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The Caffarelli–Kohn–Nirenberg inequalities for radial functions

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Abstract. We establish the *full range* of the Caffarelli–Kohn–Nirenberg inequalities for radial functions in the Sobolev and the fractional Sobolev spaces of order $0 < s \le 1$. In particular, we show that the range of the parameters for radial functions is strictly larger than the one without symmetric assumption. Previous known results reveal only some special ranges of parameters even in the case $s = 1$. The known proofs used the Riesz potential and inequalities for fractional integrations. Our proof is new, elementary, and is based on one-dimensional case. Applications on compact embeddings are also mentioned.

Keywords. Caffarelli–Kohn–Nirenberg inequality, radial functions, compact embedding.

2020 Mathematics Subject Classification. 26D10, 26A54.

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1. Introduction

Let $d \ge 1$, $p \ge 1$, $q \ge 1$, $\tau \ge 1$, $0 < a \le 1$, $\alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$\frac{1}{\tau} + \frac{\gamma}{d}, \ \frac{1}{p} + \frac{\alpha}{d}, \ \frac{1}{q} + \frac{\beta}{d} > 0, \tag{1}$$

and the following balance law holds

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\left(\frac{1}{p} + \frac{\alpha - 1}{d}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{d}\right). \tag{2}$$

Define σ by

$$\gamma = a\sigma + (1-a)\beta. \tag{3}$$

Assume that

$$0 \le \alpha - \sigma \tag{4}$$

and

$$\alpha - \sigma \le 1 \text{ if } \frac{1}{\tau} + \frac{\gamma}{d} = \frac{1}{p} + \frac{\alpha - 1}{d}. \tag{5}$$

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Caffarelli, Kohn, and Nirenberg [12] (see also [11]) established the following famous Caffarelli, Kohn and Nirenberg (CKN) inequalities, for $u \in C_c^1(\mathbb{R}^d)$,

$$\big\| |x|^\gamma u \big\|_{L^\tau(\mathbb{R}^d)} \le C \, \big\| |x|^\alpha \nabla u \big\|_{L^p(\mathbb{R}^d)}^a \, \Big\| |x|^\beta u \Big\|_{L^q(\mathbb{R}^d)}^{1-a}, \tag{6}$$

for some positive constant C independent of u. Quite recently, the full range of the CKN inequalities has been derived by Nguyen and Squassina [24] for the fractional Sobolev spaces $W^{s,p}(\mathbb{R}^d)$ with $0 < s < 1$ and $p > 1$. More precisely, let $d \ge 1$, $0 < s < 1$, $p > 1$, $q \ge 1$, $\tau \ge 1$, $0 < a \le 1$, and $\alpha_1, \alpha_2, \beta, \gamma \in \mathbb{R}$. Set $\alpha = \alpha_1 + \alpha_2$ and define σ by (3). Assume that

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\left(\frac{1}{p} + \frac{\alpha - s}{d}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{d}\right), \tag{7}$$

and the following conditions hold

$$0 \le \alpha - \sigma \tag{8}$$

and

$$\alpha - \sigma \le s \text{ if } \frac{1}{\tau} + \frac{\gamma}{d} = \frac{1}{p} + \frac{\alpha - s}{d}. \tag{9}$$

Nguyen and Squassina [24, Theorem 1.1] proved, for some positive constant C,

(i) if $\frac{1}{\tau} + \frac{\gamma}{d} > 0$, then for all $u \in C_c^1(\mathbb{R}^d)$, it holds

$$\big\| |x|^\gamma u \big\|_{L^\tau(\mathbb{R}^d)} \le C \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p |x|^{\alpha_1 p} |y|^{\alpha_2 p}}{|x-y|^{d+sp}} \mathrm{d}x \mathrm{d}y \right)^{\frac{a}{p}} \Big\| |x|^\beta u \Big\|_{L^q(\mathbb{R}^d)}^{1-a}, \tag{10}$$

(ii) if $\frac{1}{\tau} + \frac{\gamma}{d} < 0$, then for all $u \in C_c^1(\mathbb{R}^d \setminus \{0\})$, it holds

$$\big\| |x|^\gamma u \big\|_{L^\tau(\mathbb{R}^d)} \le C \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p |x|^{\alpha_1 p} |y|^{\alpha_2 p}}{|x-y|^{d+sp}} \mathrm{d}x \mathrm{d}y \right)^{\frac{a}{p}} \Big\| |x|^\beta u \Big\|_{L^q(\mathbb{R}^d)}^{1-a}, \tag{11}$$

In the case $\frac{1}{\tau} + \frac{\gamma}{d} = 0$, a log-correction is required, and the conditions (8) and (9) are replaced by

$$0 \le \alpha - \sigma \le s. \tag{12}$$

Denote B_R the open ball centered at the origin with radius R. Assume *additionally* that $\tau > 1$. Nguyen and Squassina [24, Theorem 3.1] showed that there exists a positive constant C such that for all $u \in C_c^1(\mathbb{R}^d)$ and for all $R_1, R_2 > 0$, we have

(i) if $\frac{1}{\tau} + \frac{\gamma}{d} = 0$ and $\operatorname{supp} u \subset B_{R_2}$, then

$$\left(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^\tau (2R_2/|x|)} |u|^\tau \mathrm{d}x \right)^{\frac{1}{\tau}} \le C \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p |x|^{\alpha_1 p} |y|^{\alpha_2 p}}{|x-y|^{d+sp}} \mathrm{d}x \mathrm{d}y \right)^{\frac{a}{p}} \Big\| |x|^\beta u \Big\|_{L^q(\mathbb{R}^d)}^{1-a}, \tag{13}$$

(ii) if $\frac{1}{\tau} + \frac{\gamma}{d} = 0$, and $\operatorname{supp} u \cap B_{R_1} = \emptyset$, then

$$\left(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^\tau (2|x|/R_1)} |u|^\tau \mathrm{d}x \right)^{\frac{1}{\tau}} \le C \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p |x|^{\alpha_1 p} |y|^{\alpha_2 p}}{|x-y|^{d+sp}} \mathrm{d}x \mathrm{d}y \right)^{\frac{a}{p}} \Big\| |x|^\beta u \Big\|_{L^q(\mathbb{R}^d)}^{1-a}. \tag{14}$$

Note that the conditions $\frac{1}{p} + \frac{\alpha}{d}$, $\frac{1}{q} + \frac{\beta}{d} > 0$ are not required in these inequalities. In the case $a = 1$ and $1/\tau + \gamma/d > 0$, several special ranges of parameters were previously derived in [1,17,21]. These works are partially motivated from new characterizations of Sobolev spaces using non-local, convex functionals proposed by Bourgain, Brezis, and Mironescu [6] (see also [8]). Related characterizations of Sobolev spaces with non-local, non-convex functionals can be found in [7, 9, 10, 22] and the references therein. The proof given in [24] (see also [25]) is new. It is based on the dyadic decomposition of the real space, Gagliardo–Nirenberg's inequalities for annulus, and a trick on summation processes to bring the information from a family of annulus to the whole space. Combining these ideas with the techniques in [23], which are used to prove new Sobolev's inequalities, we established the full range of Coulomb–Sobolev inequalities [20]. In the case $s = 1$, inequality (6) also holds in the case $1/\tau+\gamma/d<0$, and similar results as in (13) and (14) are valid in the case $1/\tau+\gamma/d=0$. We present these results in Section 4 (see Theorem 16 and Theorem 17).

In this paper, we investigate the CKN inequalities for radial functions. We show that the previous results also hold for some negative range of $\alpha-\sigma$ (compare with (4) and (8)). The fact that the range of the parameters of a family of inequalities can be larger when a symmetry condition is imposed is a well-known phenomenon, e.g., in the context of Stein-Weis inequalities [15, 26] and Coulomb–Sobolev inequalities [3, 4]. Various compactness results can be established using the extended range and are useful in the proof of the existence of minimizers of variational problems. Also, these compactness results play important roles in the analysis of various interesting physical phenomena, see, e.g., [5, 18, 19, 28], and the references therein. It is quite surprising that very few results have been known for the extended range of the CKN inequalities for radial functions. The goal of this paper is to completely fill this gap for $0<s\le 1$.

We first concentrate on the setting of the fractional Sobolev spaces. The following notation is used. For $p>1$, $0<s<1$, $\alpha\in\mathbb{R}$, $\Lambda>1$, open $\Omega\subset\mathbb{R}^d$, and a measurable function g defined in Ω, we set

$$\|g\|^p_{\dot{W}^{s,p,\alpha,\Lambda}(\Omega)}=\int_\Omega\int_\Omega\frac{|g(x)-g(y)|^p|x|^{\alpha p}}{|x-y|^{d+sp}}\chi_\Lambda(|x|,|y|)\,\mathrm{d}x\,\mathrm{d}y, \tag{15}$$

where, for $r_1,r_2\ge 0$, we denote

$$\chi_\Lambda(r_1,r_2)=\begin{cases}1 & \text{for } \Lambda^{-1}r_1\le r_2\le\Lambda r_1,\\ 0 & \text{otherwise.}\end{cases} \tag{16}$$

The dot in the LHS of (15) means that only the information of the "semi-norm" is considered.

