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Geometry of nondegenerate polynomials: Motivic nearby cycles and Cohomology of contact loci

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Abstract. We study polynomials with complex coefficients which are nondegenerate in two senses, one of Kouchnirenko and the other with respect to its Newton polyhedron, through data on contact loci and motivic nearby cycles. Introducing an explicit description of these quantities we can answer in part the question concerning the motivic nearby cycles of restriction functions in the context of Newton nondegenerate polynomials. Furthermore, in the nondegeneracy in the sense of Kouchnirenko, we give calculations on cohomology groups of the contact loci.

Keywords. arc spaces, contact loci, motivic zeta function, motivic Milnor fiber, motivic nearby cycles, Newton polyhedron, nondegeneracy, sheaf cohomology with compact support.

2020 Mathematics Subject Classification. 14B05, 14B07, 14J17, 32S05, 32S30, 32S55.

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1. Introduction

Let f be a nondegenerate $\mathbb{C}$-polynomial in the sense of Kouchnirenko (cf. Section 3.1) vanishing at the origin O of $\mathbb{C}^d$. The problem of computing the motivic Milnor fiber $\mathscr{S}_{f,O}$ in terms of the Newton polyhedron Γ of f was early mentioned in the works [1] and [6] with materials coming from [4] (see also [7] for a generalization). Recently, Steenbrink and Bultot–Nicaise obtain solutions in terms of toric geometry ([14]), or of log smooth models ([3]). Their formula for $\mathscr{S}_{f,O}$ together with the additivity of the Hodge spectrum operator allows to reduce the computation of the Hodge spectrum of (f,O) to that of quasi-homogeneous singularities. In this article, we will show that the formula also provides a way to explore the following problem for Newton nondegenerate polynomials.

Problem 1. *Let f be in $\mathbb{C}[x_1,\dots,x_d]$ with $f(O)=0$, and let H be a linear hyperplane in $\mathbb{C}^d$. What is the relation between $\mathscr{S}_{f,O}$ and $\mathscr{S}_{f|_H,O}$?*

The question concerns a motivic analogue of a monodromy relation of a complex singularity and its restriction to a generic hyperplane studied early in [9, 10]. For $n\in\mathbb{N}^*$, the n-iterated contact locus $\mathscr{X}_{n,O}(f)$ (cf. Section 2.3) admits a decomposition as a disjoint union into its μ_n-invariant $\mathbb{C}$-subvarieties $\mathscr{X}_{J,a}^{(n)}$ along $a\in(\mathbb{N}^*)^J$ and $J\subseteq[d]:=\{1,\dots,d\}$. The nondegeneracy of f allows to describe $\mathscr{X}_{J,a}^{(n)}$ via Γ, as in Theorem 9, which is the key step to compute the motivic zeta function $Z_{f,O}(T)$ and the motivic Milnor fiber $\mathscr{S}_{f,O}$, which yields a proof of Theorem 12. Note that this theorem is well known as mentioned above (see [1, 6, 7]). For every face γ of Γ, let J_γ be the unique subset of $[d]$ such that γ is contained in the hyperplanes $x_j=0$ for all $j\not\in J_\gamma$ and not contained in the other coordinate hyperplanes, and let $X_\gamma(0)$ (resp. $X_\gamma(1)$) be the $\mathbb{C}$-subvariety of $\mathbb{G}_{m,\mathbb{C}}^{J_\gamma}$ defined by the face function f_γ (resp. $f_\gamma-1$).

Theorem (see Theorem 12). *Let f be in $\mathbb{C}[x_1,\dots,x_d]$ with $O\in X_0:=f^{-1}(0)$, let d_1 and d_2 be in $\mathbb{N}$ such that $d=d_1+d_2$. The below hold in $\mathscr{M}_{X_0}^{\hat{\mu}}$ for* (i)*, in $\mathscr{M}_{\mathbb{A}_{\mathbb{C}}^{d_1}}^{\hat{\mu}}$ for* (ii)*, and in $\mathscr{M}_{\mathbb{C}}^{\hat{\mu}}$ for* (iii)*.*

(i) *If f is Newton nondegenerate, then*

$$\mathscr{S}_f=-\sum_{\gamma\in F\setminus\widetilde{F}}\lambda_\gamma\left[X_\gamma(1)\to X_0\right]+\sum_{\gamma\in F}\lambda_\gamma\left[X_\gamma(0)\to X_0\right].$$

(ii) *If f is Newton nondegenerate and $\iota:\mathbb{A}_{\mathbb{C}}^{d_1}\equiv\mathbb{A}_{\mathbb{C}}^{d_1}\times_{\mathbb{C}}\{0\}^{d_2}\hookrightarrow X_0$ is an inclusion, then*

$$\iota^*\mathscr{S}_f=-\sum_{\gamma\in F(d_1)\setminus\widetilde{F}}\lambda_\gamma\left[X_\gamma(1)\times_{X_0}\mathbb{A}_{\mathbb{C}}^{d_1}\to\mathbb{A}_{\mathbb{C}}^{d_1}\right]+\sum_{\gamma\in F(d_1)}\lambda_\gamma\left[X_\gamma(0)\times_{X_0}\mathbb{A}_{\mathbb{C}}^{d_1}\to\mathbb{A}_{\mathbb{C}}^{d_1}\right].$$

(iii) *If f is nondegenerate in the sense of Kouchnirenko, then*

$$\mathscr{S}_{f,O}=\sum_{\gamma\in K}(-1)^{|J_\gamma|+1-\dim(\gamma)}\left([X_\gamma(1)]-[X_\gamma(0)]\right).$$

Here, ι^*, F, $\widetilde{F}$, $F(d_1)$, K, and λ_γ are defined in Sections 2.1, 3.1 and 3.3.

We choose the hyperplane defined by $x_d=0$ to be H in Problem 1, and consider for any $n\geq m$ in $\mathbb{N}^*$ the so-called (n,m)-iterated contact locus $\mathscr{X}_{n,m,O}(f,x_d)$ of the pair (f,x_d). It is a μ_n-invariant $\mathbb{C}$-subvariety of $\mathscr{X}_{n,O}(f)$. Then we show in this article that the formal series

$$Z_{f,x_d,O}^{\Delta}(T):=\sum_{n\geq m\geq 1}\left[\mathscr{X}_{n,m,O}(f,x_d)\right]\mathbb{L}^{-(n+m)d}T^n$$

is rational and it can be described via data of Γ. Here, Δ stands for $\{(n,m)\in(\mathbb{R}_{>0})^2\mid n\geq m\}$ and the sum runs over $\Delta\cap(\mathbb{N}^*)^2$. Put $\mathscr{S}_{f,x_d,O}^{\Delta}:=-\lim_{T\to\infty}Z_{f,x_d,O}^{\Delta}(T)$. Using the description of $\mathscr{S}_{f,x_d,O}^{\Delta}$ together with Theorem 12, a solution to Problem 1 for the nondegeneracy in the sense of Kouchnirenko can be realized as in the following theorem.

Theorem (see Theorem 17). *With f as previous, the identity $\mathscr{S}_{f,O} = \mathscr{S}_{f|_H,O} + \mathscr{S}^{\Delta}_{f,x_d,O}$ holds in the monodromic Grothendieck ring of $\mathbb{C}$-varieties with $\widehat{\mu}$-action. A similar result also holds for the motivic nearby cycles.*

According to [2, Conjecture 1.5], it is expected that the singular cohomology groups with compact support of the $\mathbb{C}$-points of the contact loci are nothing but the Floer cohomology groups of the powers of the monodromy of the singularity (cf. [11]). Here, we are interested in a smaller problem on the computation of cohomology groups of $\mathscr{X}_{n,O}(f)$ (the reader may compare this with [2, Theorem 1.1]).

Problem 2. *Let f be a polynomial over $\mathbb{C}$ vanishing at the origin O. Compute the cohomology groups with compact support $H^m_c(\mathscr{X}_{n,O}(f),\mathbb{C})$ for all $n \in \mathbb{N}^*$ and $m \in \mathbb{N}$.*

We devote Section 4 to study this problem for nondegenerate singularities in the sense of Kouchnirenko not only using sheaf cohomology with compact support but also the Borel–Moore homology H^{BM}_*. Write $\mathscr{X}_{n,O}(f) = \bigsqcup_{(J,a)\in\widetilde{\mathscr{P}}_n} \mathscr{X}^{(n)}_{J,a}$ as in (2) with $\widetilde{\mathscr{P}}_n$ described in Lemma 8 (ii). Let $\eta : \widetilde{\mathscr{P}}_n \to \mathbb{Z}$ be the function defined by $\eta(J,a) = \dim_{\mathbb{C}} \mathscr{X}^{(n)}_{J,a}$.

Theorem (see Theorems 20, 22). *For f as in Problem 2 and nondegenerate in the sense of Kouchnirenko, for every $p, q \in \mathbb{N}$, there exist spectral sequences*

$$E^1_{p,q} := \bigoplus_{\eta(J,a)=p} H^{\mathrm{BM}}_{p+q}(\mathscr{X}^{(n)}_{J,a}) \Longrightarrow H^{\mathrm{BM}}_{p+q}(\mathscr{X}_{n,O}(f)),$$
$$E^{p,q}_1 := \bigoplus_{\eta(J,a)=p} H^{p+q}_c(\mathscr{X}^{(n)}_{J,a},\mathscr{F}) \Longrightarrow H^{p+q}_c(\mathscr{X}_{n,O}(f),\mathscr{F}),$$

for any sheaf of abelian groups $\mathscr{F}$ on $\mathscr{X}_{n,O}(f)$.

In particular, by applying the second spectral sequence with $\mathscr{F}$ being a constant sheaf, we obtain a spectral sequence converging to the compact support cohomology groups of contact loci with complex coefficients whose first page is a direct sum of (singular) homology of the spaces defined by the vanishing of the functions f_γ and $f_\gamma - 1$ (see Corollary 26).

2. Preliminaries

2.1. *Monodromic Grothendieck ring of varieties*

Let S be an algebraic $\mathbb{C}$-variety. Let Var_S be the category of S-varieties, with objects being morphisms of algebraic $\mathbb{C}$-varieties $X \to S$ and a morphism in Var_S from $X \to S$ to $Y \to S$ being a morphism of algebraic $\mathbb{C}$-varieties $X \to Y$ commuting with $X \to S$ and $Y \to S$. Denote by $\widehat{\mu}$ the limit of the projective system $\mu_{nm} \to \mu_n$ given by $x \mapsto x^m$, with for any $n \geq 1$, $\mu_n = \operatorname{Spec} \mathbb{C}[\xi]/(\xi^n - 1)$ the group scheme over $\mathbb{C}$ of nth roots of unity. Notice that any action of $\widehat{\mu}$ on a variety X in the present article is assumed to factorize through an action of μ_n for some $n \in \mathbb{N}^*$. An action on X is good if every orbit is contained in an affine open subset of X. By definition, an action of $\widehat{\mu}$ on an affine Zariski bundle $X \to B$ is affine if it is a lifting of a good action on B and its restriction to all fibers is affine.

The Grothendieck group $K^{\widehat{\mu}}_0(\mathrm{Var}_S)$ is defined to be an abelian group generated by symbols $[X \to S]$, X endowed with a good $\widehat{\mu}$-action and $X \to S$ in Var_S, such that:

(i) $[X \to S] = [Y \to S]$ if X and Y are $\widehat{\mu}$-equivariant S-isomorphic;
(ii) $[X \to S] = [Y \to S] + [X \setminus Y \to S]$ if Y is a $\widehat{\mu}$-invariant closed subvariety in X; and
(iii) $[X \times \mathbb{A}^n_{\mathbb{C}}, \sigma] = [X \times \mathbb{A}^n_{\mathbb{C}}, \sigma']$ if σ and σ' are liftings of the same $\widehat{\mu}$-action on X to $X \times \mathbb{A}^n_{\mathbb{C}}$.

