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Surfaces of infinite-type are non-Hopfian

Les surfaces de type infini sont non-Hopfian

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Abstract. We show that finite-type surfaces are characterized by a topological analogue of the Hopf property. Namely, an oriented surface Σ is of finite-type if and only if every proper map $f\colon \Sigma \to \Sigma$ of degree one is homotopic to a homeomorphism.

Résumé. Nous montrons que les surfaces de type fini sont caractérisées par un analogue topologique de la propriété de Hopf. A savoir, une surface orientée Σ est de type fini si et seulement si toute application propre $f\colon \Sigma \to \Sigma$ de degré un est homotope à un homéomorphisme.

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1. Introduction

All surfaces will be assumed to be connected and orientable throughout this note. We will say a surface is of *finite-type* if its fundamental group is finitely generated; otherwise, we will say it is of *infinite-type*.

Recall that a group G is said to be *Hopfian* if every surjective homomorphism $\varphi\colon G \twoheadrightarrow G$ is an isomorphism. It is well known that a finitely generated free group is Hopfian, for instance, as a consequence of Grushko's theorem. On the other hand, a free group generated by an infinite set S is not Hopfian as a surjective function $f\colon S \to S$ that is not injective extends to a surjective homeomorphism on the free group generated by S which is not injective.

In this note, we show that there is an analogous characterization for orientable surfaces of *finite-type*. The natural topological analog of a surjective homomorphism is a proper map of degree one, and that of an isomorphism is a homotopy equivalence.

One-half of this characterization is classical, namely that any proper map of degree one from a surface of finite-type to itself is a homotopy equivalence. For instance, a theorem of Olum (see [2, Corollary 3.4]) says that every proper map of degree one between two oriented manifolds of the same dimension is π_1-surjective. Now, the fundamental group of any surface is residually finite (see [4]). Also, any finitely generated residually finite group is Hopfian. Thus, every degree one

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self map of a finite-type surface is a weak homotopy equivalence, hence a homotopy equivalence by Whitehead's theorem.

Our main result is that infinite-type surfaces are not Hopfian.

Theorem 1. *Let Σ be any infinite-type surface. Then there exists a proper map $f\colon \Sigma \to \Sigma$ of degree one such that $\pi_1(f)\colon \pi_1(\Sigma) \to \pi_1(\Sigma)$ is not injective. In particular, f is not a homotopy equivalence.*

2. Background

A *surface* is a connected, orientable two-dimensional manifold without boundary and a *bordered surfaces* is a connected, orientable two-dimensional manifold wit non-empty boundary. A (possibly bordered) subsurface Σ' of a surface Σ is an embedded submanifold of codimension zero.

Let Σ be a non-compact surface. A *boundary component* of Σ is a nested sequence $P_1 \supseteq P_2 \supseteq \cdots$ of open, connected subsets of Σ such that the followings hold:

- the closure (in Σ) of each P_n is non-compact,
- the boundary of each P_n is compact, and
- for any subset A with compact closure (in Σ), we have $P_n \cap A = \varnothing$ for all large n.

We say that two boundary components $P_1 \supseteq P_2 \supseteq \cdots$ and $P'_1 \supseteq P'_2 \supseteq \cdots$ of Σ are *equivalent* if for any positive integer n there are positive integers k_n, ℓ_n such that $P_{k_n} \subseteq P'_n$ and $P'_{\ell_n} \subseteq P_n$. For a boundary component $\mathscr{P} = P_1 \supseteq P_2 \supseteq \cdots$, we let $[\mathscr{P}]$ to denote the equivalence class of $\mathscr{P}$.

The *space of ends* $\operatorname{Ends}(\Sigma)$ of Σ is the topological space having equivalence class of boundary components of Σ as elements, i.e., as a set $\operatorname{Ends}(\Sigma) := \big\{[\mathscr{P}] \big| \mathscr{P} \text{ is a boundary component}\big\}$; with the following topology: For any set X with compact boundary, at first, define

$$X^\dagger := \big\{[\mathscr{P} = P_1 \supseteq P_2 \supseteq \cdots] \big| X \supseteq P_n \supseteq P_{n+1} \supseteq \cdots \text{ for some large } n\big\}.$$

Now, take the set of all such $X^\dagger$ as a basis for the topology of $\operatorname{Ends}(\Sigma)$. The topological space $\operatorname{Ends}(\Sigma)$ is compact, separable, totally disconnected, and metrizable, i.e., homeomorphic to a non-empty closed subset of the Cantor set.

