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Homological Methods / *Méthodes homologiques*

# Relative global dimensions and stable homotopy categories

**Li Liang [a] and Junpeng Wang [*], [b]**

[a] School of Mathematics and Physics, Lanzhou Jiaotong University, Lanzhou 730070, China
*URL:* https://sites.google.com/site/lliangnju

[b] College of Economics, Northwest Normal University, Lanzhou 730070, China.

*E-mails:* lliangnju@gmail.com, wangjunpeng1218@163.com.

**Abstract.** In this paper we study the finiteness of global Gorenstein AC-homological dimensions for rings, and answer the questions posed by Becerril, Mendoza, Pérez and Santiago. As an application, we show that any left (or right) coherent and left Gorenstein ring has a projective and injective stable homotopy category, which improves the known result by Beligiannis.

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## 1. Introduction

Throughout this work, all rings are assumed to be associative. Let $R$ be a ring; we adopt the convention that an $R$-module is a left $R$-module, and we refer to right $R$-modules as modules over the opposite ring $R^{\circ}$.

Building from Auslander and Bridger's work [1] on modules of finite G-dimension, Enochs, Jenda and Torrecillas [15, 16] introduced and studied Gorenstein projective, Gorenstein injective and Gorenstein flat modules, and developed "Gorenstein homological algebra". Such a relative homological algebra theory has been developed rapidly during the past several years and becomes a rich theory; we refer the reader to, for example, [6, 7, 14–16, 23, 33] for related works.

[*] Corresponding author.

For a quasi-Frobenius ring $R$, the category $\mathsf{Mod}$ of $R$-modules is a Frobenius category with projective-injective objects all projective (or injective) $R$-modules. So the stable category $\underline{\mathsf{Mod}}$ modulo projectives is a triangulated category. Furthermore, it is compactly generated; see Krause [28, Sec. 1.5]. It is well known that over an arbitrary ring $R$ the subcategory $\mathsf{GP}$ (resp., $\mathsf{GI}$) of Gorenstein projective (resp., Gorenstein injective) $R$-modules is a Frobenius category with projective-injective objects projective (resp. injective) $R$-modules. Also, from a theorem by Christensen, Estrada and Thompson [12, Thm. 4.5] the subcategory $\mathsf{GF}\cap\mathsf{Cot}$ of Gorenstein flat and cotorsion $R$-modules is a Frobenius category with projective-injective objects all flat and cotorsion $R$-modules. Hence, the stable categories $\underline{\mathsf{GP}}$, $\underline{\mathsf{GI}}$ and $\underline{\mathsf{GF}\cap\mathsf{Cot}}$ are triangulated categories. It is a natural question when these stable categories are compactly generated. It follows from Beligiannis [4, Lem. 6.6 and thm. 6.7] that if $R$ is a right coherent and left perfect or left Morita ring with $\operatorname{Ggldim} R<\infty$ then $\underline{\mathsf{GP}}\simeq\underline{\mathsf{GI}}$ are compactly generated. The same conclusion holds if $R$ is Iwanaga–Gorenstein; see Hovey [26, Thm. 9.4] or Chen [11, Thm. 4.1]. One of the main results in this paper is the next improved result; see Corollaries 34 and 38.

**Theorem 1.** *Let $R$ be a ring with* $\operatorname{Ggldim} R<\infty$.

(a) *If $R$ is right coherent, then* $\underline{\mathsf{GP}}\simeq\underline{\mathsf{GI}}\simeq\underline{\mathsf{GF}\cap\mathsf{Cot}}$ *are compactly generated.*
(b) *If $R$ is left coherent, then* $\underline{\mathsf{GP}}\simeq\underline{\mathsf{GI}}$ *are compactly generated.*

Here $\operatorname{Ggldim} R$ is the global Gorenstein dimension, which is defined as $\operatorname{Ggldim} R=\sup\{\operatorname{Gpd}_R M \mid M \text{ is an } R\text{-module}\}$. We notice that $R$ satisfies $\operatorname{Ggldim} R<\infty$ if and only if $R$ is left Gorenstein[1] ; see §8. So as an immediate consequence of Theorem 1, by using [4, Lem. 6.6], we get the next result that improves [4, Thm. 6.7] by removing the assumption that the ring $R$ should be left perfect or left Morita.

**Corollary 2.** *Any left (or right) coherent and left Gorenstein ring has a projective and injective stable homotopy category.*

We refer the reader to [4, Def. 6.2] for the definition of projective/injective stable homotopy category.

An example is given to show that coherent rings of finite global Gorenstein dimension (or equivalently, left Gorenstein) may not be Iwanaga–Gorenstein nor perfect nor Morita; see Example 39.

According to [3, Thm. 6.9] and [14, Thm. 4.1], the finiteness of $\operatorname{Ggldim} R$ can be characterized by the existence of the triangulated equivalences $\underline{\mathsf{GP}}\simeq \mathsf{D}^{\mathrm{b}}(R)/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj})$ and/or $\underline{\mathsf{GI}}\simeq \mathsf{D}^{\mathrm{b}}(R)/\mathsf{K}^{\mathrm{b}}(\mathsf{Inj})$, where $\mathsf{D}^{\mathrm{b}}(R)$ denotes the bounded derived category of $R$, and $\mathsf{K}^{\mathrm{b}}(\mathsf{Prj})$ (resp., $\mathsf{K}^{\mathrm{b}}(\mathsf{Inj})$) denotes the bounded homotopy category of projective (resp., injective) $R$-modules. The Verdier quotient triangulated category $\mathsf{D}^{\mathrm{b}}(R)/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj})$ was first studied by Buchweitz [10] under the name of “stable derived category”; it is named by “singularity category” to emphasize certain homological singularity of the ring $R$ reflected by this quotient category (see Orlov [31] and Chen [11]). As another immediate consequence of Theorem 1, we see that the singularity categories $\mathsf{D}^{\mathrm{b}}(R)/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj})\simeq \mathsf{D}^{\mathrm{b}}(\mathsf{R})/\mathsf{K}^{\mathrm{b}}(\mathsf{Inj})$ are compactly generated over left (or right) coherent rings of finite global Gorenstein dimension (or equivalently, left Gorenstein); see Corollaries 34 and 38.

We prove Theorem 1 above by using the finiteness of global Gorenstein AC-homological dimensions.

Gorenstein AC-projective (resp., Gorenstein AC-injective) dimension is defined in terms of resolutions by Gorenstein AC-projective (resp., Gorenstein AC-injective) modules that were initially

[1] From Beligiannis [3], a ring $R$ is called left Gorenstein if any projective $R$-module has finite injective dimension and any injective $R$-module has finite projective dimension.

introduced by Bravo, Gillespie and Hovey [9] as a natural way to extend the notion of Gorenstein projective (resp., Gorenstein injective) modules. We let $\mathsf{GPac\text{-}gldim}\,R$ and $\mathsf{GIac\text{-}gldim}\,R$ denote the global Gorenstein AC-projective and global Gorenstein AC-injective dimension of $R$, respectively. That is, $\mathsf{GPac\text{-}gldim}\,R = \{\text{Gac-pd}_R\, M \mid M \text{ is an } R\text{-module}\}$ and $\mathsf{GIac\text{-}gldim}\,R = \{\text{Gac-id}_R\, M \mid M \text{ is an } R\text{-module}\}$. Recently, Becerril, Mendoza, Pérez and Santiago [2, 6.15] asked under which conditions on $R$ the following statements are true:

- All $R$-modules have finite Gorenstein AC-projective dimension.
- Any $R$-module has finite Gorenstein AC-projective dimension if and only if it has finite Gorenstein AC-injective dimension.

In Section 3 we focus on the above two questions. Our main results in this section are the next two theorems, where the first one is used in the proof of Theorem 1.

