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The γ-support as a micro-support

Tomohiro Asano [a], Stéphane Guillermou [b], Vincent Humilière [*, c], Yuichi Ike [d] and Claude Viterbo [e]

[a] Faculty of Electrical, Information and Communication Engineering, Institute of Science and Engineering, Kanazawa University, Kakumamachi, Kanazawa, 920-1192, Japan

[b] UMR CNRS 6629 du CNRS Laboratoire de Mathématiques Jean LERAY 2 Chemin de la Houssinière, BP 92208, F-44322 NANTES Cedex 3 France

[c] Sorbonne Université and Université de Paris, CNRS, IMJ-PRG, F-75005 Paris, France and Institut Universitaire de France

[d] Graduate School of Information Science and Technology, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, Japan

[e] Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405, Orsay, France

E-mails: tasano@se.kanazawa-u.ac.jp, tomoh.asano@gmail.com, stephane.guillermou@univ-nantes.fr, stephane.guillermou@univ-nantes.fr, vincent.humiliere@imj-prg.fr, ike@mist.i.u-tokyo.ac.jp, yuichi.ike.1990@gmail.com, claude.viterbo@universite-paris-saclay.fr

Abstract. We prove that for any element L in the completion of the space of smooth compact exact Lagrangian submanifolds of a cotangent bundle equipped with the spectral distance, the γ-support of L coincides with the reduced micro-support of its sheaf quantization. As an application, we give a characterization of the Vichery subdifferential in terms of γ-support.

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1. Introduction

Let M be a C^∞ closed manifold. The space $\mathfrak{L}(T^*M)$ of smooth compact exact Lagrangian submanifolds of T^*M carries a distance, γ, called the spectral distance. This was introduced by Viterbo [21] for the class of Lagrangians that are Hamiltonian isotopic to the zero section, and later extended[1] to $\mathfrak{L}(T^*M)$. The metric space $(\mathfrak{L}(T^*M),\gamma)$ is not complete (not even a Baires space [24, App. A]), and we are interested in this note in its completion. Its study was

* Corresponding author.

[1] In the Floer context, the extension follows from the spectral invariants defined in [11, 13, 14] by using the main result of [4]. One can also use later results in the sheaf framework, by using the spectral invariants defined by Vichery in [19] and the quantization of exact Lagrangians from [5] (see also [23]).

initiated in [10], pursued further in [24], and has applications to Hamilton-Jacobi equations [10], Symplectic Homogenization theory [22], and to conformally symplectic dynamics [1].

The elements of the completion $\widehat{\mathfrak{L}}(T^*M)$ are by definition certain equivalence classes of Cauchy sequences with respect to the spectral norm γ. Despite their very abstract nature, they admit a geometric incarnation first introduced by Humilière in [10] and in a different version called *γ-support* much more recently by Viterbo in [24]. For a smooth Lagrangian $L \in \mathfrak{L}(T^*M)$ we have $\gamma\text{-supp}(L) = L$. We refer the reader to Section 2.1 for the precise definition of the γ-support.

To each element of $\mathfrak{L}(T^*M)$, it is also possible to associate an object F_L in the derived category of sheaves $\mathsf{D}(k_{M\times\mathbb{R}})$, which more precisely belongs to the so-called Tamarkin category. This was proved by Guillermou–Kashiwara–Schapira [7] in the class of Lagrangians that are Hamiltonian isotopic to the zero section and later extended to $\mathfrak{L}(T^*M)$ by Guillermou [5] and Viterbo [23]. The object F_L is called *sheaf quantization* of L. Conversely, to an object F in the Tamarkin category, one can associate a closed subset of T^*M which we call *reduced micro-support* and denote $\mathrm{RS}(F)$, in such a way[2] that $\mathrm{RS}(F_L) = L$. The sheaf quantization can often be used instead of a generating function, and it is known to exist in more general cases (in particular for any exact embedded Lagrangian). This approach allows, on one hand, to get rid of the "Hamiltonian isotopic to the zero section" condition required to use generating functions quadratic at infinity, and allows to prove or reprove a number of results in symplectic topology (see, for example, [6]).

