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Control theory / *Théorie du contrôle*

On Binary Optimal Control in $H^s(0,T)$, $s < 1/2$

Sur le Contrôle Optimal Binaire en $H^s(0,T)$, $s < 1/2$

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Abstract. The function space $H^s(0,T)$, $s < 1/2$, allows for functions with jump discontinuities and is thus attractive for treating optimal control problems with discrete-valued control functions. We show that while arbitrary chattering controls are impossible, there exist feasible controls in $H^s(0,T)$ that have countably jump discontinuities with jump height one in each of countably many pairwise disjoint intervals. However, under mild assumptions, we show that certain types of jump discontinuities cannot be optimal. The derivation of meaningful optimality conditions via a direct variational argument using simple feasible perturbations remains a major challenge; as illustrated by an example.

Résumé. L'espace fonctionnel $H^s(0,T)$, $s < 1/2$, est compatible avec les discontinuités et est par conséquent un candidat de choix pour résoudre des problèmes de contrôle optimal avec des fonctions de contrôle à valeurs discrètes. Nous montrons que, bien que les contrôles fortement oscillants soient impossibles, il existe des contrôles admissibles dans $H^s(0,T)$ ayant un nombre fini de discontinuités avec un saut de 1 pour chacune des paires dénombrables d'intervalles disjoints. Cependant, sous des hypothèses raisonnables, nous montrons que certaines de ces discontinuités ne peuvent pas être optimaux. Établir des conditions d'optimalité pertinentes via un argument variationnel avec des perturbations admissibles simples constitue un défi majeur, ce que nous illustrons par un exemple.

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1. Introduction

A number of recents works have demonstrated the need for time-dependent, discrete-valued decision variables in the context of optimal control of (partial) differential equations, see [5–7]. This presents a unique challenge for optimal control in which the control variables are often taken in L^p-spaces. The latter is typically easier to work with both theoretically and numerically and it allows for jump discontinuities in the optimal controls. However, if our feasible set has the form

$$U_{\mathrm{ad}} \coloneqq \{u : [0,T] \to \mathbb{R} : u \text{ measurable}, u(t) \in \{0,1,\dots,N\}, t \in (0,T)\},$$

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where $T > 0$ and $N \in \mathbb{N}$, then it is not weakly closed in L^2. If we seek more regular controls, e.g., $H^1(0,T)$, then the feasible set will only contain constant functions due to absolute continuity of $H^1(0,T)$ functions. We may also look beyond the Hilbert space setting as in [8] and formulate an optimal control problem with controls in $BV(0,T)$. There are several theoretical advantages. For instance, if the total variation seminorm is used as a regularizer in the objective, then any optimal control will have a finite number of jumps. As shown in [8], this yields an intuitive stationarity concept. However, $BV(0,T)$ is not a Hilbert space, so the derivation of a convergent function-space-based numerical method is rather delicate. In addition, the TV-seminorm is non-smooth.

In light of these issues, we propose to use controls in Sobolev–Slobodeckij spaces $H^s(0,T)$ with $s < 1/2$. Our goal is to investigate whether this provides a viable alternative Hilbert space setting that straddles the inadequacy of $L^2(0,T)$ and restrictiveness of $H^1(0,T)$. Moreover, $H^s(0,T)$ allows for a wavelet expansion based on classical Haar wavelets and thus a natural discretization for the control problem, cf. [2] The advantages over the other approaches are

(1) $H^s(0,T)$ functions with $s < 1/2$ allow for jump discontinuities. In particular, the continuous embedding of $H^s(0,T)$ into $C([0,T])$ for $s > 1/2$, [3, Theorem 8.2], does not hold.
(2) $H^s(0,T)$ with $s < 1/2$ is a real separable Hilbert space.
(3) The feasible set U_{ad} is weakly closed in $H^s(0,T)$, $s < 1/2$, and allows for more than constant functions.
(4) $H^s(0,T)$ functions admit a wavelet expansion based on classical Haar wavelets, which provides a natural discretization scheme and easy treatment of U_{ad}.

