = f(x)$. Then the morphism $\Phi_a$ is nothing but the identity morphism.

Similarly, we also consider the morphism $\Psi_a : X_{J, Y}(0) \to X_0$ sending $(x_i)_{i \in J}$ to $(\hat{x}_1, \ldots, \hat{x}_d)$, which commutes with $\pi$ in Theorem 9 and the morphism $\varphi(t) \to \varphi(0)$ for $a \in \delta_{J, Y} \cap \Delta_f^{(n, k)}$ and $k \geq 1$. Recall that the $\mu_n$-action on $X_{J, Y}(0)$ is trivial. As above, we also have

Lemma 11. If $a, b \in \delta_{J, Y} \cap \Delta_f^{(n, 0)}$, then $[\Psi_a : X_{J, Y}(0) \to X_0] = [\Psi_b : X_{J, Y}(0) \to X_0]$ in $\mathcal{M}_X^H$.

Notice that, for $J \subseteq [d]$ and $J \subseteq I$, we identify $(\mathbb{R}_{\geq 0})^I$ with the set of $(x_i)_{i \in J}$ in $(\mathbb{R}_{\geq 0})^J$ such that $x_i > 0$ for all $i \in I$ and $x_j = 0$ for all $j \in J \setminus I$.

For $0 \leq m \leq d$, denote by $F(m)$ the set of $\gamma \in F$ such that for all $J \supseteq J_Y$ and $\gamma = (\gamma_{a})$ we have $a_i \geq 1$ for all $i \in J \cap [m + 1, d]$. Note that $F(0) = K$ and $F(d) = F$.

The last part of the following result is known in [1, 6, 7], we provide new formulas in i), ii) as follows.

Theorem 12. Let $f$ be in $\mathbb{C}[x_1, \ldots, x_d]$ with $f(0) = 0$, let $d_1, d_2$ be in $\mathbb{N}$ with $d = d_1 + d_2$. For any $\gamma \in F$, put $\Lambda_{\gamma} := \{I \subseteq J_Y | \hat{\delta}_{J, Y} \cap \delta_{J_Y} \neq \emptyset\}$ and

$$\lambda_{\gamma} := \sum_{I \in \Lambda_{\gamma}} (-1)^{\text{dim}(\delta_{J, Y} \cap \delta_{J_Y})}.$$

(Hence, as $\gamma \in K$, $\lambda_{\gamma} = (-1)^{\text{dim}(\delta_{J, Y})} = (-1)^{|J_Y| - \text{dim}(\gamma)}$.) The below identities hold in $\mathcal{M}_X^H$ for (i), in $\mathcal{M}_C^{\Delta_{d_1}}$ for (ii), and in $\mathcal{M}_C^{\Delta_{d_1}}$ for (iii).

(i) If $f$ is Newton nondegenerate, then

$$\mathcal{J}_f = - \sum_{\gamma \in F \setminus \bar{F}} \lambda_{\gamma} \left[ X_{\gamma}(1) \to X_0 \right] + \sum_{\gamma \in F} \lambda_{\gamma} \left[ X_{\gamma}(0) \to X_0 \right].$$

(ii) If $f$ is Newton nondegenerate and $\iota : A_{d_1} \subseteq A_{d_1} \times \mathbb{C} \to X_0$ is an inclusion, then

$$\iota^* \mathcal{J}_f = - \sum_{\gamma \in F(d_1) \setminus \bar{F}} \lambda_{\gamma} \left[ X_{\gamma}(1) \times X_0 A_{d_1} \to A_{d_1} \right] + \sum_{\gamma \in F(d_1)} \lambda_{\gamma} \left[ X_{\gamma}(0) \times X_0 A_{d_1} \to A_{d_1} \right].$$

(iii) If $f$ is nondegenerate in the sense of Kouchnirenko, then

$$\mathcal{J}_{f, 0} = \sum_{\gamma \in K} (-1)^{|J_Y| + 1 - \text{dim}(\gamma)} \left[ \left[ X_{\gamma}(1) \right] - \left[ X_{\gamma}(0) \right] \right].$$

Proof. Notice that (i) is not a particular case of (ii) in general, but our proof method of (i) is similar to that of (ii); while (iii) is really a consequence of (ii) (when $d_1 = 0$); so it suffices to prove (ii). By the decomposition (1) and Lemma 8(ii), we have

$$\mathcal{J}_n(f) = \bigsqcup_{\gamma \in F \setminus \bar{F}} \bigsqcup_{k \in \mathbb{N}} \bigsqcup_{a \in \delta_{J, Y} \cap \Delta_f^{(n, k)}} \mathcal{J}_{J, a}^{(n)}.$$
Take the fiber product on both sides with \( \iota : \mathcal{D}_{\mathcal{C}}^{d_1} \to X_0 \). If there is an \( i \in J \cap [d_1 + 1, \ldots, d] \) with \( a_i = 0 \) (i.e. \( \gamma \not\in F(d_i) \)), then \( \varphi_i(0) \) is in \( \mathcal{S}_m \), thus the image of \( \iota \) is disjoint with the image of \( \mathcal{D}_{f,a}^n \) in \( X_0 \), so \( \iota^* \mathcal{D}_{f,a}^n = 0 \). It follows that

\[
\iota^* \mathcal{D}_{f,a}^n(f) = \sum_{\gamma \in F(d_i)} \sum_{J \supset J_f} \sum_{k \in \mathbb{N}} \sum_{a \in \partial_{\gamma} \cap \Delta} \iota^* \mathcal{D}_{f,a}^n.
\]

