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Transcendence of $L(1,\chi_s)/\Pi$ in positive characteristic. A simple automata-style proof

Transcendance de $L(1,\chi_s)/\Pi$ en caractéristique positive. Une preuve simple avec automates finis

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Abstract. For the field of formal Laurent series over a finite field, L. Carlitz defined Π, an analog of the real number π, and D. Goss defined $L(s,\chi)$, analogs of Dirichlet L-functions. G. Damamme proved in 1999 the transcendence of $L(1,\chi_s)/\Pi$ via a criterion of de Mathan. Then Y. Hu gave in 2018 an automata-style proof of the above result. In this work, we present another and much simpler automata-style proof.

Résumé. Pour le corps des séries formelles de Laurent sur un corps fini, L. Carlitz a défini Π, un analogue du nombre réel π, et D. Goss a défini $L(s,\chi)$, analogues des fonctions L de Dirichlet. G. Damamme a démontré en 1999 la transcendance de $L(1,\chi_s)/\Pi$ à l'aide d'un critère de de Mathan. Ensuite Y. Hu a donné en 2018 une preuve à l'aide des automates finis du résultat précédent. Dans ce travail, nous présentons également avec des automates finis une autre preuve plus simple.

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Soit $p \geqslant 2$ un nombre premier et $q = p^\theta$ avec $\theta \geqslant 1$ un entier. Nous désignons par $\mathbb{F}_q$ le corps fini à q éléments, par $\mathbb{F}_q[T]$ l'anneau intègre des polynômes en T à coefficients dans $\mathbb{F}_q$, et par $\mathbb{F}_q(T)$ le corps de fractions de $\mathbb{F}_q[T]$. Pour tous les $P, Q \in \mathbb{F}_q[T]$ avec $Q \neq 0$, nous définissons $|P/Q|_\infty := q^{\deg P - \deg Q}$, et appelons $|\cdot|_\infty$ la valeur absolue ∞-adique sur $\mathbb{F}_q(T)$. Nous désignons par $\mathbb{F}_q((T^{-1}))$ le complété topologique de $\mathbb{F}_q(T)$ pour $|\cdot|_\infty$, et par $\mathbf{C}_\infty$ le complété topologique d'une clôture algébrique fixée de $\mathbb{F}_q((T^{-1}))$.

Soit $L_0 = 1$. Pour tous les entiers j ($j \geqslant 1$), définissons

$$[j] = T^{q^j} - T, \text{ et } L_j = \prod_{k=1}^{j} [k] = \prod_{k=1}^{j} (T^{q^k} - T).$$

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Fixons $a \in \mathbb{F}_q$, $M = T - a$, et $s \in \mathbb{N}$ $(1 < s < q)$. Désignons par χ_s le caractère modulo M attaché à s. Rappelons que $L(1,\chi_s)/\Pi = \left(\frac{T^q - T}{M^q}\right)^{\frac{1}{q-1}} \beta \in \mathbf{C}_\infty$, où

$$\beta = \lim_{n\to\infty} \frac{L_n}{M^{q(q^n-1)/(q-1)}} \sum_{k=0}^{n} (-1)^{k(s-1)} \frac{M^{s(q^k-1)/(q-1)}}{L_k} \in \mathbb{F}_q((T^{-1})).$$

À l'aide du critère de de Mathan, G. Damamme a démontré en 1999 le résultat suivant.

Théorème 1. *Soit $q > 2$. Alors $L(1,\chi_s)/\Pi$ est transcendante sur $\mathbb{F}_q(T)$ pour tous les entiers s $(1 < s < q)$.*

Ensuite Y. Hu a donné en 2018 une preuve à l'aide des automates finis du résultat précédent. Dans ce travail, nous présentons également avec des automates finis une preuve nouvelle et plus simple (en fait, deux preuves, mais seulement différentes dans les détails).