The purpose of this short note is to address the fundamental differences that may arise for binary or integer-valued controls in $H^s(0,T)$ as opposed to L^2, H^1 or BV by way of examples in the following sections:

- in §2 Feasible sets of the type U_{ad} are weakly closed in $H^s(0,T)$.
- in §2 Certain types of *chattering controls*, [1,4], are ruled out in the H^s-setting.
- in §3 There exist $H^s(0,T)$ functions with countably many jump discontinuities.
- in §4 There exist $H^s(0,T)$ functions with countably many pairwise disjoint neighborhoods containing countable many discontinuities.
- in §5 Controls with certain types of jump discontinuities are generally not optimal.
- in §6 The stationarity concept defined in [8] for controls in BV cannot be transferred directly.

Standing Assumptions. We will always assume that $s < 1/2$ and $q = 1 - 2s$ in this article. For $T > 0$ and a measurable function $f : [0,T] \to \mathbb{R}$, we introduce the Sobolev–Slobodeckij seminorm:

$$|f|_{H^s(0,T)} := \left(\int_{\tau \in (0,T)} \int_{t \in (0,T)} \frac{|f(t) - f(\tau)|^2}{|t - \tau|^{1+2s}} \,\mathrm{d}t \,\mathrm{d}\tau \right)^{\frac{1}{2}}. \tag{1}$$

The norm on $H^s(0,T)$ is given by $\|f\|_{H^s(0,T)} := (\|f\|^2_{L^2(0,T)} + |f|^2_{H^s(0,T)})^{\frac{1}{2}}$. All integrals in this article are Lebesgue integrals with respect to the Lebesgue measure.

2. Weak closedness of U_{ad} and prevention of chattering

We recall a classical example for chattering and the fact that U_{ad} is not weakly closed in $L^2(0,T)$ and then prove a general result that excludes such phenomena in $H^s(0,T)$.

Example 1. Let $T = 1$ and $N = 1$. A classical example of chattering for binary functions is the sequence $(u^n)_n \subset U_{\text{ad}}$ with $u^n := \sum_{i=0}^{n-1} \chi_{[2i/(2n),(2i+1)/(2n))}$ for $n \in \mathbb{N}$, where u^n alternates between 0 and 1 on subsequent uniform intervals that uniformly partition $(0,1)$ and the interval lengths, reciprocal to the frequency of the alternation, tend to zero for $n \to \infty$. The sequence $(u^n)_n \subset U_{\text{ad}}$ is bounded and an approximation argument yields $u^n \rightharpoonup \bar{u} \equiv 0.5$ in $L^2(0,1)$, implying that $\bar{u} \notin U_{\text{ad}}$. Moreover, $\liminf_n \|u^n\|_{L^2} = \sqrt{0.5} > 0.5\sqrt{0.5} = \|\bar{u}\|_{L^2}$ and thus $u^n \not\to \bar{u}$ in $L^2(0,1)$.

Such sequences of functions are unbounded in $H^s(0,T)$ and U_{ad} is weakly closed in $H^s(0,T)$.

Proposition 2. *Let $(u^n)_n \subset U_{\mathrm{ad}}$ be given. Then it follows that*

(1) *if $u^n \rightharpoonup u$ in $H^s(0,T)$ for some $u \in H^s(0,T)$, then $u \in U_{\mathrm{ad}}$, and*
(2) *if $u^n \rightharpoonup u$ and $u^n \not\to u$ in $L^2(0,T)$, then $|u^n|_{H^s(0,T)} \to \infty$.*

Proof. (1). Given $u^n \rightharpoonup u$ in $H^s(0,T)$ and the compact embedding $H^s(0,T) \hookrightarrow^c L^2(0,T)$, [3, Theorem 7.1], we have $u^n \to u$ in $L^2(0,T)$. Consequently, $(u^n)_n$ has a subsequence that converges pointwise a.e. to the same limit u. Because the discrete set $\{1,\ldots,N\}$ is closed, this implies that the limit is a.e. $\{1,\ldots,N\}$-valued and thus, $u \in U_{\mathrm{ad}}$.