The correspondence $L \mapsto F_L$ was recently extended in [9] to the completion of $\mathfrak{L}(T^*M)$ (see also [3] for a similar result in different settings). We therefore obtain two notions of support for an element L in $\widehat{\mathfrak{L}}(T^*M)$, namely the γ-support of L and the reduced micro-support F_L. These two notions coincide on $\mathfrak{L}(T^*M)$, and it was asked by Guillermou and Viterbo ([9, Problem 9.10]) whether they coincide in general. Our main result below answers positively this question.

Theorem 1. *For any $L \in \widehat{\mathfrak{L}}(T^*M)$, one has*

$$\gamma\text{-supp}(L) = \mathrm{RS}(F_L). \tag{1}$$

This allows us to understand the γ-support in terms of the more classical micro-support from sheaf theory. However, in some instances, we can also get information on the micro-support of a sheaf using γ-support (see Remark 6(3)).

The main theorem is proved in Section 3. In Section 4, we provide an application of this result to a characterization of the Vichery subdifferential defined in [20].

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2. Preliminaries

Let M be a C^∞ manifold and $\pi\colon T^*M \to M$ its cotangent bundle. We write $(x;\xi)$ for local coordinates of T^*M, the Liouville form λ is then defined by $\lambda = \sum_i \xi_i dx_i$. We denote the zero section of T^*M by 0_M.

2.1. *The γ-support of elements in $\widehat{\mathfrak{L}}(T^*M)$*

Let $\mathscr{L}(T^*M)$ denote the set of compact exact Lagrangian branes, i.e., triples $(L, f_L, \widetilde{G})$, where L is a compact exact Lagrangian submanifold of T^*M, $f_L\colon L \to \mathbb{R}$ is a function satisfying $df_L = \lambda_{|L}$, and

[2] In fact, the object F_L is only defined up to shift, but $\mathrm{RS}(F_L)$ is well-defined. See Section 2.2.

$\widetilde{G}$ is a grading of L (see [16, 24]). By abuse of notation, we simply write L for an element $(L, f_L, \widetilde{G})$ of $\mathscr{L}(T^*M)$. The action of $\mathbb{R}$ on $\mathscr{L}(T^*M)$ given by $(L, f_L, \widetilde{G}) \mapsto (L, f_L - c, \widetilde{G})$ is denoted by T_c. For L_1, L_2 in $\mathscr{L}(T^*M)$ we define as in [24] the spectral invariants $c_+(L_1, L_2)$ and $c_-(L_1, L_2)$ and finally

$$\mathrm{c}(L_1, L_2) = |c_+(L_1, L_2)| + |c_-(L_1, L_2)|.$$

Set $\mathfrak{L}(T^*M)$ to be the set of compact exact Lagrangians, where we do not record the primitive or grading. For L_1, L_2 in $\mathfrak{L}(T^*M)$, we define

$$\gamma(L_1, L_2) = \inf_{c \in \mathbb{R}} \mathrm{c}(L_1, T_c L_2) = c_+(L_1, L_2) - c_-(L_1, L_2).$$

Denote by $\widehat{\mathfrak{L}}(T^*M)$ (resp. $\widehat{\mathscr{L}}(T^*M)$) the completion of $\mathfrak{L}(T^*M)$ (resp. $\mathscr{L}(T^*M)$) with respect to γ (resp. c). We use the same symbol T_c to mean the action on $\widehat{\mathscr{L}}(T^*M)$ extending that on $\mathscr{L}(T^*M)$.