For a boundary component $[\mathscr{P}]$ with $\mathscr{P} = P_1 \supseteq P_2 \supseteq \cdots$, we say $[\mathscr{P}]$ is *planar* if P_n are homeomorphic to open subsets $\mathbb{R}^2$ for all large n. Define $\operatorname{Ends}_{\mathrm{np}}(\Sigma) := \big\{[\mathscr{P}] : [\mathscr{P}] \text{ is not planar}\big\}$. Thus, $\operatorname{Ends}_{\mathrm{np}}(\Sigma)$ is a closed subset of $\operatorname{Ends}(\Sigma)$. Also, define the *genus* of Σ as $g(\Sigma) := \sup g(\mathbf{S})$, where $\mathbf{S}$ is a compact bordered subsurface of Σ.

Theorem 2 (Kerékjártó's classification theorem [7, Theorem 1]). *Let Σ_1, Σ_2 be two non-compact surfaces. Then Σ_1 is homeomorphic to Σ_2 if and only if $g(\Sigma_1) = g(\Sigma_2)$, and there is a homeomorphism $\Phi\colon \operatorname{Ends}(\Sigma_1) \to \operatorname{Ends}(\Sigma_2)$ with $\Phi\big(\operatorname{Ends}_{\mathrm{np}}(\Sigma_1)\big) = \operatorname{Ends}_{\mathrm{np}}(\Sigma_2)$.*

Let Σ be a non-compact surface, and let $\mathscr{E}_{\mathrm{np}}(\Sigma) \subseteq \mathscr{E}(\Sigma)$ be two closed, totally-disconnected subsets of $\mathbb{S}^2$ such that the pair $\operatorname{Ends}_{\mathrm{np}}(\Sigma) \subseteq \operatorname{Ends}(\Sigma)$ is homeomorphic to the pair $\mathscr{E}_{\mathrm{np}}(\Sigma) \subseteq \mathscr{E}(\Sigma)$. Consider a pairwise disjoint collection $\{D_i \subseteq \mathbb{S}^2 \setminus \mathscr{E}(\Sigma) : i \in \mathscr{A}\}$ of closed disks, where $0 \le |\mathscr{A}| \le g(\Sigma)$, such that the following holds: For $p \in \mathbb{S}^2$, any open neighborhood (in $\mathbb{S}^2$) of p contains infinitely many D_i if and only if $p \in \mathscr{E}_{\mathrm{np}}(\Sigma)$. [7, Theorem 2] describes constructing such a collection of disks.

Now, let $M := (\mathbb{S}^2 \setminus \mathscr{E}(\Sigma)) \setminus \bigsqcup_{i\in\mathscr{A}} \operatorname{int}(D_i)$ and $N := \bigsqcup_{i\in\mathscr{A}} S_{1,1}$, where $S_{1,1}$ is the genus one compact bordered surface with one boundary component. Define a non-compact surface Σ_{handle} as follows: $\Sigma_{\text{handle}} := M \bigsqcup_{\partial M \equiv \partial N} N$. Then we have the following theorem.

Theorem 3 (Richards' representation theorem [7, Theorems 2 and 3]). *The surface Σ_{handle} is homeomorphic to Σ.*

3. Proof of Theorem 1

Let M and N be two non-compact, oriented, connected, boundaryless smooth n-manifolds. Then the singular cohomology groups with compact support $H^n_{\mathbf{c}}(M;\mathbb{Z})$ and $H^n_{\mathbf{c}}(N;\mathbb{Z})$ are infinite cyclic with preferred generators $[M]$ and $[N]$. If $f\colon M \to N$ is a proper map then the degree of f is the unique integer $\deg(f)$ defined as follows: $H^n_{\mathbf{c}}(f)([N]) = \deg(f)\cdot[M]$. Note that deg is proper-homotopy invariant and multiplicative. See [2, Section 1] for more details.

We will use the following well-known characterization of degree.

Lemma 4 ([2, Lemma 2.1b.]). *Let $f\colon M \to N$ be a proper map between two non-compact, oriented, connected, boundaryless smooth n-manifolds. Let D be a smoothly embedded closed disk in N and suppose $f^{-1}(D)$ is a smoothly embedded closed disk in M such that f maps $f^{-1}(D)$ homeomorphically onto D. Then* $\deg(f) = +1$ *or* -1 *according as $f|f^{-1}(D) \to D$ is orientation-preserving or orientation-reversing.*

We will prove Theorem 1 by considering the following three cases:

(1) Σ has infinite genus.
(2) Σ has finite genus and the set of isolated points $\mathscr{I}(\Sigma)$ of $\mathscr{E}(\Sigma)$ is finite.
(3) Σ has finite genus and the set of isolated points $\mathscr{I}(\Sigma)$ of $\mathscr{E}(\Sigma)$ is infinite.

Remark 5. If Σ is an infinite-type surface of a finite genus, then $\mathscr{E}(\Sigma)$ is an infinite set.

Our first result proves Theorem 1 in the case with infinite genus.

