INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique

Damir Yeliussizov

Stability of the Levi-Civita tensors and an Alon–Tarsi type theorem

Volume 361 (2023), p. 1367-1373

Published online: 31 October 2023

https://doi.org/10.5802/crmath.505

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2023, Vol. 361, p. 1367-1373
https://doi.org/10.5802/crmath.505

Algebra, Combinatorics / *Algèbre, Combinatoire*

Stability of the Levi-Civita tensors and an Alon–Tarsi type theorem

Damir Yeliussizov [a, b]

[a] Kazakh-British Technical University, Almaty, Kazakhstan
[b] Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
E-mail: yeldamir@gmail.com

Abstract. We show that the Levi-Civita tensors are semistable in the sense of Geometric Invariant Theory, which is equivalent to an analogue of the Alon–Tarsi conjecture on Latin squares. The proof uses the connection of Tao's slice rank with semistable tensors. We also show an application to an asymptotic saturation-type version of Rota's basis conjecture.

2020 Mathematics Subject Classification. 14L24, 15A72, 13A50, 05E14, 05B15, 05B35.

Funding. This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grants No. AP09259551).

Manuscript received 5 February 2022, accepted 10 April 2023.

1. Introduction

The *Levi-Civita symbol* ε is defined for $i_1,\ldots,i_n \in [n] := \{1,\ldots,n\}$ as follows

$$\varepsilon(i_1,\ldots,i_n) := \begin{cases} \operatorname{sgn}(i_1,\ldots,i_n), & \text{if } (i_1,\ldots,i_n) \in S_n \text{ is a permutation,} \\ 0, & \text{otherwise.} \end{cases}$$

For a map $I : [M] \to [n]$, where M is divisible by n, we denote

$$\varepsilon(I) := \varepsilon(I(1),\ldots,I(n)) \cdot \varepsilon(I(n+1),\ldots,I(2n)) \cdot \ldots \cdot \varepsilon(I(M-n+1),\ldots,I(M)) \in \{0,\pm 1\}.$$

Consider the following multidimensional generalizations of determinants for d-*tensors* (viewed as functions cf. Section 2) $X : [n]^d \to \mathbb{C}$

$$\Delta_{M,\vec{\pi}}(X) := \sum_{J_1,\ldots,J_d : [M]\to[n]} \varepsilon(J_1 \circ \pi_1) \cdots \varepsilon(J_d \circ \pi_d) \prod_{i=1}^{M} X(J_1(i),\ldots,J_d(i)), \tag{1}$$

where M is divisible by n and $\vec{\pi} = (\pi_1,\ldots,\pi_d) \in (S_M)^d$ is a d-tuple of permutations from S_M. In [8, Prop. 3.10] it is shown that these functions span the space of $\mathrm{SL}(n)^d$-invariant homogeneous degree M polynomials on d-tensors. For the minimal $M = n$ degree, $\Delta_{n,\vec{\pi}}(X)$ is (up to a sign) *Cayley's first hyperdeterminant* [9], a simple generalization of determinants for tensors (here $\vec{\pi}$ affect $\Delta_{n,\vec{\pi}}(X)$ only by a sign; and $\Delta_{n,\vec{\pi}}(X)$ is a nonzero function only for even d).

The *Levi-Civita n-tensor* $\mathsf{E}_n : [n]^n \to \mathbb{C}$ is the function given by $\mathsf{E}_n(i_1,\dots,i_n) = \varepsilon(i_1,\dots,i_n)$. The *Alon–Tarsi conjecture* on Latin squares [3] can be reformulated via hyperdeterminants as follows, see also Remark 15.

Conjecture 1 (Alon–Tarsi). *For every n even,* $\Delta_{n,\vec{\pi}}(\mathsf{E}_n) \neq 0$.

In this note we prove the following Alon–Tarsi-type result.

Theorem 2. *For every n, there exist M divisible by n and* $\vec{\pi} \in (S_M)^n$ *such that* $\Delta_{M,\vec{\pi}}(\mathsf{E}_n) \neq 0$.

This result is equivalent to the statement that the Levi-Civita tensor E_n is *semistable* in the sense of Geometric Invariant Theory, see Section 4. We prove it using *Tao's slice rank* [22] and its connection with semistable tensors as developed in [4, 8].