There is a natural ring structure on $K_0^{\hat\mu}(\mathrm{Var}_S)$ in which the product is induced by the fiber product over S. The unit 1_S for the product is the class of the identity morphism $S \to S$ with S endowed with trivial $\hat\mu$-action. Denote by $\mathbb{L}$ (or $\mathbb{L}_S$) the class of the trivial line bundle $S \times \mathbb{A}^1 \to S$, and define the localized ring $\mathscr{M}_S^{\hat\mu}$ to be $K_0^{\hat\mu}(\mathrm{Var}_S)[\mathbb{L}^{-1}]$.

Let $f : S \to S'$ be a morphism of algebraic $\mathbb{C}$-varieties. Then we have two important morphisms associated to f, which are the ring homomorphism $f^* : \mathscr{M}_{S'}^{\hat\mu} \to \mathscr{M}_S^{\hat\mu}$ induced from the fiber product (the pullback morphism) and the $\mathscr{M}_{\mathbb{C}}$-linear homomorphism $f_! : \mathscr{M}_S^{\hat\mu} \to \mathscr{M}_{S'}^{\hat\mu}$ defined by the composition with f (the push-forward morphism).

2.2. *Rational series and limit*

Let $\mathscr{A}$ be either $\mathbb{Z}[\mathbb{L},\mathbb{L}^{-1}]$ or $\mathscr{M}_S^{\hat\mu}$ as a ring. Let $\mathscr{A}[\![T]\!]_{\mathrm{sr}}$ be the $\mathscr{A}$-submodule of $\mathscr{A}[\![T]\!]$ generated by 1 and by finite products of elements of the form $\frac{\mathbb{L}^a T^b}{1-\mathbb{L}^a T^b}$ with (a,b) in $\mathbb{Z}\times\mathbb{N}^*$. Each element of $\mathscr{A}[\![T]\!]_{\mathrm{sr}}$ is called a *rational series*. By [5], there is a unique $\mathscr{A}$-linear morphism $\lim_{T\to\infty} : \mathscr{A}[\![T]\!]_{\mathrm{sr}} \to \mathscr{A}$ which sends $\frac{\mathbb{L}^a T^b}{1-\mathbb{L}^a T^b}$ to -1.

For J contained in $[d]$, we denote by $(\mathbb{R}_{\geq 0})^J$ the set of $(a_j)_{j\in J}$ with a_j in $\mathbb{R}_{\geq 0}$ for all $j \in J$, and by $(\mathbb{R}_{>0})^J$ the subset of $(\mathbb{R}_{\geq 0})^J$ consisting of $(a_j)_{j\in J}$ with $a_j > 0$ for all $j \in J$. Similarly, one can define the sets $(\mathbb{Z}_{\geq 0})^J$, $(\mathbb{Z}_{>0})^J$ and $(\mathbb{N}^*)^J$. Let σ be a rational polyhedral convex cone in $(\mathbb{R}_{>0})^J$ and let $\overline{\sigma}$ denote its closure in $(\mathbb{R}_{\geq 0})^J$ with J a finite set. Let ℓ and ℓ' be two integer linear forms on $\mathbb{Z}^J$ positive on $\overline{\sigma} \setminus \{(0,\dots,0)\}$. Then the series

$$S_{\sigma,\ell,\ell'}(T) := \sum_{a\in\sigma\cap(\mathbb{N}^*)^J} \mathbb{L}^{-\ell'(a)} T^{\ell(a)}$$

is in $\mathbb{Z}[\mathbb{L},\mathbb{L}^{-1}][\![T]\!]_{\mathrm{sr}}$ and $\lim_{T\to\infty} S_{\sigma,\ell,\ell'}(T) = \chi(\sigma)$, the Euler characteristic with compact supports of σ. If σ is relatively open, then $\lim_{T\to\infty} S_{\sigma,\ell,\ell'}(T) = (-1)^{\dim(\sigma)}$ (see [6, Lemma 2.1.5]). We have the following technique lemma.

Lemma 3. *Let σ be a relatively open rational polyhedral convex cone in $(\mathbb{R}_{>0})^I$. Let K and L be disjoint nonempty subsets of I. Consider half spaces H_j (with $j \in L$) in $\mathbb{R}^I$ defined by*

$$x_j \leq \sum_{i\in K} \alpha_i x_i,$$

where $\alpha_i \geq 0$ for all $i \in K$, such that for any disjoint subsets L_1, L_2 of L and any $j \in L \setminus (L_1 \cup L_2)$, the set

$$\sigma \cap \bigcap_{s\in L_1} \left\{ (x_i)_{i\in I} \in \mathbb{R}^I \;\middle|\; x_s < \sum_{i\in K} \alpha_i x_i \right\} \cap \bigcap_{t\in L_2} \left\{ (x_i)_{i\in I} \in \mathbb{R}^I \;\middle|\; x_t = \sum_{i\in K} \alpha_i x_i \right\}$$

either is empty or has nonempty intersection with $\mathbb{R}^I \setminus H_j$.

Then the Euler characteristic with compact supports of the set

$$\sigma_L := \sigma \cap \bigcap_{j\in L} H_j$$

is equal to zero.

Proof. We prove this lemma by induction on $|L|$. For $L = \{j\}$ a one-point set, we have

$$\sigma = \sigma_L \sqcup \left\{ (x_i)_{i\in I} \in \sigma \;\middle|\; x_j > \sum_{i\in K} \alpha_i x_i \right\}.$$

Since the second term on the right hand side and σ have the same Euler characteristic with compact supports $(-1)^{|I|}$, we get $\chi(\sigma_L) = 0$. For the case $|L| > 1$, let $j_0 \in L$ and $L' := L \setminus \{j_0\}$. Then σ_L is the disjoint union of the following two sets

$$\sigma_{L'} \cap \left\{ (x_i)_{i\in I} \in \mathbb{R}^I \;\middle|\; x_{j_0} < \sum_{i\in K} \alpha_i x_i \right\}$$

and

$$\sigma_{L'} \cap \left\{ (x_i)_{i\in I} \in \mathbb{R}^I \,\middle|\, x_{j_0} = \sum_{i\in K} \alpha_i x_i \right\}.$$

By induction, the lemma holds true for L', thus the Euler characteristic with compact supports of these two sets is zero. □

2.3. *Motivic nearby cycles of regular functions*

For any $\mathbb{C}$-variety X, let $\mathscr{L}_n(X)$ be the space of n-jets on X, and $\mathscr{L}(X)$ the arc space on X, which is the limit of the projective system of spaces $\mathscr{L}_n(X)$ and canonical morphisms $\mathscr{L}_m(X) \to \mathscr{L}_n(X)$ for $m \geq n$. The group $\hat{\mu}$ acts on $\mathscr{L}_n(X)$ via μ_n in such a natural way that $\xi \cdot \varphi(t) = \varphi(\xi t)$ for $\xi \in \mu_n$.

From now on, we assume that the $\mathbb{C}$-variety X is smooth and of pure dimension d. Consider a regular function $f : X \to \mathbb{A}^1_{\mathbb{C}}$, with the zero locus X_0. For $n \geq 1$ one defines the *n-iterated contact locus* of f as follows

$$\mathscr{X}_n(f) = \left\{ \varphi \in \mathscr{L}_n(X) \,\middle|\, f(\varphi) = t^n \bmod t^{n+1} \right\}.$$

Clearly, this variety is invariant by the $\hat{\mu}$-action on $\mathscr{L}_n(X)$ and admits a morphism to X_0 given by $\varphi(t) \mapsto \varphi(0)$, which defines an element $[\mathscr{X}_n(f)] := [\mathscr{X}_n(f) \to X_0]$ in $\mathscr{M}_{X_0}^{\hat{\mu}}$. We consider Denef–Loeser's motivic zeta function $Z_f(T) = \sum_{n\geq 1}[\mathscr{X}_n(f)]\mathbb{L}^{-nd}T^n$. They prove in [5] that $Z_f(T)$ is in $\mathscr{M}_{X_0}^{\hat{\mu}}[\![T]\!]_{\mathrm{sr}}$, and call the limit $\mathscr{S}_f := -\lim_{T\to\infty} Z_f(T)$ in $\mathscr{M}_{X_0}^{\hat{\mu}}$ the *motivic nearby cycles* of f. If x is a closed point of X_0, the $\mathbb{C}$-variety

$$\mathscr{X}_{n,x}(f) = \left\{ \varphi \in \mathscr{L}_n(X) \,\middle|\, f(\varphi) = t^n \bmod t^{n+1}, \varphi(0) = x \right\},$$

is also invariant by the $\hat{\mu}$-action on $\mathscr{L}_n(X)$, called the *n-iterated contact locus* of f at x. It is also proved that the zeta function $Z_{f,x}(T) = \sum_{n\geq 1}[\mathscr{X}_{n,x}(f)]\mathbb{L}^{-nd}T^n$ is in $\mathscr{M}_{\mathbb{C}}^{\hat{\mu}}[\![T]\!]_{\mathrm{sr}}$. The limit $\mathscr{S}_{f,x} = -\lim_{T\to\infty} Z_{f,x}(T)$ is called the *motivic Milnor fiber* of f at x. Obviously, if ι is the inclusion of $\{x\}$ in X_0, then $\mathscr{S}_{f,x} = \iota^*\mathscr{S}_f$ in $\mathscr{M}_{\mathbb{C}}^{\hat{\mu}}$.

We now modify slightly the motivic zeta functions of several functions in [6] and [7]. For a pair of regular functions (f,g) on X, we denote by $X_0 := X_0(f,g)$ their common zero locus. For $n \geq m$ in $\mathbb{N}^*$, we define

$$\mathscr{X}_{n,m}(f,g) := \left\{ \varphi \in \mathscr{L}_n(X) \,\middle|\, f(\varphi) = t^n \bmod t^{n+1}, \mathrm{ord}_t\, g(\varphi) = m \right\}.$$

We can check that $\mathscr{X}_{n,m}(f,g)$ is invariant under the natural μ_n-action on $\mathscr{L}_n(X)$, and that there is an obvious morphism of $\mathbb{C}$-varieties $\mathscr{X}_{n,m}(f,g) \to X_0$; from which we obtain the class $[\mathscr{X}_{n,m}(f,g)]$ of that morphism in $\mathscr{M}_{X_0}^{\hat{\mu}}$. Consider the series

$$Z_{f,g}^{\Delta}(T) := \sum_{n\geq m\geq 1} \left[\mathscr{X}_{n,m}(f,g)\right]\mathbb{L}^{-nd}T^n$$

in $\mathscr{M}_{X_0}^{\hat{\mu}}[\![T]\!]$. For any closed point $x \in X_0$, we can define $Z_{f,g,x}^{\Delta}(T)$ in $\mathscr{M}_{\mathbb{C}}^{\hat{\mu}}[\![T]\!]$ as above with $\mathscr{X}_{n,m}(f,g)$ replaced by its μ_n-invariant subvariety $\mathscr{X}_{n,m,x}(f,g) := \{\varphi \in \mathscr{X}_{n,m}(f,g) \mid \varphi(0) = x\}$. The rationality of the series $Z_{f,g}^{\Delta}(T)$ and $Z_{f,g,x}^{\Delta}(T)$ are stated in [6, Théorème 4.1.2] and [7, Section 2.9], up to the isomorphism of rings $\mathscr{M}_{X_0}^{\hat{\mu}} \cong \mathscr{M}_{X_0\times\mathbb{G}_m}^{\mathbb{G}_m}$ (see [8, Proposition 2.6]), where Guibert–Loeser–Merle's result is done in the framework $\mathscr{M}_{X_0\times\mathbb{G}_m}^{\mathbb{G}_m}$. Put $\mathscr{S}_{f,g}^{\Delta} := -\lim_{T\to\infty} Z_{f,g}^{\Delta}(T)$ and $\mathscr{S}_{f,g,x}^{\Delta} := -\lim_{T\to\infty} Z_{f,g,x}^{\Delta}(T)$.