Using Lemma 6, the diagram (7) for \( X_{J,Y}(1) \to X_0 \) and a similar one for \( X_{J,Y}(0) \to X_0 \) we have

\[
\sum_{n \geq 1} \iota^* \mathcal{D}_{f,a}^n(f) \leq \mathcal{L}^{-nd} T^n
\]

where

\[
\mathcal{S}_{J,Y}^0(T) = \sum_{a \in \partial_{\gamma} \cap \mathbb{N}^l} \mathcal{L}^{|J| - \ell_j(a) - |a|} T^\ell_j(a)
\]

and

\[
\mathcal{S}_{J,Y}^\ast(T) = \sum_{k \geq 1} \sum_{a \in \partial_{\gamma} \cap \mathbb{N}^l} \mathcal{L}^{|J| - \ell_j(a) + k} T^\ell_j(a) + k.
\]

The conclusion then follows from Lemma 14. \( \square \)

We need the following lemmas.

**Lemma 13.** If \( \gamma \in F, J \supseteq J_f \) and \( I \subseteq J_f \), then \( \partial_{J_f} \cap (\mathbb{N}^n)^l \neq \emptyset \).

**Proof.** We first claim that if \( a \in \sigma_{J_f} \) and \( j \in J \setminus J_f \) then for all \( t \geq 0 \) we have \( a + te^j \in \sigma_{J_f} \), where \( e^j \) is defined in the proof of Lemma 6. Indeed, for any \( b \in \gamma \), we have

\[
\langle a + te^j, b \rangle = \langle a, b \rangle \leq \langle a, c \rangle \leq \langle a + te^j, c \rangle \quad \text{for all }c \in \Gamma(f^j).
\]

That implies \( \langle a + te^j, b \rangle = \ell_j(a + te^j) \). Hence \( a + te^j \in \sigma_{J_f} \).

We assume by contradiction that there exists some point \( a \in \partial_{J_f} \cap (\mathbb{N}^n)^l \). Take \( m \in J \setminus J_f \). One can write the cone \( \sigma_{J_f} \) as

\[
\sigma_{J_f} = \{ a \in \mathbb{R}^l | h_i(a) = 0, k_j(a) \geq 0, l_s(a) > 0, i \in I_1, j \in I_2, s \in I_3 \},
\]

where \( h_i, k_j, l_s \) are linear forms on \( \mathbb{R}^l \). Since \( a \in \partial_{J_f} \) we get \( k_j(a) > 0 \) for all \( j \in I_2 \). Hence, for \( t > 0 \) small, we get \( k_j(a - te^m) > 0 \). Similarly, we get \( l_s(a - te^m) > 0 \) for \( t > 0 \) small and for all \( s \in I_3 \). By the above argument, for \( t > 0 \) we have \( a + te^m \in \sigma_{J_f} \), hence \( h_i(a + te^m) = h_i(a) = 0 \), so \( h_i(a - te^m) = 0 \), for every \( i \in I_1 \). It implies that \( a - te^m \in \sigma_{J_f} \) for \( t > 0 \) small. On the other hand, because \( m \in J \setminus J_f \), \( I \subseteq J_f \) and \( a \in (\mathbb{N}^n)^l \), we have \( a_m = 0 \), so \( a - te^m \in \mathbb{R}^l_{\geq 0} \) for any \( t > 0 \). This is a contradiction, and the lemma is proved. \( \square \)

**Lemma 14.** Use the notation in Theorem 12 and its proof, and let \( \gamma \in F \). If \( J \supseteq J_f \), then

\[
\lim_{T \to \infty} S_{J_f}^0(T) = \lim_{T \to \infty} S_{J_f}^\ast(T) = 0.
\]

If \( J = J_f \), then

\[
\lim_{T \to \infty} S_{J_f}^0(T) = - \lim_{T \to \infty} S_{J_f}^\ast(T) = \lambda_f.
\]

**Proof.** Assume that \( J \supseteq J_f \). Because

- \( (\mathbb{R}_{\geq 0})^l = \bigcup_{k \in J} (\mathbb{R}_{\geq 0})^l \), where \( (\mathbb{R}_{\geq 0})^\emptyset = \{(0, \ldots, 0)\} \) by convention,
- \( \ell_j(a) = \ell_j(a) \) for \( a \in \partial_{J_f} \cap \mathbb{N}^l \),
- if \( I \subseteq J_f \), then \( \partial_{J_f} \cap (\mathbb{N}^n)^l = \emptyset \) (by Lemma 13),
we have $S^0_{I,J}(T) = \sum_{I \subseteq J, I \not\subseteq J_0} S^0_{I,J,I}(T)$, where

$$S^0_{I,J,I}(T) := \sum_{a \in \sigma_{I,J} \cap (\mathbb{N}^*)^I} \mathbb{L}^{-\|a\|_{I} - d}T^{\ell_f(a)} = \sum_{a \in \sigma_{I,J} \cap (\mathbb{N}^*)^I} \mathbb{L}^{-|a| \|a\|_{I} - d}T^{\ell_f(a)}$$

(the last equality comes because the inequality $a_j \leq \ell_f(a)$ is automatic for every $j \in J_0$). Denote by $H_J$ the half space of $\mathbb{R}^I$ defined by $a_j \leq \ell_f(a)$. Then $\sigma_{I,J} \cap (\mathbb{R}^*_>)^I \not\subseteq H_J$ for all $J \not\subseteq J_0$, because if there exists an $a \in \sigma_{I,J} \cap H_J$, then for $t > 0$ large enough, we have $a + te^t \in \sigma_{I,J}$ but $a + te^t \not\in H_J$.