(2). Indeed, if $(u^n)_n$ were bounded in $H^s(0,T)$, then it would admit a subsequence that converges in the norm in $L^2(0,T)$ by means of the same compact embedding used above. Passing to further subsequences we obtain $u^n \to u$ in $L^2(0,T)$ for the whole sequence, which yields a contradiction. □

3. A function in $H^s(0,T)$ with countably many discontinuities

We first give a representation of the H^s-seminorm that meets our needs and then provide our example function.

Lemma 3. *Let $\bar t > 0$. Let $\{I_j \subset (0,\bar t) \mid j \in J\}$ for an ordered index set $J = \{1,2,\ldots\} \subset \mathbb{N}$ be a set of pairwise disjoint intervals with $\bar t = \sup_{j\in J}\sup_{t\in I_j} t$. The lower and upper bounds of the intervals I_j are $\ell_j := \inf_{t\in I_j} t$ and $u_j := \sup_{t\in I_j}$ for all $j \in J$. In order to facilitate the notation below, we define $u_0 := 0$ and $\ell_{|J|+1} := \bar t$ if $|J| < \infty$. Let $f = \sum_{i\in J}\chi_{I_i}$. Then*

$$|f|_{H^s(0,\bar t)} = \left(\frac{1}{sq}\sum_{i\in J}\left(\sum_{j=0}^{i-1}\check f_{ij} + \sum_{j=i}^{|J|}\widehat f_{ij}\right)\right)^{\frac12},$$

where

$$0 \le \check f_{ij} = (\ell_i - u_j)^q + (u_i - \ell_{j+1})^q - (\ell_i - \ell_{j+1})^q - (u_i - u_j)^q \quad \text{for } 0 \le j < i$$
$$\text{and}\quad 0 \le \widehat f_{ij} = (\ell_{j+1} - u_i)^q + (u_j - \ell_i)^q - (u_j - u_i)^q - (\ell_{j+1} - \ell_i)^q \quad \text{for } 1 \le i \le j$$

for all $i \in J$.

Proof. We reformulate the integral formulation of the squared Sobolev–Slobodeckij seminorm (1) and apply $|f(t) - f(\tau)| \in \{0,1\}$ as well as the fundamental theorem of calculus to deduce:

$$\begin{aligned}
|f|^2_{H^s(0,\bar t)} &= \int_{\tau\in(0,\bar t)}\int_{t\in(0,\bar t)} \frac{|f(t)-f(\tau)|}{|t-\tau|^{1+2s}}\,\mathrm{d}t\,\mathrm{d}\tau \\
&= 2\sum_{i\in J}\sum_{j\in J\cup\{0\}}\int_{\tau\in I_i}\int_{t\in(u_j,\ell_{j+1})}\frac{1}{|t-\tau|^{1+2s}}\,\mathrm{d}t\,\mathrm{d}\tau \\
&= \frac{1}{s}\sum_{i\in J}\sum_{j\in J\cup\{0\}}\int_{(u_j,\ell_{j+1})}\underbrace{\left||u_i - t|^{-2s} - |\ell_i - t|^{-2s}\right|}_{=:f_{ij}(t)\ge 0}\,\mathrm{d}t,
\end{aligned}$$

We distinguish the (by definition) mutually exclusive and exhaustive cases $u_j \le \ell_{j+1} \le \ell_i \le u_i$ and $\ell_{j+1} \ge u_j \ge u_i \ge \ell_i$ in order to determine the value of $\int_{u_j}^{\ell_{j+1}} f_{ij}$ for all $i \in J$ and $j \in J \cup \{0\}$.