Note that the Alon–Tarsi conjecture is known to hold for specific values $n = p \pm 1$ where p is any prime [11, 12]. The Alon–Tarsi conjecture also implies *Rota's basis conjecture* [15, 20], and we discuss a similar connection derived from Theorem 17 in Section 5.

2. Tensors

Let $V = \mathbb{C}^n$. We consider *tensors* as elements of the space $V^{\otimes d} = V \otimes \cdots \otimes V$ (d times). Each tensor of $V^{\otimes d}$ can be represented in coordinates as

$$\sum_{1 \le i_1,\dots,i_d \le n} T(i_1,\dots,i_d)\, \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_d},$$

where $T : [n]^d \to \mathbb{C}$ which we call a *d-tensor*, and $(\mathbf{e}_i)$ is the standard basis of V. We denote by $\mathsf{T}^d(n) := \{T : [n]^d \to \mathbb{C}\}$ the set of d-tensors.

Let $A_1,\dots A_d \in \mathsf{T}^2(n)$ viewed as $n \times n$ matrices and $X \in \mathsf{T}^d(n)$ be a d-tensor. The *multilinear product* is defined as follows

$$(A_1,\dots,A_d)\cdot X = Y \in \mathsf{T}^d(n),$$

where

$$Y(i_1,\dots,i_d) = \sum_{j_1,\dots,j_d \in [n]} A_1(i_1,j_1)\cdots A_d(i_d,j_d)\, X(j_1,\dots,j_d).$$

The multilinear product defines the natural $\mathrm{GL}(V)^d$ action[1] on $\mathsf{T}^d(n)$, and simply expresses change of bases of V for a tensor. Note that for matrices $B_1,\dots,B_d \in \mathsf{T}^2(n)$ we have

$$(A_1B_1,\dots,A_dB_d)\cdot X = (A_1,\dots,A_d)\cdot((B_1,\dots,B_d)\cdot X).$$

The *tensor product* of $X \in \mathsf{T}^d(n)$, $Y \in \mathsf{T}^d(m)$ is defined as $T = X \otimes Y \in \mathsf{T}^d(nm)$ given by

$$T(k_1,\dots,k_d) = X(i_1,\dots,i_d)\cdot Y(j_1,\dots,j_d), \quad k_\ell = i_\ell(m-1) + j_\ell.$$

Alternatively, we can view the ℓ-th coordinate of T as a pair $(i_\ell, j_\ell) \mapsto k_\ell$ ordered lexicographically, for $\ell \in [d]$. For $X \in \mathsf{T}^d(n)$, the tensor $X^{\otimes k} = X \otimes \cdots \otimes X \in \mathsf{T}^d(n^k)$ denotes the k-th tensor power of k copies of X.

3. The slice rank

A nonzero d-tensor $T \in \mathsf{T}^d(n)$ has *slice rank* 1 if it can be decomposed in a form

$$T(i_1,\dots,i_d) = \mathbf{v}(i_k)\cdot T_1(i_1,\dots,i_{k-1},i_{k+1},\dots,i_d),$$

for some $k \in [d]$, a vector $\mathbf{v} \in V$ and a $(d-1)$-tensor $T_1 \in \mathsf{T}^{d-1}(n)$. The *slice rank* of $T \in \mathsf{T}^d(n)$, denoted by slice-rank(T), is then the minimal r such that

$$T = T_1 + \ldots + T_r,$$

[1] We use the notation $G^d := G \times \cdots \times G$ (d times) for a group G.

where each summand T_i has slice rank 1. (Note that each T_i can be decomposed differently and along different coordinates k.)

For $T \in \mathsf{T}^d(n)$ we have the inequality

$$\text{slice-rank}(T) \le n,$$

since T can always be expressed as the sum of slice rank 1 tensors as follows

$$T(i_1,\dots,i_d) = \sum_{\ell=1}^{n} \delta(i_1,\ell)\cdot T(\ell,i_2,\dots,i_d),$$

where δ is the Kronecker delta function.

The following lemma is useful for finding the slice rank of some sparse tensors.

Lemma 3 ([23]). *Equip the set $[n]$ with d total orderings $\le_i$ for $i \in [d]$, which define the product partial order $\le$ on $[n]^d$. Let $T \in \mathsf{T}^d(n)$ whose support $\Gamma = \{(i_1,\dots,i_d) : T(i_1,\dots,i_d) \ne 0\}$ is an antichain w.r.t. $\le$. Then*

$$\text{slice-rank}(T) = \min_{\Gamma=\Gamma_1\cup\cdots\cup\Gamma_d} |\pi_1(\Gamma_1)| + \ldots + |\pi_d(\Gamma_d)|,$$

where the minimum is over set partitions $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_d$ and $\pi_i : [n]^d \to [n]$ is the projection map on the i-th coordinate.