This agrees with the hypothesis of Lemma 3. Hence, by Lemma 3, $\lim_{T \to \infty} S^0_{I,J,I}(T) = 0$ for any $I \subseteq J$ and $I \not\subseteq J_0$. Hence $\lim_{T \to \infty} S^0_{I,J,I}(T) = 0$. Similarly, we have $\lim_{T \to \infty} S^0_{J_0,J}(T) = 0$.

For the rest statement, it follows from [6, Lemme 2.1.5] that

$$\lim_{T \to \infty} S^0_{J_0,J}(T) = \lambda_{\gamma},$$

and similarly, $\lim_{T \to \infty} S^0_{J_0,J}(T) = -\lambda_{\gamma}$. \hfill \Box

**Remark 15.** This result revisits Guibert’s work in [6, Section 2.1] for Newton nondegenerate polynomials $f$ in a more general setting. Indeed, in [6] Guibert requires $f$ to have the form $\sum_{a \in (\mathbb{N}^*)^I} a \cdot x^a$, while we do not. Recently, Bultot–Nicaise in [3, Theorems 7.3.2, 7.3.5] provide a new approach to the motivic zeta functions $Z_f(T)$ and $Z_{f,O}(T)$, for $f$ being Newton nondegenerate, using log smooth models.

**Example 16.** Consider the function $f(x,y) = y^2 - x^3$ on $\mathbb{A}^2_\mathbb{C}$, which is well known to be non-degenerate with respect to its Newton polyhedron $\Gamma$. If $\gamma$ is either the face $[3, +\infty) \times \{0\}$ or the face $\{0\} \times [2, +\infty)$ of $\Gamma$, then $\gamma$ is a coordinate face and $X_\gamma(0) = \emptyset$. If $\gamma$ is the compact face $\{3,0\}$ or $\emptyset$, then $X_\gamma(1) = \emptyset$, $X_\gamma(0) = \emptyset$, and $\lambda_\gamma = -1$. If $\gamma$ is the compact face connecting $\{3,0\}$ and $\emptyset$, then $X_\gamma(\emptyset) = \emptyset$, as well as $\lambda_\gamma = -1$. Finally, if $\gamma = \Gamma$, it is a coordinate face and contributes $X_\Gamma(0) \subseteq \mathbb{G}_{m,\mathbb{C}}$, as well as $\lambda_\Gamma = 1$ (since $\sigma_{\{1,2,1\},\Gamma} = \{(0,0)\}$).

By Theorem 12 (we skip arrows to $X_0$ for simplicity),

$$\mathcal{S}_f = \left\{(x,y) \in \mathbb{G}_{m,\mathbb{C}}^2 \mid y^2 - x^3 = 1\right\} + \left\{(x,y) \in \mathbb{A}^2_\mathbb{C} \mid y^2 - x^3 = 1\right\} \in \mathfrak{M}^B \mathcal{H}_X.$$

This also agrees with Davison–Meinhardt’s conjecture on motivic nearby fibers of weighted homogeneous polynomials mentioned in [12]. Also by Theorem 12 we have

$$\mathcal{S}_{f,O} = \left\{(x,y) \in \mathbb{G}_{m,\mathbb{C}}^2 \mid y^2 - x^3 = 1\right\} + \left\{(x,y) \in \mathbb{A}^2_\mathbb{C} \mid y^2 - x^3 = 1\right\} \in \mathfrak{M}^B \mathcal{H}_X.$$

### 3.4. Relation between motivic nearby cycles of $f$ and $f^{[d-1]}$

Let $w$ be a linear function on $\mathbb{C}^d$ generic to $f$. In [9, 10], Lê Dũng Tráng introduced the relative monodromy concerning $(f, O)$ and $w$. We refer to [10, Theorem 2.4] for the following. Denote by $B_\varepsilon$ the closed $\varepsilon$-balls of radius $\varepsilon$ about $O$, by $D_\eta$ the closed disk of radius $\eta$ about 0 in $\mathbb{C}$, and by $D_\eta^x$ the punctured disk $D_\eta \setminus \{0\}$. Let $\Phi$ be the restriction of the map $(w,f) : \mathbb{C}^d \to \mathbb{C}^2$ to $B_\varepsilon \cap (w,f)^{-1}(D_\eta^x)$. Lê proved that, for $0 < \eta \ll \varepsilon \ll 1$, the map $\Phi^{-1}(D_\eta^x \setminus \{0\}) \to D_\eta^x$ is a smooth fibration which is fiber isomorphic to the Milnor fibration of $(f,O)$ with monodromy $M : \Phi^{-1}(D_\eta \times \{\eta\}) \to \Phi^{-1}(D_\eta \times \{\eta\})$, and that $w^{-1}(0) \cap \Phi^{-1}(D_\eta^x \setminus \{0\}) \to D_\eta^x$ is also a smooth fibration which is fiber isomorphic to the Milnor fibration of $(f|_{w=0}, O)$. Then $M$ induces the monodromy of the Milnor fibration of $(f|_{w=0}, O)$ and it lifts a diffeomorphism (which is a carousel) $D_\eta \times \{\eta\} \to D_\eta \times \{\eta\}$ along the mapping $\Phi|_{\Phi^{-1}(D_\eta \times \{\eta\})}$. 