1. Statements of main results

Fix $p \geqslant 2$ a prime number and $q = p^\theta$ with $\theta \geqslant 1$ an integer. Let $\mathbb{F}_q$ be the finite field with q elements, $\mathbb{F}_q[T]$ be the ring of polynomials in T over $\mathbb{F}_q$, and $\mathbb{F}_q(T)$ be the fraction field of $\mathbb{F}_q[T]$. For all $P, Q \in \mathbb{F}_q[T]$ and $Q \neq 0$, set $|P/Q|_\infty := q^{\deg P - \deg Q}$. We denote by $\mathbb{F}_q((T^{-1}))$ the topological completion of $\mathbb{F}_q(T)$ with respect to $|\cdot|_\infty$, and by $\mathbf{C}_\infty$ the topological completion of a fixed algebraic closure of $\mathbb{F}_q((T^{-1}))$.

Put $L_0 = 1$. For all integers j $(j \geqslant 1)$, define

$$[j] = T^{q^j} - T, \quad \text{and} \quad L_j = \prod_{k=1}^{j} [k] = \prod_{k=1}^{j} (T^{q^k} - T).$$

Fix $a \in \mathbb{F}_q$, $M = T - a$, and $s \in \mathbb{N}$ $(1 < s < q)$. Let χ_s be the character modulo M attached to s. Recall that we have $L(1,\chi_s)/\Pi = \left(\frac{T^q - T}{M^q}\right)^{\frac{1}{q-1}} \beta \in \mathbf{C}_\infty$, where

$$\beta = \lim_{n\to\infty} \frac{L_n}{M^{q(q^n-1)/(q-1)}} \sum_{k=0}^{n} (-1)^{k(s-1)} \frac{M^{s(q^k-1)/(q-1)}}{L_k} \in \mathbb{F}_q((T^{-1})).$$

For more on the above formula, see for example [3, pp. 379–380].

Via a criterion of De Mathan, G. Damamme showed in [3, p. 379, Corollaire 2] the following result.

Theorem 1. *Let $q > 2$. Then $L(1,\chi_s)/\Pi$ is transcendental over $\mathbb{F}_q(T)$ for all integers s $(1 < s < q)$.*

An automata-style proof was then obtained by Y. Hu [4]. As usual, such a proof is elementary, but often involved and technical in combinatorial computations. Below we shall present another and much simpler automata-style proof (indeed two proofs, but only different in the details).

We begin by recalling some details concerning automatic sequences.

Definition 2. *A sequence $u = (u(n))_{n\geqslant 0}$ is called q-automatic if its q-kernel*

$$\mathcal{N}_q(u) := \left\{(u(q^k n + d))_{n\geqslant 0} \,\middle|\, k \geqslant 0,\ 0 \leqslant d < q^k\right\}$$

is a finite set.

The following theorem reveals a surprising relationship between automatic sequences and algebraic formal Laurent series over $\mathbb{F}_q$ (see [1]. See also [2]), and it is the base of all automata-style proofs.

Theorem 3. *Let $u = (u(n))_{n\geqslant 0}$ be a sequence with terms in $\mathbb{F}_q$. Then u is q-automatic if and only if the formal Laurent series $\sum_{n=0}^{\infty} \frac{u(n)}{T^n}$ is algebraic over $\mathbb{F}_q(T)$.*

In the next section, we shall show that Theorem 1 can be deduced from Theorem 3 and the following result due to J.-Y. Yao [5, p. 241, Théorème 6].

Theorem 4. *Let $u = (u(n))_{n\geqslant 0}$ be a sequence with terms in $\mathbb{F}_q$. Let $(\ell(n))_{n\geqslant 0}$ and $(h(n))_{n\geqslant 0}$ be two strictly increasing sequences of nonnegative integers such that for all integers $n \geqslant 0$, we have $h(n) \leqslant \ell(n)$,*

$$u(h(n)-1) \neq 0 \quad \textit{and} \quad u(m) = 0 \ \big(h(n) \leqslant m \leqslant \ell(n)\big). \tag{1}$$

If for all integers $N \geqslant 1$, we have

$$\lim_{n\to\infty} \big(q^N \ell(n-N) - h(n)\big) = +\infty, \quad \textit{and} \quad \lim_{n\to\infty} \big(h(n) - q^N h(n-N)\big) = +\infty, \tag{2}$$

then u is not q-automatic.

The proof is simple and can be found in [5, p. 241]. For completeness, we reproduce it below.