Our first main result is the following one dealing with the case where $1/\tau+\gamma/d\ne 0$.

Theorem 1. *Let $d\ge 2$, $0<s<1$, $p>1$, $q\ge 1$, $\tau\ge 1$, $0<a\le 1$, $\alpha,\beta,\gamma\in\mathbb{R}$, and $\Lambda>1$. Define σ by* (3). *Assume* (7) *and*

$$-(d-1)s\le\alpha-\sigma<0. \tag{17}$$

We have, for some positive constant C,

(i) *if $\frac{1}{\tau}+\frac{\gamma}{d}>0$, then for all radial $u\in L^1_{loc}(\mathbb{R}^d\setminus\{0\})$ with compact support in $\mathbb{R}^d$, it holds*

$$\big\||x|^\gamma u\big\|_{L^\tau(\mathbb{R}^d)}\le C\|u\|^a_{\dot{W}^{s,p,\alpha,\Lambda}(\mathbb{R}^d)}\Big\||x|^\beta u\Big\|^{1-a}_{L^q(\mathbb{R}^d)}, \tag{18}$$

(ii) *if $\frac{1}{\tau}+\frac{\gamma}{d}<0$, then for all radial $u\in L^1_{loc}(\mathbb{R}^d\setminus\{0\})$ which is 0 in a neighborhood of 0, it holds*

$$\big\||x|^\gamma u\big\|_{L^\tau(\mathbb{R}^d)}\le C\|u\|^a_{\dot{W}^{s,p,\alpha,\Lambda}(\mathbb{R}^d)}\Big\||x|^\beta u\Big\|^{1-a}_{L^q(\mathbb{R}^d)}. \tag{19}$$

Remark 2. In (18) and (19), the following convention is used: $+\infty.0=0.(+\infty)=0$, $(+\infty)^0=1$ (this corresponds to the case $a=1$), and $+\infty\le+\infty$.

Remark 3. The condition $\alpha-\sigma\ge-(d-1)s$ is in fact optimal, see Proposition 15. In this paper, we only discuss standard parameters in the L^p-scale of the integrability. With this in mind, when we mention L^q and L^τ, then q and τ are assumed to be greater than or equal to 1 and when we mention $W^{s,p}$ the parameter s is assumed in $(0,1)$ and p is assumed to be greater than 1 then. The full range is understood under this circumstance. Part of the arguments can be extended for example to the case $0<\tau<1$ as in [24]. Nevertheless, to keep the presentation simple and to avoid the confusion, we do not pursue this direction.

Remark 4. Combining (10), (11), and Theorem 1 yields that, in the radial case, (10) and (11) hold if one replaces (8) and (9) by the condition $-(d-1)s\le\alpha-\sigma$ and (9).

Concerning the limiting case $1/\tau+\gamma/d=0$, we obtain the following result.

Theorem 5. *Let $d \geq 2$, $0 < s < 1$, $p > 1$, $q \geq 1$, $\tau \geq 1$, $0 < a \leq 1$, $\alpha, \beta, \gamma \in \mathbb{R}$, $\mu > 1$, and $\Lambda > 1$. Assume that $\tau \leq \mu$. Define σ by* (3). *Assume* (7) *and* (17). *There exists a positive constant C such that for all radial $u \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ and for all $R_1, R_2 > 0$, we have*

(i) *if $\frac{1}{\tau} + \frac{\gamma}{d} = 0$ and $\operatorname{supp} u \subset B_{R_2}$, then it holds;*

$$\left(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^{\mu}(2R_2/|x|)}|u|^{\tau}\,\mathrm{d}x\right)^{\frac{1}{\tau}} \leq C\,\|u\|^{a}_{\dot{W}^{s,p,\alpha,\Lambda}(\mathbb{R}^d)}\left\||x|^{\beta}u\right\|^{1-a}_{L^q(\mathbb{R}^d)}, \tag{20}$$

(ii) *if $\frac{1}{\tau} + \frac{\gamma}{d} = 0$, and $\operatorname{supp} u \cap B_{R_1} = \emptyset$, then it holds*

$$\left(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^{\mu}(2|x|/R_1)}|u|^{\tau}\,\mathrm{d}x\right)^{\frac{1}{\tau}} \leq C\,\|u\|^{a}_{\dot{W}^{s,p,\alpha,\Lambda}(\mathbb{R}^d)}\left\||x|^{\beta}u\right\|^{1-a}_{L^q(\mathbb{R}^d)}. \tag{21}$$

Remark 6. The convention in Remark 2 is also used in Theorem 5.

Remark 7. If $1/q + \beta/d > 0$, by considering a smooth function u which is 1 in a neighborhood of 0 we can establish the necessity of the log-term in i) of Theorem 5. Similarly, if $1/q + \beta/d < 0$, by considering a smooth function u which is 1 outside B_R for some large R the necessity of the log-term in ii) can be established.

Remark 8. Combining (13), (14), and Theorem 5 yields that, in the radial case, (13), (14) hold if one replaces (12) by the condition $-(d-1)s \leq \alpha - \sigma \leq s$.

There are very few results known for the extended range of the CKN inequalities in the fractional Sobolev spaces for radial functions (the case $s = 1$ will be discussed in the last paragraph of Section 4). It was shown by Rubin [26] (see also [3, Theorem 4.3]) that (10) holds under the assumption (17) and $1/\tau + \gamma/d > 0$ in the case where $a = 1$, $\tau \geq p = 2$, and $\alpha = 0$. The same result was proved in [15, Theorem 1.2]. These proofs are based on inequalities for fractional integrations. Our proof is different and quite elementary. It is based on an improvement of the fractional CKN inequalities in one dimensional case and a simple use of polar coordinates. This strategy can be easily extended to other contexts. The improvement was implicitly appeared in [24] and will be described briefly later. The same idea can be applied to the case $s = 1$ and will be presented in Section 4. Applications to the compact embedding will be given in Section 5. In particular, we derive the compact embedding of $W^{s,p}(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$ for radial functions if $p < q < \frac{dp}{d-sp}$ for $0 < s \leq 1$ and $sp < d$. This result was previously obtained via various technique such as Strauss' lemma, Riesz-potential, fractional integration, Rubin's lemma, atomic decomposition, etc.

It is worth noting that whether radial functions are optimal in Caffarelli–Kohn–Nirenberg inequalities is an old question that goes back to the nineties. see e.g., [13,14,16] and it may happen that the optimal functions are not radial for certain choices of the parameters, and that loss of compactness may occur along sequences of non- radial functions.

The paper is organized as follows. The improvement of the fractional CKN inequalities are given in Section 2. The proofs of Theorem 1 and Theorem 5 are given in Section 3. The results in the case $s = 1$ are given in Section 4. Section 5 is devoted to the compactness results.

2. Improvements of the fractional Caffarelli–Kohn–Nirenberg inequalities

In this section, we will establish slightly more general versions of the fractional CKN inequalities. These improvements appear very naturally in the proof of Theorem 1 and Theorem 5 when polar coordinates are used.

We begin with an improvement of (10) and (11).

Theorem 9. *Let $d \ge 1$, $0 < s < 1$, $p > 1$, $q \ge 1$, $\tau \ge 1$, $0 < a \le 1$, $\alpha, \beta, \gamma \in \mathbb{R}$, and $\Lambda > 1$. Define σ by* (3)*. Assume* (7), (8), *and* (9)*. There exists a positive constant C such that*

(i) *if $\frac{1}{\tau} + \frac{\gamma}{d} > 0$, then for all $u \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ with compact support in $\mathbb{R}^d$, it holds*

$$\left\| |x|^\gamma u \right\|_{L^\tau(\mathbb{R}^d)} \le C \| u \|^a_{\dot{W}^{s,p,\alpha,\Lambda}(\mathbb{R}^d)} \left\| |x|^\beta u \right\|^{1-a}_{L^q(\mathbb{R}^d)}, \tag{22}$$

(ii) *if $\frac{1}{\tau} + \frac{\gamma}{d} < 0$, then for all $u \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ which is 0 in a neighborhood of 0, it holds*

$$\left\| |x|^\gamma u \right\|_{L^\tau(\mathbb{R}^d)} \le C \| u \|^a_{\dot{W}^{s,p,\alpha,\Lambda}(\mathbb{R}^d)} \left\| |x|^\beta u \right\|^{1-a}_{L^q(\mathbb{R}^d)}. \tag{23}$$

Concerning an improvement of (13) and (14), we have the following result.