**Theorem 3.** *Let $R$ be a ring with* $\text{Ggldim}\,R < \infty$.

(a) *If $R$ is right coherent, then* $\mathsf{GPac\text{-}gldim}\,R < \infty$.
(b) *If $R$ is left coherent, then* $\mathsf{GIac\text{-}gldim}\,R < \infty$.

This result is proved in Theorem 25. The converses of the above statements are not true in general; see Example 27.

**Theorem 4.** *If $R$ is a commutative ring, then* $\mathsf{GIac\text{-}gldim}\,R = \mathsf{GPac\text{-}gldim}\,R$.

This result is proved in Corollary 19. However, to the best of our knowledge, we don't know whether the equality $\mathsf{GIac\text{-}gldim}\,R = \mathsf{GPac\text{-}gldim}\,R$ holds for an arbitrary ring $R$.

## 2. Preliminaries

We begin with some notation and terminology for use throughout this paper.

**5.** By an $R$-complex $M$ we mean a complex of $R$-modules as follows:

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\partial^M_{i+1}} M_i \xrightarrow{\partial^M_i} M_{i-1} \longrightarrow \cdots .$$

We frequently (and without warning) identify $R$-modules with $R$-complexes concentrated in degree 0. For an $R$-complex $M$, we set $\sup M = \sup\{i \in \mathbb{Z} \mid M_i \neq 0\}$ and $\inf M = \inf\{i \in \mathbb{Z} \mid M_i \neq 0\}$. An $R$-complex $M$ is called *bounded* if $\sup M < \infty$ and $\inf M > -\infty$. The symbol $\mathrm{H}_n(M)$ denotes the $n$th *homology* of $M$, i.e., $\operatorname{Ker}\partial^M_n / \operatorname{Im}\partial^M_{n+1}$. An $R$-complex $M$ is called *homology bounded* if $\sup \mathrm{H}(M) < \infty$ and $\inf \mathrm{H}(M) > -\infty$. For an $R$-complex $M$, the symbol $M_{\leq n}$ denotes the subcomplex of $M$ with $(M_{\leq n})_i = M_i$ for $i \leq n$ and $(M_{\leq n})_i = 0$ for $i > n$, and the symbol $M_{\geq n}$ denotes the quotient complex of $M$ with $(M_{\geq n})_i = M_i$ for $i \geq n$ and $(M_{\geq n})_i = 0$ for $i < n$.

We denote by $\mathsf{D}^{\mathrm{b}}(R)$ the bounded derived category of $R$-modules, by $\mathsf{Prj}$ (resp., $\mathsf{Inj}$, $\mathsf{Flat}$, and $\mathsf{Cot}$) the subcategory of projective (resp., injective, flat, and cotorsion) $R$-modules, and by $\mathsf{K}^{\mathrm{b}}(\mathsf{Prj})$ (resp., $\mathsf{K}^{\mathrm{b}}(\mathsf{Inj})$, and $\mathsf{K}^{\mathrm{b}}(\mathsf{FlatCot})$) the bounded homotopy category of projective (resp., injective, and flat and cotorsion) $R$-modules.

**6.** An $R$-module $M$ is called *Gorenstein projective* [15] if there exists an exact sequence $\cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$ of projective $R$-modules such that $M \cong \operatorname{Coker}(P_1 \to P_0)$, and it remains exact after applying the functor $\operatorname{Hom}_R(-, P)$ for each projective $R$-module $P$. Dually, one has the definition of Gorenstein injective $R$-modules. An $R$-module $M$ is called *Gorenstein flat* [16] if there exists an exact sequence $\cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$ of flat $R$-modules such that $M \cong \operatorname{Coker}(F_1 \to F_0)$, and it remains exact after applying the functor $I \otimes_R -$ for each injective $R^{\circ}$-module $I$. We let $\mathsf{GP}$ (resp., $\mathsf{GI}$, and $\mathsf{GF}$) denote the subcategory of Gorenstein projective (resp., Gorenstein injective, and Gorenstein flat) $R$-modules.

The Gorenstein projective dimension of an $R$-module $M$, $\operatorname{Gpd}_R M$, is defined by declaring that $\operatorname{Gpd}_R M \le n$ if and only if $M$ has a Gorenstein projective resolution of length $n$, that is, there is an exact sequence $0 \to G_n \to \cdots \to G_0 \to M \to 0$ with each $G_i$ Gorenstein projective. The definition of Gorenstein injective dimension, $\operatorname{Gid}_R M$, can be defined dually. We let $\operatorname{Ggldim} R$ denote the global Gorenstein projective dimension of $R$, that is, $\operatorname{Ggldim} R = \sup\{\operatorname{Gpd}_R M \mid M \text{ is an } R\text{-module}\}$. The next result is proved by Bennis and Mahdou [7, Thm. 1.1], which is used frequently in the paper.

**Lemma 7.** *For any ring $R$,* $\operatorname{Ggldim} R = \sup\{\operatorname{Gid}_R M \mid M$ *is an $R$-module*$\}$.

**8.** Let $\operatorname{silp} R$ denote the supremum of the injective lengths of projective $R$-modules, and $\operatorname{spli} R$ the supremum of the projective lengths of injective $R$-modules. Since an arbitrary direct sum of projective $R$-modules is projective, the invariant $\operatorname{silp} R$ is finite if and only if every projective $R$-module has finite injective dimension. It follows from Beligiannis and Reiten [5, Thm. VII. 2.2] that every injective $R$-module has finite projective dimension and $\operatorname{silp} R$ is finite if and only if both $\operatorname{spli} R$ and $\operatorname{silp} R$ are finite. Thus by Emmanouil [14, Thm. 4.1] one gets that the global Gorenstein projective dimension $\operatorname{Ggldim} R$ is finite if and only if $R$ is left Gorenstein.

**9.** Recall from [9] that an $R$-module $F$ is *type* $\mathit{FP}_\infty$ if $F$ has a degree-wise finitely generated projective resolution. An $R$-module $A$ is called *absolutely clean* if $\operatorname{Ext}^1_R(F, A) = 0$ for all $R$-modules $F$ of type $\mathrm{FP}_\infty$, and an $R^\circ$-module $L$ is called *level* if $\operatorname{Tor}^R_1(L, F) = 0$ for all $R$-modules $F$ of type $\mathrm{FP}_\infty$.

Recall from [9] that an $R$-module $M$ is *Gorenstein AC-projective* if there exists an exact sequence $\cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$ of projective $R$-modules such that $M \cong \operatorname{Coker}(P_1 \to P_0)$, and it remains exact after applying the functor $\operatorname{Hom}_R(\cdot, L)$ for each level $R$-module $L$.

Dually, an $R$-module $M$ is called *Gorenstein AC-injective* if there exists an exact sequence $\cdots \to I_1 \to I_0 \to I_{-1} \to \cdots$ of injective $R$-modules such that $M \cong \operatorname{Coker}(I_1 \to I_0)$, and it remains exact after applying the functor $\operatorname{Hom}_R(A, \cdot)$ for each absolutely clean $R$-module $A$.

Recall from [8] that an $R$-module $M$ is *Gorenstein AC-flat* if there exists an exact sequence $\cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$ of flat $R$-modules such that $M \cong \operatorname{Coker}(F_1 \to F_0)$, and it remains exact after applying the functor $A \otimes_R -$ for each absolutely clean $R^\circ$-module $A$.

The symbol $\mathsf{GPac}$ (resp., $\mathsf{GIac}$, and $\mathsf{GFac}$) denotes the subcategory of Gorenstein AC-projective (resp., Gorenstein AC-injective, and Gorenstein AC-flat) $R$-modules. It is easy to see that $\mathsf{GPac} \subseteq \mathsf{GP}$ and $\mathsf{GIac} \subseteq \mathsf{GI}$.

From [9, Thm. A.6], one has the next lemma.