For all integers d, k, n ($k, n \geqslant 0$, $0 \leqslant d < q^k$), put $u^{k,d}(n) = u(q^k n + d)$. By the hypothesis (1), we have

$$u(h(n)-1) \neq 0 \quad \text{and} \quad h(m) = 0 \ \big(h(n) \leqslant m \leqslant \ell(n)\big).$$

Consider the q-ary expansion of $h(n) - 1$:

$$h(n) - 1 = \sum_{j=0}^{b(n)} h_j(n) q^j.$$

For all integers m $\big(0 \leqslant m \leqslant b(n)\big)$, put

$$c_m(n) = \sum_{j=0}^{b(n)-m} h_{j+m} q^j \quad \text{and} \quad d_m(n) = \sum_{j=0}^{m-1} h_j(n) q^j.$$

Then $h(n) - 1 = c_m(n) q^m + d_m(n)$, and $u^{m,d_m(n)}\big(c_m(n)\big) = u(h(n)-1) \neq 0$.

Now fix $N \geqslant 1$ an integer. By the hypothesis (2), we can find an integer $n > N$ such that

$$q^{-k} h(n) + q^N \leqslant \ell(n-k) \quad \text{and} \quad h(n-k) + q^N \leqslant q^{-k}(h(n)-1),$$

for all integers k ($1 \leqslant k \leqslant N$). Fix k, m two integers such that $1 \leqslant k < m \leqslant N$. Then

$$h(n+k-m) \leqslant q^k c_m(n) + d_k(n) = q^{k-m}(h(n)-1) + d_k(n) - q^{k-m} d_m(n) \leqslant \ell(n+k-m),$$

from which we deduce $u^{k,d_k(n)}(c_m(n)) = 0$, by using the hypothesis (1) with $n+k-m$ in the place of n. But $u^{m,d_m(n)}(c_m(n)) \neq 0$, hence $\mathrm{Card}(\mathcal{N}_q(u)) \geqslant N$, for all integers $N \geqslant 1$. So u is not q-automatic.

2. Proof of Theorem 1

Since $\left(\frac{T^q - T}{M^q}\right)^{\frac{1}{q-1}}$ is algebraic over $\mathbb{F}_q(T)$, it suffices to show that β is transcendental. Note that for all integers $j \geqslant 1$, we have $a^{q^j} = a$, hence $[j] = T^{q^j} - T = M^{q^j} - M$, and then

$$\begin{aligned}
\beta &= \lim_{n\to\infty} \left(\frac{1}{M}\right)^{q(q^n-1)/(q-1)} \sum_{k=0}^{n} (-1)^{k(s-1)} M^{s(q^k-1)/(q-1)} \prod_{j=k+1}^{n} \big(M^{q^j} - M\big) \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} (-1)^{k(s-1)} \left(\frac{1}{M}\right)^{(q-s)(q^k-1)/(q-1)} \prod_{j=k+1}^{n} \left(1 - \left(\frac{1}{M}\right)^{q^j-1}\right) \\
&= \sum_{k=0}^{\infty} (-1)^{k(s-1)} \left(\frac{1}{M}\right)^{(q-s)(q^k-1)/(q-1)} \prod_{j=k+1}^{\infty} \left(1 - \left(\frac{1}{M}\right)^{q^j-1}\right) \qquad (3) \\
&= \sum_{m=0}^{\infty} \frac{u(m)}{M^m},
\end{aligned}$$

with $u(m) \in \mathbb{F}_q$, for all integers $m \geqslant 0$. For all integers $n \geqslant 1$, put $\ell(n) = \frac{q(q^n-1)}{q-1}$,

$$P_n = \sum_{k=0}^{n} (-1)^{k(s-1)} M^{s(q^k-1)/(q-1)} \prod_{j=k+1}^{n} (M^{q^j} - M), \quad \text{and} \quad Q_n = M^{\ell(n)},$$

$$\beta_n = \sum_{k=n+1}^{\infty} (-1)^{k(s-1)} \left(\frac{1}{M}\right)^{(q-s)(q^k-1)/(q-1)} \prod_{j=k+1}^{\infty} \left(1 - \left(\frac{1}{M}\right)^{q^j-1}\right),$$

$$\gamma_n = \beta - \beta_n - \frac{P_n}{Q_n}.$$

Then we can write $\frac{P_n}{Q_n} = \sum_{k=0}^{\ell(n)} \frac{u_n(k)}{M^k}$. By the ultrametric inequality, we have