3. Motivic nearby cycles of a nondegenerate polynomial and applications

3.1. *Newton polyhedron of a polynomial*

Recall that $[d]$ stands for $\{1,\dots,d\}$, $d\in\mathbb{N}^*$. Let $x=(x_1,\dots,x_d)$ be a set of d variables, and let $f(x)=\sum_{\alpha\in\mathbb{N}^d}c_\alpha x^\alpha$ be in $\mathbb{C}[x]$ with $f(O)=0$, with O the origin of $\mathbb{C}^d$. Let Γ be the Newton polyhedron of f, i.e., the convex hull of the set $\bigcup_{c_\alpha\neq 0}(\alpha+(\mathbb{R}_{\geq 0})^d)$ in $(\mathbb{R}_{\geq 0})^d$. For every face γ of Γ (not necessarily compact, the case $\gamma=\Gamma$ included), define by $f_\gamma(x)=\sum_{\alpha\in\gamma}c_\alpha x^\alpha$ the *face function* of f with respect to γ.

Note that for every face γ of Γ (including Γ itself), there exists a unique set $J_\gamma\subseteq[d]$ such that γ is contained in the hyperplanes $x_j=0$ for all $j\notin J_\gamma$ and not contained in other coordinate hyperplanes.

Definition 4. *The polynomial f is called* nondegenerate on a face γ *of* Γ *if the hypersurface $f_\gamma^{-1}(0)$ has no singular point in $\mathbb{G}_{m,\mathbb{C}}^{J_\gamma}$. We say that f is* nondegenerate in the sense of Kouchnirenko *if it is nondegenerate on every compact face γ. If f is nondegenerate on every face of Γ (including non-compact faces, and Γ itself), we say that f is* nondegenerate in the sense of Newton polyhedron *or simply* Newton nondegenerate.

Consider the function $\ell=\ell_\Gamma:(\mathbb{R}_{\geq 0})^d\to\mathbb{R}$ which sends a in $(\mathbb{R}_{\geq 0})^d$ to $\min_{b\in\Gamma}\langle a,b\rangle$, where $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^d$. For a in $(\mathbb{R}_{\geq 0})^d$, we denote by γ_a the maximal face of Γ to which the restriction of the function $\langle a,\cdot\rangle$ gets its minimum. Note that γ_a is a compact face if and only if a is in $(\mathbb{R}_{>0})^d$ (cf. [4, Property 2.3]). This comes from the fact that $\gamma_a=\{b\in\Gamma\mid\langle a,b\rangle=\ell(a)\}$. Moreover, $\gamma_a=\Gamma$ when $a=(0,\dots,0)$ in $\mathbb{R}^d$, and γ_a is a proper face of Γ otherwise. For every proper face γ of Γ, we define

$$\sigma_\gamma:=\sigma_{[d],\gamma}:=\{a\in(\mathbb{R}_{\geq 0})^d \,\big|\, \gamma=\gamma_a\}.$$

It is clear that σ_γ is a rational polyhedral convex cone of dimension $d-\dim(\gamma)$.

For any $J\subseteq[d]$, denote by f^J the polynomial in $\mathbb{C}[(x_j)_{j\in J}]$ obtained from $f(x)$ substituting x_i by 0 for all $i\in[d]\setminus J$. If f is nondegenerate in the sense of Kouchnirenko (resp. Newton nondegenerate) then f^J is also nondegenerate in the sense of Kouchnirenko (resp. Newton polyhedron).

Let ℓ_J stand for $\ell_{\Gamma(f^J)}$. For $a\in(\mathbb{R}_{\geq 0})^J$, we define the face γ_a^J similarly as above, i.e.

$$\gamma_a^J:=\{b\in\Gamma(f^J)\,\big|\,\langle a,b\rangle=\ell_J(a)\}.$$

If γ is a face of the Newton polyhedron $\Gamma(f^J)$, denote by $\sigma_{J,\gamma}$ the cone $\{a\in(\mathbb{R}_{\geq 0})^J\mid\gamma=\gamma_a^J\}$ and by $\mathring{\sigma}_{J,\gamma}$ the relative interior of $\sigma_{J,\gamma}$, both of which have dimension $|J|-\dim(\gamma)$. The following lemma is trivial to prove.

Lemma 5. *There exists a canonical partition of $(\mathbb{R}_{\geq 0})^J$ into rational polyhedral convex cones $\mathring{\sigma}_{J,\gamma}$ with γ being all faces of $\Gamma(f^J)$.*

A face γ of Γ is called a *coordinate face* if $\gamma=\Gamma(f^J)$ for some $J\subseteq[d]$. We have the following description for the coordinate faces.

Lemma 6. *A face γ of Γ is a coordinate face if and only if for all $J\supseteq J_\gamma$, the restriction of ℓ_J to $\mathring{\sigma}_{J,\gamma}$ is the zero function.*

Proof. For each $j\in J$, we denote by e^j the vector $(0,\dots,0,1,0,\dots,0)\in\mathbb{R}^J$ with 1 in the j-th coordinate. Assume that $\gamma=\Gamma(f^{J_0})$ for some $J_0\subseteq[d]$. Then, for any $i\in J_0$, $t\geq 0$ and for any $b\in\gamma$, the point $b+te^i$ is also in γ. For $J\supseteq J_0=J_\gamma$ and $a=(a_j)_{j\in J}\in\mathring{\sigma}_{J,\gamma}$, we have $\ell_J(a)=\langle a,b+te^i\rangle$ for every $i\in J_0$ and $t\geq 0$. As a consequence, we get $a_j=0$ for all $j\in J_0$. Hence $\ell_J(a)=\langle a,b\rangle=0$ for any $b\in\gamma=\Gamma(f^{J_0})$.

Now, we assume that restriction of ℓ_J to $\mathring{\sigma}_{J,\gamma}$ is the zero function for some $J \supseteq J_\gamma$. Take $a = (a_j)_{j\in J} \in \mathring{\sigma}_{J,\gamma}$, we have $\langle a, b\rangle = \ell_J(a) = 0$ for any $b \in \gamma$. Since γ is not contained in any hyperplane $x_k = 0$ for any $k \in J_\gamma$, we have $a_j = 0$ for all $j \in J_\gamma$. This together with the description of γ, namely,

$$\gamma = \{b \in \Gamma(f^J) \mid \langle a, b\rangle = 0\},$$

implies that $\Gamma(f^{J_\gamma}) \subseteq \gamma$. Therefore $\gamma = \Gamma(f^{J_\gamma})$. □

Notation 7. In the rest of this article, let F (resp. K) denote the set of all the faces (resp. the compact faces) of Γ, and let $\widetilde{F}$ denote the set of all the coordinate faces of Γ.

3.2. *Contact loci*

Let $(x_1,\dots,x_d)$ be the standard coordinates of $\mathbb{A}^d_{\mathbb{C}}$, and let $f(x_1,\dots,x_d)$ be as above. For $n \in \mathbb{N}^*$, $k \in \mathbb{N}$ and $J \subseteq [d]$, denote by $\Delta_J^{(n,k)}$ (resp. $\widetilde{\Delta}_J^{(n,k)}$) the set of $a \in \{0,\dots,n\}^J$ (resp. $a \in [n]^J$) such that $\ell_J(a) + k = n$. Clearly, $\widetilde{\Delta}_J^{(n,k)} \subseteq \Delta_J^{(n,k)}$.

For $a \in \Delta_J^{(n,k)}$, put

$$\mathscr{X}_{J,a}^{(n)} := \{\varphi \in \mathscr{X}_n(f) \mid \mathrm{ord}_t\, x_j(\varphi) = a_j \ \ \forall\, j \in J,\ x_i(\varphi) \equiv 0 \ \ \forall\, i \notin J\}.$$

This subvariety of $\mathscr{X}_n(f)$ is invariant by the μ_n-action given by $\xi \cdot \varphi(t) = \varphi(\xi t)$, and it defines an element $[\mathscr{X}_{J,a}^{(n)}] := [\mathscr{X}_{J,a}^{(n)} \to X_0]$ in $K_0^{\hat{\mu}}(\mathrm{Var}_{X_0})$, where the structure map is given by $\varphi \mapsto \varphi(0)$. Let $\mathscr{P}_n$ and $\widetilde{\mathscr{P}}_n$ be the index sets consisting of all such pairs (J,a) such that

$$\mathscr{X}_n(f) = \bigsqcup_{(J,a)\in\mathscr{P}_n} \mathscr{X}_{J,a}^{(n)} \tag{1}$$

and

$$\mathscr{X}_{n,O}(f) = \bigsqcup_{(J,a)\in\widetilde{\mathscr{P}}_n} \mathscr{X}_{J,a}^{(n)}. \tag{2}$$

Lemma 8.

(i) *$\mathscr{P}_n$ is the set of all the pairs (J,a) such that $J \supseteq J_\gamma$, $a \in \bigsqcup_{k\in\mathbb{N}} \big(\mathring{\sigma}_{J,\gamma} \cap \Delta_J^{(n,k)}\big)$ and $\gamma \in F$.*

(ii) *$\widetilde{\mathscr{P}}_n$ is the set of all the pairs (J,a) such that $J \supseteq J_\gamma$, $a \in \bigsqcup_{k\in\mathbb{N}} \big(\mathring{\sigma}_{J,\gamma} \cap \widetilde{\Delta}_J^{(n,k)}\big)$ and $\gamma \in K$.*

Proof. For any φ in $\mathscr{X}_n(f)$, there exists a unique subset J of $\{1,\dots,d\}$ such that $x_i(\varphi) \equiv 0$ for all $i \notin J$ and that $x_j(\varphi) \not\equiv 0$ for all $j \in J$. Put $a := (\mathrm{ord}_t\, x_j(\varphi))_{j\in J} \in \{0,\dots,n\}^J$, and put $\gamma := \gamma_a^J$. Then we have $J_\gamma \subseteq J$ and

$$f(\varphi) = f_\gamma(\widetilde{\varphi}(0))\, t^{\ell_J(a)} + \text{higher terms},$$

where $\widetilde{\varphi} := (t^{-a_j} x_j(\varphi))_{j\in J}$, thus $\ell_J(a) \le n$ and $\varphi \in \mathscr{X}_{J,a}^{(n)}$. The proof for (i) is completed by using Lemma 5. Similar arguments work for (ii). □

For $\gamma \in \widetilde{F}$, if $a \in \mathring{\sigma}_{J,\gamma} \cap \Delta_J^{(n,k)}$ and $k \neq n$, then $\mathscr{X}_{J,a}^{(n)} = \emptyset$.

For every face $\gamma \in F$ of $\Gamma(f^J)$, let us consider the $\mathbb{C}$-varieties

$$X_{J,\gamma}(1) := \{x \in \mathbb{G}_{m,\mathbb{C}}^J \mid f_\gamma(x) = 1\}, \quad X_{J,\gamma}(0) := \{x \in \mathbb{G}_{m,\mathbb{C}}^J \mid f_\gamma(x) = 0\}.$$

When $J = J_\gamma$ we write simply $X_\gamma(\varepsilon)$ instead of $X_{J_\gamma,\gamma}(\varepsilon)$, for $\varepsilon = 0, 1$. We always consider the trivial action of $\hat{\mu}$ on the variety $X_{J,\gamma}(0)$. Let a be in $\mathring{\sigma}_{J,\gamma}$. Then the variety $X_{J,\gamma}(1)$ admits a natural $\mu_{\ell_J(a)}$-action as follows

$$e^{2\pi i r/\ell_J(a)} \cdot (x_j)_{j\in J} := \big(e^{2\pi i r a_j/\ell_J(a)} x_j\big)_{j\in J}, \tag{3}$$

for $r \in [\ell_J(a)]$. Note that the class $[X_{J,\gamma}(1)]$ in $\mathscr{M}_{\mathbb{C}}^{\mu_n}$ does not depend on a provided a is in $\mathring{\sigma}_{J,\gamma}$ and $\ell_J(a) = n$, which follows from the construction of the Grothendieck ring (see [13, Proposition 3.13]).