We now consider the “non-generic” hyperplane $x_d = 0$. We write $\tilde{f}$ for $f^{[d-1]}$, that is,
$$\tilde{f}(x_1, \ldots, x_{d-1}) = f(x_1, \ldots, x_{d-1}, 0),$$
and write $\tilde{O}$ for the origin of $C^{d-1}$. Let $\tilde{X}_0$ be the zero locus of $\tilde{f}$, which may be included into $X_0$. The following theorem may be partially considered as a motivic analogue of the work mentioned above. Using realizations, it would be interesting to compare the motivic result to the topological result of Lê Dũng Tráng.

**Theorem 17.** Let $f \in C[x_1, \ldots, x_d]$, and let $d_1, d_2 \in \mathbb{N}$ such that $d = d_1 + d_2$. The below identities hold in $\mathcal{H}^{d}_{\mathbb{A}^d_C}$ for (i) and in $\mathcal{H}^{d}_{\mathbb{A}^{d_1}_C}$ for (ii).

(i) Suppose that $f$ is Newton nondegenerate, $\tilde{X}_0 \subseteq X_0$ and that $\mathbb{A}^{d_1}_C$ is embedded into $\tilde{X}_0$ with the inclusions of $\mathbb{A}^{d_1}_C$ in both $X_0$ and $\tilde{X}_0$ denoted by the same symbol $i$. Then
$$t^* \mathcal{F}_f = t^* \mathcal{F}_{\tilde{f}} + t^* \mathcal{F}_f^{\Delta}. \quad \text{(i)}$$

(ii) If $f$ is nondegenerate in the sense of Kouchnirenko and $f(O) = 0$, then
$$\mathcal{F}_{f,0} = \mathcal{F}_{\tilde{f},0} + \mathcal{F}_f^{\Delta}. \quad \text{(ii)}$$

**Proof.** It suffices to prove (i). By the definition of $(n, m)$-iterated contact loci, we have
$$\mathcal{X}_{n,m}(f, x_d) = \bigcup_{(J, a) \in \mathcal{F}_{n,a_d = m}} \mathcal{X}^{(n)}_{J, a}. \quad \text{(1)}$$

We deduce from Section 2.3 and the method in the proof of Theorem 12 that
$$\sum_{n \geq m \geq 1} t^* [\mathcal{X}_{n,m}(f, x_d)] L^{-nd} T^n$$
$$= \sum_{\gamma \in F(d_1)} \sum_{J \supseteq J_f} \sum_{a \in \sigma f, a \cap \Delta_f} t^* [\mathcal{X}^{(f J_f, a)}_{J, a}] \cdot L^{-d J_f(a)} T^{J_f(a)}$$
$$\quad + \sum_{\gamma \in F(d_1)} \sum_{J \supseteq J_f} \sum_{k \geq 1} \sum_{a \in \sigma f, a \cap \Delta_f} t^* [\mathcal{X}^{(f J_f, a)}_{J, a}] \cdot L^{-d (J_f(a) + k)} T^{J_f(a) + k}.$$

We apply Theorem 9 to $\gamma \in F(d_1)$ and $a \in \sigma f, a$. If $d \notin J_f$, then $J_f(a) + k \geq a_d \geq 1$ automatically for any $k \in \mathbb{N}$. If $d \notin J_f$, then the inequalities $J_f(a) + k \geq a_d \geq 1$ is in the situation of [8, Lemma 2.10], in which the corresponding series has the limit zero. Therefore, taking $\lim_{T \to \infty}$ and using Theorem 9, Lemma 14 and the proof of Theorem 12, we get
$$t^* \mathcal{F}_f^{\Delta} = \sum_{\gamma \in F(d_1) \setminus F_f} \sum_{d \notin J_f} \lambda_f \cdot [X_{d_1}(1) \times X_0 \mathbb{A}^{d_1}_C \rightarrow \mathbb{A}^{d_1}_C] + \sum_{\gamma \in F(d_1) \setminus F_f} \sum_{d \notin J_f} \lambda_f \cdot [X_{d_1}(0) \times X_0 \mathbb{A}^{d_1}_C \rightarrow \mathbb{A}^{d_1}_C].$$

By Theorem 12,
$$t^* \mathcal{F}_f = \sum_{\gamma \in F(d_1) \setminus F_f} \sum_{d \notin J_f} \lambda_f \cdot [X_{d_1}(1) \times X_0 \mathbb{A}^{d_1}_C \rightarrow \mathbb{A}^{d_1}_C] + \sum_{\gamma \in F(d_1) \setminus F_f} \sum_{d \notin J_f} \lambda_f \cdot [X_{d_1}(0) \times X_0 \mathbb{A}^{d_1}_C \rightarrow \mathbb{A}^{d_1}_C].$$