**Lemma 10.** *All Gorenstein AC-projective $R$-modules are Gorenstein AC-flat. That is,* $\mathsf{GPac} \subseteq \mathsf{GFac}$.

**11.** The Gorenstein AC-projective dimension of $R$-module $M$, $\text{Gac-pd}_R M$, is defined by declaring that $\text{Gac-pd}_R M \le n$ if and only if $M$ has a Gorenstein AC-projective resolution of length $n$, that is, there is an exact sequence $0 \to G_n \to \cdots \to G_0 \to M \to 0$ with each $G_i$ Gorenstein AC-projective. The Gorenstein AC-injective and Gorenstein AC-flat dimensions are defined similarly, which are denoted $\text{Gac-id}_R M$ and $\text{Gac-fd}_R M$, respectively.

Let $\mathsf{GPac}\text{-gldim} R$ (resp., $\mathsf{GIac}\text{-gldim} R$, and $\mathsf{GFac}\text{-gldim} R$) denote the global Gorenstein AC-projective (resp., global Gorenstein AC-injective, and global Gorenstein AC-flat) dimension of $R$. For example,

$$\mathsf{GPac}\text{-gldim} R = \{\text{Gac-pd}_R M \mid M \text{ is an } R\text{-module}\}.$$

By Lemma 7, one has

$$\operatorname{Ggldim} R \le \min\{\mathsf{GPac}\text{-gldim} R, \mathsf{GIac}\text{-gldim} R\}. \tag{1}$$

**12.** A pair $(\mathsf{X}, \mathsf{Y})$ of subcategories of $R$-modules is called a *cotorsion pair* if $\mathsf{X}^\perp = \mathsf{Y}$ and $\mathsf{Y} = {}^\perp\mathsf{X}$. Here $\mathsf{X}^\perp = \{A \mid \operatorname{Ext}^1_R(X, A) = 0 \text{ for all } X \in \mathsf{X}\}$, and similarly one can define ${}^\perp\mathsf{X}$. A cotorsion pair

$(\mathsf{X},\mathsf{Y})$ is said to be *hereditary* if $\mathrm{Ext}_R^n(X,Y)=0$ for all $X\in\mathsf{X}$, $Y\in\mathsf{Y}$ and $n\geq 1$, or equivalently, if $\mathsf{Y}$ is injectively coresolving (that is, whenever $0\to L'\to L\to L''\to 0$ is exact with $L',L\in\mathsf{Y}$ then $L''$ is also in $\mathsf{Y}$). A cotorsion pair $(\mathsf{X},\mathsf{Y})$ is called *complete* if for any $R$-module $A$, there exist exact sequences $0\to Y\to X\to A\to 0$ and/or $0\to A\to Y'\to X'\to 0$ with $X,X'\in\mathsf{X}$ and $Y,Y'\in\mathsf{Y}$.

## 3. Global Gorenstein AC-homological dimensions

In this section we focus on the global Gorenstein AC-projective/injective dimension. We let $\operatorname{silac} R=\sup\{\mathrm{id}_R M\mid M \text{ is an absolutely clean } R\text{-module}\}$.

**Lemma 13.** *Let R be a ring. Then there exists an equality*

$$\max\{\operatorname{Ggldim} R,\operatorname{silac} R\}=\mathsf{GIac}\text{-}\mathrm{gldim}\, R.$$

**Proof.** For the inequality "$\geq$", we let $\max\{\operatorname{Ggldim} R,\operatorname{silac} R\}=m<\infty$. Then all absolutely clean $R$-module have finite injective dimension. This implies that $\mathsf{GIac}=\mathsf{GI}$. Thus we have $\mathsf{GIac}\text{-}\mathrm{gldim}\, R=\operatorname{Ggldim} R\leq m$.

For the inequality "$\leq$", we let $\mathsf{GIac}\text{-}\mathrm{gldim}\, R=n<\infty$. It is easy to see that $\operatorname{Ggldim} R\leq \mathsf{GIac}\text{-}\mathrm{gldim}\, R=n$. Next we prove that $\operatorname{silac} R\leq n$. Let $A$ be an absolutely clean $R$-module. For each $R$-module $M$, one has $\mathsf{GIac}\text{-}\mathrm{id}_R M\leq n$. So there is an exact sequence $0\to M\to G^0\to\cdots\to G^n\to 0$ with each $G^i$ Gorenstein AC-injective. Thus $\mathrm{Ext}_R^{n+1}(A,M)\cong\mathrm{Ext}_R^1(A,G^n)=0$. This yields that $A$ has finite projective dimension at most $n$. So $A$ has finite injective dimension at most $n$ by [7, Cor. 2.7], as $\operatorname{Ggldim} R\leq\mathsf{GIac}\text{-}\mathrm{gldim}\, R=n$; see (1). Thus one has $\operatorname{silac} R\leq n$. □

The next result is immediate by Lemma 13.

**Lemma 14.** *Let R be a ring with* $\mathsf{GIac}\text{-}\mathrm{gldim}\, R$ *finite. Then all absolutely clean R-modules have finite injective dimension at most* $\mathsf{GIac}\text{-}\mathrm{gldim}\, R$. *Hence, all Gorenstein injective R-modules are Gorenstein AC-injective.*

The following two results are proved dually, where we let

$$\operatorname{spll} R=\sup\{\mathrm{pd}_R M\mid M \text{ is a level } R\text{-module}\}.$$

**Lemma 15.** *Let R be a ring. Then there exists an equality*

$$\max\{\operatorname{Ggldim} R,\operatorname{spll} R\}=\mathsf{GPac}\text{-}\mathrm{gldim}\, R.$$

**Lemma 16.** *Let R be a ring with* $\mathsf{GPac}\text{-}\mathrm{gldim}\, R$ *finite. Then all level R-modules have finite projective dimension at most* $\mathsf{GPac}\text{-}\mathrm{gldim}\, R$. *Hence, all Gorenstein projective R-modules are Gorenstein AC-projective.*

**Theorem 17.** *The following statements hold:*

(a) *If* $\mathsf{GIac}\text{-}\mathrm{gldim}\, R<\infty$, *then there is an inequality* $\mathsf{GIac}\text{-}\mathrm{gldim}\, R\leq\mathsf{GPac}\text{-}\mathrm{gldim}\, R$.
(b) *If* $\mathsf{GIac}\text{-}\mathrm{gldim}\, R^\circ<\infty$, *then there is an inequality* $\mathsf{GPac}\text{-}\mathrm{gldim}\, R\leq\mathsf{GIac}\text{-}\mathrm{gldim}\, R$.

**Proof.** (a). By Lemma 14 all Gorenstein injective $R$-modules are Gorenstein AC-injective, so one has $\mathsf{GIac}\text{-}\mathrm{gldim}\, R=\operatorname{Ggldim} R\leq\mathsf{GPac}\text{-}\mathrm{gldim}\, R$.

(b). We may assume that $\mathsf{GIac}\text{-}\mathrm{gldim}\, R$ is finite. By Lemma 14 all absolutely clean $R^\circ$-modules have finite injective dimension, as $\mathsf{GIac}\text{-}\mathrm{gldim}\, R^\circ<\infty$. Let $L$ be a level $R$-module. Then by [9, Thm. 2.12] $L^+$ is a absolutely clean $R^\circ$-module, and so $\mathrm{fd}_R L=\mathrm{id}_R L^+<\infty$. Thus $\mathrm{pd}_R L<\infty$ by [7, Cor. 2.7] as $\operatorname{Ggldim} R<\infty$. Hence, one has $\mathsf{GP}=\mathsf{GPac}$; see Lemma 16. Thus $\mathsf{GPac}\text{-}\mathrm{gldim}\, R=\operatorname{Ggldim} R\leq\mathsf{GIac}\text{-}\mathrm{gldim}\, R$; see 11. □

The next result is proved dually.