Case $u_j \le \ell_{j+1} \le \ell_i \le u_i$. All $t \in (u_j,\ell_{j+1})$ satisfy $(u_i - t) \ge (\ell_i - t) \ge 0$, which gives $(\ell_i - t)^{-2s} - (u_i - t)^{-2s} \ge 0$. Then the fundamental theorem of calculus gives

$$\int_{(u_j,\ell_{j+1})} f_{ij}(t)\,\mathrm{d}t = \frac{-1}{q}\left((\ell_i - \ell_{j+1})^q - (\ell_i - u_j)^q - (u_i - \ell_{j+1})^q + (u_i - u_j)^q\right).$$

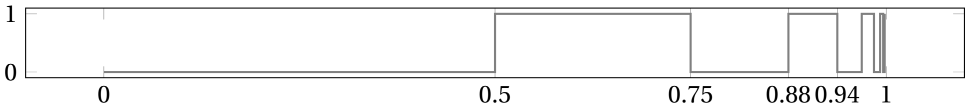

Figure 1. Visualization of f as constructed in Example 4.

Case $\ell_{j+1} \geq u_j \geq u_i \geq \ell_i$. All $t \in (u_j, \ell_{j+1})$ satisfy $0 \leq (t - u_i) \leq (t - \ell_i)$, which gives $(t-u_i)^{-2s} - (t-\ell_i)^{-2s} \geq 0$. Then the fundamental theorem of calculus gives

$$\int_{(u_j,\ell_{j+1})} f_{ij}(t)\,\mathrm{d}t = \frac{1}{q}\left((\ell_{j+1}-u_i)^q - (u_j - u_i)^q - (\ell_{j+1}-\ell_i)^q + (u_j - \ell_i)^q\right).$$

Combining both cases with the reformulation of the $|f|^2_{H^s(0,\bar{t})}$ yields the claim. □

The following example provides a binary-valued function that is a sum of characteristic functions of countably many pairwise disjoint intervals. We will see that our choice of the intervals implies that the value of its $H^s(0,T)$-seminorm is finite.

Example 4. Let $T = 1$. We define interval boundaries ℓ_i and u_i for $i \in \mathbb{N}$ as follows:

$$u_i := 1 - 2^{-2i} \text{ for all } i \in \mathbb{N} \quad \text{and} \quad \ell_i := 1 - 2^{-(2i-1)} \text{ for all } i \in \mathbb{N},$$

that is $u_1 = \frac{3}{4}$, $u_2 = \frac{15}{16}$, ..., and $\ell_1 = \frac{1}{2}$, $\ell_2 = \frac{7}{8}$, ... Moreover, we define $I_i := [\ell_i, u_i)$ for all $i \in \mathbb{N}$. Our function of interest is $f := \sum_{i=1}^\infty \chi_{I_i}$. Lebesgue's dominated convergence theorem gives $f \in L^2(0,T)$. By construction, the intervals I_i are pairwise disjoint (with $\ell_1 < u_1 < \ell_2 < u_2 < \ldots$) and Lemma 3 may be applied with the choices $\bar{t} = T$ and $J = \mathbb{N}$. In order to determine $|f|_{H^s(0,T)}$, we need to compute the terms $\check{f}_{ij}$ and $\widehat{f}_{ij}$. We distinguish the two cases regarding the order of i and j.

Case $\check{f}_{ij}$ for $0 \leq j \leq i-1$. We consider the terms (before taking their power to q) that make up $\check{f}_{ij}$ one by one:

$$\ell_i - u_j = \frac{1-2^{2(j-i)+1}}{2^{2j}},\ u_i - \ell_{j+1} = \frac{2^{-1}-2^{2(j-i)}}{2^{2j}},\ \ell_i - \ell_{j+1} = \frac{2^{-1}-2^{2(j-i)+1}}{2^{2j}},\ u_i - u_j = \frac{1-2^{2(j-i)}}{2^{2j}}.$$

Case $\widehat{f}_{ij}$ for $1 \leq i \leq j$. We consider the terms (before taking their power to q) that make up $\widehat{f}_{ij}$ one by one:

$$\ell_{j+1} - u_i = \frac{1-2^{2(i-j)-1}}{2^{2i}},\ u_j - u_i = \frac{1-2^{2(i-j)}}{2^{2i}},\ \ell_{j+1} - \ell_i = \frac{2-2^{2(i-j)-1}}{2^{2i}},\ u_j - \ell_i = \frac{2-2^{2(i-j)}}{2^{2i}}.$$

We visualize f in Figure 1. We prove that its H^s-norm is bounded in Theorem 5 below.