The condition $d \notin J_f$ means that $J_f \subseteq [d - 1]$, hence, again by Theorem 12,
$$t^* \mathcal{F}_f = \sum_{\gamma \in F(d_1) \setminus F_f} \sum_{d \notin J_f} \lambda_f \cdot [X_{d_1}(1) \times X_0 \mathbb{A}^{d_1}_C \rightarrow \mathbb{A}^{d_1}_C] + \sum_{\gamma \in F(d_1) \setminus F_f} \sum_{d \notin J_f} \lambda_f \cdot [X_{d_1}(0) \times X_0 \mathbb{A}^{d_1}_C \rightarrow \mathbb{A}^{d_1}_C].$$

The theorem is completely proved. \[ \square \]
4. Cohomology groups of contact loci of nondegenerate singularities

As before, let \( f \) be in \( \mathbb{C}[x_1, \ldots, x_d] \) which vanishes at \( O \). In this section, we always assume that \( f \) is nondegenerate in the sense of Kouchnirenko (say for short that \( f \) is nondegenerate).

4.1. Borel–Moore homology groups of contact loci

Consider the decomposition of \( \mathcal{X}_{n,0}(f) \) shown in (2) and Lemma 8 (ii):

\[
\mathcal{X}_{n,0}(f) = \bigcup_{(J, a) \in \mathcal{P}_n} \mathcal{X}^{-}(n)_{J, a},
\]

where \( \mathcal{P}_n \) is the set of all the pairs \((J, a)\) such that \( J \supseteq I_\gamma, \ a \in \cup_{k \in \mathbb{N}} \{ \mathcal{S}_J \cap \mathcal{X}^{-}(n,k) \} \) and \( \gamma \in K \). We consider an ordering in \( \mathcal{P}_n \) defined as follows: for \((J, a)\) and \((J', a')\) in \( \mathcal{P}_n \), \((J', a') \leq (J, a)\) if and only if \( J' \subseteq J \) and \( a_j \leq a'_j \) for all \( j \in J' \), where \( a = (a_j)_{j \in J} \) and \( a' = (a'_j)_{j \in J'} \).

**Lemma 18.** Let \( n \) be in \( \mathbb{N}^* \). For all \((J, a)\) and \((J', a')\) in \( \mathcal{P}_n \) such that \( \mathcal{X}^{-}(n)_{J, a} \) and \( \mathcal{X}^{-}(n)_{J', a'} \) are nonempty, the following are equivalent:

(i) \((J', a') \leq (J, a)\),

(ii) \( \mathcal{X}^{-}(n)_{J', a'} \subseteq \mathcal{X}^{-}(n)_{J, a} \),

(iii) \( \mathcal{X}^{-}(n)_{J, a} \cap \mathcal{X}^{-}(n)_{J', a'} \neq \emptyset \),

the closure taken in the usual topology. Consequently, for all \((J, a)\) in \( \mathcal{P}_n \) such that \( \mathcal{X}^{-}(n)_{J, a} \neq \emptyset \) we have

\[
\overline{\mathcal{X}^{-}(n)_{J, a}} = \bigcup_{(J', a') \leq (J, a), (J', a') \neq (J, a)} \mathcal{X}^{-}(n)_{J', a'}.
\]

**Proof.** We prove here that (iii) implies (i), the rest are straightforward. Observe firstly that, due to the definition of \( \mathcal{X}^{-}(n)_{J, a} \), if it is nonempty then

\[
\overline{\mathcal{X}^{-}(n)_{J, a}} = \{ \varphi \in \mathcal{X}^{-}(n)_{J, a} \mid \operatorname{ord}_J x_j(\varphi) \geq a_j \ \forall \ j \in J, \ x_i(\varphi) = 0 \ \forall \ i \notin J \},
\]

where \( a = (a_j)_{j \in J} \), and \( a_j > 0 \) for all \( j \in J \). Assume that there exists \( \varphi^0 \in \mathcal{X}^{-}(n)_{J', a'} \cap \overline{\mathcal{X}^{-}(n)_{J, a}} \neq \emptyset \). Then we have \( \operatorname{ord}_J x_j(\varphi^0) = a'_j > 0 \) for all \( j \in J' \), and \( \operatorname{ord}_J x_j(\varphi^0) \geq a_j > 0 \) for all \( j \in J \). If \( i \notin J \), then we have \( \operatorname{ord}_I x_i(\varphi^0) = +\infty \), thus \( i \notin J' \), so \( J' \subseteq J \). Clearly, \( a_j \leq a'_j \) for all \( j \in J' \). Therefore, \((J', a') \leq (J, a)\). \( \square \)

Consider the function \( \eta: \mathcal{P}_n \to \mathbb{Z} \) given by \( \eta(J, a) = \dim_{\mathcal{C}} \mathcal{X}^{-}(n)_{J, a} \), for every \( n \in \mathbb{N}^* \). Put

\[
S_p := \bigcup_{(J, a) \in \mathcal{P}_n, \eta(J, a) \leq p} \mathcal{X}^{-}(n)_{J, a},
\]

for \( p \in \mathbb{N} \). The below are some properties of \( \eta \) and \( S_p \)‘s.