**Theorem 18.** *The following statements hold:*

(a) *If* GPac-gldim $R < \infty$*, then there is an inequality* GPac-gldim $R \leq$ GIac-gldim $R$.
(b) *If* GPac-gldim $R^{\circ} < \infty$*, then there is an inequality* GIac-gldim $R \leq$ GPac-gldim $R$.

The next corollary advertised in the introduction is immediate by Theorems 17 and 18.

**Corollary 19.** *If $R$ is a commutative ring, then* GIac-gldim $R =$ GPac-gldim $R$.

**Lemma 20.** *Let $M$ be an $R$-module and $n$ a nonnegative integer. Then the following conditions are equivalent.*

(i) GFac-pd$_R\,M \leq n$.
(ii) *There is an exact sequence $0 \to M \to F \to N \to 0$ of $R$-modules with $\mathrm{fd}_R\,F \leq n$ and* $N \in$ GPac.

**Proof.** (i) $\Longrightarrow$ (ii)**.** We prove the result by induction on $n$. The case where $n = 0$ holds by Proposition 40. Now let $n > 0$. Consider an exact sequence $0 \to K \to H \to M \to 0$ of $R$-modules with $H$ flat. Then one has GFac-pd$_R\,K \leq n-1$, and so by induction, there is an exact sequence $0 \to K \to H' \to G \to 0$ of $R$-modules with $\mathrm{fd}_R\,H' \leq n-1$ and $G \in$ GPac. Consider the following pushout diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & H & \longrightarrow & M & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H' & \longrightarrow & H'' & \longrightarrow & M & \longrightarrow & 0.\\
 & & \downarrow & & \downarrow & & & & \\
 & & G & = & G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

In the middle columnn, by Lemma 10 both $H$ and $G$ are in GFac, so is $H''$. Whence, by Proposition 40, there is an exact sequence $0 \to H'' \to L \to N \to 0$ of $R$-modules with $L \in$ Flat and $N \in$ GPac. Now we obtain another pushout diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H' & \longrightarrow & H'' & \longrightarrow & M & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H' & \longrightarrow & L & \longrightarrow & F & \longrightarrow & 0.\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N & = & N & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since, in the middle row, $L \in$ Flat and $\mathrm{fd}_R\,H' \leq n-1$, it follows that $\mathrm{fd}_R\,F \leq n$. So the condition (ii) holds by the rightmost non-zero column.

(ii) $\Longrightarrow$ (i). Assume that there is an exact sequence $0 \to M \to F \to N \to 0$ of $R$-modules with $\mathrm{fd}_R F \leq n$ and $N \in \mathsf{GPac}$. Since $(\mathsf{GPac}, \mathsf{GPac}^{\perp})$ is a hereditary complete cotorsion pair by Gillespie [21, fact. 10.2], there is an exact sequence $0 \to E \to L \to F \to 0$ of $R$-modules with $L \in \mathsf{GPac}$ and $E \in \mathsf{GPac}^{\perp}$. Consider the next pullback diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & E & = & E & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & Q & \to & L & \to & N & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M & \to & F & \to & N & \to & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By the middle column one gets $\mathsf{GFac}\text{-}\mathrm{pd}_R E \leq n-1$ since $\mathsf{GFac}\text{-}\mathrm{pd}_R F \leq \mathrm{fd}_R F \leq n$ and $L \in \mathsf{GPac} \subseteq \mathsf{GFac}$; see Lemma 10. By the middle row one gets that $Q$ is in $\mathsf{GPac} \subseteq \mathsf{GFac}$ since $N$ and $L$ are in $\mathsf{GPac}$. Thus, by the first non-zero column one has $\mathsf{GFac}\text{-}\mathrm{pd}_R M \leq n$. □

**Proposition 21.** *Let $R$ be a ring with $\mathsf{GFac}\text{-}\mathrm{gldim}\, R < \infty$. Then all Gorenstein flat $R$-modules are Gorenstein AC-flat.*

**Proof.** We assume that $\mathsf{GFac}\text{-}\mathrm{gldim}\, R = n < \infty$. Let $M$ be a Gorenstein flat $R$-module. Then one has $\mathsf{GFac}\text{-}\mathrm{pd}_R M \leq n$, and so by Lemma 20 there is an exact sequence $0 \to M \to F \to N \to 0$ of $R$-modules with $\mathrm{fd}_R F \leq n$ and $N \in \mathsf{GPac} \subseteq \mathsf{GFac}$. Since $M$ and $N$ are Gorenstein flat, a recent result by Šaroch and Šťovíček [32, Thm. 3.11] yields that $F$ is Gorenstein flat. Thus $F$ is flat, and hence $M$ is Gorenstein AC-flat by Proposition 40. □

**22.** The cotorsion dimension of $R$-module $M$, $\mathsf{Cot}\text{-}\mathrm{id}_R M$, is defined by declaring that $\mathsf{Cot}\text{-}\mathrm{id}_R M \leq n$ if and only if $M$ has a cotorsion coresolution of length $n$, that is, there is an exact sequence $0 \to M \to C^0 \cdots \to C^n \to 0$ with each $C^i$ cotorsion. We let $\mathsf{Cot}\text{-}\mathrm{gldim} R = \sup\{\mathsf{Cot}\text{-}\mathrm{id}_R M \mid M \text{ is an } R\text{-module}\}$.

The next result was proved by Mao and Ding in [30, Thm. 19.2.14].

**Lemma 23.** *For each $R$-module $M$ there exists an inequality*

$$\mathrm{pd}_R M \leq \mathrm{fd}_R M + \mathsf{Cot}\text{-}\mathrm{gldim} R.$$

The next result is used in the proofs of Corollaries 33 and 34.

**Theorem 24.** *Let $R$ be a ring. Then there exist inequalities*

$$\max\{\mathsf{GFac}\text{-}\mathrm{gldim}\, R, \mathsf{Cot}\text{-}\mathrm{gldim} R\} \leq \mathsf{GPac}\text{-}\mathrm{gldim}\, R \leq \mathsf{GFac}\text{-}\mathrm{gldim}\, R + \mathsf{Cot}\text{-}\mathrm{gldim} R.$$

*In particular,* $\mathsf{GPac}\text{-}\mathrm{gldim}\, R$ *is finite if and only if* $\mathsf{GFac}\text{-}\mathrm{gldim}\, R$ *and* $\mathsf{Cot}\text{-}\mathrm{gldim} R$ *are finite.*

**Proof.** For the first inequality one let $\mathsf{GPac}\text{-}\mathrm{gldim}\, R = n < \infty$. We notice that all Gorenstein AC-projective modules are Gorenstein AC-flat. So one has $\mathsf{GFac}\text{-}\mathrm{gldim}\, R \leq n$. Let $F$ be an flat $R$-module. Since $\mathrm{Ggldim}\, R \leq \mathsf{GPac}\text{-}\mathrm{gldim}\, R = n$, one has $\mathrm{pd}_R F \leq n$ by [7, Cor. 2.7]. Thus [30, Cor. 7.2.6] yields $\mathsf{Cot}\text{-}\mathrm{gldim} R \leq n$.

For the second inequality we let $\mathsf{GFac}\text{-}\mathrm{gldim}\, R = n < \infty$ and $\mathsf{Cot}\text{-}\mathrm{gldim} R = m < \infty$. Let $M$ be an $R$-module. Then $\mathsf{GFac}\text{-}\mathrm{pd}_R M \leq n$. By Lemma 20, one gets an exact sequence $0 \to M \to F \to N \to 0$ of $R$-modules with $\mathrm{fd}_R F \leq n$ and $N \in \mathsf{GPac}$. So $\mathrm{pd}_R H \leq n+m$; see Lemma 23. Similar to the proof of (ii) $\Longrightarrow$ (i) in Lemma 20 one gets $\mathsf{GPac}\text{-}\mathrm{pd}_R M \leq n+m$. Thus $\mathsf{GPac}\text{-}\mathrm{gldim}\, R \leq n+m$. □

Next we give some rings that have finite global Gorenstein AC-projective/injective dimension.