Theorem 10. *Let $d \ge 1$, $0 < s < 1$, $p > 1$, $q \ge 1$, $\tau \ge 1$, $0 < a \le 1$, $\alpha, \beta, \gamma \in \mathbb{R}$, $\mu > 1$, and $\Lambda > 1$. Assume that $\tau \le \mu$. Define σ by* (3)*. Assume* (7) *and* (12)*. There exists a positive constant C such that for all $u \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ and for all $R_1, R_2 > 0$, we have*

(i) *if $\frac{1}{\tau} + \frac{\gamma}{d} = 0$ and $\operatorname{supp} u \subset B_{R_2}$, then it holds*

$$\left(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^\mu(2R_2/|x|)} |u|^\tau \mathrm{d}x \right)^{\frac{1}{\tau}} \le C \| u \|^a_{\dot{W}^{s,p,\alpha,\Lambda}(\mathbb{R}^d)} \left\| |x|^\beta u \right\|^{1-a}_{L^q(\mathbb{R}^d)}, \tag{24}$$

(ii) *if $\frac{1}{\tau} + \frac{\gamma}{d} = 0$, and $\operatorname{supp} u \cap B_{R_1} = \emptyset$, then it holds*

$$\left(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^\mu(2|x|/R_1)} |u|^\tau \mathrm{d}x \right)^{\frac{1}{\tau}} \le C \| u \|^a_{\dot{W}^{s,p,\alpha,\Lambda}(\mathbb{R}^d)} \left\| |x|^\beta u \right\|^{1-a}_{L^q(\mathbb{R}^d)}. \tag{25}$$

It is clear that Theorem 9 implies (10) and (11) and Theorem 10 yields (13) and (14). Theorem 9 and Theorem 10 were already implicitly contained in [24] where (10), (11), (13), and (14) were established. For the convenience of the reader, we will describe briefly the proofs of Theorem 9 and Theorem 10 in the next two sections respectively.

2.1. *Proof of Theorem 9*

The proof is divided into two steps where we prove (i) and (ii) respectively.

Step 1: Proof of (i). For simplicity of arguments, we assume that $\Lambda > 4$ from later on[1].

We first consider the case $0 \le \alpha - \sigma \le s$. As in [24], for $k \in \mathbb{Z}$ set

$$\mathscr{A}_k := \left\{ x \in \mathbb{R}^d;\ 2^k \le |x| < 2^{k+1} \right\}. \tag{26}$$

Since $\alpha - \sigma \ge 0$, by Gagliardo–Nirenberg inequality [24, Lemma 2.2][2], we derive that

$$\left(\fint_{\mathscr{A}_k} \left| u - \fint_{\mathscr{A}_k} u \right|^\tau \right)^{\frac{1}{\tau}} \le C \left(2^{-(d-sp)k} \int_{\mathscr{A}_k} \int_{\mathscr{A}_k} \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} \mathrm{d}x \mathrm{d}y \right)^{a/p} \left(\fint_{\mathscr{A}_k} |u(x)|^q \mathrm{d}x \right)^{(1-a)/q}. \tag{27}$$

Here and in what follows in the proof of Theorem 9, C denotes a positive constant independent of u and k (and also independent of m, and n, which appear later), and $\fint_\Omega := \frac{1}{|\Omega|} \int_\Omega$. Since

$$2^{\tau\gamma k} \int_{\mathscr{A}_k} |u|^\tau \le C 2^{(\tau\gamma + d)k} \fint_{\mathscr{A}_k} \left| u - \fint_{\mathscr{A}_k} u \right|^\tau + C 2^{(\tau\gamma+d)k} \left| \fint_{\mathscr{A}_k} u \right|^\tau,$$

---

[1] In the general case, one just needs to define $\mathscr{A}_k$ by $\left\{ x \in \mathbb{R}^d;\ \lambda^k \le |x| < \lambda^{k+1} \right\}$ with $\lambda^2 = \Lambda$ instead of (26).

[2] [24, Lemma 2.2] states for functions of class C^1 up to the boundary but the same result holds for our setting by using the standard convolution technique.

using (7), we derive from (27) that

$$\int_{\mathscr{A}_k} |u|^\tau |x|^{\tau\gamma}\,\mathrm{d}x \le C2^{(\gamma\tau+d)k}\left|\fint_{\mathscr{A}_k} u\right|^\tau$$
$$+C\left(\int_{\mathscr{A}_k}\int_{\mathscr{A}_k} \frac{|u(x)-u(y)|^p|x|^{\alpha p}}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{a\tau}{p}}\left(\int_{\mathscr{A}_k}|u(x)|^q|x|^{\beta q}\,\mathrm{d}x\right)^{\frac{(1-a)\tau}{q}}. \quad (28)$$

Let $m, n \in \mathbb{Z}$ be such that $m \le n-2$ and $\operatorname{supp} u \subset B_{2^n}$. Summing (28) with respect to k from m to n, we get

$$\int_{\{2^m<|x|<2^{n+1}\}} |u|^\tau |x|^{\tau\gamma} \le C\sum_{k=m}^{n} 2^{(\gamma\tau+d)k}\left|\fint_{\mathscr{A}_k} u\right|^\tau$$
$$+C\sum_{k=m}^{n}\left(\int_{\mathscr{A}_k}\int_{\mathscr{A}_k} \frac{|u(x)-u(y)|^p|x|^{\alpha p}}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{a\tau}{p}}\left\||x|^\beta u\right\|_{L^q(\mathscr{A}_k)}^{(1-a)\tau}. \quad (29)$$

Applying Lemma 11 below with $\kappa = a\tau/p$ and $\eta = (1-a)\tau/q$ after using the condition $\alpha-\sigma \le s$ to check that $\kappa+\eta \ge 1$, we derive that

$$\sum_{k=m}^{n}\left(\int_{\mathscr{A}_k}\int_{\mathscr{A}_k} \frac{|u(x)-u(y)|^p|x|^{\alpha p}}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{a\tau}{p}}\left\||x|^\beta u\right\|_{L^q(\mathscr{A}_k)}^{(1-a)\tau} \le \|u\|_{\dot W^{s,p,\alpha,\Lambda}(\mathbb{R}^d)}^{a\tau}\left\||x|^\beta u\right\|_{L^q(\mathbb{R}^d)}^{(1-a)\tau}. \quad (30)$$

Combining (29) and (30) yields

$$\int_{\{|x|>2^m\}} |u|^\tau |x|^{\tau\gamma} \le C\sum_{k=m}^{n} 2^{(\gamma\tau+d)k}\left|\fint_{\mathscr{A}_k} u\right|^\tau + C\|u\|_{\dot W^{s,p,\alpha,\Lambda}(\mathbb{R}^d)}^{a\tau}\left\||x|^\beta u\right\|_{L^q(\mathbb{R}^d)}^{(1-a)\tau}. \quad (31)$$

We next estimate the first term of the RHS of (31). We have, as in (27),

$$\left|\fint_{\mathscr{A}_k} u - \fint_{\mathscr{A}_{k+1}} u\right|^\tau \le C\left(2^{(d-sp)k}\int_{\mathscr{A}_k\cup\mathscr{A}_{k+1}}\int_{\mathscr{A}_k\cup\mathscr{A}_{k+1}} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{a\tau}{p}}$$
$$\times\left(\fint_{\mathscr{A}_k\cup\mathscr{A}_{k+1}} |u(x)|^q\,\mathrm{d}x\right)^{\frac{(1-a)\tau}{q}}. \quad (32)$$

With $c = 2/(1+2^{\gamma\tau+d}) < 1$, since $c2^{\gamma\tau+d} > 1$ thanks to $\gamma\tau+d>0$ we derive from (32) that

$$2^{(\gamma\tau+d)k}\left|\fint_{\mathscr{A}_k} u\right|^\tau \le c2^{(\gamma\tau+d)(k+1)}\left|\fint_{\mathscr{A}_{k+1}} u\right|^\tau$$
$$+C\left(\int_{\mathscr{A}_k\cup\mathscr{A}_{k+1}}\int_{\mathscr{A}_k\cup\mathscr{A}_{k+1}} \frac{|u(x)-u(y)|^p|x|^{\alpha p}}{|x-y|^{d+sp}}\mathrm{d}x\mathrm{d}y\right)^{\frac{a\tau}{p}}\left\||x|^\beta u\right\|_{L^q(\mathscr{A}_k\cup\mathscr{A}_{k+1})}^{(1-a)\tau}. \quad (33)$$