Theorem 6. *Let Σ be a surface of the infinite genus. Then there exists a degree one map $f\colon \Sigma \to \Sigma$ which is not π_1-injective.*

Proof. Since Σ has infinite genus, there exists a compact bordered subsurface $\mathscr{S} \subset \Sigma$ such that $\mathscr{S}$ has genus one and one boundary component. Define $\Sigma' := \Sigma/\mathscr{S}$ be the quotient of Σ with $\mathscr{S}$ pinched to a point and let $q\colon \Sigma \to \Sigma'$ be the quotient map. Thus, Σ' is also an infinite genus surface. Further, there are compact sets in $K \subset \Sigma$ and $K' \subset \Sigma'$ whose complements are homeomorphic, so the pair $(\mathscr{E}(\Sigma), \mathscr{E}_{\mathrm{np}}(\Sigma))$ is homeomorphic to the pair $(\mathscr{E}(\Sigma'), \mathscr{E}_{\mathrm{np}}(\Sigma'))$. Hence, by Theorem 2, there is a homeomorphism $\varphi\colon \Sigma' \to \Sigma$.

Let $f\colon \Sigma \to \Sigma$ be the composition $f = \varphi \circ q$. By Lemma 4, the quotient map $q\colon \Sigma \to \Sigma'$ is of degree ± 1. Thus, $\deg(f) = \pm 1$ as homeomorphisms have degree ± 1. Notice that f sends $\partial\mathscr{S}$ to a point. But $\partial\mathscr{S}$ does not bound any disk in Σ, i.e., $\partial\mathscr{S}$ represents a primitive element of $\pi_1(\Sigma)$, see [1, Theorem 1.7. and Theorem 4.2.]. Hence, f is not π_1-injective. If $\deg(f) = 1$, then we are done. Otherwise, we replace f by $f \circ f$ to get a map that has degree one and is not injective on π_1. □

For the remaining two cases, we use a map from the sphere to the sphere, which has degree ± 1 but with some disks identified. We will replace these disks with appropriate surfaces to get Σ.

Lemma 7. *There exist pairwise disjoint closed disks $\mathscr{D}_0, \mathscr{D}_1 \subseteq \mathbb{S}^2$ and a map $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ such that the following hold:*

- *$f^{-1}(\mathscr{D}_0) = \mathscr{D}_0$ and $f|_{\mathscr{D}_0}\colon \mathscr{D}_0 \to \mathscr{D}_0$ is the identity map.*
- *$f^{-1}(\mathscr{D}_1)$ is the union of pairwise-disjoint closed disks $\mathscr{D}_{1,1}, \mathscr{D}_{1,2}$, and $\mathscr{D}_{1,3}$ in $\mathbb{S}^2$; and $f|_{\mathscr{D}_{1k}}\colon \mathscr{D}_{1k} \to \mathscr{D}_1$ is a homeomorphism for each $k \in \{1,2,3\}$.*

Further, there is a loop γ in $\mathbb{S}^2 \setminus \mathrm{int}(\mathscr{D}_0 \cup \mathscr{D}_{1,1} \cup \mathscr{D}_{1,2} \cup \mathscr{D}_{1,3})$ which is not homotopically trivial in $\mathbb{S}^2 \setminus \mathrm{int}(\mathscr{D}_0 \cup \mathscr{D}_{1,1} \cup \mathscr{D}_{1,2} \cup \mathscr{D}_{1,3})$ but such that $f(\gamma)$ is null-homotopic in $\mathbb{S}^2 \setminus \mathrm{int}(\mathscr{D}_0 \cup \mathscr{D}_1)$.

Proof. For each $k \in \{0,1,2,3\}$, choose $(a_k, b_k) \in \mathbb{R}^2$ such that if we define $\mathscr{B}_k := \{(x,y) \in \mathbb{R}^2 : (x-a_k)^2 + (y-b_k)^2 \le 1\}$, then $\{\mathscr{B}_0, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3\}$ is a pairwise-disjoint collection of closed disks.

Define $X := \mathbb{S}^2 \setminus \bigcup_{i=0}^{3} \mathrm{int}(\mathscr{B}_i)$ and $Y := \mathbb{S}^2 \setminus \bigcup_{i=0}^{1} \mathrm{int}(\mathscr{B}_i)$. Next, define a map $f\colon \partial X \to Y$ as follows:

- $f|_{\partial\mathscr{B}_k}\colon \partial\mathscr{B}_k \to \partial\mathscr{B}_k$ is the identity map for each $k \in \{0,1\}$;
- $f|_{\partial\mathscr{B}_2}\colon \partial\mathscr{B}_2 \to \partial\mathscr{B}_1$ is defined as $f(x,y) := (-x+a_2+a_1, y-b_2+b_1)$ for all $(x,y) \in \partial\mathscr{B}_2$.
- $f|_{\partial\mathscr{B}_3}\colon \partial\mathscr{B}_3 \to \partial\mathscr{B}_1$ is defined as $f(x,y) := (x-a_3+a_1, y-b_3+b_1)$ for all $(x,y) \in \partial\mathscr{B}_3$.