Remark 4. The slice rank was introduced by Tao in [22] and studied in [23]. This notion found many applications especially in additive combinatorics, see [14] for a related survey.

Remark 5. For $d = 2$, the slice rank coincides with the usual matrix rank. For $d \ge 3$, it significantly differs from the more common *tensor rank* (e.g. [18]) which can be way larger.

3.1. *The slice rank of the Levi-Civita tensors*

Now we find the slice rank of the Levi-Civita n-tensor $\mathsf{E}_n \in \mathsf{T}^n(n)$ and its k-th tensor power $\mathsf{E}_n^{\otimes k} \in \mathsf{T}^n(n^k)$.

Lemma 6. *We have:* slice-rank$(\mathsf{E}_n^{\otimes k}) = n^k$ *is full for all k.*

Proof. The support of $\mathsf{E}_n^{\otimes k} \in \mathsf{T}^n(n^k)$ can be identified with the following set

$$\Gamma = \left\{(\mathbf{i}_1,\dots,\mathbf{i}_n) : \mathbf{i}_\ell = (i_{\ell,1},\dots,i_{\ell,k}) \in [n]^k \text{ for } \ell \in [n], \text{ and } (i_{1,j},\dots,i_{n,j}) \in S_n \text{ for } j \in [n]\right\}.$$

Take the lexicographic ordering $\le_\ell$ on $\mathbf{i}_\ell \in [n]^k$ for each $\ell \in [n]$, which define the product partial order $\le$ on Γ. Let us show that Γ is an antichain w.r.t. this partial order. Assume we have $(\mathbf{i}_1,\dots,\mathbf{i}_n) \le (\mathbf{i}'_1,\dots,\mathbf{i}'_n)$ for elements of Γ, which means $\mathbf{i}_\ell = (i_{\ell,1},\dots,i_{\ell,k}) \le_\ell \mathbf{i}'_\ell = (i'_{\ell,1},\dots,i'_{\ell,k})$ for all $\ell \in [n]$. In particular, $i_{\ell,1} \le i'_{\ell,1}$ for all $\ell \in [n]$ but both $(i_{1,1},\dots,i_{n,1}), (i'_{1,1},\dots,i'_{n,1}) \in S_n$ are permutations which is only possible when $(i_{1,1},\dots,i_{n,1}) = (i'_{1,1},\dots,i'_{n,1})$. Since $\le_\ell$ are lexicographic, we then have $i_{\ell,2} \le i'_{\ell,2}$ for all $\ell \in [n]$ and by the same argument we get $(i_{1,2},\dots,i_{n,2}) = (i'_{1,2},\dots,i'_{n,2})$. Proceeding the same way we obtain that $(i_{1,j},\dots,i_{n,j}) = (i'_{1,j},\dots,i'_{n,j})$ for all $j \in [n]$ and hence $(\mathbf{i}_1,\dots,\mathbf{i}_n) = (\mathbf{i}'_1,\dots,\mathbf{i}'_n)$ which shows that Γ is indeed an antichain.

Let $\rho : [n]^k \to [n]^k$ be the (bijective) cyclic shift map given by

$$\rho : (i_1,\dots,i_k) \longmapsto (i'_1,\dots,i'_k) = (i_1+1,\dots,i_k+1) \bmod n.$$

Consider the following subset of Γ

$$S = \left\{(\mathbf{i},\rho\,\mathbf{i},\dots,\rho^{n-1}\mathbf{i}) : \mathbf{i} \in [n]^k\right\} \subset \Gamma.$$

Take any partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$. Note that for each $j \in [n]$ we have

$$|\pi_j(\Gamma_j)| \ge |\pi_j(\Gamma_j \cap S)| \ge |\Gamma_j \cap S|$$

since the elements of S differ in the j-th coordinate. Hence we have

$$|\pi_1(\Gamma_1)| + \ldots + |\pi_n(\Gamma_n)| \geq |\Gamma_1 \cap S| + \ldots + |\Gamma_n \cap S| = |S| = n^k,$$

which by Lemma 3 implies that slice-rank$(\mathsf{E}_n^{\otimes k}) \geq n^k$. On the other hand, we know that slice-rank$(\mathsf{E}_n^{\otimes k}) \leq n^k$ and hence the equality follows. □

Remark 7. It was noticed in [13] that slice-rank$(\mathsf{E}_3) = 3$.