**Lemma 19.** Let \( n \) be in \( \mathbb{N}^* \).

(i) If \((J', a') \leq (J, a)\) in \( \mathcal{P}_n \) and \( \mathcal{X}^{-}(n)_{J, a} \neq \emptyset \), then \( \eta(J', a') \leq \eta(J, a) \).

(ii) For all \( p \in \mathbb{N} \), \( S_p \) are closed and \( S_p \subseteq S_{p+1} \). As a consequence, there is a filtration of \( \mathcal{X}_{n,0}(f) \) by closed subspaces:

\[
\mathcal{X}_{n,0}(f) = S_{d_0} \supseteq S_{d_0-1} \supseteq \cdots \supseteq S_{-1} = \emptyset,
\]

where \( d_0 \) denotes the \( \mathcal{C} \)-dimension of \( \mathcal{X}_{n,0}(f) \).

**Proof.** The first statement (i) is trivial. To prove (ii) we take the closure of \( S_p \); then using Lemma 18 we get

\[
\overline{S_p} = \bigcup_{\eta(J, a) \leq p} \overline{\mathcal{X}^{-}(n)_{J, a}} = \bigcup_{\eta(J, a) \leq p, \mathcal{X}^{-}(n)_{J, a} \neq \emptyset} \overline{\mathcal{X}^{-}(n)_{J, a}} = \bigcup_{\eta(J, a) \leq p} \bigcup_{(J', a') \leq (J, a)} \mathcal{X}^{-}(n)_{J', a'}.
\]
This decomposition combined with (i) implies that \( \overline{S}_p \subseteq S_p \), which proves that \( S_p \) is a closed subspace. The remaining statements of (ii) are trivial. \( \square \)

A main result of this section is the following theorem. To express the result, we work with the Borel–Moore homology \( H^*_f \).

**Theorem 20.** Let \( f \in \mathbb{C}[x_1, \ldots, x_d] \) be nondegenerate in the sense of Kouchnirenko, \( n \in \mathbb{N}^* \) and \( f(O) = 0 \). Then there is a spectral sequence

\[
E^1_{p,q} := \bigoplus_{(j,a) \in \mathcal{T}_n, \eta(j,a) = p} H^\text{BM}_{p+q}(\mathcal{X}_{j,a}^{(n)}) \Rightarrow H^\text{BM}_{p+q}(\mathcal{X}_{n,0}(f)).
\]

**Proof.** We have the following the Gysin exact sequence

\[
\cdots \to H^\text{BM}_{p+q}(S_{p-1}) \to H^\text{BM}_{p+q}(S_p) \to H^\text{BM}_{p+q}(S_p \setminus S_{p-1}) \to H^\text{BM}_{p+q-1}(S_{p-1}) \to \cdots.
\]

Put

\[
A_{p,q} := H^\text{BM}_{p+q}(S_p), \quad E_{p,q} := H^\text{BM}_{p+q}(S_p \setminus S_{p-1}).
\]

Then we have the bigraded \( \mathbb{Z} \)-modules \( A := \bigoplus_{p,q} A_{p,q} \) and \( E := \bigoplus_{p,q} E_{p,q} \). The previous exact sequence induces the exact couple \((A,E;i,j)\), where \( h : A \to A \) is induced from the inclusions \( S_m \subseteq S_{m+1} \), \( i : A \to E \) and \( j : E \to A \) are induced from the above exact sequence. Since the filtration in Lemma 19 (ii) is finite, that exact couple gives us the following spectral sequence

\[
E^1_{p,q} = E_{p,q} = H^\text{BM}_{p+q}(S_p \setminus S_{p-1}) \Rightarrow H^\text{BM}_{p+q}(\mathcal{X}_{n,0}(f)).
\]

On the other hand, we have

\[
S_p \setminus S_{p-1} = \bigcup_{(j,a) \in \mathcal{T}_n, \eta(j,a) = p} \mathcal{X}_{j,a}^{(n)}.
\]

One claims that for two different pairs \((j,a),(j',a')\) in \( \mathcal{T}_n \) with \( \eta(j,a) = \eta(j',a') = p \) then

\[
\mathcal{X}_{j,a}^{(n)} \cap \mathcal{X}_{j',a'}^{(n)} = \emptyset \quad \text{and} \quad \mathcal{X}_{j',a'}^{(n)} \cap \mathcal{X}_{j,a}^{(n)} = \emptyset.
\]

Indeed, if otherwise, suppose that

\[
\mathcal{X}_{j',a'}^{(n)} \cap \mathcal{X}_{j,a}^{(n)} \neq \emptyset.
\]

By Lemma 18, we obtain that \( \mathcal{X}_{j',a'}^{(n)} \subseteq \mathcal{X}_{j,a}^{(n)} \), but \( \mathcal{X}_{j',a'}^{(n)} \) and \( \mathcal{X}_{j,a}^{(n)} \) are two disjoint smooth manifolds, then \( \eta(j',a') < \eta(j,a) \). This is a contradiction.