Note that the standard action of the group of compactly supported Hamiltonian diffeomorphisms $\mathrm{Ham}_c(T^*M)$ on $\mathfrak{L}(T^*M)$ given by $(\phi, L) \mapsto \phi(L)$ naturally extends to an action of $\mathrm{Ham}_c(T^*M)$ on the completion $\widehat{\mathfrak{L}}(T^*M)$. We are now ready to define the γ-support.

Definition 2 (Viterbo [24]). *Let $L \in \widehat{\mathfrak{L}}(T^*M)$. The γ-support of L, denoted $\gamma\text{-supp}(L)$, is the complement of the set of all $x \in T^*M$ which admit an open neighborhood U such that $\phi(L) = L$ for any Hamiltonian diffeomorphism ϕ supported in U.*

When L is a genuine smooth Lagrangian submanifold, i.e., belongs to $\mathfrak{L}(T^*M)$, then $\gamma\text{-supp}(L) = L$ (see [24, Prop. 6.17.(1)]). In general, $\gamma\text{-supp}(L)$ is a closed subset of T^*M which can be very singular. However, not every closed subset can arise as a γ-support since γ-supports are always coisotropic in a generalized sense (called γ-coisotropic, see [24, Thm. 7.12]). We will use the following property (see [24, Prop. 6.20.(4)]): given smooth closed manifolds M_1, M_2, we have

$$\gamma\text{-supp}(L_1 \times L_2) \subset \gamma\text{-supp}(L_1) \times \gamma\text{-supp}(L_2) \tag{2}$$

for any $L_1 \in \widehat{\mathfrak{L}}(T^*M_1)$ and $L_2 \in \widehat{\mathfrak{L}}(T^*M_2)$.

We refer the interested reader to [24] for many further properties of the γ-support.

2.2. *Sheaf quantization of elements in $\widehat{\mathscr{L}}(T^*M)$*

We fix a field $\mathbf{k}$ throughout the paper. Given a C^∞-manifold without boundary X, we let $\mathsf{D}(\mathbf{k}_X)$ denote the unbounded derived category of sheaves of $\mathbf{k}$-vector spaces on X. We denote by $\mathbf{k}_X$ the constant sheaf on X with stalk $\mathbf{k}$. For an inclusion $i\colon Z \hookrightarrow X$ of a locally closed subset, we also write $\mathbf{k}_Z$ for the zero-extension to X of the constant sheaf on Z with stalk $\mathbf{k}$. For an object $F \in \mathsf{D}(\mathbf{k}_X)$, we denote by $\mathrm{SS}(F) \subset T^*X$ its micro-support, which is defined in [12] (see also Robalo–Schapira [15] for the unbounded setting).

We now recall the definition of the Tamarkin category [17] (see also [8]). We denote by $(t;\tau)$ the canonical coordinate on $T^*\mathbb{R}_t$. The Tamarkin category $\mathscr{D}(M)$ is defined as the quotient category

$$\mathsf{D}(\mathbf{k}_{M\times\mathbb{R}_t}) / \mathsf{D}_{\{\tau \le 0\}}(\mathbf{k}_{M\times\mathbb{R}_t}),$$

where $\mathsf{D}_{\{\tau\le 0\}}(\mathbf{k}_{M\times\mathbb{R}_t}) := \{F \in \mathsf{D}(\mathbf{k}_X) \mid \mathrm{SS}(F) \subset \{\tau \le 0\}\}$ is the full triangulated subcategory of $\mathsf{D}(\mathbf{k}_X)$. The category $\mathscr{D}(M)$ is equivalent to the left orthogonal ${}^{\perp}\mathsf{D}_{\{\tau\le 0\}}(\mathbf{k}_{M\times\mathbb{R}_t})$. For an object $F \in \mathscr{D}(M)$, we define its reduced micro-support $\mathrm{RS}(F) \subset T^*M$ by

$$\mathrm{RS}(F) := \overline{\rho_t(\mathrm{SS}(F) \cap \{\tau > 0\})},$$

where $\{\tau > 0\} \subset T^*(M \times \mathbb{R}_t)$ and $\rho_t\colon \{\tau > 0\} \to T^*M, (x, t; \xi, \tau) \mapsto (x; \xi/\tau)$.