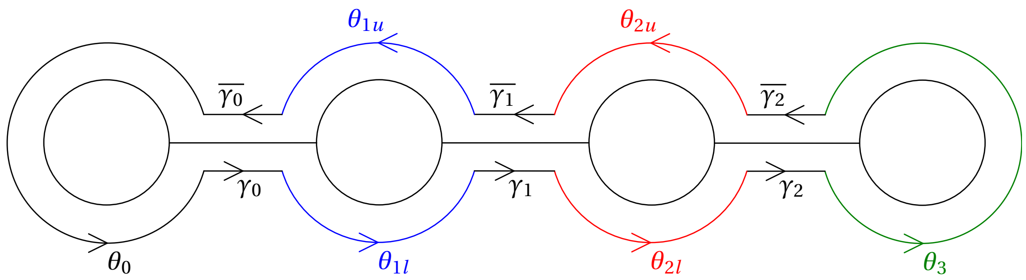

Figure 1. The four-holed sphere X by attaching a 2-cell.

For each $k \in \{0,1,2\}$, let $\gamma_k\colon [0,1] \hookrightarrow X$ be an embedding such that $\mathrm{im}(\gamma_k) \cap \partial X$ consists of $\gamma_k(0) = (a_k+1, b_k) \in \partial\mathscr{B}_k$ and $\gamma_k(1) = (a_{k+1}-1, b_{k+1}) \in \partial\mathscr{B}_{k+1}$.

Define $\Gamma_0\colon [0,1] \to Y$ as $\Gamma_0(t) := \gamma_0(t)$ for all $t \in [0,1]$. Let $\Gamma_1, \Gamma_2\colon [0,1] \to Y$ be the constant loops based at the points $(a_1+1, b_1) \in \partial Y$ and $(a_1-1, b_1) \in \partial Y$, respectively.

Next, define $X^{(1)} := \partial X \cup \mathrm{im}(\gamma_0) \cup \mathrm{im}(\gamma_1) \cup \mathrm{im}(\gamma_2)$. Extend $f\colon \partial X \to Y$ to a map $X^{(1)} \to Y$, which we again denote by $f\colon X^{(1)} \to Y$, by mapping γ_0 onto Γ_0 by the identity, and, for each $k = 1,2$, mapping γ_k to the constant loop Γ_k.

Let θ_0 (resp. θ_3) be the simple loop that traverses $\partial\mathscr{B}_0$ (resp. $\partial\mathscr{B}_3$) in the counter-clockwise direction starting from (a_0+1, b_0) (resp. (a_3-1, b_3)).

Let $\theta_{1,l}$ (resp. $\theta_{1,u}$) be the simple arc that traverses $\partial\mathscr{B}_1 \cap \{y \le b_1\}$ (resp. $\partial\mathscr{B}_1 \cap \{y \ge b_1\}$) counter-clockwise direction. Similarly, define $\theta_{2,l}$ and $\theta_{2,u}$.

Now, $X \cong X^{(1)} \cup_\varphi \mathbb{D}^2$, (see Figure 1) where the attaching map $\varphi\colon \mathbb{S}^1 \to X^{(1)}$ can be described as

$$\varphi := \theta_0 * \gamma_0 * \theta_{1,l} * \gamma_1 * \theta_{2,l} * \gamma_2 * \theta_3 * \overline{\gamma_2} * \theta_{2,u} * \overline{\gamma_1} * \theta_{1,u} * \overline{\gamma_0}.$$

Notice that $f(\gamma_1) = \Gamma_1$ and $f(\gamma_2) = \Gamma_2$ are constant loops. Also, as in Figure 2, $\overline{f \circ \theta_{1,l}} = f \circ \theta_{2,l}$ and $\overline{f \circ \theta_{1,u}} = f \circ \theta_{2,u}$. Thus, $f \circ \varphi$ is homotopic to $(f \circ \theta_0) * \Gamma_0 * (f \circ \theta_3) * \overline{\Gamma_0}$.

If $r\colon Y \cong \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1$ is the projection then $r \circ f \circ \theta_0$ and $r \circ f \circ \theta_3$ traverse $\mathbb{S}^1$ in opposite directions. Since r is a strong deformation retract, $(f \circ \theta_0) * \Gamma_0 * (f \circ \theta_3) * \overline{\Gamma_0}$, and hence $f \circ \varphi$ is null-homotopic. Now, the null-homotopic map $f \circ \varphi\colon \mathbb{S}^1 \to Y$ can be extended to a map $\mathbb{D}^2 \to Y$. Thus $f\colon X^{(1)} \to Y$ can be extended to a map $X \cong X^{(1)} \cup_\varphi \mathbb{D}^2 \to Y$, which will be again denoted by $f\colon X \to Y$.