$$\begin{aligned} |\beta_n|_\infty &= \left| \sum_{k=n+1}^{\infty} (-1)^{k(s-1)} \left(\frac{1}{M}\right)^{(q-s)(q^k-1)/(q-1)} \prod_{j=k+1}^{\infty} \left(1 - \left(\frac{1}{M}\right)^{q^j-1}\right) \right|_\infty \\ &\leqslant \max_{k>n} \left| \left(\frac{1}{M}\right)^{(q-s)(q^k-1)/(q-1)} \prod_{j=k+1}^{\infty} \left(1 - \left(\frac{1}{M}\right)^{q^j-1}\right) \right|_\infty \\ &= q^{-(q-s)(q^{n+1}-1)/(q-1)} < q^{-\ell(n)}, \\ |\gamma_n|_\infty &= \left| \sum_{k=0}^{n} (-1)^{k(s-1)} \left(\frac{1}{M}\right)^{(q-s)(q^k-1)/(q-1)} \prod_{j=k+1}^{n} \left(1 - \left(\frac{1}{M}\right)^{q^j-1}\right) \left(1 - \prod_{j=n+1}^{\infty} \left(1 - \left(\frac{1}{M}\right)^{q^j-1}\right)\right) \right|_\infty \\ &\leqslant \max_{0 \leqslant k \leqslant n} q^{-(q-s)(q^k-1)/(q-1) - q^{n+1}+1} = q^{-q^{n+1}+1} < q^{-\ell(n)}, \\ \left|\beta - \frac{P_n}{Q_n}\right|_\infty &\leqslant \max\{|\beta_n|_\infty, |\gamma_n|_\infty\} < q^{-\ell(n)}, \end{aligned}$$

from which we deduce that $u(k) = u_n(k)$ $\left(0 \leqslant k \leqslant \ell(n)\right)$. Put $h(n) = \ell(n) - n + 1$. Note that n is the greatest integer $d \geqslant 0$ such that M^d divides P_n in $\mathbb{F}_q[T]$, hence we have

$$u(h(n)-1) = u_n(h(n)-1) \neq 0, \quad \text{and} \quad u(k) = u_n(k) = 0 \ \left(h(n) \leqslant k \leqslant \ell(n)\right).$$

A direct computation shows that the conditions of Theorem 4 are satisfied, thus Theorem 1 holds by combining Theorem 3 and Theorem 4.

It is also possible to give a pure combinatorial proof without the above computations. Note that for all integers k, l $(0 \leqslant k < l)$, we have $(q-s)\frac{q^k-1}{q-1} + \sum_{j=k+1}^{l}(q^j-1) < q^{l+1}-1$, thus from the formula (3), we can deduce at once that for all integers $m \geqslant 0$, we have $u(m) = 0$ or ± 1, and $u(m) \neq 0$ if and only if there exists an integer $k \geqslant 0$, and a sequence $\varepsilon = (\varepsilon_j)_{j \geqslant k+1}$ in $\{0,1\}$ which is ultimately zero such that

$$m = (q-s)\frac{q^k-1}{q-1} + \sum_{j=k+1}^{\infty} \varepsilon_j (q^j - 1), \tag{4}$$

where the pair (k, ε) is unique if it does exist. Note that for all integers $n \geqslant 0$, we have

$$h(n) - 1 = \ell(n) - n = \sum_{j=1}^{n} (q^j - 1),$$

hence $h(n)-1$ has the form (4) with $k = 0$, $\varepsilon_j = 1$ $(1 \leqslant j \leqslant n)$, and $\varepsilon_j = 0$ for $j > n$. Thus $u(h(n)-1) \neq 0$. To conclude, it suffices to show that if $h(n) \leqslant m \leqslant \ell(n)$, then m cannot take the form (4). By contradiction, assume that the formula (4) holds. Since $q^{n+1}-1 > \ell(n)$, thus $\varepsilon_j = 0$ for $j > n$. But $s > 1$, hence

$$m \leqslant (q-2)\frac{q^k-1}{q-1} + \sum_{j=k+1}^{n} (q^j - 1) < h(n).$$

Absurd. So the desired result holds.

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