Summing this inequality with respect to k from m to n for large n, since u has a compact support in B_{2^n} and $c<1$ thanks to $\gamma\tau+d>0$, we derive that

$$\sum_{k=m}^{n} 2^{(\gamma\tau+d)k}\left|\fint_{\mathscr{A}_k} u\right|^\tau \le C\sum_{k=m}^{n}\left(\int_{\mathscr{A}_k\cup\mathscr{A}_{k+1}}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p|x|^{\alpha p}}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{a\tau}{p}}\left\||x|^\beta u\right\|_{L^q(\mathscr{A}_k\cup\mathscr{A}_{k+1})}^{(1-a)\tau}. \quad (34)$$

Applying Lemma 11 below again and letting $m \to -\infty$, we obtain

$$\sum_{k\in\mathbb{Z}} 2^{(\gamma\tau+d)k}\left|\fint_{\mathscr{A}_k} u\right|^\tau \le C\|u\|_{\dot W^{s,p,\alpha,\Lambda}(\mathbb{R}^d)}^{a\tau}\left\||x|^\beta u\right\|_{L^q(\mathbb{R}^d)}^{(1-a)\tau}. \quad (35)$$

Combining (31) and (35) and letting $m \to -\infty$, we obtain (i) of Theorem 9. The proof of $i)$ in the case $0 \le \alpha-\sigma \le s$ is complete.

The proof of $i)$ in the case $\alpha-\sigma > s$ and $\frac{1}{\tau}+\frac{\gamma}{d} \ne \frac{1}{p}+\frac{\alpha-s}{d}$ is based on the standard interpolation technique as in [12, 24]. One just notes that, for $\lambda>0$,

$$\|u(\lambda\cdot)\|_{\dot W^{s,p,\alpha,\Lambda}(\mathbb{R}^d)} = \lambda^{s-\alpha-\frac{d}{p}}\|u\|_{\dot W^{s,p,\alpha,\Lambda}(\mathbb{R}^d)}$$

since $\chi_\Lambda(x,y) = \chi_\Lambda(\lambda x, \lambda y)$, and

$$\||x|^\gamma u(\lambda\,\cdot)\|_{L^\tau(\mathbb{R}^d)} = \lambda^{-\gamma-\frac{d}{\tau}}\||x|^\gamma u\|_{L^\tau(\mathbb{R}^d)} \quad \text{and} \quad \||x|^\beta u(\lambda\,\cdot)\|_{L^q(\mathbb{R}^d)} = \lambda^{-\beta-\frac{d}{q}}\||x|^\beta u\|_{L^q(\mathbb{R}^d)}.$$

The details are omitted.

Step 2: Proof of (ii). The proof of (ii) of Theorem 9 is similar to that of (i). We only deal with the case $0 \le \alpha - \sigma \le s$ since the proof in the case where $\alpha - \sigma > s$ and $\frac{1}{\tau} + \frac{\gamma}{d} \neq \frac{1}{p} + \frac{\alpha - s}{d}$ is only by interpolation and almost unchanged.

Assume $0 \le \alpha - \sigma \le s$. Let m be such that $u = 0$ in B_{2^m}. Similar to (31), we have

$$\int_{\{|x|<2^n\}} |u|^\tau |x|^{\tau\gamma} \le C \sum_{k=m}^{n} 2^{(\gamma\tau+d)k} \left| \fint_{\mathscr{A}_k} u \right|^\tau + C\|u\|^{a\tau}_{\dot{W}^{s,p,\alpha,\Lambda}(\mathbb{R}^d)} \left\| |x|^\beta u \right\|^{(1-a)\tau}_{L^q(\mathbb{R}^d)}. \tag{36}$$

To estimate the first term in RHS of (36), one just needs to note that, instead of (33), we have with $c = (1+2^{\gamma\tau+d})/2 < 1$ thanks to $\gamma\tau + d < 0$,

$$2^{(\gamma\tau+d)(k+1)} \left| \fint_{\mathscr{A}_{k+1}} u \right|^\tau \le c 2^{(\gamma\tau+d)k} \left| \fint_{\mathscr{A}_k} u \right|^\tau + C \left(\int_{\mathscr{A}_k \cup \mathscr{A}_{k+1}} \int_{\mathscr{A}_k \cup \mathscr{A}_{k+1}} \frac{|u(x)-u(y)|^p |x|^{\alpha p}}{|x-y|^{d+sp}} \mathrm{d}x\mathrm{d}y \right)^{\frac{a\tau}{p}} \left\| |x|^\beta u \right\|^{(1-a)\tau}_{L^q(\mathscr{A}_k \cup \mathscr{A}_{k+1})}.$$

Summing with respect to k, we also obtain (35). The conclusion now follows from (35) and (36). $\square$

The following simple lemma is used in the proof of Theorem 9.

Lemma 11. *For $\kappa, \eta \ge 0$ with $\kappa + \eta \ge 1$, and $k \in \mathbb{N}$, we have*

$$\sum_{i=1}^k |a_i|^\kappa |b_i|^\eta \le \left(\sum_{i=1}^k |a_i| \right)^\kappa \left(\sum_{i=1}^k |b_i| \right)^\eta \quad \textit{for } a_i, b_i \in \mathbb{R}. \tag{37}$$

2.2. *Proof of Theorem 10*

As in the proof of Theorem 9, we assume that $\Lambda > 4$ for notational ease. In this proof, we use the notations in the proof of Theorem 9. We only prove the first assertion. The second assertion follows similarly as in the spirit of the proof of Theorem 9. Let $n \in \mathbb{N}$ be such that $2^{n-1} \le R_2 < 2^n$.

Set

$$\nu = \mu - 1 > 0. \tag{38}$$

Since $\alpha - \sigma \ge 0$, using (7), we also obtain (28). Summing (28) with respect to k from m to n, we obtain

$$\int_{\{|x|>2^m\}} \frac{1}{\ln^{1+\nu}(2R_2/|x|)} |x|^{\gamma\tau} |u|^\tau \,\mathrm{d}x \le C \sum_{k=m}^n \frac{1}{(n-k+1)^{1+\nu}} \left| \fint_{\mathscr{A}_k} u \right|^\tau + C \sum_{k=m}^n \left(\int_{\mathscr{A}_k} \int_{\mathscr{A}_k} \frac{|u(x)-u(y)|^p |x|^{\alpha p}}{|x-y|^{d+sp}} \,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{a\tau}{p}} \||x|^\beta u\|^{(1-a)\tau}_{L^q(\mathscr{A}_k)}. \tag{39}$$

As in (32), we have

$$\left| \fint_{\mathscr{A}_k} u - \fint_{\mathscr{A}_{k+1}} u \right|^\tau \le C \left(2^{(d-sp)k} \int_{\mathscr{A}_k \cup \mathscr{A}_{k+1}} \int_{\mathscr{A}_k \cup \mathscr{A}_{k+1}} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} \,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{a\tau}{p}} \times \left(\fint_{\mathscr{A}_k \cup \mathscr{A}_{k+1}} |u(x)|^q \,\mathrm{d}x \right)^{\frac{(1-a)\tau}{q}}. \tag{40}$$

Applying Lemma 12 below with $c = (n-k+1)^\nu/(n-k+1/2)^\nu$, we deduce that

$$\left|\fint_{\mathscr{A}_k} u\right|^\tau \le \frac{(n-k+1)^\nu}{(n-k+1/2)^\nu}\left|\fint_{\mathscr{A}_{k+1}} u\right|^\tau + C(n-k+1)^{\tau-1}\left|\fint_{\mathscr{A}_k} u - \fint_{\mathscr{A}_{k+1}} u\right|^\tau,$$

since, for $\nu > 0$,

$$(n-k+1)^\nu/(n-k+1/2)^\nu - 1 \sim \frac{1}{n-k+1}.$$

It follows from (7) and (40) that

$$\left|\fint_{\mathscr{A}_k} u\right|^\tau \le \frac{(n-k+1)^\nu}{(n-k+1/2)^\nu}\left|\fint_{\mathscr{A}_{k+1}} u\right|^\tau$$
$$+ C(n-k+1)^{\tau-1}\left(\int_{\mathscr{A}_k\cup\mathscr{A}_{k+1}}\int_{\mathscr{A}_k\cup\mathscr{A}_{k+1}} \frac{|u(x)-u(y)|^p|x|^{\alpha p}}{|x-y|^{d+sp}}\mathrm{d}x\mathrm{d}y\right)^{\frac{a\tau}{p}} \left\||x|^\beta u\right\|_{L^q(\mathscr{A}_k\cup\mathscr{A}_{k+1})}^{(1-a)\tau}.$$