Note that every homeomorphism $\mathbb{S}^1 \to \mathbb{S}^1$ can be extended to a homeomorphism $\mathbb{D}^2 \to \mathbb{D}^2$ naturally. Thus, we can extend $f\colon X \to Y$ to a map $\mathbb{S}^2 \to \mathbb{S}^2$, which will be again denoted by $f\colon \mathbb{S}^2 \to \mathbb{S}^2$. Let $\mathscr{D}_0$ (resp. $\mathscr{D}_1$) be any closed disk, which is contained in $\mathrm{int}(\mathscr{B}_0)$ (resp. $\mathrm{int}(\mathscr{B}_1)$).

Finally, observe that if $\gamma = \theta_{1u} * \theta_{1l} * \gamma_1 * \theta_{2l} * \theta_{2u} * \overline{\gamma_1}$, then γ is a loop in $\mathbb{S}^2 \setminus \mathrm{int}(\mathscr{D}_0 \cup \mathscr{D}_{1,1} \cup \mathscr{D}_{1,2} \cup \mathscr{D}_{1,3})$ which is not homotopically trivial in $\mathbb{S}^2 \setminus \mathrm{int}(\mathscr{D}_0 \cup \mathscr{D}_{1,1} \cup \mathscr{D}_{1,2} \cup \mathscr{D}_{1,3})$, but such that $f(\gamma)$ is null-homotopic in $\mathbb{S}^2 \setminus \mathrm{int}(\mathscr{D}_0 \cup \mathscr{D}_1)$, as claimed. □

We now prove Theorem 1 in the two remaining cases, in both of which we have a finite genus surface. Note that for a finite genus surface, all ends are planar, so in applying Theorem 2, it suffices to consider the genus and the space of ends.

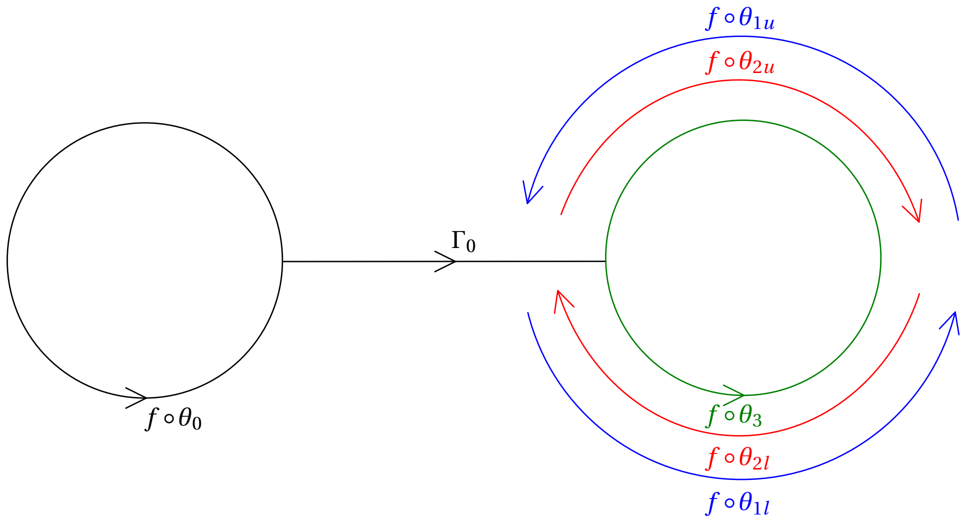

Figure 2. The map on the $X^{(1)}$

Theorem 8. *Let Σ be a finite genus infinite-type surface such that $\mathscr{E}(\Sigma)$ has finitely many isolated points. Then there is a degree one map $f: \Sigma \to \Sigma$ which is not π_1-injective.*

Proof. Let $\mathscr{I}(\Sigma)$ be the set of all isolated points of $\mathscr{E}(\Sigma)$, let $k \in \mathbb{N} \cup \{0\}$ be the cardinality of $\mathscr{I}(\Sigma)$, and let g be the genus of Σ. Then $\mathscr{C}(\Sigma) := \mathscr{E}(\Sigma) \setminus \mathscr{I}(\Sigma)$ is a non-empty, perfect, compact, totally-disconnected, metrizable space as it is infinite (by Remark 5) and has no isolated points. Thus $\mathscr{C}(\Sigma)$ is a Cantor space (see [5, Theorem 8 of Chapter 12]).

Let $\mathscr{D}_0, \mathscr{D}_1, \mathscr{D}_{1,1}, \mathscr{D}_{1,2}, \mathscr{D}_{1,3} \subseteq \mathbb{S}^2$, $f: \mathbb{S}^2 \to \mathbb{S}^2$, and let γ be as in the conclusion of Lemma 7. Let $C_1 \subset \mathrm{int}(\mathscr{D}_1)$ be a subset homeomorphic to the Cantor set and let $\mathscr{I} \subset \mathrm{int}(\mathscr{D}_0)$ be a set consisting of k points (hence homeomorphic to $\mathscr{I}(\Sigma)$). Let $C_{1,j} = f^{-1}(C_1) \cap \mathscr{D}_{1,j}$ for $j = 1,2,3$. Note that each $C_{1,j}$ is homeomorphic to the Cantor set. See Figure 3.