4. Semistable tensors

The notion of semistable tensors comes from Geometric Invariant Theory [19]. A polynomial $P(X)$ is $\mathrm{SL}(n)^d$-*invariant* on $\mathsf{T}^d(n)$ if $P(g \cdot X) = P(X)$ for all $g \in \mathrm{SL}(n)^d$ and $X \in \mathsf{T}^d(n)$. A tensor $X \in \mathsf{T}^d(n)$ is called *semistable* (for the action of $\mathrm{SL}(n)^d$) if $P(X) \neq 0$ for some nonconstant $\mathrm{SL}(n)^d$-invariant homogeneous polynomial P. The following important characterization of semistable tensors shows their connection with the slice rank.

Theorem 8 ([8, Cor. 6.5]). *A tensor $X \in \mathsf{T}^d(n)$ is semistable iff* slice-rank$(X^{\otimes k}) = n^k$ *is full for all k.*

Lemma 6 with this Theorem give the following result.

Theorem 9. *The Levi-Civita n-tensor E_n is semistable.*

To show the equivalence of Theorem 9 with Theorem 2, we use the following concrete description of SL-invariant generating polynomials.

Lemma 10 ([8, Prop. 3.10], cf. [7, Ex. 7.18]). *The space of* $\mathrm{SL}(n)^d$*-invariant homogeneous degree M polynomials on $\mathsf{T}^d(n)$ is nonzero only if M is divisible by n, in which case it is spanned by the polynomials $\{\Delta_{M,\vec{\pi}}\}$ (defined in eq.* (1)*) indexed by d-tuples of permutations $\vec{\pi} = (\pi_1, \ldots, \pi_d) \in (S_M)^d$.*

Corollary 11. *Let $X \in \mathsf{T}^d(n)$ be semistable. Then $\Delta_{M,\vec{\pi}}(X) \neq 0$ for some M divisible by n and permutations $\vec{\pi} \in (S_M)^d$.*

Remark 12. The connection of slice rank with semistable tensors was first established in [4], where it was shown that slice-rank$(X) < n$ implies X is unstable (i.e. not semistable), and if X is unstable then slice-rank$(X^{\otimes k}) < n^k$ for some k. In [4] these results are given for $d = 3$ and for any d the statements are in [8]; the proofs use the Hilbert–Mumford criterion.

Remark 13. The formula (1) is given in exactly this form in [7, Ex. 7.18], and in [8, Prop. 3.10] it is stated in a slightly different form.

Remark 14. The degree M can be bounded above using a result from [10], see [8, Lem. 7.11] for a precise statement, which gives $M \leq d^{dn^2-d} n^d$.

Remark 15. Let us see how to get the original formulation of the Alon–Tarsi conjecture via Latin squares [3]. Let $\vec{\pi}$ be the n-tuple of identity permutations from S_n. We then have

$$\Delta_{n,\vec{\pi}}(\mathsf{E}_n) = \sum_{J_1,\ldots,J_n:[n]\to[n]} \varepsilon(J_1)\cdots\varepsilon(J_n) \prod_{i=1}^{n} \varepsilon(J_1(i),\ldots,J_n(i)).$$

The maps $J_1, \ldots, J_n$ corresponding to nonzero terms in this sum are in one-to-one correspondence with the $n \times n$ Latin squares (i.e. matrices whose every row and column is a permutation from S_n) formed by the rows $J_1, \ldots, J_n$. The nonzero terms define the *signs* for these Latin squares. Then the conjecture states that for each even n, the number of Latin squares with the odd sign is *not* equal to the number of Latin squares with the even sign, i.e. $\Delta_{n,\vec{\pi}}(\mathsf{E}_n) \neq 0$. A similar formulation via Latin-type $n \times M$ matrices can be given for our result $\Delta_{M,\vec{\pi}}(\mathsf{E}_n) \neq 0$.

Remark 16. It can be shown that the results in [16, 17] combined with the eventual surjectivity of the Hadamard–Howe map [5, 6] lead to similar sums over Latin-type matrices with column signs.

5. A version of Rota's basis conjecture

As an application we show the following asymptotic version of Rota's basis conjecture.