This yields

$$\frac{1}{(n-k+1)^\nu}\left|\fint_{\mathscr{A}_k} u\right|^\tau \le \frac{1}{(n-k+1/2)^\nu}\left|\fint_{\mathscr{A}_{k+1}} u\right|^\tau$$
$$+ C(n-k+1)^{\tau-1-\nu}\left(\int_{\mathscr{A}_k\cup\mathscr{A}_{k+1}}\int_{\mathscr{A}_k\cup\mathscr{A}_{k+1}} \frac{|u(x)-u(y)|^p|x|^{\alpha p}}{|x-y|^{d+sp}}\mathrm{d}x\mathrm{d}y\right)^{\frac{a\tau}{p}} \left\||x|^\beta u\right\|_{L^q(\mathscr{A}_k\cup\mathscr{A}_{k+1})}^{(1-a)\tau}. \quad (41)$$

We have, for $\nu > 0$ and $k \le n$,

$$\frac{1}{(n-k+1)^\nu} - \frac{1}{(n-k+3/2)^\nu} \sim \frac{1}{(n-k+1)^{\nu+1}} \quad (42)$$

and, since $\tau \le 1+\nu$,

$$(n-k+1)^{\tau-1-\nu} \le 1. \quad (43)$$

Summing (41) from m to n, and using (42) and (43), we derive that

$$\sum_{k=m}^n \frac{1}{(n-k+1)^{1+\nu}}\left|\fint_{\mathscr{A}_k} u\right|^\tau$$
$$\le C\sum_{k=m}^n \left(\int_{\mathscr{A}_k\cup\mathscr{A}_{k+1}}\int_{\mathscr{A}_k\cup\mathscr{A}_{k+1}} \frac{|u(x)-u(y)|^p|x|^{\alpha p}}{|x-y|^{d+sp}}\mathrm{d}x\mathrm{d}y\right)^{\frac{a\tau}{p}} \left\||x|^\beta u\right\|_{L^q(\mathscr{A}_k\cup\mathscr{A}_{k+1})}^{(1-a)\tau}. \quad (44)$$

Combining (39) and (44), we obtain

$$\int_{\{|x|>2^m\}} \frac{|x|^{\gamma\tau}}{\ln^{1+\nu}(2^{n+1}/|x|)}|u|^\tau\,\mathrm{d}x$$
$$\le C\sum_{k=m}^n \left(\int_{\mathscr{A}_k\cup\mathscr{A}_{k+1}}\int_{\mathscr{A}_k\cup\mathscr{A}_{k+1}} \frac{|u(x)-u(y)|^p|x|^{\alpha p}}{|x-y|^{d+sp}}\mathrm{d}x\mathrm{d}y\right)^{\frac{a\tau}{p}} \left\||x|^\beta u\right\|_{L^q(\mathscr{A}_k\cup\mathscr{A}_{k+1})}^{(1-a)\tau}.$$

Applying Lemma 11 with $\kappa = a\tau/p$ and $\eta = (1-a)\tau/q$, we derive that

$$\int_{\{|x|>2^m\}} \frac{|x|^{\gamma\tau}}{\ln^{1+\nu}(2^{n+1}/|x|)}|u|^\tau\,\mathrm{d}x \le C\|u\|_{\dot W^{s,p,\alpha,\Lambda}(\mathbb{R}^d)}^{a\tau}\||x|^\beta u\|_{L^q(\bigcup_{k=m}^\infty \mathscr{A}_k)}^{(1-a)\tau}.$$

This yields the conclusion. □

In the proof of Theorem 10, we used the following elementary lemma which was stated in [24, Lemma 3.2]. For the completeness, we give the proof below.

Lemma 12. *Let $M > 1$ and $\tau \ge 1$. There exists $C = C(M,\tau) > 0$, depending only on M and τ such that, for all $1 < c < M$,*

$$(|a|+|b|)^\tau \le c|a|^\tau + \frac{C}{(c-1)^{\tau-1}}|b|^\tau \text{ for all } a, b \in \mathbb{R}. \quad (45)$$

Proof. The inequality is trivial when $\tau = 1$. We next only deal with the case $\tau > 1$.

Without loss of generality, one might assume that $a \geq 0$ and $b \geq 0$. Inequality (45) is clear if $a = 0$ or $b = 0$. Thus it suffices to consider the case where $a > 0$ and $b > 0$. This will be assumed from now on. Set $x = a/b$. Multiplying two sides of the inequality by $a^{-\tau}$, it is enough to prove that, for some $C > 0$,

$$(1+x)^\tau \leq c + \frac{C}{(c-1)^{\tau-1}} x^\tau \text{ for } x > 0. \tag{46}$$

There exists $x_0 > 0$ such that, for $0 < x < x_0$,

$$(1+x)^\tau \leq 1 + 2\tau x.$$

On the other hand, we have

$$c + \frac{C}{(c-1)^{\tau-1}} x^\tau = 1 + (c-1) + \frac{C}{(c-1)^{\tau-1}} x^\tau \geq 1 + \frac{\tau-1}{\tau}(c-1) + \frac{1}{\tau}\frac{C}{(c-1)^{\tau-1}} x^\tau.$$

Applying the Young inequality, we obtain

$$\frac{\tau-1}{\tau}(c-1) + \frac{1}{\tau}\frac{C}{(c-1)^{\tau-1}} x^\tau \geq (c-1)^{\frac{\tau-1}{\tau}} \frac{C^{\frac{1}{\tau}} x}{(c-1)^{\frac{\tau-1}{\tau}}} \geq 2x \quad \text{if} \quad C > C_1 := 2^\tau.$$

Thus (46) holds for $0 < x < x_0$ for $C \geq C_1$.

It is clear that there exists $C_2 > 0$ such that (46) holds for $x \geq x_0$ for $C \geq C_2$.

By choosing $C = \max\{C_1, C_2\}$, we obtain (46) and the conclusion follows. □

Remark 13. Lemma 12 is stated in [24] for $\tau > 1$. Nevertheless, the result is trivial for $\tau = 1$.

3. The Caffarelli–Kohn–Nirenberg inequalities for radial functions in the fractional Sobolev spaces

This section containing two subsections is devoted to the proofs of Theorem 1 and Theorem 5. In the first subsection, we present a lemma which brings the situation in the radial case into the one of one dimensional space via polar coordinates. The proof of Theorem 1 is given in the second subsection by applying Theorem 9 in one dimensional space and using the lemma in the first subsection.

3.1. *A useful lemma*

The improvement forms of the CKN inequalities are inspired by the following lemma.

Lemma 14. *Let $d \geq 2$, $0 < s < 1$, $1 \leq p < \infty$, $\alpha \in \mathbb{R}$, $\Lambda > 1$, and let $u \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ be radial. We have, with $\hat{u}(r) = u(r\sigma)$ for some $\sigma \in \mathbb{S}^{d-1}$ and for $r > 0$,*

$$\int_0^\infty \int_0^\infty \frac{|\hat{u}(r_1) - \hat{u}(r_2)|^p r_1^{\alpha p + (d-1)} \chi_\Lambda(r_1, r_2)}{|r_1 - r_2|^{1+sp}} \mathrm{d}r_1 \mathrm{d}r_2 \leq C \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p |x|^{\alpha p} \chi_\Lambda(|x|, |y|)}{|x-y|^{d+sp}} \mathrm{d}x \,\mathrm{d}y, \tag{47}$$

where C is a positive constant depending only on d, s, α, p, and Λ.

Proof. The proof is simply based on the use of the polar coordinates. Using these coordinates, we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x_1) - u(x_2)|^p |x|^{\alpha p} \chi_\Lambda(|x_1|, |x_2|)}{|x_1 - x_2|^{d+sp}} \mathrm{d}x_1\, dx_2 = \int_0^\infty \int_0^\infty |\hat{u}(r_1) - \hat{u}(r_2)|^p r_1^{\alpha p + (d-1)} r_2^{d-1} \chi_\Lambda(r_1, r_2) \int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} \frac{\mathrm{d}\sigma_1 \mathrm{d}\sigma_2}{|r_1\sigma_1 - r_2\sigma_2|^{d+sp}} \mathrm{d}r_1\, \mathrm{d}r_2. \tag{48}$$

Since

$$|r_1\sigma_1 - r_2\sigma_2| = |(r_1-r_2)\sigma_1 + r_2(\sigma_1-\sigma_2)| \le |r_1-r_2| + |r_2||\sigma_1-\sigma_2|,$$

it follows that, for $\Lambda r_1 \le r_2 \le \Lambda r_1$,

$$\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} \frac{\mathrm{d}\sigma_1 \mathrm{d}\sigma_2}{|r_1\sigma_1 - r_2\sigma_2|^{d+sp}} \ge C\int_0^1 \frac{s^{d-2}\mathrm{d}s}{\big(|r_1-r_2|+|r_2|s\big)^{d+sp}} \ge \frac{C}{r_2^{d-1}|r_1-r_2|^{1+sp}}. \tag{49}$$