The result and proof ideas of the following theorem are well known due to [1, 6, 7]. In the present article, we are going to contribute a detailed explanation for every step of proof. Denote $|a| := \sum_{j\in J} a_j$ for $a = (a_j)_{j\in J} \in \mathbb{R}^J$.

Theorem 9 (cf. [1, 6, 7]). *Let $f \in \mathbb{C}[x_1,\dots,x_d]$ such that $f(O) = 0$. Assume that f is nondegenerate on a face $\gamma \in F$. Let $J \subseteq [d]$ containing J_γ. If $a \in \mathring{\sigma}_{J,\gamma} \cap \Delta_J^{(n,0)}$ and $\mathscr{X}_{J,a}^{(n)}$ is nonempty, then there is a naturally μ_n-equivariant isomorphism of $\mathbb{C}$-varieties*

$$\tau : \mathscr{X}_{J,a}^{(n)} \to X_{J,\gamma}(1) \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^{|J|\ell_J(a)-|a|}.$$

If $k \in \mathbb{N}^*$, $a \in \mathring{\sigma}_{J,\gamma} \cap \Delta_J^{(n,k)}$, and if $\mathscr{X}_{J,a}^{(n)}$ is nonempty, there is a Zariski locally trivial fibration

$$\pi : \mathscr{X}_{J,a}^{(n)} \to X_{J,\gamma}(0)$$

with fiber $\mathbb{A}_{\mathbb{C}}^{|J|(\ell_J(a)+k)-|a|-k}$.

Proof. It suffices to prove the theorem for $J = [d]$. Let $a = (a_1,\dots,a_d)$ be in $\mathring{\sigma}_\gamma \cap \Delta_{[d]}^{(n,0)}$, hence $n = \ell(a)$ and $\gamma = \gamma_a$. Every element φ in $\mathscr{X}_{[d],a}^{(n)}$ has the form

$$\left(\sum_{j=a_1}^{\ell(a)} b_{1j}t^j,\dots,\sum_{j=a_d}^{\ell(a)} b_{dj}t^j\right)$$

with $b_{ia_i} \neq 0$ for $1 \le i \le d$. The coefficient of $t^{\ell(a)}$ in $f(\varphi(t))$ is nothing but $f_{\gamma_a}(b_{1a_1},\dots,b_{da_d})$, thus $(b_{1a_1},\dots,b_{da_d})$ is in $X_{[d],\gamma_a}(1)$. We deduce that $\mathscr{X}_{[d],a}^{(\ell(a))}$ is $\mu_{\ell(a)}$-equivariant isomorphic to $X_{[d],\gamma_a}(1) \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^{d\ell(a)-|a|}$ (where $\mu_{\ell(a)}$ acts trivially on $\mathbb{A}_{\mathbb{C}}^{d\ell(a)-|a|}$) via the map

$$\tau : \varphi(t) \mapsto \big((b_{ia_i})_{1\le i\le d}, (b_{ij})_{1\le i\le d, a_i<j\le \ell(a)}\big).$$

Indeed, for every ξ in $\mu_{\ell(a)}$, the element $\varphi(\xi t)$ is sent to

$$\big((\xi^{a_i} b_{ia_i})_{1\le i\le d}, (b_{ij})_{1\le i\le d, a_i<j\le \ell(a)}\big) = \xi\cdot\big((b_{ia_i})_{1\le i\le d}, (b_{ij})_{1\le i\le d, a_i<j\le \ell(a)}\big).$$

Thus τ is a $\mu_{\ell(a)}$-equivariant isomorphism.

Now we prove the second statement. Let a be in $\mathring{\sigma}_\gamma \cap \Delta_{[d]}^{(n,k)}$ for $k \in \mathbb{N}^*$, hence $n = \ell(a) + k$ and $\gamma = \gamma_a$. For φ in $\mathscr{X}_{[d],a}^{(n)}$, putting

$$\widetilde{\varphi} := \big(t^{-a_1}x_1(\varphi),\dots,t^{-a_d}x_d(\varphi)\big), \tag{4}$$

we get

$$f(\varphi) = t^{\ell(a)} f_{\gamma_a}(\widetilde{\varphi}) + \sum_{k\ge 1} t^{\ell(a)+k} \sum_{\langle\alpha,a\rangle=\ell(a)+k} c_\alpha \widetilde{\varphi}^\alpha. \tag{5}$$

Defining

$$\widetilde{f}(\widetilde{\varphi}, t) := f_{\gamma_a}(\widetilde{\varphi}) + \sum_{k\ge 1} t^k \sum_{\langle\alpha,a\rangle=\ell(a)+k} c_\alpha \widetilde{\varphi}^\alpha,$$

we obtain a function

$$\widetilde{f} : \mathscr{L}_{\ell(a)+k+1-a_1}(\mathbb{A}_{\mathbb{C}}^1) \times_{\mathbb{C}} \cdots \times_{\mathbb{C}} \mathscr{L}_{\ell(a)+k+1-a_d}(\mathbb{A}_{\mathbb{C}}^1) \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^1 \to \mathbb{A}_{\mathbb{C}}^1$$

given by

$$\widetilde{f}(\widetilde{\varphi}, t_0) := \widetilde{f}(\widetilde{\varphi}(t_0), t_0).$$

It thus follows from (5) that φ is in $\mathscr{X}_{[d],a}^{(\ell(a)+k)}$ if and only if $\widetilde{f}(\widetilde{\varphi}, t) = t^k \mod t^{k+1}$. Putting $\widetilde{\varphi}_i(t) = \sum_{j=0}^{\ell(a)-a_i+k} b_{ij}t^j$ for $1 \le i \le d$, the latter means that

$$\begin{cases} f_{\gamma_a}(b_{10},\dots,b_{d0}) = 0 & \text{with } b_{i0} \neq 0 \text{ for } 1 \le i \le d,\\ q_j(b_{1j},\dots,b_{dj}) + p_j((b_{i'j'})_{i',j'}) = 0 & \text{for } 1 \le j \le k-1,\\ q_k(b_{1k},\dots,b_{dk}) + p_k((b_{i'j'})_{i',j'}) = 1, & \end{cases}$$

where p_j, for $1 \le j \le k$, are polynomials in variables $b_{i'j'}$ with $i' \le d$ and $j' < j$, and

$$q_j(b_{1j},\dots,b_{dj}) = \sum_{i=1}^{d} \frac{\partial f_{\gamma_a}}{\partial x_i}(b_{10},\dots,b_{d0}) b_{ij}.$$

We consider the morphism

$$\pi : \mathscr{X}_{[d],a}^{(\ell(a)+k)} \to X_{[d],\gamma_a}(0)$$

which sends the φ described previously to $(b_{10},\dots,b_{d0})$. Since $\widehat{\mu}$ acts trivially on $X_{[d],\gamma_a}(0)$, we only need to prove that π is a locally trivial fibration with fiber $\mathbb{A}_{\mathbb{C}}^{d(\ell(a)+k)-|a|-k}$. For every $1 \le i \le d$, we put

$$U_i := \left\{ (x_1,\dots,x_d) \in X_{[d],\gamma_a}(0) \,\middle|\, \frac{\partial f_{\gamma_a}}{\partial x_i}(x_1,\dots,x_d) \neq 0 \right\}. \tag{6}$$

The nondegeneracy of f on the face $\gamma = \gamma_a$ gives us an open covering $\{U_1,\dots,U_d\}$ of $X_{[d],\gamma}(0)$. We construct trivializations of π as follows

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\Phi_{U_i}"] \arrow[dr, "\pi"'] & & U_i \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^{e} \arrow[dl, "\mathrm{pr}_1"] \\ & U_i & \end{tikzcd}$$

where $e = \sum_{l=1}^{d}(\ell(a) - a_l + k) - k$ and we identify $\mathbb{A}_{\mathbb{C}}^{e}$ with the subvariety of $\mathbb{A}_{\mathbb{C}}^{\sum_{l=1}^{d}(\ell(a)-a_l+k)}$ defined by the equations $\widetilde{b}_{ij} = 0$ for $1 \le j \le k-1$ and $\widetilde{b}_{ik} = 1$ in the coordinate system $(\widetilde{b}_{lj})$, and for φ as previous,

$$\Phi_{U_i}(\varphi) = \left((\widetilde{b}_{10},\dots,\widetilde{b}_{d0}), (\widetilde{b}_{lj})_{1\le l\le d, 1\le j\le \ell(a)-a_l+k}\right),$$

with $\widetilde{b}_{ij} = 0$ if $1 \le j \le k-1$, $\widetilde{b}_{ik} = 1$, and $\widetilde{b}_{lj} = b_{lj}$ otherwise. Furthermore, the inverse map $\Phi_{U_i}^{-1}$ of Φ_{U_i} is also a regular morphism given explicitly as follows

$$\Phi_{U_i}^{-1}(\widetilde{b}_{lj}) = \left(\sum_{j=0}^{\ell(a)-a_1+k} b_{1j} t^{j+a_1}, \dots, \sum_{j=0}^{\ell(a)-a_d+k} b_{dj} t^{j+a_d} \right),$$

where $b_{lj} = \widetilde{b}_{lj}$ for either that $l \neq i$ or that $l = i$ and $k < j \le \ell(a) - a_l + k$, and

$$b_{ij} = \frac{-p_j((b_{lj'})_{l\le d, j'<j}) - \sum_{l\le d, l\neq i}(\partial f_{\gamma_a}/\partial x_l)(\widetilde{b}_{10},\dots,\widetilde{b}_{d0})\widetilde{b}_{lj}}{(\partial f_{\gamma_a}/\partial x_i)(\widetilde{b}_{10},\dots,\widetilde{b}_{d0})},$$

for $1 \le j \le k-1$, and

$$b_{ik} = \frac{1 - p_k((b_{lj'})_{l\le d, j'<k}) - \sum_{l\le d, l\neq i}(\partial f_{\gamma_a}/\partial x_l)(\widetilde{b}_{10},\dots,\widetilde{b}_{d0})\widetilde{b}_{lk}}{(\partial f_{\gamma_a}/\partial x_i)(\widetilde{b}_{10},\dots,\widetilde{b}_{d0})}.$$

This proves that π is a (Zariski) locally trivial fibration with fiber $\mathbb{A}_{\mathbb{C}}^{e}$. □

3.3. *Motivic nearby cycles*

For every $\gamma = \gamma_a^J \in F$ with $a \in \mathring{\sigma}_{J,\gamma} \cap \Delta_J^{(n,0)}$, we consider the morphism $\Phi_a : X_{J,\gamma}(1) \to X_0$ which sends $(x_i)_{i\in J}$ to $(\widehat{x}_1,\dots,\widehat{x}_d)$, where $\widehat{x}_i = 0$ if either $i \in [d] \setminus J$ or $a_i \ge 1$, and $\widehat{x}_i = x_i$ if $a_i = 0$. With this morphism it follows from Theorem 9 the below commutative diagram

$$\begin{tikzcd} \mathscr{X}_{J,a}^{(n)} \arrow[rr, "\cong"] \arrow[dr, "\varphi(t)\mapsto\varphi(0)"'] & & X_{J,\gamma}(1) \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^{|J|\ell_J(a)-|a|} \arrow[dl, "\Phi_a\circ\mathrm{pr}_1"] \\ & X_0 & \end{tikzcd} \tag{7}$$

Lemma 10. *If $a, b \in \mathring{\sigma}_{J,\gamma} \cap \Delta_J^{(n,0)}$, then $[\Phi_a : X_{J,\gamma}(1) \to X_0] = [\Phi_b : X_{J,\gamma}(1) \to X_0]$ in $\mathscr{M}_{X_0}^{\mu_n}$.*

Proof. Suggested from [13, Section 3.4.3], we stratify $(\mathbb{R}_{\geq 0})^J$ into the cones

$$C_\delta := \left\{(k_i)_i \in (\mathbb{R}_{\geq 0})^J \,\middle|\, k_i > 0 \text{ if } \delta_i = 1, k_i = 0 \text{ if } \delta_i = 0\right\}$$

with $\delta \in \{0,1\}^J$. We have a stratification of $\sigma_{J,\gamma} \cap \Delta_J^{(n,0)}$ into the strata $C_\delta \cap \sigma_{J,\gamma} \cap \Delta_J^{(n,0)}$. It is a fact that $\Phi_a = \Phi_b$ if and only if a, b belong the same stratum. Now, to deduce the lemma, we use the same arguments as in the proof of [13, Proposition 3.13]. □

In particular, when $\gamma_a^J = \Gamma$ we have $a = (0, \dots, 0)$, $J = [d]$, and $f_{\gamma_a}(x) = f(x)$. Then the morphism Φ_a is nothing but the identity morphism.