We can also describe the action of $\mathrm{Ham}_c(T^*M)$ on $\mathscr{D}(M)$ as follows. Let $H\colon T^*M \times I \to \mathbb{R}$ be a compactly supported Hamiltonian function and denote by $\phi^H = (\phi^H_s)_{s\in I}\colon T^*M \times I \to T^*M$ the Hamiltonian isotopy generated by H. Then we can construct an object $K^H \in \mathsf{D}(\mathbf{k}_{(M\times\mathbb{R})^2\times I})$ whose micro-support coincides with the Lagrangian lift of the graph of ϕ^H outside the zero section (see [7] for the definition). For $s \in I$, we set $K_s^H := K^H|_{(M\times\mathbb{R})^2\times\{s\}} \in \mathsf{D}(\mathbf{k}_{(M\times\mathbb{R})^2})$. We define a functor $\Phi_s^H \colon \mathscr{D}(M) \to \mathscr{D}(M)$ $(s \in I)$ to be the composition with K_s^H. For any $F \in \mathscr{D}(M)$, we find that

$$\mathrm{RS}(\Phi_s^H(F)) = \phi_s^H(\mathrm{RS}(F)).$$

We now explain the sheaf quantization of an element of $\mathscr{L}(T^*M)$. For $L \in \mathscr{L}(T^*M)$, we define

$$\widetilde{L} := \{(x,t;\xi,\tau) \mid \tau > 0, (x;\xi/\tau) \in L, t = -f_L(x;\xi/\tau)\}.$$

Guillermou [5] (see also [6, 23]) proved the existence and the uniqueness of an object $F_L \in \mathscr{D}(M)$ that satisfies $\mathrm{SS}(F_L) \setminus 0_{M\times\mathbb{R}_t} = \widetilde{L}$ and $F_L|_{M\times(c,\infty)} \simeq \mathbf{k}_{M\times(c,\infty)}$ for a sufficiently large $c > 0$. The object F_L is called the sheaf quantization of L. We write this correspondence by $Q \colon \mathscr{L}(T^*M) \to \mathscr{D}(M), L \mapsto F_L$. Note that the grading of L specifies the grading of F_L; in other words, Q sends $L[k]$ to $F_L[k]$. However, we shall mostly forget about gradings here.

We note that ϕ^H has a canonical lift to a homogeneous Hamiltonian isotopy of $T^*(M\times\mathbb{R}) \setminus 0_{M\times\mathbb{R}}$ (use the extra variable τ to make H homogeneous), or equivalently, a contact isotopy of $J^1(M)$. In this way ϕ^H also acts on $\mathscr{L}(T^*M)$. Moreover ϕ^H commutes with T_c and it extends to $\widehat{\mathscr{L}}(T^*M)$. By the uniqueness, we find that

$$Q(\phi_1^H(L)) \simeq \Phi_1^H(Q(L)) \tag{3}$$

for any compact supported Hamiltonian function H.

We can define an interleaving-like distance $d_{\mathscr{D}(M)}$ on the Tamarkin category $\mathscr{D}(M)$ (see Asano–Ike [2, 3] and Guillermou–Viterbo [9]). Since there are several different definitions and conventions for distance on $\mathscr{D}(M)$, we give the definition here. For $c \in \mathbb{R}$, we let $T_c \colon M\times\mathbb{R}_t \to M\times\mathbb{R}_t, (x,t) \mapsto (x,t+c)$ be the translation map to the $\mathbb{R}_t$-direction by c. For an object $F \in \mathscr{D}(M)$, we simply write T_cF for $T_{c*}F$. Note that Q sends T_cL to T_cF_L. For $F \in \mathscr{D}(M)$ and $c \geq 0$, we have a canonical morphism $\tau_c(F) \colon F \to T_cF$ in $\mathscr{D}(M)$ (see [8, 17] for details). Using the canonical morphisms, we can define the (pseudo-)distance on $\mathscr{D}(M)$ as follows.