**Theorem 25.** *For a ring $R$ with* $\mathrm{Ggldim}\, R < \infty$*, the following statements hold:*

(a) *If $R$ is left coherent, then* $\mathsf{GIac}\text{-}\mathrm{gldim}\, R < \infty$.

(b) *If $R$ is right coherent, then* $\mathsf{GPac}\text{-}\mathrm{gldim}\, R < \infty$.

**Proof.** (a). By Lemma 13, it is suffices to show that $\mathrm{silac}\, R < \infty$. Let $A$ be an absolutely clean $R$-module, and let $\mathrm{Ggldim}\, R = n < \infty$. Then [9, Cor. 2.9] yields that $A$ is FP-injective since $R$ is left coherent. Hence, there is a pure exact sequence $0 \to A \to I \to C \to 0$ of $R$-modules with $I$ injective. By [7, Cor. 2.7], one has $\mathrm{fd}_R\, I \leq n$. It follows that $\mathrm{fd}_R\, A \leq \mathrm{fd}_R\, I \leq n$, and hence one has $\mathrm{id}_R\, A \leq n$ again by [7, Cor. 2.7]. This gives that $\mathrm{silac}\, R \leq n < \infty$.

(b). By Lemma 15, it is suffices to show that $\mathrm{spll}\, R < \infty$. Let $L$ be a level $R$-module, and let $\mathrm{Ggldim}\, R = n < \infty$. Then [9, Cor. 2.11] yields that $L$ is flat since $R$ is right coherent. Hence one has $\mathrm{pd}_R\, L \leq n$ by [7, Cor. 2.7]. This gives that $\mathrm{spll}\, R \leq n < \infty$. □

In the followin we give an example to show that the converses of the statements in Theorem 25 are not true in general. Before that we give some facts.

**26.** Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings. If $M_i$ is an $R_i$-module for $i = 1, 2, \cdots, n$ then $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is an $R$-module. Conversely, if $M$ is an $R$-module then it is of the form $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$, where $M_i$ is an $R_i$-module for $i = 1, 2, \cdots, n$. It is easy to see that the following equalities hold

$$\mathsf{GPac}\text{-}\mathrm{pd}_R\, M = \sup\{\mathsf{GPac}\text{-}\mathrm{pd}_{R_i}\, M_i \mid i = 1, \ldots, n\}$$

and

$$\mathsf{GIac}\text{-}\mathrm{id}_R\, M = \sup\{\mathsf{GIac}\text{-}\mathrm{id}_{R_i}\, M_i \mid i = 1, \ldots, n\},$$

which are parallel to the well-known ones about projective and injective dimension, respectively. So one gets that $R$ is of finite global Gorenstein AC-projective/injective dimension if and only if each $R_i$ is so; the same conclusion holds for global dimension. On the other hand, it is known that $R$ is left/right coherent if and only if each $R_i$ is so.

**Example 27.** Let $R = D + (x_1, x_2)K[x_1, x_2]$, where $D$ is a Dedekind domain and $K$ its quotient field. According to Kirkman and Kuzmanovich [27, Example in p. 128], $R$ is a commutative non-coherent ring of finite global dimension. On the other hand, there exists a commutative Iwanaga–Gorenstein ring $S$ of infinite global dimension; see Bennis [6, p. 857]. So $S$ has finite global Gorenstein AC-projective dimension and finite global Gorenstein AC-injective dimension; see Theorem 25. Hence, $R \times S$ has finite global Gorenstein AC-projective dimension and finite global Gorenstein AC-injective dimension. However, $R \times S$ is neither of finite global dimension nor coherent.

## 4. Compactly generatedness of singularity categories

We now turn to study the compactly generatedness of singularity categories and stable categories with respect to Gorenstein AC-homological modules, and prove Theorem 1 advertised in the introduction. We open this section with the following terminology.

**28.** Let $(\mathsf{A}, \mathsf{B})$ be a cotorsion pair in $\mathsf{Mod}(R)$, and let $X$ be an $R$-complex. From Yang and Ding [34], the $\mathsf{A}$*-projective dimension* of $X$, $\mathsf{A}\text{-}\,\mathrm{pd}_R\, X$, is defined as

$$\mathsf{A}\text{-}\,\mathrm{pd}_R\, X = \inf\{\sup A \mid X \simeq A \text{ in } \mathrm{D}(R) \text{ with } A \in \mathsf{dgA}\}.$$

The $\mathsf{B}$*-injective dimension* of $X$, $\mathsf{B}\text{-}\,\mathrm{id}_R\, X$, is defined as

$$\mathsf{B}\text{-}\,\mathrm{id}_R\, X = \inf\{-\inf B \mid X \simeq B \text{ in } \mathrm{D}(R) \text{ with } B \in \mathsf{dgB}\}.$$

Here dgA and dgB denote the subcategories of dg-A complexes and dg-B complexes, respectively; see Gillespie [20].

It is known that $(\mathsf{Flat},\mathsf{Cot})$ is a complete hereditary cotorsion pair, and by [9] and Proposition 41 $(\mathsf{GPac},\mathsf{GPac}^{\perp})$, $({}^{\perp}\mathsf{GIac},\mathsf{GIac})$ and $(\mathsf{GFac},(\mathsf{GPac})^{\perp}\cap\mathsf{Cot})$ are complete hereditary cotorsion pairs. So for an $R$-complex $X$ we have the definitions of $\mathsf{GPac}\text{-}\mathrm{pd}_R X$, $\mathsf{GFac}\text{-}\mathrm{pd}_R X$, $\mathsf{GIac}\text{-}\mathrm{id}_R X$ and $\mathsf{Cot}\text{-}\mathrm{id}_R X$. We let $\mathsf{D}^{\mathrm{b}}(R)_{\widehat{\mathsf{GPac}}}$ (resp., $\mathsf{D}^{\mathrm{b}}(R)_{\widehat{\mathsf{GFac}}\cap\widehat{\mathsf{Cot}}}$, and $\mathsf{D}^{\mathrm{b}}(R)_{\widehat{\mathsf{GIac}}}$) denote the triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(R)$ consisting of all homology bounded complexes $X$ with $\mathsf{GPac}\text{-}\mathrm{pd}_R X<\infty$ (resp, $\mathsf{GFac}\text{-}\mathrm{pd}_R X<\infty$ and $\mathsf{Cot}\text{-}\mathrm{id}_R X<\infty$, and $\mathsf{GIac}\text{-}\mathrm{id}_R X<\infty$). It is easy to see that for an $R$-module $M$ (viewed as an $R$-complex concentrated in degree 0), the definitions of $\mathsf{GPac}\text{-}\mathrm{pd}_R M$, $\mathsf{GFac}\text{-}\mathrm{pd}_R M$, $\mathsf{GIac}\text{-}\mathrm{id}_R M$ and $\mathsf{Cot}\text{-}\mathrm{id}_R M$ are the same as in §11 and §22.

**Lemma 29.** *The subcategory* $\mathsf{GPac}$ *(resp.,* $\mathsf{GFac}\cap\mathsf{Cot}$*, and* $\mathsf{GIac}$*) together with all short exact sequences in* $\mathsf{GPac}$ *(resp.,* $\mathsf{GFac}\cap\mathsf{Cot}$*, and* $\mathsf{GIac}$*) forms a Frobenius category with projective-injective objects all projective (resp., flat-cotorsion, and injective) $R$-modules.*

**Proof.** We give a straight proof for the case $\mathsf{GIac}$; see §42 for the other ones.

The subcategory $\mathsf{GIac}$, together with all short exact sequences in $\mathsf{GIac}$, forms an exact category, as $\mathsf{GIac}$ is closed under extensions by [9, Lem. 5.6].