Similarly, we also consider the morphism $\Psi_a : X_{J,\gamma}(0) \to X_0$ sending $(x_i)_{i\in J}$ to $(\widehat{x}_1, \dots, \widehat{x}_d)$, which commutes with π in Theorem 9 and the morphism $\varphi(t) \mapsto \varphi(0)$ for $a \in \mathring{\sigma}_{J,\gamma} \cap \Delta_J^{(n,k)}$ and $k \geq 1$. Recall that the μ_n-action on $X_{J,\gamma}(0)$ is trivial. As above, we also have

Lemma 11. *If $a, b \in \mathring{\sigma}_{J,\gamma} \cap \Delta_J^{(n,0)}$, then $[\Psi_a : X_{J,\gamma}(0) \to X_0] = [\Psi_b : X_{J,\gamma}(0) \to X_0]$ in $\mathscr{M}_{X_0}^{\mu_n}$.*

Notice that, for $J \subseteq [d]$ and $I \subseteq J$, we identify $(\mathbb{R}_{>0})^I$ with the set of $(x_j)_{j\in J}$ in $(\mathbb{R}_{\geq 0})^J$ such that $x_i > 0$ for all $i \in I$ and $x_j = 0$ for all $j \in J \setminus I$.

For $0 \leq m \leq d$, denote by $F(m)$ the set of $\gamma \in F$ such that for all $J \supseteq J_\gamma$ and $\gamma = \gamma_a^J$ we have $a_i \geq 1$ for all $i \in J \cap [m+1, d]$. Note that $F(0) = K$ and $F(d) = F$.

The last part of the following result is known in [1, 6, 7], we provide new formulas in i), ii) as follows.

Theorem 12. *Let f be in $\mathbb{C}[x_1, \dots, x_d]$ with $f(O) = 0$, let d_1, d_2 be in $\mathbb{N}$ with $d = d_1 + d_2$. For any $\gamma \in F$, put $\Lambda_\gamma := \{I \subseteq J_\gamma \mid \mathring{\sigma}_{J_\gamma,\gamma} \cap (\mathbb{R}_{>0})^I \neq \emptyset\}$ and*

$$\lambda_\gamma := \sum_{I\in\Lambda_\gamma} (-1)^{\dim(\mathring{\sigma}_{J_\gamma,\gamma}\cap(\mathbb{R}_{>0})^I)}.$$

(Hence, as $\gamma \in K$, $\lambda_\gamma = (-1)^{\dim(\mathring{\sigma}_{J_\gamma,\gamma})} = (-1)^{|J_\gamma|-\dim(\gamma)}$.) The below identities hold in $\mathscr{M}_{X_0}^{\hat\mu}$ for (i)*, in $\mathscr{M}_{\mathbb{A}_{\mathbb{C}}^{d_1}}^{\hat\mu}$ for* (ii)*, and in $\mathscr{M}_{\mathbb{C}}^{\hat\mu}$ for* (iii).

(i) *If f is Newton nondegenerate, then*

$$\mathscr{S}_f = -\sum_{\gamma\in F\setminus\widetilde{F}} \lambda_\gamma \left[X_\gamma(1) \to X_0\right] + \sum_{\gamma\in F} \lambda_\gamma \left[X_\gamma(0) \to X_0\right].$$

(ii) *If f is Newton nondegenerate and $\iota : \mathbb{A}_{\mathbb{C}}^{d_1} \equiv \mathbb{A}_{\mathbb{C}}^{d_1} \times_{\mathbb{C}} \{0\}^{d_2} \hookrightarrow X_0$ is an inclusion, then*

$$\iota^*\mathscr{S}_f = -\sum_{\gamma\in F(d_1)\setminus\widetilde{F}} \lambda_\gamma \left[X_\gamma(1) \times_{X_0} \mathbb{A}_{\mathbb{C}}^{d_1} \to \mathbb{A}_{\mathbb{C}}^{d_1}\right] + \sum_{\gamma\in F(d_1)} \lambda_\gamma \left[X_\gamma(0) \times_{X_0} \mathbb{A}_{\mathbb{C}}^{d_1} \to \mathbb{A}_{\mathbb{C}}^{d_1}\right].$$

(iii) *If f is nondegenerate in the sense of Kouchnirenko, then*

$$\mathscr{S}_{f,O} = \sum_{\gamma\in K} (-1)^{|J_\gamma|+1-\dim(\gamma)} \left([X_\gamma(1)] - [X_\gamma(0)]\right).$$

Proof. Notice that (i) is not a particular case of (ii) in general, but our proof method of (i) is similar to that of (ii); while (iii) is really a consequence of (ii) (when $d_1 = 0$); so it suffices to prove (ii). By the decomposition (1) and Lemma 8 (i), we have

$$\mathscr{X}_n(f) = \bigsqcup_{\gamma\in F} \bigsqcup_{J\supseteq J_\gamma} \bigsqcup_{k\in\mathbb{N}} \bigsqcup_{a\in\mathring{\sigma}_{J,\gamma}\cap\Delta_J^{(n,k)}} \mathscr{X}_{J,a}^{(n)}.$$

Take the fiber product on both sides with $\iota: \mathbb{A}^{d_1}_{\mathbb{C}} \hookrightarrow X_0$. If there is an $i \in J \cap [d_1+1,\dots,d]$ with $a_i = 0$ (i.e. $\gamma \notin F(d_1)$), then $\varphi_i(0)$ is in $\mathbb{G}_m$, thus the image of ι is disjoint with the image of $\mathscr{X}^{(n)}_{J,a}$ in X_0, so $\iota^*[\mathscr{X}^{(n)}_{J,a}] = 0$. It follows that

$$\iota^*[\mathscr{X}_n(f)] = \sum_{\gamma\in F(d_1)} \sum_{J\supseteq J_\gamma} \sum_{k\in\mathbb{N}} \sum_{a\in\mathring{\sigma}_{J,\gamma}\cap\Delta_J^{(n,k)}} \iota^*\big[\mathscr{X}^{(n)}_{J,a}\big].$$

Using Lemma 6, the diagram (7) for $X_{J,\gamma}(1) \to X_0$ and a similar one for $X_{J,\gamma}(0) \to X_0$ we have

$$\begin{aligned}&\sum_{n\geq 1} \iota^*\big[\mathscr{X}_n(f)\big]\mathbb{L}^{-nd}T^n \\ &\quad= \sum_{\gamma\in F(d_1)\setminus\widetilde{F}} \sum_{J\supseteq J_\gamma} \big[X_{J,\gamma}(1)\times_{X_0}\mathbb{A}^{d_1}_{\mathbb{C}} \to \mathbb{A}^{d_1}_{\mathbb{C}}\big] S^0_{J,\gamma}(T) + \sum_{\gamma\in F(d_1)} \sum_{J\supseteq J_\gamma} \big[X_{J,\gamma}(0)\times_{X_0}\mathbb{A}^{d_1}_{\mathbb{C}} \to \mathbb{A}^{d_1}_{\mathbb{C}}\big] S^>_{J,\gamma}(T),\end{aligned}$$

where

$$S^0_{J,\gamma}(T) = \sum_{\substack{a\in\mathring{\sigma}_{J,\gamma}\cap\mathbb{N}^J\\ a_j\leq \ell_J(a),\, j\in J}} \mathbb{L}^{(|J|-d)\ell_J(a)-|a|}\,T^{\ell_J(a)}$$

and

$$S^>_{J,\gamma}(T) = \sum_{k\geq 1} \sum_{\substack{a\in\mathring{\sigma}_{J,\gamma}\cap\mathbb{N}^J\\ a_j\leq \ell_J(a)+k,\, j\in J}} \mathbb{L}^{(|J|-d)(\ell_J(a)+k)-|a|-k}\,T^{\ell_J(a)+k}.$$

The conclusion then follows from Lemma 14. □

We need the following lemmas.

Lemma 13. *If $\gamma \in F$, $J \supsetneq J_\gamma$ and $I \subseteq J_\gamma$, then $\mathring{\sigma}_{J,\gamma} \cap (\mathbb{N}^*)^I = \emptyset$.*

Proof. We first claim that if $a \in \sigma_{J,\gamma}$ and $j \in J \setminus J_\gamma$ then for all $t \geq 0$ we have $a + te^j \in \sigma_{J,\gamma}$, where e^j is defined in the proof of Lemma 6. Indeed, for any $b \in \gamma$, we have

$$\langle a+te^j, b\rangle = \langle a,b\rangle \leq \langle a,c\rangle \leq \langle a+te^j, c\rangle \quad \text{for all } c\in\Gamma(f^J).$$

That implies $\langle a+te^j, b\rangle = \ell_J(a+te^j)$. Hence $a+te^j \in \sigma_{J,\gamma}$.