Theorem 17. *Let $B_1,\dots,B_n$ be n bases of $V=\mathbb{C}^n$. There is $\ell\ge 1$ and $n\times \ell n$ matrix A such that:*

- *in the i-th row of A each element of B_i appears ℓ times, for $i=1,\dots,n$*
- *every column of A forms a basis of V.*[2]

Rota's basis conjecture [15,21] states that this holds for $\ell=1$ thus presenting the problem as a saturation-type[3] question.

5.1. *Relative invariance*

We use the polynomials $\Delta_{M,\vec{\pi}}$ as relative GL-invariants which is well known.

Lemma 18. *Let $X\in \mathsf{T}^d(n)$ and $A_1,\dots,A_d\in \mathrm{GL}(n)$. We have*

$$\Delta_{M,\vec{\pi}}((A_1,\dots,A_d)\cdot X)=\Delta_{M,\vec{\pi}}(X)\cdot \det(A_1)^{M/n}\cdots\det(A_d)^{M/n}.$$

Proof. It is enough to check the identity for one matrix $A=A_1$. Write $A=BD$ for $B\in \mathrm{SL}(n)$ and $D=\mathrm{diag}(\det(A),1,\dots,1)$. Then as $\Delta_{M,\vec{\pi}}$ is $\mathrm{SL}(n)^d$-invariant, we get

$$\Delta_{M,\vec{\pi}}((BD,I,\dots,I)\cdot X)=\Delta_{M,\vec{\pi}}((B,I,\dots,I)\cdot((D,I,\dots,I)\cdot X))=\Delta_{M,\vec{\pi}}((D,I,\dots,I)\cdot X).$$

Let $Y=(D,I,\dots,I)\cdot X$. We have

$$Y(i_1,\dots,i_d)=\sum_j D(i_1,j)X(j,i_2,\dots,i_d)=\begin{cases}\det(A)\cdot X(i_1,\dots,i_d), & \text{if } i_1=1,\\ X(i_1,\dots,i_d), & \text{otherwise.}\end{cases}$$

From the formula (1) we can see that each nonzero term $\prod_{k=1}^d \varepsilon(J_k\circ\pi_k)\prod_{i=1}^M X(J_1(i),\dots,J_d(i))$ of $\Delta_{M,\vec{\pi}}(X)$ has exactly M/n variables $X(1,*\dots,*)$. Hence, $\Delta_{M,\vec{\pi}}(Y)=\Delta_{M,\vec{\pi}}(X)\cdot\det(A)^{M/n}$ as needed. □

5.2. *Determinantal tensors*

For a matrix A denote by $A[i]$ the i-th column vector of A. For matrices $A_1,\dots,A_n\in\mathrm{GL}(n)$ define the *determinantal n-tensor* $\mathsf{D}=\mathsf{D}(A_1,\dots,A_n)\in\mathsf{T}^n(n)$ given by

$$\mathsf{D}(i_1,\dots,i_n):=\det(A_1[i_1],\dots,A_n[i_n]),\quad \forall\, i_1,\dots,i_n\in[n].$$

Lemma 19. *We have:*

(i) *Let $A_1,\dots,A_n,B_1,\dots,B_n\in\mathrm{GL}(n)$. Then*

$$\mathsf{D}(A_1B_1,\dots,A_nB_n)=(B_1^T,\dots,B_n^T)\cdot\mathsf{D}(A_1,\dots,A_n).$$

(ii) $\mathsf{D}(I_n,\dots,I_n)=\mathsf{E}_n$, *where I_n is the identity $n\times n$ matrix.*

---

[2]To be precise, each entry of A is a vector in V. Here V can be any n-dimensional vector space over a field of characteristic 0.

[3]By analogy with algebraic notions of saturation for monoids or ideals.

Proof.