For $I\in\mathsf{Inj}$ and $G\in\mathsf{GIac}$, one gets that $\mathrm{Ext}^1_R(G,I)=0=\mathrm{Ext}^1_R(I,G)$, which yields that all injective $R$-modules are both projectives and injectives in $\mathsf{GIac}$. Conversely, let $M$ (resp., $N$) be a injective (resp., projective) object in $\mathsf{GIac}$. Then there exist split exact sequences $0\to M\to I\to M'\to 0$ and $0\to N'\to H\to N\to 0$ with $I,H\in\mathsf{Inj}$ and $M',N'\in\mathsf{GIac}$. So both $M$ and $N$ are in $\mathsf{Inj}$. Thus projectives and injectives in $\mathsf{GPac}$ are exactly injective $R$-modules.

Finally, for every $G\in\mathsf{GIac}$ there exist exact sequences $0\to G\to I'\to G'\to 0$ and $0\to G''\to I''\to G\to 0$ with $I',I''\in\mathsf{Inj}$ and $G',G''\in\mathsf{GIac}$, so the subcategory $\mathsf{GIac}$ has enough injectives and enough projectives. □

**30.** By Lemma 29, the stable category $\underline{\mathsf{GPac}}$ (resp., $\underline{\mathsf{GFac}\cap\mathsf{Cot}}$, and $\underline{\mathsf{GIac}}$) modulo projectives (resp., flat-cotorsions, and injectives) is a triangulated category.

**Theorem 31.** *The following conditions are equivalent.*

(i) $\mathsf{GPac}\text{-}\mathrm{gldim}\, R<\infty$.
(ii) *There is an equality* $\mathsf{D}^{\mathrm{b}}(R)_{\widehat{\mathsf{GPac}}}=\mathsf{D}^{\mathrm{b}}(R)$.
(iii) *The natural functor* $\mathrm{F}:\underline{\mathsf{GPac}}\to\mathsf{D}^{\mathrm{b}}(R)/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj})$ *induced by the compositions*

$$\mathsf{GPac}\hookrightarrow\mathsf{D}^{\mathrm{b}}(\mathrm{R})_{\widehat{\mathsf{GPac}}}\to\mathsf{D}^{\mathrm{b}}(\mathrm{R})_{\widehat{\mathsf{GPac}}}/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj})\hookrightarrow\mathsf{D}^{\mathrm{b}}(\mathrm{R})/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj})$$

*is a triangulated equivalence.*

**Proof.** (i) $\Longrightarrow$ (ii)**.** Fix $P\in\mathsf{D}^{\mathrm{b}}(R)$. It suffices to show that $\mathsf{GPac}\text{-}\mathrm{pd}_R P<\infty$. Without loss of generality, we may assume that $P$ is bounded as follows:

$$P=0\to P_k\to P_{k-1}\to\cdots\to P_1\to P_0\to 0.$$

Consider the exact sequence $0\to P_0\to P\to P_{\geq 1}\to 0$ of $R$-complexes. Since $P_0$ and $P_{\geq 1}$ have finite Gorenstein AC-projective dimension by (i) and induction on $k$, respectively, so does $P$.

(ii) $\Longrightarrow$ (i)**.** Each $R$-module $M$, viewed as an $R$-complex concentrated in degree 0, is in $\mathsf{D}^{\mathrm{b}}(R)$. Thus $M\in\mathsf{D}^{\mathrm{b}}(R)_{\widehat{\mathsf{GPac}}}$, and so $\mathsf{GPac}\text{-}\mathrm{pd}_R M<\infty$; see §28. Note that $(\mathsf{GPac},\mathsf{GPac}^{\perp})$ is a complete hereditary cotorsion pair. It is a standard way to see that for any family $(M_i)_{i\in\Lambda}$ of $R$-modules there is an equality

$$\mathsf{GPac}\text{-}\mathrm{pd}_R(\oplus_{i\in\Lambda}M_i)=\sup\{\mathsf{GPac}\text{-}\mathrm{pd}_R M_i\mid i\in\Lambda\}.$$

Thus it is easy to verify that the condition (i) holds.

(ii) $\Longrightarrow$ (iii). From a result by Di, Liu, Yang and Zhang [13, Cor. 5.9], the induced natural functor $\mathrm{F}: \underline{\mathsf{GPac}} \to \mathsf{D}^{\mathrm{b}}(R)_{\widehat{\mathsf{GPac}}}/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj})$ is a triangulated equivalence, so the statement (iii) follows from (ii).

(iii) $\Longrightarrow$ (ii). It is clear that $\mathsf{D}^{\mathrm{b}}(R)_{\widehat{\mathsf{GPac}}} \subseteq \mathsf{D}^{\mathrm{b}}(R)$. Conversely, we let $X \in \mathsf{D}^{\mathrm{b}}(R)$ ($X$ is also an object of $\mathsf{D}^{\mathrm{b}}(R)/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj})$). By (iii) and [13, Cor. 5.9], the functor

$$\mathsf{D}^{\mathrm{b}}(R)_{\widehat{\mathsf{GPac}}}/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj}) \hookrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{R})/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj})$$

is a triangulated equivalence. We notice that each triangulated equivalence is dense. So $X$ is isomorphic to an $R$-complex in $\mathsf{D}^{\mathrm{b}}(R)_{\widehat{\mathsf{GPac}}}/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj})$. It follows that $X$ is in $\mathsf{D}^{\mathrm{b}}(R)_{\widehat{\mathsf{GPac}}}$. □

**Theorem 32.** *The following conditions are equivalent.*

(i) $\mathsf{GFac}\text{-gldim}\, R < \infty$ *and* $\mathsf{Cot}\text{-gldim}\, R < \infty$.
(ii) *There is an equality* $\mathsf{D}^{\mathrm{b}}(R)_{\widehat{\mathsf{GFac}}\cap\widehat{\mathsf{Cot}}} = \mathsf{D}^{\mathrm{b}}(R)$.
(iii) *There is a triangulated equivalence*

$$\underline{\mathsf{GFac} \cap \mathsf{Cot}} \simeq \mathsf{D}^{\mathrm{b}}(R)/\mathsf{K}^{\mathrm{b}}(\mathsf{FlatCot}).$$

(iv) *The natural functor* $\mathrm{F}: \underline{\mathsf{GFac} \cap \mathsf{Cot}} \to \mathsf{D}^{\mathrm{b}}(R)/\mathsf{K}^{\mathrm{b}}(\mathsf{FlatCot})$ *induced by the compositions* $\mathsf{GFac} \cap \mathsf{Cot} \hookrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{R})_{\widehat{\mathsf{GFac}}\cap\widehat{\mathsf{Cot}}} \to \mathsf{D}^{\mathrm{b}}(\mathrm{R})_{\widehat{\mathsf{GFac}}\cap\widehat{\mathsf{Cot}}}/\mathsf{K}^{\mathrm{b}}(\mathsf{FlatCot}) \hookrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{R})/\mathsf{K}^{\mathrm{b}}(\mathsf{FlatCot})$ *is a triangulated equivalence.*

**Proof.** Analogous to the proof of Theorem 31, using Corollary 44 instead of [13, Cor. 5.9]. □

**Corollary 33.** *Let $R$ be a ring with* $\mathsf{GPac}\text{-gldim}\, R$ *finite. Then*

$$\mathsf{D}^{\mathrm{b}}(R)/\mathsf{K}^{\mathrm{b}}(\mathsf{Inj}) \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{R})/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj}) \simeq \underline{\mathsf{GPac}} \simeq \underline{\mathsf{GFac} \cap \mathsf{Cot}} \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{R})/\mathsf{K}^{\mathrm{b}}(\mathsf{FlatCot})$$

*are compactly generated.*

**Proof.** The first equivalence in the statement holds by [3, Thm. 6.9] since $\mathrm{Ggldim}\, R$ is finite; see (1). The second equivalence follows from Theorem 31, the third one holds by Corollary 43, and the last one follows from Theorems 24 and 32. By a careful reading of the proof of Gillespie [25, Thm. 6.2], one gets that $\underline{\mathsf{GPac}}$ is compactly generated. □

It is from [4, lem. 6.6 and thm. 6.7] that if $R$ is a right coherent and left perfect or left Morita ring with $\mathrm{Ggldim}\, R < \infty$ then $\underline{\mathsf{GP}} \simeq \underline{\mathsf{GI}}$ are compactly generated. The same conclusion holds if $R$ is Iwanaga–Gorenstein; see [26, Thm. 9.4] or [11, Thm. 4.1]. We have the next improved result.