The conclusion now follows from (48) and (49). $\square$

3.2. *Proof of Theorem 1*

Denote $\widehat{u}(r) = u(r\sigma)$ with $r>0$ and $\sigma \in \mathbb{S}^{d-1}$. We have, by polar coordinates,

$$\||x|^\gamma u\|_{L^\tau(\mathbb{R}^d)} = |\mathbb{S}^{d-1}|^{\frac{1}{\tau}} \|r^{\gamma+\frac{d-1}{\tau}}\widehat{u}\|_{L^\tau(0,\infty)}, \tag{50}$$

$$\||x|^\beta u\|_{L^q(\mathbb{R}^d)} = |\mathbb{S}^{d-1}|^{\frac{1}{q}} \|r^{\beta+\frac{d-1}{q}}\widehat{u}\|_{L^q(0,\infty)}, \tag{51}$$

and by Lemma 14,

$$\int_0^\infty\int_0^\infty \frac{|\widehat{u}(r_1)-\widehat{u}(r_2)|^p r_1^{\alpha p+d-1}\chi_\Lambda(r_1,r_2)}{|r_1-r_2|^{1+sp}}\,\mathrm{d}r_1\,\mathrm{d}r_2 \le C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^p|x|^{\alpha p}\chi_\Lambda(|x|,|y|)}{|x-y|^{d+sp}}\mathrm{d}x\mathrm{d}y. \tag{52}$$

Extend $\widehat{u}$ in $\mathbb{R}$ as an even function and still denote the extension by $\widehat{u}$. We have

$$\||\xi|^{\gamma+\frac{d-1}{\tau}}\widehat{u}\|_{L^\tau(\mathbb{R})} \sim \|r^{\gamma+\frac{d-1}{\tau}}\widehat{u}\|_{L^\tau(0,\infty)} \tag{53}$$

$$\||\xi|^{\beta+\frac{d-1}{q}}\widehat{u}\|_{L^\tau(\mathbb{R})} \sim \|r^{\beta+\frac{d-1}{q}}\widehat{u}\|_{L^q(0,\infty)}, \tag{54}$$

and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\widehat{u}(\xi_1)-\widehat{u}(\xi_2)|^p|\xi_1|^{\alpha p+d-1}\chi_\Lambda(|\xi_1|,|\xi_2|)}{|\xi_1-\xi_2|^{1+sp}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \le 4\int_0^\infty\int_0^\infty \frac{|\widehat{u}(r_1)-\widehat{u}(r_2)|^p r_1^{\alpha p+d-1}\chi_\Lambda(r_1,r_2)}{|r_1-r_2|^{1+sp}}\,\mathrm{d}r_1\,\mathrm{d}r_2. \tag{55}$$

Hereafter in this proof, two quantities are $\sim$ if each one is bounded by the other up to a positive constant depending only on the parameters.

It thus suffices to prove

$$\||\xi|^{\gamma+\frac{d-1}{\tau}}\widehat{u}\|_{L^\tau(\mathbb{R})} \le C\|\widehat{u}\|^a_{\dot{W}^{s,p,\alpha+\frac{d-1}{p},\Lambda}(\mathbb{R})} \||\xi|^{\beta+\frac{d-1}{q}}\widehat{u}\|^{1-a}_{L^\tau(\mathbb{R})}. \tag{56}$$

This is in fact a consequence of Theorem 9 in one dimensional case. To this end, let first rewrite the conclusion of Theorem 9 in one dimensional case. Let $0<s'<1$, $p'>1$, $q'\ge 1$ $\tau'\ge 1$, $0<a'\le 1$, $\alpha',\beta',\gamma' \in \mathbb{R}$ and define σ' by $\sigma' \in \mathbb{R}$ by $\gamma' = a'\sigma' + (1-a')\beta'$. Assume that

$$\frac{1}{\tau'}+\gamma' = a'\left(\frac{1}{p'}+\alpha'-s'\right)+(1-a')\left(\frac{1}{q'}+\beta'\right), \tag{57}$$

$$0 \le \alpha'-\sigma', \tag{58}$$

and

$$\alpha'-\sigma' \le s' \quad \text{if } \frac{1}{\tau'}+\gamma' = \frac{1}{p'}+\alpha'-s'. \tag{59}$$

Then, if $\frac{1}{\tau'}+\gamma'>0$, it holds

$$\left\||x|^{\gamma'}g\right\|_{L^{\tau'}(\mathbb{R})} \le C\left\|g\right\|^{a'}_{\dot{W}^{s',p',\alpha',4}(\mathbb{R})}\left\||x|^{\beta'}g\right\|^{1-a'}_{L^{q'}(\mathbb{R})} \text{ for } g\in L^1_{loc}(\mathbb{R}\setminus\{0\}), \text{ with compact support in } \mathbb{R}, \tag{60}$$

and if $\frac{1}{\tau'} + \gamma' < 0$, it holds

$$\left\| |x|^{\gamma'} g \right\|_{L^{\tau'}(\mathbb{R})} \le C \| g \|^{a'}_{\dot{W}^{s',p',\alpha',4}(\mathbb{R})} \left\| |x|^{\beta'} g \right\|^{1-a'}_{L^{q'}(\mathbb{R})} \text{ for } g \in L^1_{loc}(\mathbb{R}) \text{ with } 0 \notin \operatorname{supp} g. \tag{61}$$

We are applying (60) and (61) with $s' = s$, $a' = a$, $p' = p$, $q' = q$, $\tau' = \tau$, $\alpha' = \alpha + \frac{d-1}{p}$, $\beta' = \beta + \frac{d-1}{q}$, $\gamma' = \gamma + \frac{d-1}{\tau}$, $a\sigma' + (1-a)\beta' = \gamma'$. Then clearly,

$$\frac{1}{\tau'} + \gamma' = \frac{d}{\tau} + \gamma, \quad \frac{1}{p'} + \alpha' - s' = \frac{d}{p} + \alpha - s, \quad \frac{1}{q'} + \beta' = \frac{d}{q} + \beta.$$

Hence (57) follows from (7).

We next compute $\alpha' - \sigma'$. Since $a\sigma' + (1-a)\beta' = \gamma' = \gamma + \frac{d-1}{\tau}$ and $a\sigma + (1-a)\beta = \gamma$, it follows that

$$\begin{aligned} a(\sigma' - \sigma) = \frac{d-1}{\tau} - (1-a)(\beta' - \beta) &= \frac{d-1}{\tau} - \frac{(1-a)(d-1)}{q} \\ &= (d-1)\left(\frac{1}{\tau} - \frac{1-a}{q}\right) \overset{(7)}{=} a(d-1)\left(\frac{1}{p} + \frac{\alpha - \sigma - s}{d}\right). \end{aligned}$$

It follows that

$$\alpha' - \sigma' = \alpha + \frac{d-1}{p} - \sigma - (d-1)\left(\frac{1}{p} + \frac{\alpha - \sigma - s}{d}\right) = \frac{\alpha - \sigma}{d} + \frac{s(d-1)}{d}.$$

This yields that $\alpha' - \sigma' \ge 0$ if and only if $\alpha - \sigma \ge -s(d-1)$.

The conclusion now follows from (60) and (61). □

3.3. *Proof of Theorem 10*

The proof is in the same spirit of the one of Theorem 9. For the convenience of the reader, we briefly describe the main lines. Denote $\widehat{u}(r) = u(r\sigma)$ with $r > 0$ and $\sigma \in \mathbb{S}^{d-1}$. We have, by polar coordinates,

$$\left(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^{\mu}(2R_2/|x|)} |u|^{\tau} \mathrm{d}x\right)^{1/\tau} = |\mathbb{S}^{d-1}|^{1/\tau} \left(\int_0^{\infty} \frac{r^{\gamma\tau + d-1}}{\ln^{\mu}(2R_2 r)} |\widehat{u}|^{\tau} \mathrm{d}r\right)^{1/\tau} \tag{62}$$

and

$$\left(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^{\mu}(2|x|/R_1)} |u|^{\tau} \mathrm{d}x\right)^{1/\tau} = |\mathbb{S}^{d-1}|^{1/\tau} \left(\int_0^{\infty} \frac{r^{\gamma\tau + (d-1)}}{\ln^{\mu}(2r/R_1)} |\widehat{u}|^{\tau} \mathrm{d}r\right)^{1/\tau}. \tag{63}$$