We assume by contradiction that there exists some point $a \in \mathring{\sigma}_{J,\gamma} \cap (\mathbb{N}^*)^I$. Take $m \in J \setminus J_\gamma$. One can write the cone $\sigma_{J,\gamma}$ as

$$\sigma_{J,\gamma} = \big\{\alpha \in \mathbb{R}^J \,\big|\, h_i(\alpha) = 0, k_j(\alpha) \geq 0, l_s(\alpha) > 0, i \in I_1, j \in I_2, s \in I_3\big\},$$

where h_i, k_j, l_s are linear forms on $\mathbb{R}^J$. Since $a \in \mathring{\sigma}_{J,\gamma}$ we get $k_j(a) > 0$ for all $j \in I_2$. Hence, for $t > 0$ small, we get $k_j(a - te^m) > 0$. Similarly, we get $l_s(a - te^m) > 0$ for $t > 0$ small and for all $s \in I_3$. By the above argument, for $t > 0$ we have $a + te^m \in \sigma_{J,\gamma}$, hence $h_i(a+te^m) = h_i(a) = 0$, so $h_i(a - te^m) = 0$, for every $i \in I_1$. It implies that $a - te^m \in \sigma_{J,\gamma}$ for $t > 0$ small. On the other hand, because $m \in J \setminus J_\gamma$, $I \subseteq J_\gamma$ and $a \in (\mathbb{N}^*)^I$, we have $a_m = 0$, so $a - te^m \notin \mathbb{R}^J_{\geq 0}$ for any $t > 0$. This is a contradiction, and the lemma is proved. □

Lemma 14. *Use the notation in Theorem 12 and its proof, and let $\gamma \in F$. If $J \supsetneq J_\gamma$, then*

$$\lim_{T\to\infty} S^0_{J,\gamma}(T) = \lim_{T\to\infty} S^>_{J,\gamma}(T) = 0.$$

If $J = J_\gamma$, then

$$\lim_{T\to\infty} S^0_{J_\gamma,\gamma}(T) = -\lim_{T\to\infty} S^>_{J_\gamma,\gamma}(T) = \lambda_\gamma.$$

Proof. Assume that $J \supsetneq J_\gamma$. Because

- $(\mathbb{R}_{\geq 0})^J = \bigsqcup_{I\subseteq J} (\mathbb{R}_{>0})^I$, where $(\mathbb{R}_{>0})^\emptyset = \{(0,\dots,0)\}$ by convention,
- $\ell_J(a) = \ell_{J_\gamma}(a)$ for $a \in \mathring{\sigma}_{J,\gamma} \cap \mathbb{N}^J$,
- if $I \subseteq J_\gamma$, then $\mathring{\sigma}_{J,\gamma} \cap (\mathbb{N}^*)^I = \emptyset$ (by Lemma 13),

we have $S^0_{J,\gamma}(T)=\sum_{I\subseteq J, I\not\subseteq J_\gamma} S^0_{J,I,\gamma}(T)$, where

$$S^0_{J,I,\gamma}(T) := \sum_{\substack{a\in\mathring{\sigma}_{J,\gamma}\cap(\mathbb{N}^*)^I\\ a_j\le \ell_{J_\gamma}(a), j\in J}} \mathbb{L}^{-|a|}(\mathbb{L}^{|J|-d}T)^{\ell_{J_\gamma}(a)} = \sum_{\substack{a\in\mathring{\sigma}_{J,\gamma}\cap(\mathbb{N}^*)^I\\ a_j\le \ell_{J_\gamma}(a), j\in I\setminus J_\gamma}} \mathbb{L}^{-|a|}(\mathbb{L}^{|J|-d}T)^{\ell_{J_\gamma}(a)}$$

(the last equality comes because the inequality $a_j\le \ell_{J_\gamma}(a)$ is automatic for every $j\in J_\gamma$). Denote by H_j the half space of $\mathbb{R}^I$ defined by $a_j\le\ell_{J_\gamma}(a)$. Then $\mathring{\sigma}_{J,\gamma}\cap(\mathbb{R}_{>0})^I\not\subseteq H_j$ for all $j\in I\setminus J_\gamma$, because if there exists an $a\in\sigma_{J,\gamma}\cap H_j$, then for $t>0$ large enough, we have $a+te^j\in\sigma_{J,\gamma}$ but $a+te^j\notin H_j$. This agrees with the hypothesis of Lemma 3. Hence, by Lemma 3, $\lim_{T\to\infty}S^0_{J,I,\gamma}(T)=0$ for any $I\subseteq J$ and $I\not\subseteq J_\gamma$. Hence $\lim_{T\to\infty}S^0_{J,\gamma}(T)=0$. Similarly, we have $\lim_{T\to\infty}S^>_{J,\gamma}(T)=0$.

For the rest statement, it follows from [6, Lemme 2.1.5] that

$$\lim_{T\to\infty} S^0_{J_\gamma,\gamma}(T)=\sum_{I\subseteq J_\gamma}\lim_{T\to\infty}\sum_{a\in\mathring{\sigma}_{J_\gamma,\gamma}\cap(\mathbb{N}^*)^I}\mathbb{L}^{-|a|}(\mathbb{L}^{|J|-d}T)^{\ell_{J_\gamma}(a)}=\lambda_\gamma,$$

and similarly, $\lim_{T\to\infty}S^>_{J_\gamma,\gamma}(T)=-\lambda_\gamma$. □

Remark 15. This result revisits Guibert's work in [6, Section 2.1] for Newton nondegenerate polynomials f in a more general setting. Indeed, in [6] Guibert requires f to have the form $\sum_{\alpha\in(\mathbb{N}^*)^d} a_\alpha x^\alpha$, while we do not. Recently, Bultot–Nicaise in [3, Theorems 7.3.2, 7.3.5] provide a new approach to the motivic zeta functions $Z_f(T)$ and $Z_{f,O}(T)$, for f being Newton nondegenerate, using log smooth models.

Example 16. Consider the function $f(x,y)=y^2-x^3$ on $\mathbb{A}^2_{\mathbb{C}}$, which is well known to be nondegenerate with respect to its Newton polyhedron Γ. If γ is either the face $[3,+\infty)\times\{0\}$ or the face $\{0\}\times[2,+\infty)$ of Γ, then γ is a coordinate face and $X_\gamma(0)=\emptyset$. If γ is the compact face $\{(3,0)\}$ (resp. $\{(0,2)\}$), it contributes $X_\gamma(1)=\mu_3$ and $X_\gamma(0)=\emptyset$ (resp. $X_\gamma(1)=\mu_2$ and $X_\gamma(0)=\emptyset$), as well as $\lambda_\gamma=-1$. If γ is the compact face connecting $(3,0)$ and $(0,2)$, it contributes $X_\gamma(1)=\{(x,y)\in\mathbb{G}^2_{m,\mathbb{C}}\mid y^2-x^3=1\}$ and $X_\gamma(0)=\{(x,y)\in\mathbb{G}^2_{m,\mathbb{C}}\mid y^2-x^3=0\}\cong\mathbb{G}_{m,\mathbb{C}}$, as well as $\lambda_\gamma=-1$. Finally, if $\gamma=\Gamma$, it is a coordinate face and contributes $X_\Gamma(0)\cong\mathbb{G}_{m,\mathbb{C}}$, as well as $\lambda_\Gamma=1$ (since $\mathring{\sigma}_{\{1,2\},\Gamma}=\{(0,0)\}$). By Theorem 12 (we skip arrows to X_0 for simplicity),

$$\begin{aligned}\mathscr{S}_f&=\big[\{(x,y)\in\mathbb{G}^2_{m,\mathbb{C}}\mid y^2-x^3=1\}\big]+[\mu_3]+[\mu_2]-(\mathbb{L}-1)+(\mathbb{L}-1)\\ &=\big[\{(x,y)\in\mathbb{A}^2_{\mathbb{C}}\mid y^2-x^3=1\}\big]\quad(\in\mathscr{M}^{\hat\mu}_{X_0}).\end{aligned}$$

This also agrees with Davison–Meinhardt's conjecture on motivic nearby fibers of weighted homogeneous polynomials mentioned in [12]. Also by Theorem 12 we have

$$\begin{aligned}\mathscr{S}_{f,O}&=\big[\{(x,y)\in\mathbb{G}^2_{m,\mathbb{C}}\mid y^2-x^3=1\}\big]+[\mu_3]+[\mu_2]-(\mathbb{L}-1)\\ &=\big[\{(x,y)\in\mathbb{A}^2_{\mathbb{C}}\mid y^2-x^3=1\}\big]-(\mathbb{L}-1)\quad(\in\mathscr{M}^{\hat\mu}_{\mathbb{C}}).\end{aligned}$$

3.4. *Relation between motivic nearby cycles of f and $f^{[d-1]}$*

Let w be a linear function on $\mathbb{C}^d$ generic to f. In [9, 10], Lê Dũng Tráng introduced the relative monodromy concerning (f,O) and w. We refer to [10, Theorem 2.4] for the following. Denote by B_ε the closed d-balls of radius ε about O, by D_η the closed disk of radius η about 0 in $\mathbb{C}$, and by $D^\times_\eta$ the punctured disk $D_\eta\setminus\{0\}$. Let Φ be the restriction of the map $(w,f):\mathbb{C}^d\to\mathbb{C}^2$ to $B_\varepsilon\cap(w,f)^{-1}(D^2_\eta)$. Lê proved that, for $0<\eta\ll\epsilon\ll1$, the map $\Phi^{-1}(D^2_\eta\setminus(D_\eta\times\{0\}))\to D^\times_\eta$ is a smooth fibration which is fiber isomorphic to the Milnor fibration of (f,O) with monodromy $M:\Phi^{-1}(D_\eta\times\{\eta\})\to\Phi^{-1}(D_\eta\times\{\eta\})$, and that $w^{-1}(0)\cap\Phi^{-1}(D^2_\eta\setminus(D_\eta\times\{0\}))\to D^\times_\eta$ is also a smooth fibration which is fiber isomorphic to the Milnor fibration of $(f|_{w=0},O)$. Then M induces the monodromy of the Milnor fibration of $(f|_{w=0},O)$ and it lifts a diffeomorphism (which is a carousel) $D_\eta\times\{\eta\}\to D_\eta\times\{\eta\}$ along the mapping $\Phi|_{\Phi^{-1}(D_\eta\times\{\eta\})}$.

We now consider the "non-generic" hyperplane $x_d = 0$. We write $\widetilde{f}$ for $f^{[d-1]}$, that is,

$$\widetilde{f}(x_1,\ldots,x_{d-1}) = f(x_1,\ldots,x_{d-1},0),$$

and write $\widetilde{O}$ for the origin of $\mathbb{C}^{d-1}$. Let $\widetilde{X}_0$ be the zero locus of $\widetilde{f}$, which may be included into X_0. The following theorem may be partially considered as a motivic analogue of the work mentioned above. Using realizations, it would be interesting to compare the motivic result to the topological result of Lê Dũng Tráng.

Theorem 17. *Let $f \in \mathbb{C}[x_1,\ldots,x_d]$, and let $d_1, d_2 \in \mathbb{N}$ such that $d = d_1 + d_2$. The below identities hold in $\mathscr{M}^{\hat{\mu}}_{\mathbb{A}^{d_1}_{\mathbb{C}}}$ for* (i) *and in $\mathscr{M}^{\hat{\mu}}_{\mathbb{C}}$ for* (ii)*.*

(i) *Suppose that f is Newton nondegenerate, $\widetilde{X}_0 \subseteq X_0$ and that $\mathbb{A}^{d_1}_{\mathbb{C}}$ is embedded into $\widetilde{X}_0$ with the inclusions of $\mathbb{A}^{d_1}_{\mathbb{C}}$ in both X_0 and $\widetilde{X}_0$ denoted by the same symbol ι. Then*

$$\iota^*\mathscr{S}_f = \iota^*\mathscr{S}_{\widetilde{f}} + \iota^*\mathscr{S}^{\Delta}_{f,x_d}.$$

(ii) *If f is nondegenerate in the sense of Kouchnirenko and $f(O) = 0$, then*

$$\mathscr{S}_{f,O} = \mathscr{S}_{\widetilde{f},\widetilde{O}} + \mathscr{S}^{\Delta}_{f,x_d,O}.$$

Proof. It suffices to prove (i). By the definition of (n,m)-iterated contact loci, we have

$$\mathscr{X}_{n,m}(f,x_d) = \bigsqcup_{(J,a)\in\mathscr{P}_n.a_d=m} \mathscr{X}^{(n)}_{J_\gamma,a},$$

We deduce from Section 2.3 and the method in the proof of Theorem 12 that

$$\begin{aligned}\sum_{n\geq m\geq 1} \iota^*[\mathscr{X}_{n,m}(f,x_d)]\mathbb{L}^{-nd}T^n \\ &= \sum_{\gamma\in F(d_1)}\sum_{J\supseteq J_\gamma}\sum_{\substack{a\in\mathring{\sigma}_{J,\gamma}\cap\mathbb{N}^J\\ a_j\leq \ell_J(a),j\in J\\ 1\leq a_d\leq \ell_J(a)}} \iota^*\big[\mathscr{X}^{(\ell_J(a))}_{J,a}\big]\mathbb{L}^{-d\ell_J(a)}T^{\ell_J(a)} \\ &\quad + \sum_{\gamma\in F(d_1)}\sum_{J\supseteq J_\gamma}\sum_{k\geq 1}\sum_{\substack{a\in\mathring{\sigma}_{J,\gamma}\cap\mathbb{N}^J\\ a_j\leq \ell_J(a),j\in J\\ 1\leq a_d\leq \ell_J(a)+k}} \iota^*\big[\mathscr{X}^{(\ell_J(a)+k)}_{J,a}\big]\mathbb{L}^{-d(\ell_J(a)+k)}T^{\ell_J(a)+k}.\end{aligned}$$