**Corollary 34.** *Let $R$ be a right coherent ring with* $\mathrm{Ggldim}\, R < \infty$. *Then*

$$\mathsf{D}^{\mathrm{b}}(R)/\mathsf{K}^{\mathrm{b}}(\mathsf{Inj}) \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{R})/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj}) \simeq \underline{\mathsf{GP}} \simeq \underline{\mathsf{GI}} \simeq \underline{\mathsf{GF} \cap \mathsf{Cot}} \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{R})/\mathsf{K}^{\mathrm{b}}(\mathsf{FlatCot})$$

*are compactly generated.*

**Proof.** By Theorem 25 one has $\mathsf{GPac}\text{-gldim}\, R$ finite. We notice that $\underline{\mathsf{GP}} \simeq \underline{\mathsf{GI}}$ by [3, Thm. 6.9] as $\mathrm{Ggldim}\, R < \infty$. On the other hand, by Lemma 16 one has $\mathsf{GPac} = \mathsf{GP}$, and the equality $\mathsf{GFac} = \mathsf{GF}$ holds by Proposition 21 and Theorem 24. So the desired result in the statement follows from Corollary 33. □

Dual to the proof of Theorem 31, we have the following result.

**Theorem 35.** *The following conditions are equivalent.*

(i) $\mathsf{GIac}\text{-gldim}\, R < \infty$.
(ii) *There is an equality* $\mathsf{D}^{\mathrm{b}}(R)_{\widehat{\mathsf{GIac}}} = \mathsf{D}^{\mathrm{b}}(R)$
(iii) *The natural functor* $\mathrm{F}: \underline{\mathsf{GIac}} \to \mathsf{D}^{\mathrm{b}}(R)/\mathsf{K}^{\mathrm{b}}(\mathsf{Inj})$ *induced by the compositions*

$$\mathsf{GIac} \hookrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{R})_{\widehat{\mathsf{GIac}}} \to \mathsf{D}^{\mathrm{b}}(\mathrm{R})_{\widehat{\mathsf{GIac}}}/\mathsf{K}^{\mathrm{b}}(\mathsf{Inj}) \hookrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{R})/\mathsf{K}^{\mathrm{b}}(\mathsf{Inj})$$

*is a triangulated equivalence.*

From Gillespie [24, Def. 5.1], a complex $I$ of injective $R$-modules is called *AC-injective* if each chain map into $I$ from an acyclic complex with each cycle absolutely clean is null homotopic.

**Proposition 36.** *Let $R$ be a ring with* $\mathsf{Glac}\text{-gldim}\,R$ *finite. Then all complexes of injective $R$-modules are AC-injective.*

**Proof.** We let $\mathsf{dw}\widetilde{\mathsf{Inj}}$ denote the subcategory of complexes of injective $R$-modules. Let $I \in \mathsf{dw}\widetilde{\mathsf{Inj}}$, and let $\alpha : X \to I$ be an homomorphisms of $R$-complexes with $X$ acyclic and each cycle $\mathrm{Z}_i(X)$ absolutely clean. Next we prove that $\alpha$ is null homotopic. Set $n = \mathsf{Glac}\text{-gldim}\,R < \infty$. Then by Lemma 14 each cycle $\mathrm{Z}_i(X)$ has finite injective dimension $\le n$, and hence has finite flat dimension $\le n$ as $\mathrm{Ggldim}\,R \le n$. On the other hand, by [21, Prop. 7.2] the pair $({}^{\perp}\mathsf{dw}\widetilde{\mathsf{Inj}}, \mathsf{dw}\widetilde{\mathsf{Inj}})$ is an injective cotorsion pair in $\mathsf{Ch}(R)$. Then it follows from Gillespie [23, Cor. 3.3] that $X$ is in ${}^{\perp}\mathsf{dw}\widetilde{\mathsf{Inj}}$, and so is $\Sigma\mathsf{X}$. Thus one has $\mathrm{Ext}^1_{\mathsf{Ch}(R)}(\Sigma X, I) = 0$. This yields that the exact sequence $0 \to I \to \mathsf{Cone}\alpha \to \Sigma\mathsf{X} \to 0$ is split. So $\alpha$ is null homotopic; see Enochs, Jenda and Xu [17, Lem. 3.2]. □

**Corollary 37.** *Let $R$ be a ring with* $\mathsf{Glac}\text{-gldim}\,R$ *finite. Then*

$$\underline{\mathsf{Glac}} \simeq \mathsf{D}^{\mathrm{b}}(R)/\mathsf{K}^{\mathrm{b}}(\mathsf{Inj}) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{R})/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj})$$

*are compactly generated.*

**Proof.** The first equivalence in the statement holds by Theorem 35, and the second one follows from [3, Thm. 6.9] as $\mathrm{Ggldim}\,R < \infty$. Next we prove that $\underline{\mathsf{Glac}}$ is compactly generated. Let $\mathsf{S}(\mathsf{ACInj})$ denote the homotopy category of all acyclic AC-injective $R$-complexes, and let $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj})$ (resp., $\mathsf{K}_{\mathsf{tac}}(\mathsf{Inj})$) denote the homotopy category of acyclic (resp., totally acyclic) complexes of injective $R$-modules. Consider the following equivalences:

$$\underline{\mathsf{Glac}} = \underline{\mathsf{GI}} \simeq \mathsf{K}_{\mathsf{tac}}(\mathsf{Inj}) = \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}) = \mathsf{S}(\mathsf{ACInj})\,.$$

Here the first equality holds by Lemma 14. Since $\mathrm{Ggldim}\,R$ is finite by (1), all $R$-module have finite Gorenstein injective dimension by Lemma 7. It follows that every acyclic complex of injective $R$-modules has Gorenstein injective cycles and so it is totally acyclic. This yields that the second equality holds. The last equality follows from Lemma 36; while the equivalence holds by Krause [29, Prop. 7.2]. Finally, from [24, Thm. 5.8 and 4.6] that $\mathsf{S}(\mathsf{ACInj})$ is compactly generated. □

Let $R$ be a ring with $\mathsf{Glac}\text{-gldim}\,R$ finite. Then one has $\underline{\mathsf{GP}} \simeq \underline{\mathsf{GI}}$ by [3, Thm. 6.9] as $\mathrm{Ggldim}\,R < \infty$; see §11. On the other hand, by Lemma 14, the equality $\mathsf{Glac} = \mathsf{GI}$ holds. So the next result is immediate by Theorem 25 and Corollary 37.

**Corollary 38.** *Let $R$ be a left coherent ring with* $\mathrm{Ggldim}\,R < \infty$. *Then*

$$\mathsf{D}^{\mathrm{b}}(R)/\mathsf{K}^{\mathrm{b}}(\mathsf{Inj}) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{R})/\mathsf{K}^{\mathrm{b}}(\mathsf{Prj}) \simeq \underline{\mathsf{GP}} \simeq \underline{\mathsf{GI}}$$

*are compactly generated.*

We close this section with the following example; it shows that coherent rings of finite global Gorenstein dimension may not be Iwanaga–Gorenstein nor perfect nor Morita[2]. Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings (see §26). It is easy to see that $R$ is Iwanaga–Gorenstein (resp., left perfect and left Morita) if and only if each $R_i$ is Iwanaga–Gorenstein (resp., left perfect and left Morita).