Extend $\widehat{u}$ in $\mathbb{R}$ as an even function and still denote the extension by $\widehat{u}$. Using (51) and (52), as in (56), it suffices to prove that if $\operatorname{supp} \widehat{u} \subset B_{R_2} \subset \mathbb{R}$, then it holds

$$\left(\int_{\mathbb{R}} \frac{|\xi|^{(\gamma + \frac{d-1}{\tau})\tau}}{\ln^{\mu}(2R_2/|\xi|)} |\widehat{u}|^{\tau} \mathrm{d}\xi\right)^{\frac{1}{\tau}} \le C \|\widehat{u}\|^{a}_{\dot{W}^{s,p,\alpha + \frac{d-1}{p},\Lambda}(\mathbb{R})} \left\| |\xi|^{\beta + \frac{d-1}{q}} |\widehat{u}|^{1-a} \right\|_{L^q(\mathbb{R})}, \tag{64}$$

and if $\operatorname{supp} \widehat{u} \cap B_{R_1} = \emptyset$, then it holds

$$\left(\int_{\mathbb{R}} \frac{|\xi|^{(\gamma + \frac{d-1}{\tau})\tau}}{\ln^{\mu}(2|\xi|/R_1)} |\widehat{u}|^{\tau} \mathrm{d}\xi\right)^{\frac{1}{\tau}} \le C \|\widehat{u}\|^{a}_{\dot{W}^{s,p,\alpha + \frac{d-1}{p},\Lambda}(\mathbb{R})} \left\| |\xi|^{\beta + \frac{d-1}{q}} |\widehat{u}|^{1-a} \right\|_{L^q(\mathbb{R})}. \tag{65}$$

The conclusion now follows from Theorem 10 as in the proof of Theorem 5. The details are omitted. □

We next show the optimality of condition $\alpha - \sigma \ge -(d-1)s$ given in (17).

Proposition 15. *The condition $\alpha - \sigma \ge -(d-1)s$ in* (17) *is necessary for the assertions in Theorem 1 to hold.*

Proof. Let $v \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} v \subset (0,1)$. For large $R > 0$ define $u_R(x) := v(|x| - R)$, for $x \in \mathbb{R}^d$. Clearly, $u_R \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} u_R \subset \mathscr{A}_{R,R+1}$, where for any $b, c \in (0,\infty)$, with $b < c$, the set $\mathscr{A}_{b,c}$ is defined by

$$\mathscr{A}_{b,c} := \{x \in \mathbb{R}^d : b < |x| < c\}.$$

We denote

$$\gamma' := \frac{d-1}{\tau} + \gamma,\ \alpha' := \frac{d-1}{p} + \alpha,\ \beta' := \frac{d-1}{q} + \beta.$$

One can check that

$$\|u_R\|_{\dot W^{s,p,0,\Lambda}(\mathbb{R}^d)} \le C\|u_R\|_{W^{1,p}(\mathbb{R}^d)} \le CR^{\frac{d-1}{p}}, \tag{66}$$

and since $\operatorname{supp} u_R \subset \mathscr{A}_{R,R+1}$,

$$\|u_R\|_{\dot W^{s,p,\alpha,\Lambda}(\mathbb{R}^d)} \le \|u_R\|_{\dot W^{s,p,\alpha,\Lambda}\left(\mathscr{A}_{R\Lambda^{-1},(R+1)\Lambda}\times\mathscr{A}_{R\Lambda^{-1},(R+1)\Lambda}\right)} \le CR^{\alpha}\|u_R\|_{\dot W^{s,p,0,\Lambda}(\mathbb{R}^d)}. \tag{67}$$

Combining (66) and (67) yields

$$\|u_R\|^a_{\dot W^{s,p,\alpha,\Lambda}(\mathbb{R}^d)} \le CR^{a\alpha'}. \tag{68}$$

On the other hand, one can check that

$$\left\||x|^\gamma u_R\right\|_{L^\tau(\mathbb{R}^d)} \sim R^{\gamma'} \text{ and } \left\||x|^\beta u_R\right\|^{1-a}_{L^q(\mathbb{R}^d)} \sim R^{(1-a)\beta'}. \tag{69}$$

Therefore, if either (18) or (19) holds then using them for $u = u_R$, we conclude form (68) and (69)

$$R^{\gamma'} \le CR^{a\alpha'+(1-a)\beta'} \text{ for } R \text{ large and } C > 0 \text{ independent of } R,$$

which is possible only when $\alpha - \sigma \ge -(d-1)s$. The proof is complete. □

4. The Caffarelli–Kohn–Nirenberg inequalities for radial functions in the Sobolev spaces

In this section, we present the result in the case $s = 1$. We first state variants/improvements of the CKN inequalities in the Sobolev spaces which follows directly from the approach given in [24] (see also the proof of Theorem 9). We begin with the case $1/\tau + \gamma/d \neq 0$.

Theorem 16. *Let $d \ge 1$, $p \ge 1$, $q \ge 1$, $\tau \ge 1$, $0 < a \le 1$, and $\alpha, \beta, \gamma \in \mathbb{R}$. Define σ by* (3). *Assume* (2), (4), *and* (5). *We have, for some positive constant C,*

(i) *if $\frac{1}{\tau} + \frac{\gamma}{d} > 0$, then for all $u \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ with compact support in $\mathbb{R}^d$, it holds*

$$\left\||x|^\tau u\right\|_{L^\tau(\mathbb{R}^d)} \le C\left\||x|^\alpha \nabla u\right\|^a_{L^p(\mathbb{R}^d\setminus\{0\})}\left\||x|^\beta u\right\|^{1-a}_{L^q(\mathbb{R}^d)}, \tag{70}$$

(ii) *if $\frac{1}{\tau} + \frac{\gamma}{d} < 0$, then for all $u \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ which is 0 in a neighborhood of* 0, (70) *holds.*

Concerning the limiting case $1/\tau + \gamma/d = 0$, one has the following result.

Theorem 17. *Let $d \ge 1$, $p \ge 1$, $q \ge 1$, $\tau \ge 1$, $0 < a \le 1$, and $\alpha, \beta, \gamma \in \mathbb{R}$, and $\mu > 1$. Assume that $\tau \le \mu$. Define σ by* (3). *Assume* (2) *and*

$$0 \le \alpha - \sigma \le 1. \tag{71}$$

There exists a positive constant C such that for all $u \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ and for all $R_1, R_2 > 0$, we have

(i) *if $\frac{1}{\tau} + \frac{\gamma}{d} = 0$ and $\operatorname{supp} u \subset B_{R_2}$, then*

$$\left(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^\mu(2R_2/|x|)}|u|^\tau \mathrm{d}x\right)^{\frac{1}{\tau}} \le C\||x|^\alpha \nabla u\|^a_{L^p(\mathbb{R}^d\setminus\{0\})}\left\||x|^\beta u\right\|^{1-a}_{L^q(\mathbb{R}^d)}, \tag{72}$$

(ii) *if $\frac{1}{\tau} + \frac{\gamma}{d} = 0$, and $\operatorname{supp} u \cap B_{R_1} = \emptyset$, then*

$$\left(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^\mu(2|x|/R_1)}|u|^\tau \mathrm{d}x\right)^{\frac{1}{\tau}} \le C\||x|^\alpha \nabla u\|^a_{L^p(\mathbb{R}^d\setminus\{0\})}\left\||x|^\beta u\right\|^{1-a}_{L^q(\mathbb{R}^d)}. \tag{73}$$

We are ready to state the corresponding results in the radial case. We begin with the case $1/\tau+\gamma/d \neq 0$.

Theorem 18. *Let $d \geq 2$, $p \geq 1$, $q \geq 1$, $\tau \geq 1$, $0 < a \leq 1$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Define σ by* (3)*. Assume* (2) *and*

$$-(d-1) \leq \alpha-\sigma < 0. \tag{74}$$

We have, for some positive constant C,

(i) *if $\frac{1}{\tau}+\frac{\gamma}{d} > 0$, then for all radial $u \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ with compact support in $\mathbb{R}^d$,* (70) *holds;*
(ii) *if $\frac{1}{\tau}+\frac{\gamma}{d} < 0$, then for all radial $u \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ which is 0 in a neighborhood of* 0, (70) *holds.*

Concerning the limiting case $1/\tau+\gamma/d = 0$, we obtain the following result.