Definition 3. *Let $F, G \in \mathscr{D}(M)$ and $a, b \geq 0$.*

(1) *The pair (F,G) is said to be (a,b)*-isomorphic *if there exist morphisms $\alpha \colon F \to T_aG$ and $\beta \colon G \to T_bF$ in $\mathscr{D}(M)$ such that*

$$\begin{cases} \left[F \xrightarrow{\alpha} T_aG \xrightarrow{T_a\beta} T_{a+b}F\right] = \tau_{a+b}(F), \\ \left[G \xrightarrow{\beta} T_bF \xrightarrow{T_b\alpha} T_{a+b}G\right] = \tau_{a+b}(G). \end{cases}$$

(2) *We define*

$$d_{\mathscr{D}(M)}(F,G) := \inf\left\{a+b \,\middle|\, (F,G) \textit{ is } (a,b)\textit{-isomorphic}\right\}.$$

In Asano–Ike [3] and Guillermou–Viterbo [9], it is shown that $d_{\mathscr{D}(M)}$ is complete[3]. In Guillermou–Viterbo [9], Remark 6.12, it is also proved that for $L_1, L_2 \in \mathscr{L}(T^*M)$

$$d_{\mathscr{D}(M)}(F_{L_1}, F_{L_2}) \leq \mathrm{c}(L_1, L_2) \leq 2d_{\mathscr{D}(M)}(F_{L_1}, F_{L_2}). \tag{4}$$

Hence, using the completeness and the non-degeneracy of the distance for limits of constructible sheaves ([9, Prop. B.7], we can extend $Q \colon \mathscr{L}(T^*M) \to \mathscr{D}(M)$ as

$$\widehat{Q} \colon \widehat{\mathscr{L}}(T^*M) \to \mathscr{D}(M). \tag{5}$$

We still write $F_L = \widehat{Q}(L)$ for $L \in \widehat{\mathscr{L}}(T^*M)$. Note that $\widehat{Q}$ also satisfies $\widehat{Q}(T_cL) \simeq T_c\widehat{Q}(L)$ for $L \in \widehat{\mathscr{L}}(T^*M)$. By a result of Viterbo [24, Prop. 5.5], the canonical map $\widehat{\mathscr{L}}(T^*M) \to \widehat{\mathfrak{L}}(T^*M)$ is surjective, and two elements $L_1, L_2 \in \widehat{\mathscr{L}}(T^*M)$ have the same image if and only if they coincide up to shift. Hence, for $L \in \widehat{\mathfrak{L}}(T^*M)$, the object $F_L \in \mathscr{D}(M)$ is well-defined up to shift. In particular, $\mathrm{RS}(F_L)$ is well-defined for $L \in \widehat{\mathfrak{L}}(T^*M)$.

[3] Since $d_{\mathscr{D}(M)}$ is a pseudo-distance, the limit is not necessarily unique.

Since the action of Φ^H on $\mathscr{L}(T^*M)$ (or also $\mathscr{D}(M)$) commutes with T_c, it is an isometry. It follows that the extension of this action to $\widehat{\mathscr{L}}(T^*M)$ still satisfies (3):

$$\widehat{Q}(\phi_1^H(L)) \simeq \Phi_1^H(\widehat{Q}(L)). \tag{6}$$

3. Proof of the main result

Our proof of Theorem 1 will use the following lemma.