As $f^{-1}(\mathscr{D}_0) = \mathscr{D}_0$ and $f|_{\mathscr{D}_0}: \mathscr{D}_0 \to \mathscr{D}_0$ is the identity map, we can say that $f^{-1}(\mathscr{I}) = \mathscr{I}$. Let Σ_1 be the surface obtained from $\mathbb{S}^2 \setminus (\mathscr{I} \cup C_1)$ by attaching g handles along disjoint disks $\Delta_k \subset \mathrm{int}(\mathscr{D}_0) \setminus \mathscr{I}$, $1 \le k \le g$ and let Σ_2 be the surface obtained from $\mathbb{S}^2 \setminus (\mathscr{I} \cup C_{1,1} \cup C_{1,2} \cup C_{1,3})$ by attaching g handles along the (same) disks Δ_k, $1 \le k \le g$. Then f induces a proper map, which we also call f, from Σ_2 to Σ_1. By Lemma 4, $\deg(f) = \pm 1$.

Further, we claim that $f: \Sigma_2 \to \Sigma_1$ is not injective on π_1. Namely, the fundamental group of Σ_2 is the amalgamated free product of four groups, one of which is $\pi_1(\mathbb{S}^2 \setminus \mathrm{int}(\mathscr{D}_0 \cup \mathscr{D}_{1,1} \cup \mathscr{D}_{1,2} \cup \mathscr{D}_{1,3}))$. As γ is not homotopic to the trivial loop in $\mathbb{S}^2 \setminus \mathrm{int}(\mathscr{D}_0 \cup \mathscr{D}_{1,1} \cup \mathscr{D}_{1,2} \cup \mathscr{D}_{1,3})$, and components of an amalgamated free product inject, γ is not homotopic to the trivial loop in Σ_2. However, $f(\gamma)$ is homotopic to the trivial loop in $\mathbb{S}^2 \setminus \mathrm{int}(\mathscr{D}_0 \cup \mathscr{D}_1)$ and hence in Σ_1. Hence, f is not injective on π_1.

Both Σ_1 and Σ_2 have genus the same as Σ, and the space of ends homeomorphic to that of Σ (as a finite disjoint union of Cantor spaces is a Cantor space by the universality of the Cantor set) with all ends planar. Hence, by Theorem 2 both Σ_1 and Σ_2 are homeomorphic to Σ.

Identifying Σ_1 and Σ_2 with Σ by homeomorphisms, we get a proper map $f: \Sigma \to \Sigma$ which is not injective on π_1. As homeomorphisms have degree ± 1, it follows that $\deg(f) = \pm 1$. Replacing f by $f \circ f$ if necessary, we obtain a proper map of degree one that is not injective on π_1. $\square$

Theorem 9. *Let Σ be a finite genus surface such that $\mathscr{E}(\Sigma)$ has infinitely many isolated points. Then there is a degree one map $f: \Sigma \to \Sigma$ which is not π_1-injective.*