Theorem 5. *The function f from Example 4 has countably many discontinuities with jump height* 1 *and* $|f|_{H^s(0,T)} < \infty$.

Proof. The countably many support intervals of f are pairwise disjoint and have positive distance between each other. Combining this with the fact that f is binary-valued, we obtain that f has countably many discontinuities with jump height 1. It remains to show that $|f|_{H^s(0,T)} < \infty$.

We use the structure of $|\cdot|_{H^s(0,T)}$ given by Lemma 3. We estimate the terms $\check{f}_{ij}$ (for $0 \leq j \leq i-1$) and $\widehat{f}_{ij}$ (for $1 \leq i \leq j$):

$$0 \leq \check{f}_{ij} = \underbrace{(\ell_i - u_j)^q - (u_i - u_j)^q}_{\leq 0} + \underbrace{(u_i - \ell_{j+1})^q - (\ell_i - \ell_{j+1})^q}_{\geq 0} \leq (u_i - \ell_{j+1})^q - (\ell_i - \ell_{j+1})^q,$$

$$0 \leq \widehat{f}_{ij} = \underbrace{(u_j - \ell_i)^q - (\ell_{j+1} - \ell_i)^q}_{\leq 0} + \underbrace{(\ell_{j+1} - u_i)^q - (u_j - u_i)^q}_{\geq 0} \leq (\ell_{j+1} - u_i)^q - (u_j - u_i)^q.$$

We insert the formulas for the differences stated in Example 4 and employ the fact that $(a+b)^r \le a^r + b^r$ holds for all $a, b \ge 0$ and $r \in (0,1]$ to deduce $\check{f}_{ij} \le 2^{-2iq}$ and $\widehat{f}_{ij} \le 2^{-(2j+1)q}$. This implies the sums

$$\sum_{j=0}^{i-1} \check{f}_{ij} \le \frac{i}{4^{iq}} \quad \text{and} \quad \sum_{j=i}^{\infty} \widehat{f}_{ij} \le \frac{1}{2^q}\sum_{j=i}^{\infty}\frac{1}{4^{jq}} = \frac{1}{2^q}\left(\sum_{j=0}^{\infty}\frac{1}{4^{jq}} - \sum_{j=0}^{i-1}\frac{1}{4^{jq}}\right) = \frac{1}{2^q}\frac{1}{1-4^{-q}}\frac{1}{4^{qi}}.$$

We obtain

$$\sum_{i=1}^{\infty}\left(\sum_{j=0}^{i-1}\check{f}_{ij} + \sum_{j=i}^{\infty}\widehat{f}_{ij}\right) \le \sum_{i=0}^{\infty}\frac{i}{4^{iq}} + \frac{1}{2^q}\frac{1}{1-4^{-q}}\left(\frac{1}{1-4^{-q}} - 1\right) = \frac{4^{-q}}{(1-4^{-q})^2} + \frac{1}{2^q}\frac{4^{-q}}{(1-4^{-q})^2},$$

which in turn gives $|f|_{H^s(0,T)} < \infty$ by means of the formula from Lemma 3. □

4. A function in $H^s(0,T)$ with countably many pairwise disjoint neighborhoods containing countably many discontinuities

We start from the function constructed above to further construct a function that has countably many discontinuities in each of countably many pairwise disjoint intervals. Let f be a binary function such that the infimum of its support is strictly greater than zero. We first bound $|f|_{H^s(0,T)}$ by $|f|_{H^s(0,\bar{t})}$, where $\bar{t}$ is the supremum of the support of f, and a constant that only depends on $\bar{t}$. Then we show that it is possible to scale the end points of the support intervals such that a) the support of the function with scaled interval end points is strictly left of the support of f and b) the $H^s(0,T)$-norm of the function with scaled interval end points is less than $q|f|_{H^s(0,T)}$. Then we construct a sequence of such scalings recursively to obtain the desired function.