(i). It is enough to check the identity for one matrix $B_1 = B$. By definition and multilinearity of determinants we have

$$\begin{aligned}\mathsf{D}(A_1B, A_2, \dots, A_n)(i_1, \dots, i_n) &= \det(A_1B[i_1], A_2[i_2], \dots, A_n[i_n]) \\ &= \det\left(\sum_{j=1}^{n} A_1[j] \cdot B(j, i_1), A_2[i_2], \dots, A_n[i_n]\right) \\ &= \sum_{j=1}^{n} B(j, i_1) \cdot \det(A_1[j], A_2[i_2], \dots, A_n[i_n]) \\ &= (B^T, I_n, \dots, I_n) \cdot \mathsf{D}(A_1, \dots, A_n)(i_1, \dots, i_n).\end{aligned}$$

(ii). We have

$$\mathsf{D}(I_n, \dots, I_n)(i_1, \dots, i_n) = \det(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n}) = \varepsilon(i_1, \dots, i_n)$$

and the equality follows. □

Corollary 20. *Let* $B_1, \dots, B_n \in \mathrm{GL}(n)$. *We have*

$$\mathsf{D}(B_1, \dots, B_n) = (B_1^T, \dots, B_n^T) \cdot \mathsf{D}(I_n, \dots, I_n) = (B_1^T, \dots, B_n^T) \cdot \mathsf{E}_n.$$

Remark 21. Determinantal tensors are implicitly used in [20]; an explicit formulation appears in [2].

Proof of Theorem 17

We have $B_1, \dots, B_n \in \mathrm{GL}(n)$ whose elements are given by column vectors. Consider the determinantal tensor

$$\mathsf{D} = \mathsf{D}(B_1, \dots, B_n) = (B_1^T, \dots, B_n^T) \cdot \mathsf{E}_n.$$

Since E_n is semistable, there exist $M = \ell n$ and $\vec{\pi} \in (S_M)^n$ such that $\Delta_{M,\vec{\pi}}(\mathsf{E}_n) \neq 0$ (Corollary 11). By Lemmas 18 and 19 we have

$$\Delta_{M,\vec{\pi}}(\mathsf{D}) = \Delta_{M,\vec{\pi}}(\mathsf{E}_n) \cdot \det(B_1)^{\ell} \cdots \det(B_n)^{\ell} \neq 0.$$

On the other hand, let us check the expansion of this polynomial, which is given by

$$\begin{aligned}\Delta_{M,\vec{\pi}}(\mathsf{D}) &= \sum_{J_1, \dots, J_n : [M] \to [n]} \prod_{k=1}^{n} \varepsilon(J_k \circ \pi_k) \prod_{i=1}^{M} \mathsf{D}(J_1(i), \dots, J_n(i)) \\ &= \sum_{J_1, \dots, J_n : [M] \to [n]} \prod_{k=1}^{n} \varepsilon(J_k \circ \pi_k) \prod_{i=1}^{M} \det(B_1[J_1(i)], \dots, B_n[J_n(i)]).\end{aligned}$$

Since $\Delta_{M,\vec{\pi}}(\mathsf{D}) \neq 0$, at least one term in this expansion is also nonzero, which will give a desired arrangement. Indeed, if

$$\prod_{k=1}^{n} \varepsilon(J_k \circ \pi_k) \prod_{i=1}^{M} \det(B_1[J_1(i)], \dots, B_n[J_n(i)]) \neq 0$$

then we can arrange the columns of $B_1, \dots, B_n$ into an $n \times M$ matrix A w.r.t. the maps $J_1, \dots, J_n : [M] \to [n]$ such that the i-th column of A has the entries $B_1[J_1(i)], \dots, B_n[J_n(i)]$ of the corresponding columns of $B_1, \dots, B_n$. Since $\det(B_1[J_1(i)], \dots, B_n[J_n(i)]) \neq 0$ they are all bases as needed. The rows of A also satisfy the needed property, i.e. each entry appears exactly ℓ times, since $\varepsilon(J_k \circ \pi_k) \neq 0$ for all $k = 1, \dots, n$ which is clear from the definition of the sign $\varepsilon(J)$. □

Remark 22. In the case $M = n$, this is the method of [20] showing how the Alon–Tarsi conjecture implies Rota's basis conjecture.

Remark 23. From Remark 14, we can see that an upper bound on the multiplicity $\ell = M/n$ is large, it gives $\ell \le n^{n^3}$.

Remark 24. It can be shown that Theorem 17 is equivalent to a *fractional* version of Rota's basis conjecture, which holds for matroids and follows from a more general result in [1, Thm. 10.4] on fractional coloring.

Note: This paper supersedes author's preprint [24].

Acknowledgements

I am grateful to Alimzhan Amanov for useful comments and many interesting conversations. I thank Asaf Ferber, Matthew Kwan, and Lisa Sauermann for telling me about Remark 24. I am also grateful to a referee for useful comments.