Therefore, in the set \( S_p \setminus S_{p-1} \) with the induced topology, each set \( \mathcal{X}_{j,a}^{(n)} \) which \( \eta(j,a) = p \) is open, hence, is also closed. This implies that

\[
H^\text{BM}_{p+q}(S_p \setminus S_{p-1}) = \bigoplus_{(j,a) \in \mathcal{T}_n, \eta(j,a) = p} H^\text{BM}_{p+q}(\mathcal{X}_{j,a}^{(n)}).
\]

The theorem is then proved. \( \square \)

**Corollary 21.** With the hypothesis as in Theorem 20, there is an isomorphism of groups

\[
H^\text{BM}_{2d_0}(\mathcal{X}_{n,0}(f)) \cong \mathbb{Z}^s,
\]

where \( s \) is the number of connected components of \( \mathcal{X}_{n,0}(f) \) which have the same complex dimension \( d_0 \) as \( \mathcal{X}_{n,0}(f) \).
4.2. Sheaf cohomology groups of contact loci

In this subsection, we are going to prove the following theorem.

**Theorem 22.** Let \( f \in \mathbb{C}[x_1, \ldots, x_d] \) be nondegenerate in the sense of Kouchnirenko, \( n \in \mathbb{N}^* \) and \( f(O) = 0 \). Let \( \mathcal{F} \) be an arbitrary sheaf of abelian groups on \( \mathcal{X}_{n,0}(f) \). Then, there is a spectral sequence

\[
E_1^{p,q} := \bigoplus_{(j,a) \in \mathcal{F}_{n,\eta}(j,a) = p} H^p_c(\mathcal{X}_{j,a}^{(n)}, \mathcal{F}) \Rightarrow H^{p+q}_c(\mathcal{X}_{n,0}(f), \mathcal{F}). \tag{8}
\]

**Proof.** We use the notation in Lemma 19. For simplicity, we write \( S \) for \( S_{d_0} = \mathcal{X}_{n,0}(f) \). For any \( 0 \leq p \leq d_0 \), we put \( S_p = S \setminus S_{p-1} \), which is a \( \mu_n \)-invariant subset of \( S \). Consider the inclusions \( j_p : S_p \to S \), \( k_p : S \setminus S_p \to S \) and \( i_p : S_p \to S \). Put \( \mathcal{F}_p := (j_p)! (i_p)^{-1} (i_p)^{-1} \mathcal{F} \) and \( F_p^p(\mathcal{F}) := (k_p^{-1}) (k_p^{-1})^{-1} \mathcal{F} \) for every \( p \geq 1 \), with the convention \( F^0(\mathcal{F}) := \mathcal{F} \). Then we have the exact sequences

\[
0 \to F_p^{p+1}(\mathcal{F}) \to F_p(\mathcal{F}) \quad \text{and} \quad 0 \to (i_p)_* \mathcal{F}_p \to \mathcal{F}|_{S_p} \to \mathcal{F}|_{S_{p-1}},
\]

in which by \( \mathcal{F}|_{S_p} \) we mean \((i_p)_* (i_p)^{-1} \mathcal{F} \). Therefore we have the following diagram

\[
\begin{array}{cccccc}
0 & \to & F^{p+1}(\mathcal{F}) & \to & \mathcal{F} & \to & \mathcal{F}|_{S_p} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & F^{p}(\mathcal{F}) & \to & \mathcal{F} & \to & \mathcal{F}|_{S_{p-1}} & \to & 0 \\
\end{array}
\]

It implies from the snake lemma that \( F^p_p(\mathcal{F})/F^{p+1}_p(\mathcal{F}) \equiv (i_p)_* \mathcal{F}_p \). Thus there is a filtration of \( \mathcal{F} \) by “skeleta”: \( \mathcal{F} = F^0(\mathcal{F}) \supseteq F^1(\mathcal{F}) \supseteq \cdots \). It gives the following spectral sequence of cohomology groups with compact support

\[
E_1^{p,q}(S, \mathcal{F}) := H^p_c(S, (i_p)_* \mathcal{F}_p) \Rightarrow H^p_c(S, \mathcal{F}). \tag{9}
\]

Since \( S_p \) is a closed subset of \( S \), \( H^m_c(S, (i_p)_* \mathcal{F}_p) \equiv H^m_c(S_p, \mathcal{F}_p) \) for any \( m \in \mathbb{N} \). Also, by the isomorphisms given by the extension by zero sheaf, we have

\[
H^m_c(S_p, \mathcal{F}_p) = H^m_c(S_p, (j_p)! (j_p)^{-1} (i_p)^{-1} \mathcal{F}) \equiv H^m_c(S_{p_0}, (i_p)^{-1} \mathcal{F}).
\]

We have that

\[
S_{p_0} = \bigcup_{(j,a) \in \mathcal{F}_{n,\eta}(j,a) = p} \mathcal{X}_{j,a}^{(n)}.
\]

Then by the reason as in the proof of Theorem 20, we get

\[
H^m_c(S_{p_0}, (i_p)^{-1} \mathcal{F}) = \bigoplus_{(j,a) \in \mathcal{F}_{n,\eta}(j,a) = p} H^{p+q}_c(\mathcal{X}_{j,a}^{(n)}, (l_j)_* (i_j)_1^{-1} (i_j)^{-1} \mathcal{F}),
\]

where \( l_j, a \) is the inclusion of \( \mathcal{X}_{j,a}^{(n)} \) in \( S_p \). For simplicity of notation, we write \( H^{p+q}_c(\mathcal{X}_{j,a}^{(n)}, \mathcal{F}) \) instead of \( H^{p+q}_c(\mathcal{X}_{j,a}^{(n)}, (l_j)_* (i_j)_1^{-1} (i_j)^{-1} \mathcal{F}) \). The proof is completed. \( \Box \)