Lemma 4. *Let $F \in \mathscr{D}(M)$. We assume that a Hamiltonian function $H\colon T^*M \times I \to \mathbb{R}$ satisfies* $\operatorname{supp}(H_s) \cap \operatorname{RS}(F) = \emptyset$ *for all $s \in I$. Then $F \simeq \Phi_1^H(F)$.*

Proof. We recall how to construct K^H. We first lift H to a homogeneous Hamiltonian function $\widetilde{H}\colon (T^*(M\times\mathbb{R}) \setminus 0_{M\times\mathbb{R}}) \times I \to \mathbb{R}$ by setting $\widetilde{H}_s(x,t;\xi,\tau) := \tau H_s(x;\xi/\tau)$ for $\tau \neq 0$ and $\widetilde{H}_s = 0$ when $\tau = 0$. Then we apply the results for homogeneous Hamiltonian isotopies in [7]. Composing K^H with F yields a sheaf G on $M\times\mathbb{R}\times I$ whose micro-support, outside the zero section, is given by

$$\operatorname{SS}(G) = \left\{ (x,t,s;\xi,\tau,\sigma) \;\middle|\; \begin{array}{l} \exists\,(x',t';\xi',\tau') \in \operatorname{SS}(F), \tau' \neq 0, \\ (x,t;\xi,\tau) = \phi_s^{\widetilde{H}}(x',t';\xi',\tau'), \\ \sigma = -\widetilde{H}_s(x,t;\xi,\tau) = -\tau H_s(x;\xi/\tau) \end{array} \right\}.$$

Since $\operatorname{supp}(H_s) \cap \operatorname{RS}(F) = \emptyset$ for all s, we see that the fiber variable σ vanishes on $\operatorname{SS}(G)$. By [12, Prop. 5.4.5] this implies that G is the pull-back of a sheaf on $M\times\mathbb{R}$. In particular $G|_{M\times\mathbb{R}\times\{0\}} \simeq G|_{M\times\mathbb{R}\times\{1\}}$, which is the claimed result. □

We now turn to the proof of Theorem 1.

Proof of Theorem 1. We first prove the inclusion $\gamma\text{-supp}(L) \subset \operatorname{RS}(F_L)$. Let U be an open subset such that $U \cap \operatorname{RS}(F_L) = \emptyset$. For any H such that $\operatorname{supp}(H_s) \subset U$ for any $s \in I$, by Lemma 4 we get $d_{\mathscr{D}(M)}(F_L, \Phi_1^H(F_L)) = 0$. By (4), we deduce

$$\gamma(L, \phi_1^H(L)) \leq 2 d_{\mathscr{D}(M)}(F_L, \Phi_1^H(F_L)) = 0,$$

hence $\phi_1^H(L) = L$. This proves that $U \cap \gamma\text{-supp}(L) = \emptyset$ for any such open subset U. As a consequence $\gamma\text{-supp}(L) \subset \operatorname{RS}(F_L)$.

We next prove $\operatorname{RS}(F_L) \subset \gamma\text{-supp}(L)$. As a first step, we establish the following.

Lemma 5. *For $L \in \widehat{\mathfrak{L}}(T^*M)$, one has*

$$\partial \operatorname{RS}(F_L) \subset \gamma\text{-supp}(L), \tag{7}$$

where ∂ means topological boundary, i.e., $\partial \operatorname{RS}(F_L) = \operatorname{RS}(F_L) \cap \operatorname{Int}(\operatorname{RS}(F_L))^c$.

Proof. Let U be an open subset such that $U \cap \gamma\text{-supp}(L) = \emptyset$. Then for any H such that $\operatorname{supp}(H_s) \subset U$ for any $s \in I$, we have $L = \phi_1^H(L)$. As recalled after (5) we can lift L to $L' \in \widehat{\mathscr{L}}(T^*M)$ and we have $\phi_1^H(L') = T_c(L')$ for some c. By (6) we deduce $T_c(F_L) \simeq \Phi_1^H(F_L)$, hence $\operatorname{RS}(F_L) = \phi_1^H(\operatorname{RS}(F_L))$. Thus, either $U \cap \operatorname{RS}(F_L) = \emptyset$ or $U \subset \operatorname{Int}(\operatorname{RS}(F))$, which shows (7). □