References

[1] R. Aharoni, E. Berger, "The intersection of a matroid and a simplicial complex", *Trans. Am. Math. Soc.* **358** (2006), no. 11, p. 4895-4917.

[2] R. Aharoni, M. Loebl, "The odd case of Rota's bases conjecture", *Adv. Math.* **282** (2015), p. 427-442.

[3] N. Alon, M. Tarsi, "Colorings and orientations of graphs", *Combinatorica* **12** (1992), no. 2, p. 125-143.

[4] J. Blasiak, T. Church, H. Cohn, J. A. Grochow, E. Naslund, W. Sawin, C. Umans, "On cap sets and the group-theoretic approach to matrix multiplication", *Discrete Anal.* **2017** (2017), article no. 3 (27 pages).

[5] M. Brion, "Stable properties of plethysm: on two conjectures of Foulkes", *Manuscr. Math.* **80** (1993), no. 4, p. 347-371.

[6] ———, "Sur certains modules gradués associés aux produits symétriques", in *Algèbre non commutative, groupes quantiques et invariants (Reims, 1995)*, Séminaires et Congrès, vol. 2, Société Mathématique de France, 1995, p. 157-183.

[7] P. Bürgisser, C. Franks, A. Garg, R. Oliveira, M. Walter, A. Wigderson, "Towards a theory of non-commutative optimization: geodesic first and second order methods for moment maps and polytopes", in *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*, IEEE Computer Society, 2019, p. 845-861.

[8] P. Bürgisser, A. Garg, R. Oliveira, M. Walter, A. Wigderson, "Alternating minimization, scaling algorithms, and the null-cone problem from invariant theory", 2017, https://arxiv.org/abs/1711.08039.

[9] A. Cayley, "On the theory of determinants", *Trans. Camb. Philos. Soc.* **8** (1843), p. 1-16.

[10] H. Derksen, "Polynomial bounds for rings of invariants", *Proc. Am. Math. Soc.* **129** (2001), no. 4, p. 955-963.

[11] A. A. Drisko, "On the number of even and odd Latin squares of order $p+1$", *Adv. Math.* **128** (1997), no. 1, p. 20-35.

[12] D. G. Glynn, "The conjectures of Alon–Tarsi and Rota in dimension prime minus one", *SIAM J. Discrete Math.* **24** (2010), no. 2, p. 394-399.

[13] W. T. Gowers, "The slice rank of a direct sum", 2021, https://arxiv.org/abs/2105.08394.

[14] J. A. Grochow, "New applications of the polynomial method: The cap set conjecture and beyond", *Bull. Am. Math. Soc.* **56** (2019), p. 29-64.

[15] R. Huang, G.-C. Rota, "On the relations of various conjectures on Latin squares and straightening coefficients", *Discrete Math.* **128** (1994), no. 1-3, p. 225-236.

[16] S. Kumar, "A study of the representations supported by the orbit closure of the determinant", *Compos. Math.* **151** (2015), no. 2, p. 292-312.

[17] S. Kumar, J. M. Landsberg, "Connections between conjectures of Alon–Tarsi, Hadamard–Howe, and integrals over the special unitary group", *Discrete Math.* **338** (2015), no. 7, p. 1232-1238.

[18] J. M. Landsberg, *Tensors: geometry and applications*, Graduate Studies in Mathematics, vol. 128, American Mathematical Society, 2012.

[19] D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory*, 3rd ed., vol. 34, Springer, 1994.

[20] S. Onn, "A colorful determinantal identity, a conjecture of Rota, and Latin squares", *Am. Math. Mon.* **104** (1997), no. 2, p. 156-159.

[21] G.-C. Rota, "Ten mathematics problems I will never solve", *Mitt. Dtsch. Math.-Ver.* **6** (1998), p. 45-52.

[22] T. Tao, "A symmetric formulation of the Croot-Lev-Pach-Ellenberg-Gijswijt capset bound", available at https://terrytao.wordpress.com/2016/05/18/a-symmetric-formulation-of-the-croot-lev-pach-ellenberg-gijswijt-capset-bound/, 2016.

[23] T. Tao, W. Sawin, "Notes on the "slice rank" of tensors", available at https://terrytao.wordpress.com/2016/08/24/notes-on-the-slice-rank-of-tensors/, 2016.

[24] D. Yeliussizov, "Saturation of Rota's basis conjecture", 2021, https://arxiv.org/abs/2107.12926.