[2] See [4] for the definition of Morita rings. It is known that a ring $R$ is left Morita if and only if $R$ is left Artinian and $\mathsf{Mod}(R)$ has a finitely generated injective cogenerator.

**Example 39.** Let $R=\mathbb{Z}$ and $S=\begin{pmatrix}\mathbb{Q} & \mathbb{R}\\ 0 & \mathbb{Q}\end{pmatrix}$. Then $R$ is a commutative Iwanaga–Gorenstein ring that is neither perfect nor Artin; hence $R$ is commutative coherent with $\mathrm{Ggldim}\, R<\infty$. According to Wang [33, Exa. 3.4] $S$ is a commutative perfect coherent (non-noetherian) ring with $\mathrm{Ggldim}\, S<\infty$. Then the direct product $R\times S$ is a commutative coherent ring with $\mathrm{Ggldim}(R\times S)=\sup\{\mathrm{Ggldim}\, R,\mathrm{Ggldim}\, S\}<\infty$, which is neither Iwanaga–Gorenstein nor perfect nor Morita.

## Appendix. Gorenstein AC-flat modules

In this section we give some properties of Gorenstein AC-flat modules. We notice that all Gorenstein AC-projective $R$-modules are Gorenstein AC-flat. Actually, by [9, Thm. A.6], an $R$-module $M$ is Gorenstein AC-projective if and only if there exists an exact sequence $\cdots\to P_1\to P_0\to P_{-1}\to\cdots$ of projective $R$-modules such that $M\cong \mathrm{Coker}(P_1\to P_0)$, and it remains exact after applying the functor $A\otimes_R -$ for each absolutely clean $R^\circ$-module $A$. The next two results are from Estrada, Iacob and Pérez [19, Thm. 2.12] and [19, Exa. 2.17 (2)].

**Proposition 40.** *The following conditions are equivalent for an $R$-module $M$.*

(i) *$M$ is Gorenstein AC-flat.*
(ii) *There is a short exact sequence $0\to K\to L\to M\to 0$ of $R$-modules with $K\in\mathsf{Flat}$ and $L\in\mathsf{GPac}$, and it remains exact after applying the functor $\mathrm{Hom}_R(-,C)$ for any (flat) cotorsion $R$-module $C$.*
(iii) *$\mathrm{Ext}^1_R(M,C)=0$ holds for all cotorsion $R$-modules $C\in(\mathsf{GPac})^\perp$.*
(iv) *There is a short exact sequence $0\to M\to F\to N\to 0$ of $R$-modules with $F\in\mathsf{Flat}$ and $N\in\mathsf{GPac}$.*

**Proposition 41.** *The pair $(\mathsf{GFac},(\mathsf{GPac})^\perp\cap\mathsf{Cot})$ is a complete and hereditary cotorsion pair with the equality $\mathsf{GFac}\cap(\mathsf{GPac})^\perp=\mathsf{Flat}$.*

Recall that a triple $(\mathsf{Q},\mathsf{W},\mathsf{R})$ of classes of $R$-modules is Hovey triple if $\mathsf{W}$ is thick and $(\mathsf{Q}\cap\mathsf{W},\mathsf{R})$ and $(\mathsf{Q},\mathsf{W}\cap\mathsf{R})$ are complete cotorsion pairs. If furthermore the above two cotorsion pairs are hereditary then the Hovey triple $(\mathsf{Q},\mathsf{W},\mathsf{R})$ is called hereditary. From Hovey [26, Thm. 2.2], an abelian model structure on $\mathsf{Mod}(R)$ is equivalent to a Hovey triple. This fact is known as "Hovey correspondence" in the literature. We hence always denote an abelian model structure $\mathscr{M}$ as a Hovey triple $\mathscr{M}=(\mathsf{Q},\mathsf{W},\mathsf{R})$.

For an abelian model structure $\mathscr{M}=(\mathsf{Q},\mathsf{W},\mathsf{R})$, we denote by $\mathrm{Ho}(\mathscr{M})$ the homotopy category of $\mathscr{M}$. By Gillespie [22, Sec. 4 and 5], for any hereditary Hovey triple $\mathscr{M}=(\mathsf{Q},\mathsf{W},\mathsf{R})$, there is a Frobenius exact category $\mathsf{Q}\cap\mathsf{R}$ whose projective-injective objects are precisely those in $\mathsf{Q}\cap\mathsf{R}\cap\mathsf{W}$. Furthermore, the stable category $\underline{\mathsf{Q}\cap\mathsf{R}}$ is triangulated equivalent to $\mathrm{Ho}(\mathscr{M})$. This triangulated equivalence is known as the fundamental theorem of model categories in the literature.

**42.** By [9], the triple $\mathscr{M}=(\mathsf{GPac},\mathsf{GPac}^\perp,\mathsf{Mod(R)})$ is a hereditary Hovey triple. As an immediate consequence of Proposition 41 one gets that the triple $\mathscr{M}'=(\mathsf{GFac},\mathsf{GPac}^\perp,\mathsf{Cot})$ is a hereditary Hovey triple, which can also be found in [19, Cor. 4.3]. Thus the category $\mathsf{GPac}$ (resp., $\mathsf{GFac}\cap\mathsf{Cot}$) is a Frobenius category with projective-injective objects all projective (resp., flat-cotorsion) $R$-modules. By the fundamental theorem of model categories, $\underline{\mathsf{GPac}}$ (resp., $\underline{\mathsf{GFac}\cap\mathsf{Cot}}$) is triangulated equivalent to $\mathrm{Ho}(\mathscr{M})$ (resp., $\mathrm{Ho}(\mathscr{M}')$).

It follows from Estrada and Gillespie [18, Lem. 5.4] that if two hereditary Hovey triples $\mathscr{M}=(\mathsf{Q},\mathsf{W},\mathsf{R})$ and $\mathscr{M}'=(\mathsf{Q}',\mathsf{W},\mathsf{R}')$ on $\mathsf{Mod}(R)$ have the same class $\mathsf{W}$ of trivial objects and if $\mathsf{Q}\subseteq\mathsf{Q}'$ (or equivalently, $\mathsf{R}'\subseteq\mathsf{R}$), then there is a triangulated equivalence $\mathrm{Ho}(\mathscr{M})\simeq\mathrm{Ho}(\mathscr{M}')$. Applying this fact to the hereditary Hovey triples $\mathscr{M}=(\mathsf{GPac},\mathsf{GPac}^\perp,\mathsf{Mod(R)})$ and $\mathscr{M}'=(\mathsf{GFac},\mathsf{GPac}^\perp,\mathsf{Cot})$ (in view of 42) we get

**Corollary 43.** *There exits a triangulated equivalence* $\underline{\mathsf{GPac}} \simeq \underline{\mathsf{GFac} \cap \mathsf{Cot}}$.

Note that the pairs $(\mathsf{Flat}, \mathsf{Cot})$ and $(\mathsf{GFac}, (\mathsf{GPac})^{\perp} \cap \mathsf{Cot})$ are complete hereditary cotorsion pairs with $\mathsf{GFac} \cap (\mathsf{GPac})^{\perp} \cap \mathsf{Cot} = \mathsf{Flat} \cap \mathsf{Cot}$; see Proposition 41. The following result is immediate by [13, Thm. 4.5].

**Corollary 44.** *There exists a triangle equivalence*

$$\underline{\mathsf{GFac} \cap \mathsf{Cot}} \simeq \mathsf{D}^{\mathsf{b}}(R)_{\widehat{\mathsf{GFac}} \cap \widehat{\mathsf{Cot}}} / \mathsf{K}^{\mathsf{b}}(\mathsf{FlatCot}).$$

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