Theorem 19. *Let $d \geq 2$, $p \geq 1$, $q \geq 1$, $\tau \geq 1$, $0 < a \leq 1$, $\alpha, \beta, \gamma \in \mathbb{R}$, and $\mu > 1$. Assume that $\tau \leq \mu$. Define σ by* (3)*. Assume* (2) *and* (74)*. There exists a positive constant C such that for all $u \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ and for all $0 < R_1 < R_2$, we have*

(i) *if $\frac{1}{\tau}+\frac{\gamma}{d} = 0$ and $\operatorname{supp} u \subset B_{R_2}$, then* (72) *holds.*
(ii) *if $\frac{1}{\tau}+\frac{\gamma}{d} = 0$, and $\operatorname{supp} u \cap B_{R_1} = \emptyset$, then* (73) *holds.*

Remark 20. The convention in Remark 2 is also used in Theorem 16, Theorem 17, Theorem 18, and Theorem 19. In these theorems, the quantity $\| |x|^{\alpha} \nabla u\|_{L^p(\mathbb{R}^d \setminus \{0\})}$ is also considered as infinity if $\nabla u \notin [L^p_{loc}(\mathbb{R}^d \setminus \{0\})]^d$.

Remark 21. By similar considerations as in Remark 7, we can conclude that the log-term is necessary in Theorem 19.

Theorem 18 and Theorem 19 are direct consequences of Theorem 16 and Theorem 17 in the one dimensional case. The proofs are as in the spirit of the proof of Theorem 1 and Theorem 5 but simpler where a variant of Lemma 14 is not required. The details are left to the reader.

Concerning the optimality of the condition of (74), we have the following result whose proof is similar to the one of Proposition 15 and omitted.

Proposition 22. *The condition $\alpha-\sigma \geq -(d-1)$ in* (74) *is necessary for the assertions in Theorem 18 to hold.*

We end this section by mentioning what has been proved previously. In the case $1/\tau+\gamma/d > 0$, under the following additional requirement (see [15, the first inequality in (1.8) and (1.10)])

$$\frac{a(\alpha-1-\sigma)}{d}+\frac{1-a}{q} \geq 0 \quad \text{and} \quad \frac{1}{p}+\frac{\alpha-1}{d} > 0,$$

assertion i) of Theorem 18 was previously proved in [15] by a different approach via the Riesz potential and inequalities on fractional integrations.

5. Applications to the compactness

In this section, we derive several compactness results from previous inequalities for radial case. We only consider the case $1/\tau+\gamma/d > 0$. We begin with the following result.

Proposition 23. *Let $d \geq 1$, $0 < s < 1$, $p > 1$, $q \geq 1$, $\tau \geq 1$, $0 < a < 1$, α, β, $\gamma \in \mathbb{R}$, and $\Lambda > 1$ be such that $1/\tau+\gamma/d > 0$. Define σ by* (3)*. Assume* (7)*,*

$$\alpha-\sigma > 0, \quad \text{and} \quad \frac{1}{p}+\frac{\alpha-s}{d} \neq \frac{1}{q}+\frac{\beta}{d}.$$

Assume that the embedding $W^{s,p}(B_1) \cap L^q(B_1)$ into $L^\tau(B_1)$ is compact. Let $(u_n)_n \subset L^1_{loc}(\mathbb{R}^d)$ with compact support be such that the sequences $\big(\|u_n\|_{\dot{W}^{s,p,\alpha,\Lambda}(\mathbb{R}^d)}\big)_n$ and $\big(\||x|^\beta u_n\|_{L^q(\mathbb{R}^d)}\big)_n$ are bounded. Then, up to a subsequence, $(|x|^\gamma u_n)_n$ converges in $L^\tau(\mathbb{R}^d)$.

Proof. One just notes that for γ' sufficiently close to γ, one can choose $0 < a' < 1$ close to a such that the assumptions of Proposition 23 hold with (a,γ) being replaced by (a',γ'). This implies, by Theorem 9 (see also (10)) that, for $\varepsilon > 0$ sufficiently small,

$$(\||x|^{\gamma+\varepsilon} u_n\|_{L^\tau}) \text{ and } (\||x|^{\gamma-\varepsilon} u_n\|_{L^\tau}) \text{ are bounded.}$$

The conclusion follows since the embedding $W^{s,p}(B_R) \cap L^q(B_R)$ into $L^\tau(B_R)$ is compact for $R > 0$. □

In the case $s = 1$, one has the following result, whose proof is almost identical and omitted.

Proposition 24. *Let $d \ge 1$, $p \ge 1$, $q \ge 1$, $\tau \ge 1$, $0 < a < 1$, and $\alpha,\beta,\gamma \in \mathbb{R}$ be such that $1/\tau + \gamma/d > 0$. Define σ by (3). Assume (2),*

$$\alpha - \sigma > 0, \quad \text{and} \quad \frac{1}{p} + \frac{\alpha - 1}{d} \neq \frac{1}{q} + \frac{\beta}{d}.$$

Assume that the embedding $W^{1,p}(B_1) \cap L^q(B_1)$ into $L^\tau(B_1)$ is compact. Let $(u_n)_n \subset L^1_{loc}(\mathbb{R}^d)$ with compact support be such that the sequences $\big(\||x|^\alpha \nabla u_n\|_{L^p(\mathbb{R}^d)}\big)_n$ and $\big(\||x|^\beta u_n\|_{L^q(\mathbb{R}^d)}\big)_n$ are bounded. Then, up to a subsequence, $\big(|x|^\gamma u_n\big)_n$ converges in $L^\tau(\mathbb{R}^d)$.

Here are the variants for radial functions, whose proof are almost the same and omitted.

Proposition 25. *Let $d \ge 2$, $0 < s < 1$, $p > 1$, $q \ge 1$, $\tau \ge 1$, $0 < a < 1$, α, β, $\gamma \in \mathbb{R}$, and $\Lambda > 1$ be such that $1/\tau + \gamma/d > 0$. Define σ by (3). Assume (7),*

$$\alpha - \sigma > -(d-1)s, \quad \text{and} \quad \frac{1}{p} + \frac{\alpha - s}{d} \neq \frac{1}{q} + \frac{\beta}{d}.$$

Assume that the embedding $W^{s,p}(B_1) \cap L^q(B_1)$ into $L^\tau(B_1)$ is compact. Let $(u_n)_n \subset L^1_{loc}(\mathbb{R}^d)$ be radial such that the sequences $\big(\|u_n\|_{\dot{W}^{s,p,\alpha,\Lambda}(\mathbb{R}^d)}\big)_n$ and $\big(\||x|^\beta u_n\|_{L^q(\mathbb{R}^d)}\big)_n$ are bounded. Then, up to a subsequence, $\big(|x|^\gamma u_n\big)_n$ converges in $L^\tau(\mathbb{R}^d)$.

Proposition 26. *Let $d \ge 2$, $p \ge 1$, $q \ge 1$, $\tau \ge 1$, $0 < a < 1$, and $\alpha,\beta,\gamma \in \mathbb{R}$. Define σ by (3). Assume (2),*

$$\alpha - \sigma > -(d-1), \quad \text{and} \quad \frac{1}{p} + \frac{\alpha - 1}{d} \neq \frac{1}{q} + \frac{\beta}{d}.$$

Assume that the embedding $W^{1,p}(B_1) \cap L^q(B_1)$ into $L^\tau(B_1)$ is compact. Let $(u_n)_n \subset L^1_{loc}(\mathbb{R}^d)$ be radial with compact support such that the sequences $\big(\||x|^\alpha \nabla u_n\|_{L^p(\mathbb{R}^d)}\big)_n$ and $\big(\||x|^\beta u_n\|_{L^q(\mathbb{R}^d)}\big)_n$ are bounded. Then, up to a subsequence, $\big(|x|^\gamma u_n\big)_n$ converges in $L^\tau(\mathbb{R}^d)$.

We obtain the following corollary after using the density of the radial functions $C_c^\infty(\mathbb{R}^d)$ in the class of radial functions in $W^{s,p}(\mathbb{R}^d)$.

Corollary 27. *Let $d \ge 2$, $0 < s \le 1$, $p \ge 1$ and $sp < d$. Assume that $p < \tau < pd/(d - sp)$ and ($p > 1$ if $s < 1$). Let $\gamma_1 > 0$ and $\gamma_2 < 0$ be such that, for $j = 1,2$,*

$$\frac{1}{p} - \frac{s}{d} < \frac{1}{\tau} + \frac{\gamma_j}{d} < \frac{1}{p}.$$

Then the embedding $W^{s,p}(\mathbb{R}^d)$ into $L^\tau(|x|^{\gamma_j},\mathbb{R}^d)$ for radial functions is compact. As a consequence, the embedding $W^{s,p}(\mathbb{R}^d)$ into $L^\tau(\mathbb{R}^d)$ in the class of radial functions is compact.

Remark 28. The fact that the embedding $W^{s,p}(\mathbb{R}^d)$ into $L^\tau(\mathbb{R}^d)$ in the class of radial functions is compact is known, see e.g., [5, 19, 28] in the case $s = 1$ and [27] in the case $0 < s < 1$ (whose proof is based on the atomic decomposition). The ideas to derive the compactness as presented here are quite standard, see, e.g., [2].

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