We apply Theorem 9 to $\gamma \in F(d_1)$ and $a \in \mathring{\sigma}_{J,\gamma}$. If $d \in J_\gamma$, then $\ell_{J_\gamma}(a) + k \geq a_d \geq 1$ automatically for any $k \in \mathbb{N}$. If $d \notin J_\gamma$, then the inequalities $\ell_{J_\gamma}(a) + k \geq a_d \geq 1$ is in the situation of [8, Lemma 2.10], in which the corresponding series has the limit zero. Therefore, taking $\lim_{T\to\infty}$ and using Theorem 9, Lemma 14 and the proof of Theorem 12, we get

$$\iota^*\mathscr{S}^{\Delta}_{f,x_d} = -\sum_{\substack{\gamma\in F(d_1)\setminus\widetilde{F}\\ d\in J_\gamma}} \lambda_\gamma\big[X_\gamma(1)\times_{X_0}\mathbb{A}^{d_1}_{\mathbb{C}}\to\mathbb{A}^{d_1}_{\mathbb{C}}\big] + \sum_{\substack{\gamma\in F(d_1)\\ d\in J_\gamma}} \lambda_\gamma\big[X_\gamma(0)\times_{X_0}\mathbb{A}^{d_1}_{\mathbb{C}}\to\mathbb{A}^{d_1}_{\mathbb{C}}\big].$$

By Theorem 12,

$$\iota^*\mathscr{S}_f = -\sum_{\substack{\gamma\in F(d_1)\setminus\widetilde{F}\\ d\notin J_\gamma}} \lambda_\gamma\big[X_\gamma(1)\times_{X_0}\mathbb{A}^{d_1}_{\mathbb{C}}\to\mathbb{A}^{d_1}_{\mathbb{C}}\big] + \sum_{\substack{\gamma\in F(d_1)\\ d\notin J_\gamma}} \lambda_\gamma\big[X_\gamma(0)\times_{X_0}\mathbb{A}^{d_1}_{\mathbb{C}}\to\mathbb{A}^{d_1}_{\mathbb{C}}\big] + \iota^*\mathscr{S}^{\Delta}_{f,x_d}.$$

The condition $d \notin J_\gamma$ means that $J_\gamma \subseteq [d-1]$, hence, again by Theorem 12,

$$\iota^*\mathscr{S}_{\widetilde{f}} = -\sum_{\substack{\gamma\in F(d_1)\setminus\widetilde{F}\\ d\notin J_\gamma}} \lambda_\gamma\big[X_\gamma(1)\times_{X_0}\mathbb{A}^{d_1}_{\mathbb{C}}\to\mathbb{A}^{d_1}_{\mathbb{C}}\big] + \sum_{\substack{\gamma\in F(d_1)\\ d\notin J_\gamma}} \lambda_\gamma\big[X_\gamma(0)\times_{X_0}\mathbb{A}^{d_1}_{\mathbb{C}}\to\mathbb{A}^{d_1}_{\mathbb{C}}\big].$$

The theorem is completely proved. □

4. Cohomology groups of contact loci of nondegenerate singularities

As before, let f be in $\mathbb{C}[x_1,\dots,x_d]$ which vanishes at O. In this section, we always assume that f is nondegenerate in the sense of Kouchnirenko (say for short that f is *nondegenerate*).

4.1. *Borel–Moore homology groups of contact loci*

Consider the decomposition of $\mathscr{X}_{n,O}(f)$ shown in (2) and Lemma 8 (ii):

$$\mathscr{X}_{n,O}(f) = \bigsqcup_{(J,a)\in\widetilde{\mathscr{P}}_n} \mathscr{X}_{J,a}^{(n)},$$

where $\widetilde{\mathscr{P}}_n$ is the set of all the pairs (J,a) such that $J \supseteq J_\gamma$, $a \in \bigsqcup_{k\in\mathbb{N}}\big(\mathring{\sigma}_{J,\gamma} \cap \widetilde{\Delta}_J^{(n,k)}\big)$ and $\gamma \in K$. We consider an ordering in $\widetilde{\mathscr{P}}_n$ defined as follows: for (J,a) and (J',a') in $\widetilde{\mathscr{P}}_n$, $(J',a') \le (J,a)$ if and only if $J' \subseteq J$ and $a_j \le a'_j$ for all $j \in J'$, where $a = (a_j)_{j\in J}$ and $a' = (a'_j)_{j\in J'}$.

Lemma 18. *Let n be in $\mathbb{N}^*$. For all (J,a) and (J',a') in $\widetilde{\mathscr{P}}_n$ such that $\mathscr{X}_{J',a'}^{(n)}$ and $\mathscr{X}_{J,a}^{(n)}$ are nonempty, the following are equivalent:*

(i) $(J',a') \le (J,a)$,
(ii) $\mathscr{X}_{J',a'}^{(n)} \subseteq \overline{\mathscr{X}_{J,a}^{(n)}}$,
(iii) $\mathscr{X}_{J',a'}^{(n)} \cap \overline{\mathscr{X}_{J,a}^{(n)}} \neq \emptyset$,

the closure taken in the usual topology. Consequently, for all $(J,a) \in \widetilde{\mathscr{P}}_n$ such that $\mathscr{X}_{J,a}^{(n)} \neq \emptyset$ we have

$$\overline{\mathscr{X}_{J,a}^{(n)}} = \bigsqcup_{(J',a')\le(J,a)} \mathscr{X}_{J',a'}^{(n)}.$$

Proof. We prove here that (iii) implies (i), the rest are straighforward. Observe firstly that, due to the definition of $\mathscr{X}_{J,a}^{(n)}$, if it is nonempty then

$$\overline{\mathscr{X}_{J,a}^{(n)}} = \big\{\varphi \in \mathscr{X}_n(f) \mid \mathrm{ord}_t\, x_j(\varphi) \ge a_j \ \ \forall\, j \in J,\ x_i(\varphi) \equiv 0 \ \ \forall\, i \notin J\big\},$$

where $a = (a_j)_{j\in J}$, and $a_j > 0$ for all $j \in J$. Assume that there exists $\varphi^0 \in \mathscr{X}_{J',a'}^{(n)} \cap \overline{\mathscr{X}_{J,a}^{(n)}} \neq \emptyset$. Then we have $\mathrm{ord}_t\, x_j(\varphi^0) = a'_j > 0$ for all $j \in J'$, and $\mathrm{ord}_t\, x_j(\varphi^0) \ge a_j > 0$ for all $j \in J$. If $i \notin J$, then we have $\mathrm{ord}_t\, x_i(\varphi^0) = +\infty$, thus $i \notin J'$, so $J' \subseteq J$. Clearly, $a_j \le a'_j$ for all $j \in J'$. Therefore, $(J',a') \le (J,a)$. ☐

Consider the function $\eta : \widetilde{\mathscr{P}}_n \to \mathbb{Z}$ given by $\eta(J,a) = \dim_{\mathbb{C}} \mathscr{X}_{J,a}^{(n)}$, for every $n \in \mathbb{N}^*$. Put

$$S_p := \bigsqcup_{(J,a)\in\widetilde{\mathscr{P}}_n,\,\eta(J,a)\le p} \mathscr{X}_{J,a}^{(n)},$$

for $p \in \mathbb{N}$. The below are some properties of η and S_p's.

Lemma 19. *Let n be in $\mathbb{N}^*$.*

(i) *If $(J',a') \le (J,a)$ in $\widetilde{\mathscr{P}}_n$ and $\mathscr{X}_{J,a}^{(n)} \neq \emptyset$, then $\eta(J',a') \le \eta(J,a)$.*
(ii) *For all $p \in \mathbb{N}$, S_p are closed and $S_p \subseteq S_{p+1}$. As a consequence, there is a filtration of $\mathscr{X}_{n,O}(f)$ by closed subspaces:*

$$\mathscr{X}_{n,O}(f) = S_{d_0} \supseteq S_{d_0-1} \supseteq \cdots \supseteq S_{-1} = \emptyset,$$

where d_0 denotes the $\mathbb{C}$-dimension of $\mathscr{X}_{n,O}(f)$.

Proof. The first statement (i) is trivial. To prove (ii) we take the closure of S_p; then using Lemma 18 we get

$$\overline{S}_p = \bigcup_{\eta(J,a)\le p} \overline{\mathscr{X}_{J,a}^{(n)}} = \bigcup_{\eta(J,a)\le p,\,\mathscr{X}_{J,a}^{(n)}\neq\emptyset} \overline{\mathscr{X}_{J,a}^{(n)}} = \bigcup_{\eta(J,a)\le p,\,\mathscr{X}_{J,a}^{(n)}\neq\emptyset}\ \bigsqcup_{(J',a')\le(J,a)} \mathscr{X}_{J',a'}^{(n)}.$$

This decomposition combined with (i) implies that $\overline{S}_p \subseteq S_p$, which proves that S_p is a closed subspace. The remaining statements of (ii) are trivial. □

A main result of this section is the following theorem. To express the result, we work with the Borel–Moore homology H_*^{BM}.

Theorem 20. *Let $f \in \mathbb{C}[x_1,\dots,x_d]$ be nondegenerate in the sense of Kouchnirenko, $n \in \mathbb{N}^*$ and $f(O) = 0$. Then there is a spectral sequence*

$$E^1_{p,q} := \bigoplus_{(J,a)\in\widetilde{\mathscr{P}}_n,\eta(J,a)=p} H^{\mathrm{BM}}_{p+q}(\mathscr{X}^{(n)}_{J,a}) \Longrightarrow H^{\mathrm{BM}}_{p+q}(\mathscr{X}_{n,O}(f)).$$

Proof. We have the following the Gysin exact sequence

$$\cdots \to H^{\mathrm{BM}}_{p+q}(S_{p-1}) \to H^{\mathrm{BM}}_{p+q}(S_p) \to H^{\mathrm{BM}}_{p+q}(S_p \setminus S_{p-1}) \to H^{\mathrm{BM}}_{p+q-1}(S_{p-1}) \to \cdots.$$

Put

$$A_{p,q} := H^{\mathrm{BM}}_{p+q}(S_p), \quad E_{p,q} := H^{\mathrm{BM}}_{p+q}(S_p \setminus S_{p-1}).$$

Then we have the bigraded $\mathbb{Z}$-modules $A := \bigoplus_{p,q} A_{p,q}$ and $E := \bigoplus_{p,q} E_{p,q}$. The previous exact sequence induces the exact couple $\langle A, E; h, i, j\rangle$, where $h : A \to A$ is induced from the inclusions $S_m \subseteq S_{m+1}$, $i : A \to E$ and $j : E \to A$ are induced from the above exact sequence. Since the filtration in Lemma 19 (ii) is finite, that exact couple gives us the following spectral sequence

$$E^1_{p,q} := E_{p,q} = H^{\mathrm{BM}}_{p+q}(S_p \setminus S_{p-1}) \Longrightarrow H^{\mathrm{BM}}_{p+q}(\mathscr{X}_{n,O}(f)).$$

On the other hand, we have

$$S_p \setminus S_{p-1} = \bigsqcup_{(J,a)\in\widetilde{\mathscr{P}}_n,\eta(J,a)=p} \mathscr{X}^{(n)}_{J,a}.$$

One claims that for two different pairs $(J,a), (J^{'},a^{'})$ in $\widetilde{\mathscr{P}}_n$ which $\eta(J,a) = \eta(J^{'},a^{'}) = p$ then

$$\mathscr{X}^{(n)}_{J,a} \cap \overline{\mathscr{X}^{(n)}_{J^{'},a^{'}}} = \emptyset \quad \text{and} \quad \mathscr{X}^{(n)}_{J^{'},a^{'}} \cap \overline{\mathscr{X}^{(n)}_{J,a}} = \emptyset.$$

Indeed, if otherwise, suppose that

$$\mathscr{X}^{(n)}_{J^{'},a^{'}} \cap \overline{\mathscr{X}^{(n)}_{J,a}} \neq \emptyset.$$

By Lemma 18, we obtain that $\mathscr{X}^{(n)}_{J^{'},a^{'}} \subseteq \overline{\mathscr{X}^{(n)}_{J,a}}$, but $\mathscr{X}^{(n)}_{J^{'},a^{'}}$ and $\mathscr{X}^{(n)}_{J,a}$ are two disjoint smooth manifolds, then $\eta(J^{'},a^{'}) < \eta(J,a)$. This is a contradiction.