Now, let us consider the spectral sequence (8) for a constant sheaf. We need some notation, for each \( \gamma \in K, \ k \in \mathbb{N}, n \in \mathbb{N}^*, \ p \in \mathbb{Z} \) and \( J \supset J_\gamma \), we denote by \( D_{j,\gamma, k, p}^{(n)} \) the set of all \( a \in \tilde{\mathcal{X}}_{j, \gamma} \cap \Delta_{j}^{(n,k)} \) such that \( \dim_c \mathcal{X}_{j,a}^{(n)} = p \).

**Lemma 23.** For any \( \gamma \in K, \ k \in \mathbb{N}, n \in \mathbb{N}^*, \ p \in \mathbb{Z} \) and \( J \supset J_\gamma \), the set \( D_{j,\gamma, k, p}^{(n)} \) is finite.
Proof. Notice that $\dim \mathcal{A}^{(n)}_{f,a} = d - 1 + |J|/n - |a| - k$. The finiteness of $D^{(n)}_{f,\gamma,k,p}$ follows from the fact that the system of equations $d - 1 + |J|/n - |a| - k = p$, $\ell_j(a) + k = n$ in variables $a$ only has finite solutions in $\mathbb{N}^d$. □

The summands in the spectral sequence (8) are described more explicitly in case of constant sheaf as below.

Lemma 24. Let $\gamma \in K$, $n \in \mathbb{N}^*$, $p, q \in \mathbb{Z}$ and $J \supseteq J_\gamma$. Then, for any $a \in D^{(n)}_{f,\gamma,0,p}$ we have

$$H^p_{c}^{p+q}(\mathcal{A}^{(n)}_{f,a}, \mathbb{C}) \cong H_{p-q}(X_{f,\gamma}(1), \mathbb{C}).$$

Proof. Since $a \in D^{(n)}_{f,\gamma,0,p}$, it follows from Theorem 9 that $\mathcal{A}^{(n)}_{f,a}$ is a complex manifold of real dimension $2p$ and is homeomorphic to $X_{f,\gamma}(1) \times \mathbb{C}^{(|J|/\ell_j(a) - |a|)}$. Then, by combining the duality and the Kunneth formula we get the conclusion. □

We also have the following description for the cohomology of $\mathcal{A}^{(n)}_{f,a}$ for $J \supseteq J_\gamma$ and $a \in D^{(n)}_{f,\gamma,k,p}$ with $k \in \mathbb{N}^*$.

Lemma 25. Let $\gamma \in K$, $n \in \mathbb{N}^*$, $p, q \in \mathbb{Z}$ and $J \supseteq J_\gamma$. Then, for any $a \in D^{(n)}_{f,\gamma,k,p}$ we have

$$H^p_{c}^{p+q}(\mathcal{A}^{(n)}_{f,a}, \mathbb{C}) \cong H_{p-q}(X_{f,\gamma}(0), \mathbb{C}).$$

Proof. Since $\mathcal{A}^{(n)}_{f,a}$ is a complex manifold of real dimension $2p$, then by duality, we have

$$H^p_{c}^{p+q}(\mathcal{A}^{(n)}_{f,a}, \mathbb{C}) \cong H_{p-q}(\mathcal{A}^{(n)}_{f,a}, \mathbb{C}).$$

On the other hand, by Theorem 9, $\mathcal{A}^{(n)}_{f,a}$ is a locally trivial fibration on $X_{f,\gamma}(0)$ with fiber $\mathbb{C}^{(|J|/(\ell_j(a) - k) - |a|)}$ which is contractible. Hence, by the spectral sequence for (Serre) fibration, we obtain that $H_{p-q}(\mathcal{A}^{(n)}_{f,a}, \mathbb{C}) \cong H_{p-q}(X_{f,\gamma}(0), \mathbb{C})$. The proof is completed. □

We have the following result concerning cohomology of contact loci.

Corollary 26. Let $f \in \mathbb{C}[x_1, \ldots, x_q]$ be nondegenerate in the sense of Kouchnirenko, $n \in \mathbb{N}^*$ and $f(O) = 0$. Then, there is a spectral sequence

$$E^{p,q}_1 \Rightarrow H^p_{c}^{p+q}(\mathcal{A}^{n,0}(f), \mathbb{C}),$$

where

$$E^{p,q}_1 = \bigoplus_{\gamma \in k, J} \left( H_{p-q}(X_{J,\gamma}(1), \mathbb{C})^{D^{(n)}_{f,J,\gamma,k,p}} \oplus H_{p-q}(X_{J,\gamma}(0), \mathbb{C})^{D^{(n)}_{f,J,\gamma,k,p}} \right).$$

Proof. Apply Theorem 22 for $\mathcal{A}$ to be the constant sheaf on $\mathcal{A}^{n,0}(f)$ associated to the field of complex numbers $\mathbb{C}$, since the inverse image of constant sheaf is a constant sheaf, the Corollary is a direct consequence of Theorem 22 and Lemmas 24 and 25. □

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