We can now conclude the proof of Theorem 1. To prove $\operatorname{RS}(F_L) \subset \gamma\text{-supp}(L)$, it is enough to show that $\operatorname{RS}(F_L) \times 0_{\mathbb{S}^1} \subset \gamma\text{-supp}(L) \times 0_{\mathbb{S}^1}$. Now we consider $L \times 0_{\mathbb{S}^1} \in \widehat{\mathscr{L}}(T^*(M\times\mathbb{S}^1))$. Then we get $F_{L\times 0_{\mathbb{S}^1}} \simeq F_L \boxtimes \mathbf{k}_{\mathbb{S}^1}$, and hence $\operatorname{RS}(F_{L\times 0_{\mathbb{S}^1}}) = \operatorname{RS}(F_L) \times 0_{\mathbb{S}^1}$, whose interior is empty. By Lemma 5, we get

$$\operatorname{RS}(F_L) \times 0_{\mathbb{S}^1} = \operatorname{RS}(F_{L\times 0_{\mathbb{S}^1}}) \subset \gamma\text{-supp}(L \times 0_{\mathbb{S}^1}) \subset \gamma\text{-supp}(L) \times 0_{\mathbb{S}^1},$$

where the last inclusion follows from (2). □

Remarks 6.

(1) Let X, Y be two sets in a symplectic manifold. Define the following variant of Usher's distance (see [18]) between X and Y as
$$d_\gamma(X,Y) = \inf\{\gamma(\varphi) \mid \varphi(X) = Y\}.$$
Then we have for any $L, L' \in \widehat{\mathfrak{L}}(T^*M)$
$$d_\gamma(\gamma\text{-supp}(L), \gamma\text{-supp}(L')) \le \widehat{\gamma}(L,L'),$$
where $\widehat{\gamma}(L,L') = \inf\{\gamma(\varphi) \mid \varphi(L) = L'\}$.
In general $\gamma(L,L') \le \widehat{\gamma}(L,L')$, but if we had an equality, this would mean (as in the proof of the theorem)
$$d_\gamma(\mathrm{RS}(F_L), \mathrm{RS}(F_{L'})) \le 2d_{\mathscr{D}(M)}(F_L, F_{L'}).$$
Then, we may conjecture that in general for all $F, G \in D(M)$ we have
$$d_\gamma(\mathrm{RS}(F), \mathrm{RS}(G)) \le 2d_{\mathscr{D}(M)}(F,G).$$

(2) In [9], it is proved that the micro-support (hence also the reduced micro-support by [24, Prop. 9.4]) is γ-coisotropic.

(3) In [1], it is proved that there are indecomposable sets (i.e., compact connected sets that cannot be written as the union of two nontrivial compact connected) that appear as γ-supports. They can thus also appear as singular support of sheaves, and we may even impose that these are limits of constructible sheaves.

4. An application to subdifferentials

Let f be a continuous function on M. In [20], Vichery defined a subdifferential of f at x as follows.

Definition 7 ([20, Def. 3.4]). *The epigraph of f is the set $Z_f = \{(x,t) \in M \times \mathbb{R} \mid f(x) \le t\}$. Then ∂f is defined as $-\mathrm{RS}(\mathbf{k}_{Z_f})$ and $\partial f(x) = -\mathrm{RS}(\mathbf{k}_{Z_f}) \cap T_x^*M$ where $-A = \{(x,-p) \mid (x,p) \in A\}$.*

A more elementary definition from the same paper by Vichery is the following ([20, Def. 4.6]).

Proposition 8. *The vector $\xi \in T_x^*M$ belongs to $\partial f(x)$ if and only (x,ξ) belongs to the closure of the set of pairs $(y;\eta)$ such that setting $a = f(y)$ and $f_\eta(z) = f(z) - \langle \eta, z\rangle$ the map*
$$\lim_{\substack{U \ni x \\ \varepsilon \to 0}} H^*(U \cap f_\eta^{<a+\varepsilon}) \longrightarrow \lim_{\substack{U \ni x \\ \varepsilon \to 0}} H^*(U \cap f_\eta^{<a})$$
is not an isomorphism.