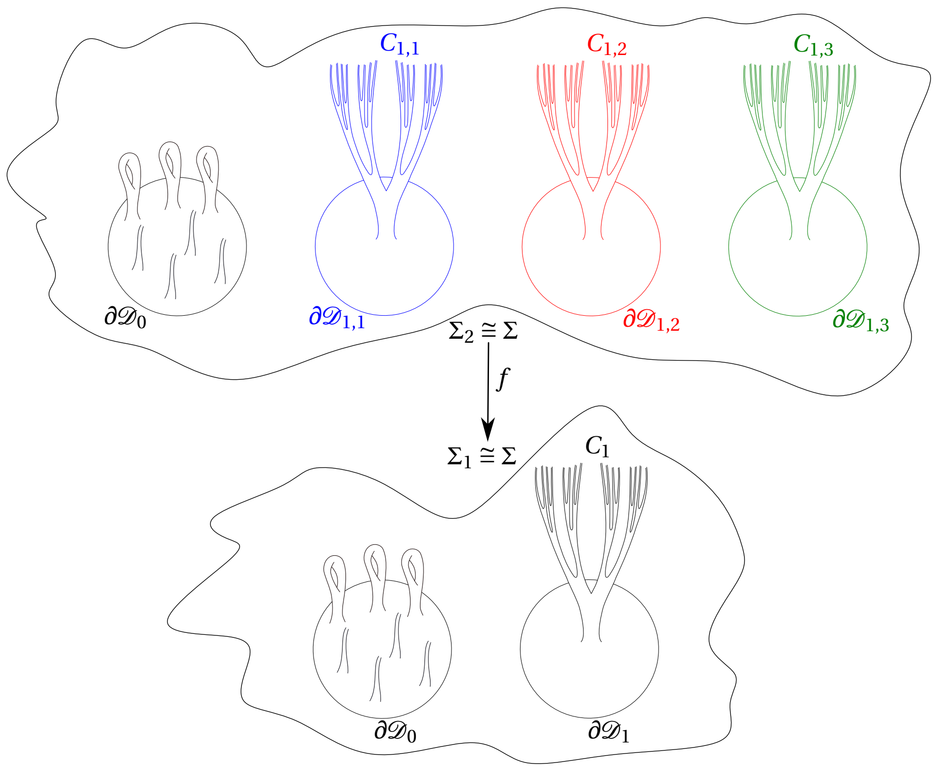

Figure 3. A non π_1-injective degree ± 1 map $f\colon \Sigma \to \Sigma$, where $g = 3$ and $|\mathscr{I}| = 4$.

Proof. Let $\mathscr{I}(\Sigma)$ be the set of all isolated points of $\mathscr{E}(\Sigma)$ and let g be the genus of Σ. Let $\mathscr{D}_0, \mathscr{D}_1, \mathscr{D}_{1,1}, \mathscr{D}_{1,2}, \mathscr{D}_{1,3} \subseteq \mathbb{S}^2$, $f\colon \mathbb{S}^2 \to \mathbb{S}^2$, and let γ be as in the conclusion of Lemma 7. Let $\mathscr{E}$ be a subset of $\operatorname{int}(\mathscr{D}_0)$ such that $\mathscr{E}$ is homeomorphic to $\mathscr{E}(\Sigma)$. Also, let $p_1 \in \operatorname{int}(\mathscr{D}_1)$ and $p_{1,i} \in \operatorname{int}(\mathscr{D}_{1,i})$, $i = 1,2,3$ be points such that $f(p_{1,i}) = p_1$ for each $i = 1,2,3$. See Figure 4.

Recall that $f^{-1}(\mathscr{D}_0) = \mathscr{D}_0$ and $f|_{\mathscr{D}_0}\colon \mathscr{D}_0 \to \mathscr{D}_0$ is the identity map. Thus $f^{-1}(\mathscr{E}) = \mathscr{E}$. Now, let Σ_1 be the surface obtained from $\mathbb{S}^2 \setminus (\mathscr{E} \cup \{p_1\})$ by attaching g handles along disjoint disks $\Delta_k \subset \operatorname{int}(\mathscr{D}_0) \setminus \mathscr{I}$, $1 \le k \le g$ and let Σ_2 be the surface obtained from $\mathbb{S}^2 \setminus (\mathscr{E} \cup \{p_{1,1}, p_{1,2}, p_{1,3}\})$ by attaching g handles along the same disks Δ_k, $1 \le k \le g$. Then f induces a proper map, which we also call f, from Σ_2 to Σ_1. By Lemma 4, $\deg(f) = \pm 1$.

Further, we claim that $f\colon \Sigma_2 \to \Sigma_1$ is not injective on π_1. Namely, the fundamental group of Σ_2 is the amalgamated free product of four groups, one of which is $\pi_1(\mathbb{S}^2 \setminus \operatorname{int}(\mathscr{D}_0 \cup \mathscr{D}_{1,1} \cup \mathscr{D}_{1,2} \cup \mathscr{D}_{1,3}))$. As γ is not homotopic to the trivial loop in $\mathbb{S}^2 \setminus \operatorname{int}(\mathscr{D}_0 \cup \mathscr{D}_{1,1} \cup \mathscr{D}_{1,2} \cup \mathscr{D}_{1,3})$, and components of an amalgamated free product inject, γ is not homotopic to the trivial loop in Σ_2. However, $f(\gamma)$ is homotopic to the trivial loop in $\mathbb{S}^2 \setminus \operatorname{int}(\mathscr{D}_0 \cup \mathscr{D}_1)$ and hence in Σ_1. Hence, f is not injective on π_1.

Both Σ_1 and Σ_2 have genus the same as Σ and, by Lemma 10 below, $\mathscr{E}(\Sigma_1)$ and $\mathscr{E}(\Sigma_2)$ are homeomorphic to $\mathscr{E}(\Sigma)$. Further, all ends of Σ, Σ_1 and Σ_2 are planar. Hence, by Theorem 2 both Σ_1 and Σ_2 are homeomorphic to Σ.