Therefore, in the set $S_p \setminus S_{p-1}$ with the induced topology, each set $\mathscr{X}^{(n)}_{J,a}$ which $\eta(J,a) = p$ is open, hence, is also closed. This implies that

$$H^{\mathrm{BM}}_{p+q}(S_p \setminus S_{p-1}) = \bigoplus_{(J,a)\in\widetilde{\mathscr{P}}_n,\eta(J,a)=p} H^{\mathrm{BM}}_{p+q}(\mathscr{X}^{(n)}_{J,a}).$$

The theorem is then proved. □

Corollary 21. *With the hypothesis as in Theorem 20, there is an isomorphism of groups*

$$H^{\mathrm{BM}}_{2d_0}(\mathscr{X}_{n,O}(f)) \cong \mathbb{Z}^s,$$

where s is the number of connected components of $\mathscr{X}_{n,O}(f)$ which have the same complex dimension d_0 as $\mathscr{X}_{n,O}(f)$.

4.2. *Sheaf cohomology groups of contact loci*

In this subsection, we are going to prove the following theorem.

Theorem 22. *Let $f \in \mathbb{C}[x_1,\dots,x_d]$ be nondegenerate in the sense of Kouchnirenko, $n \in \mathbb{N}^*$ and $f(O)=0$. Let $\mathscr{F}$ be an arbitrary sheaf of abelian groups on $\mathscr{X}_{n,O}(f)$. Then, there is a spectral sequence*

$$E_1^{p,q} := \bigoplus_{(J,a)\in\widetilde{\mathscr{P}}_n,\eta(J,a)=p} H_c^{p+q}(\mathscr{X}_{J,a}^{(n)},\mathscr{F}) \Longrightarrow H_c^{p+q}(\mathscr{X}_{n,O}(f),\mathscr{F}). \tag{8}$$

Proof. We use the notation in Lemma 19. For simplicity, we write S for $S_{d_0} = \mathscr{X}_{n,O}(f)$. For any $0 \le p \le d_0$, we put $S_p^\circ := S_p \setminus S_{p-1}$, which is a μ_n-invariant subset of S. Consider the inclusions $j_p : S_p^\circ \hookrightarrow S_p$, $k_p : S\setminus S_p \hookrightarrow S$ and $i_p : S_p \hookrightarrow S$. Put $\mathscr{F}_p := (j_p)_!(j_p)^{-1}(i_p)^{-1}\mathscr{F}$ and $F^p(\mathscr{F}) := (k_{p-1})_!(k_{p-1})^{-1}\mathscr{F}$ for every $p \ge 1$, with the convention $F^0(\mathscr{F}) := \mathscr{F}$. Then we have the exact sequences

$$0 \to F^{p+1}(\mathscr{F}) \to F^p(\mathscr{F}) \quad \text{and} \quad 0 \to (i_p)_*\mathscr{F}_p \to \mathscr{F}|_{S_p} \to \mathscr{F}|_{S_{p-1}},$$

in which by $\mathscr{F}|_{S_p}$ we mean $(i_p)_*(i_p)^{-1}\mathscr{F}$. Therefore we have the following diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & (i_p)_*\mathscr{F}_p \arrow[d] & \\
0 \arrow[r] & F^{p+1}(\mathscr{F}) \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d, equal] & \mathscr{F}|_{S_p} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^p(\mathscr{F}) \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{F}|_{S_{p-1}} \arrow[r] & 0
\end{tikzcd}$$

It implies from the snake lemma that $F^p(\mathscr{F})/F^{p+1}(\mathscr{F}) \cong (i_p)_*\mathscr{F}_p$. Thus there is a filtration of $\mathscr{F}$ by "skeleta": $\mathscr{F} = F^0(\mathscr{F}) \supseteq F^1(\mathscr{F}) \supseteq \cdots$. It gives the following spectral sequence of cohomology groups with compact support

$$E_1^{p,q}(S,\mathscr{F}) := H_c^{p+q}(S,(i_p)_*\mathscr{F}_p) \Longrightarrow H_c^{p+q}(S,\mathscr{F}). \tag{9}$$

Since S_p is a closed subset of S, $H_c^m(S,(i_p)_*\mathscr{F}_p) \cong H_c^m(S_p,\mathscr{F}_p)$ for any m in $\mathbb{N}$. Also, by the isomorphisms given by the extension by zero sheaf, we have

$$H_c^m(S_p,\mathscr{F}_p) = H_c^m(S_p,(j_p)_!(j_p)^{-1}(i_p)^{-1}\mathscr{F}) \cong H_c^m(S_p^\circ,(j_p)^{-1}(i_p)^{-1}\mathscr{F}).$$

We have that

$$S_p^\circ = \bigsqcup_{(J,a)\in\widetilde{\mathscr{P}}_n,\eta(J,a)=p} \mathscr{X}_{J,a}^{(n)}.$$

Then by the reason as in the proof of Theorem 20, we get

$$H_c^m(S_p^\circ,(j_p)^{-1}(i_p)^{-1}\mathscr{F}) = \bigoplus_{(J,a)\in\widetilde{\mathscr{P}}_n,\eta(J,a)=p} H_c^{p+q}(\mathscr{X}_{J,a}^{(n)},(l_{J,a})^{-1}(j_p)^{-1}(i_p)^{-1}\mathscr{F}),$$

where $l_{J,a}$ is the inclusion of $\mathscr{X}_{J,a}^{(n)}$ in S_p°. For simplicity of notation, we write $H_c^{p+q}(\mathscr{X}_{J,a}^{(n)},\mathscr{F})$ instead of $H_c^{p+q}(\mathscr{X}_{J,a}^{(n)},(l_{J,a})^{-1}(j_p)^{-1}(i_p)^{-1}\mathscr{F})$. The proof is completed. □

Now, let us consider the spectral sequence (8) for a constant sheaf. We need some notation, for each $\gamma \in K$, $k \in \mathbb{N}$, $n \in \mathbb{N}^*$, $p \in \mathbb{Z}$ and $J \supseteq J_\gamma$, we denote by $D_{J,\gamma,k,p}^{(n)}$ the set of all $a \in \mathring{\sigma}_{J,\gamma} \cap \widetilde{\Delta}_J^{(n,k)}$ such that $\dim_{\mathbb{C}} \mathscr{X}_{J,a}^{(n)} = p$.

Lemma 23. *For any $\gamma \in K$, $k \in \mathbb{N}$, $n \in \mathbb{N}^*$, $p \in \mathbb{Z}$ and $J \supseteq J_\gamma$, the set $D_{J,\gamma,k,p}^{(n)}$ is finite.*

Proof. Notice that $\dim_{\mathbb{C}} \mathscr{X}_{J,a}^{(n)} = d-1+|J|n-|a|-k$. The finiteness of $D_{J,\gamma,k,p}^{(n)}$ follows from the fact that the system of equations $d-1+|J|n-|a|-k=p$, $\ell_J(a)+k=n$ in variables a only has finite solutions in $\mathbb{N}^J$. □

The summands in the spectral sequence (8) are described more explicitly in case of constant sheaf as below.

Lemma 24. *Let $\gamma \in K$, $n \in \mathbb{N}^*$, $p, q \in \mathbb{Z}$ and $J \supseteq J_\gamma$. Then, for any $a \in D_{J,\gamma,0,p}^{(n)}$ we have*

$$H_c^{p+q}(\mathscr{X}_{J,a}^{(n)}, \mathbb{C}) \cong H_{p-q}(X_{J,\gamma}(1), \mathbb{C}).$$

Proof. Since $a \in D_{J,\gamma,0,p}^{(n)}$, it follows from Theorem 9 that $\mathscr{X}_{J,a}^{(n)}$ is a complex manifold of real dimension $2p$ and is homeomorphic to $X_{J,\gamma}(1) \times \mathbb{C}^{|J|\ell_J(a)-|a|}$. Then, by combining the duality and the Kunneth formula we get the conclusion. □

We also have the following description for the cohomology of $\mathscr{X}_{J,a}^{(n)}$ for $J \supseteq J_\gamma$ and $a \in D_{J,\gamma,k,p}^{(n)}$ with $k \in \mathbb{N}^*$.

Lemma 25. *Let $\gamma \in K$, $n, k \in \mathbb{N}^*$, $p, q \in \mathbb{Z}$ and $J \supseteq J_\gamma$. Then, for any $a \in D_{J,\gamma,k,p}^{(n)}$ we have*

$$H_c^{p+q}(\mathscr{X}_{J,a}^{(n)}, \mathbb{C}) \cong H_{p-q}(X_{J,\gamma}(0), \mathbb{C}).$$

Proof. Since $\mathscr{X}_{J,a}^{(n)}$ is a complex manifold of real dimension $2p$, then by duality, we have

$$H_c^{p+q}(\mathscr{X}_{J,a}^{(n)}, \mathbb{C}) \cong H_{p-q}(\mathscr{X}_{J,a}^{(n)}, \mathbb{C}).$$

On the other hand, by Theorem 9, $\mathscr{X}_{J,a}^{(n)}$ is a locally trivial fibration on $X_{J,\gamma}(0)$ with fiber $\mathbb{C}^{|J|(\ell_J(a)+k)-|a|-k}$ which is contractible. Hence, by the spectral sequence for (Serre) fibration, we obtain that $H_{p-q}(\mathscr{X}_{J,a}^{(n)}, \mathbb{C}) \cong H_{p-q}(X_{J,\gamma}(0), \mathbb{C})$. The proof is completed. □

We have the following result concerning cohomology of contact loci.

Corollary 26. *Let $f \in \mathbb{C}[x_1, \ldots, x_d]$ be nondegenerate in the sense of Kouchnirenko, $n \in \mathbb{N}^*$ and $f(O) = 0$. Then, there is a spectral sequence*

$$E_1^{p,q} \Longrightarrow H_c^{p+q}(\mathscr{X}_{n,O}(f), \mathbb{C}),$$

where

$$E_1^{p,q} = \bigoplus_{\gamma \in K} \bigoplus_{J \supseteq J_\gamma} \left(H_{p-q}(X_{J,\gamma}(1), \mathbb{C})^{|D_{J,\gamma,0,p}^{(n)}|} \oplus \bigoplus_{k \geq 1} H_{p-q}(X_{J,\gamma}(0), \mathbb{C})^{|D_{J,\gamma,k,p}^{(n)}|} \right).$$

Proof. Apply Theorem 22 for $\mathscr{F}$ to be the constant sheaf on $\mathscr{X}_{n,O}(f)$ associated to the field of complex numbers $\mathbb{C}$, since the inverse image of constant sheaf is a constant sheaf, the Corollary is a direct consequence of Theorem 22 and Lemmas 24 and 25. □

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