We refer to [20, §3.4] for the proof and the connection between this "homological subdifferential" and other subdifferentials, but notice that if f is Lipschitz and $\partial_C f(x)$ is the Clarke differential at x we have $\partial f(x) \subset \partial_C f(x)$ and the inclusion can be strict.

Note that if f is smooth, the graph of df is an exact Lagrangian submanifold denoted by $\mathrm{graph}(df)$. Since $\gamma(\mathrm{graph}(df), \mathrm{graph}(dg)) = \max(f-g) - \min(f-g) = \mathrm{osc}(f-g)$, a C^0 Cauchy sequence of functions yields a Cauchy sequence in $\mathscr{L}(T^*M)$, so that $\mathrm{graph}(df)$ is well defined in $\widehat{\mathscr{L}}(T^*M)$ for any $f \in C^0(M,\mathbb{R})$.

Proposition 9. *For any continuous function $f : M \to \mathbb{R}$, we have*
$$\gamma\text{-supp}(\mathrm{graph}(df)) = \partial f.$$

Proof. By applying Theorem 1 to $F = \mathbf{k}_{Z_f}$, we get $\mathrm{RS}(\mathbf{k}_{Z_f}) = \gamma\text{-supp}(\mathrm{graph}(df))$ provided we prove that $\widehat{Q}(\mathrm{graph}(df)) = \mathbf{k}_{Z_f}$. This of course holds if f is smooth but needs to be established in the continuous case.

We first claim that for any continuous functions f, g we have:

$$d_{\mathscr{D}(M)}(\mathbf{k}_{Z_g}, \mathbf{k}_{Z_f}) \leq 2\|f - g\|_{C^0}.$$

For two open sets Z and Z', there is a non-trivial morphism from $\mathbf{k}_Z$ to $\mathbf{k}_{Z'}$ if and only if $Z' \subset Z$. We set $\varepsilon := \|f - g\|_{C^0}$, then we have $Z_f \subset Z_{g+\varepsilon}$ and $Z_g \subset Z_{f+\varepsilon}$. Since $T_c \mathbf{k}_{Z_f} \simeq \mathbf{k}_{Z_{f+c}}$ for $c \in \mathbb{R}$, these inclusions imply that there exist canonical non-trivial morphisms $\mathbf{k}_{Z_f} \to T_\varepsilon \mathbf{k}_{Z_g}$ and $\mathbf{k}_{Z_g} \to T_\varepsilon \mathbf{k}_{Z_f}$, which give an $(\varepsilon, \varepsilon)$-isomorphism for the pair $(\mathbf{k}_{Z_f}, \mathbf{k}_{Z_g})$. This proves the inequality.

Now let f_n be a sequence of smooth functions C^0 converging to a continuous function f. Then, by the above inequality $\mathbf{k}_{Z_{f_n}}$ converges to $\mathbf{k}_{Z_f}$ with respect to the distance $d_{\mathscr{D}(M)}$ as n goes to $+\infty$. This implies $\widehat{Q}(\mathrm{graph}(df)) = \mathbf{k}_{Z_f}$ and concludes our proof. □

Remark 10. If $L \in \widehat{\mathfrak{L}}_c(T^*M)$, then $\gamma\text{-supp}(L) \cap T^*_x M$ is non empty for all $x \in M$ (see [24, Def. 6.4 and Prop. 6.10]). For $L = \mathrm{graph}(df)$, this means that $\partial f(x)$ is non-empty for all x. The condition $\mathrm{graph}(df) \in \widehat{\mathfrak{L}}_c(T^*M)$ should correspond to f being Lipschitz, in which case it is easy to see that $\partial f(x) \neq \emptyset$.

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