Identifying Σ_1 and Σ_2 with Σ by homeomorphisms, we get a proper map $f\colon \Sigma \to \Sigma$ which is not injective on π_1. As homeomorphisms have degree ± 1, it follows that $\deg(f) = \pm 1$. Replacing f by $f \circ f$ if necessary, we obtain a proper map of degree one that is not injective on π_1. □

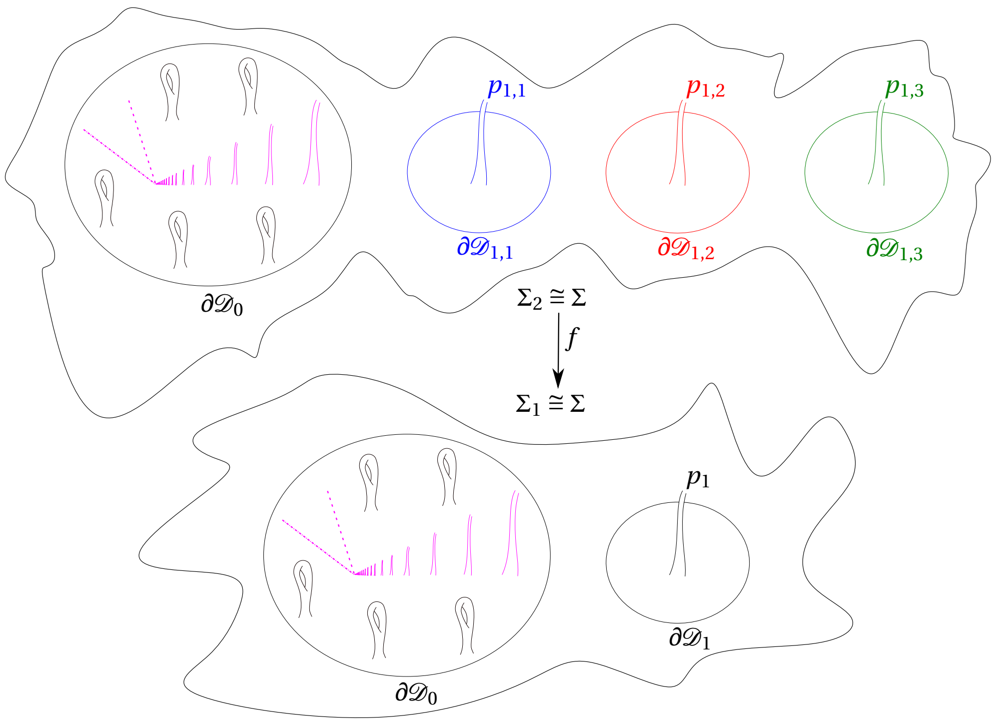

Figure 4. A non π_1-injective degree ± 1 map $f\colon \Sigma \to \Sigma$, where $g = 5$ and $\mathscr{I}$ is an infinite set.

Lemma 10. *Let $\mathscr{E}$ be a closed totally disconnected subset of $\mathbb{S}^2$. Let $\mathscr{I}$ be the set of all isolated points of $\mathscr{E}$. Assume $\mathscr{I}$ is infinite. If $\mathscr{F}$ is a finite subset of $\mathbb{S}^2 \setminus \mathscr{E}$, then $\mathscr{E} \cup \mathscr{F}$ is homeomorphic to $\mathscr{E}$.*

Proof. Let $\mathscr{A} := \{a_1, a_2, \dots\}$ be a subset of $\mathscr{I}$ such that $a_n \to \ell \in \mathscr{E}$ ($\mathscr{A}$ exists as $\mathscr{E}$ is compact and infinite). Define $\mathscr{B} := \mathscr{A} \cup \mathscr{F}$. Write $\mathscr{B}$ as $\mathscr{B} = \{b_1, b_2, \dots\}$. Then the map $g\colon \mathscr{E} \cup \mathscr{F} \to \mathscr{E}$ defined by

$$g(z) := \begin{cases} z & \text{if } z \in (\mathscr{E} \cup \mathscr{F}) \setminus \mathscr{B}, \\ a_n & \text{if } z = b_n \in \mathscr{B}, \end{cases}$$

is a homeomorphism. To prove this, note that g is a bijection from a compact space to a Hausdorff space, so it suffices to show that g is continuous. But observe that g restricted to the closed set $(\mathscr{E} \cup \mathscr{F}) \setminus \mathscr{B}$ is the identity, so g is continuous on $(\mathscr{E} \cup \mathscr{F}) \setminus \mathscr{B}$. Also g restricted to the closed set $\mathscr{B} \cup \{\ell\}$ is continuous as $b_n \to \ell$ and $g(b_n) = a_n \to \ell = g(\ell)$, and all other points of $\mathscr{B} \cup \{\ell\}$ are isolated. Thus g is continuous, as required. □

Remark 11. In the paper [3], the authors have proved that for every infinite-type surface Σ, there exists a subsurface homeomorphic to Σ such that the inclusion map is not homotopic to a homeomorphism. As our surfaces are connected, this type of inclusion map can't be proper because of the following two facts:

- Any injective map between two boundaryless topological manifolds of the same dimension is an open map. This follows from the invariance of domain.
- Any proper map between two topological manifolds is a closed map, as manifolds are compactly generated spaces, see [6].

Also, notice that all our results are related to proper maps.

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