In this section we always assume that $s < 1/2$. Let $\{I_j \subset (0,T) \,|\, j \in J\}$ for a countable index set J be a set of pairwise disjoint intervals. Let $f = \sum_{i\in J}\chi_{I_i}$. Let $\bar{t} := \sup_{j\in J}\sup_{t\in I_j} t$ and $\underline{t} := \inf_{j\in J}\inf_{t\in I_j} t$ denote the upper and lower bound of the support of f.

Lemma 6.

$$\textit{If } \bar{t} < T, \textit{ then } \quad |f|_{H^s(0,T)} \le \left(|f|^2_{H^s(0,\bar{t})} + \frac{1}{s(1-2s)}\bar{t}^{1-2s}\right)^{\frac{1}{2}}.$$

$$\textit{If } \underline{t} > 0, \textit{ then } \quad |f|_{H^s(0,T)} \le \left(|f|^2_{H^s(\underline{t},T)} + \frac{1}{s(1-2s)}\underline{t}^{1-2s}\right)^{\frac{1}{2}}.$$

Proof. The arguments for the first and second claim are completely analogous so that we only show the first claim. Thus let $\bar{t} < T$. Because the function f is binary-valued, we obtain that

$$|f|^2_{H^s(0,T)} = |f|^2_{H^s(0,\bar{t})} + \underbrace{2\int_{t\in(0,\bar{t})}\int_{\tau\in(\bar{t},T)}\frac{f(t)}{(\tau-t)^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}t}_{=:r(f)}.$$

We estimate $f(t) \le 1$ for all $t \in [0,\bar{t}]$ and deduce with the fundamental theorem of calculus:

$$r(f) \le \frac{-1}{2s}\int_{(0,\bar{t})}\left((T-t)^{-2s} - (\bar{t}-t)^{-2s}\right)\mathrm{d}t = \frac{1}{2s(1-2s)}\Big(\underbrace{(T-\bar{t})^{1-2s} - T^{1-2s}}_{\le 0} + \bar{t}^{1-2s}\Big) \le \frac{\bar{t}^{1-2s}}{2s(1-2s)}.$$ □

For a set $A \subset \mathbb{R}$ and a positve scalar $\alpha > 0$, we introduce the notation $\alpha A := \{\alpha a \,|\, a \in A\}$.

Lemma 7. *Let $\underline{t} > 0$ and $\bar{t} \le T$. Let $p \in (0,1)$. Then there exists $\alpha_0 > 0$ such that for all $\alpha \in (0,\alpha_0)$ the quantities $f^{(\alpha)} := \sum_{i\in J}\chi_{\alpha I_i}$, $\underline{t}^{(\alpha)} := \inf_{j\in J}\inf_{t\in\alpha I_j} t$, and $\bar{t}^{(\alpha)} := \sup_{j\in J}\sup_{t\in\alpha I_j} t$ satisfy*

(1) $\bar{t}^{(\alpha)} < \underline{t}$,

(2) $\left(\frac{1}{s(1-2s)}(\bar{t}^{(\alpha)})^{1-2s}\right)^{\frac{1}{2}} < p\left(\frac{1}{s(1-2s)}\bar{t}^{1-2s}\right)^{\frac{1}{2}}$, *and*

(3) $|f^{(\alpha)}|_{H^s(0,\bar{t}^{(\alpha)})} \le p|f|_{H^s(0,\bar{t})}$.

Proof. Clearly, $\bar{t}^\alpha = \alpha\bar{t}$, which implies (1) and (2) for all α that are sufficiently small. Lemma 3 gives that $|f|^2_{H^s(0,\bar{t})}$ can be written as a sum of terms of the form $a(b-c)^{1-2s}$ such that a, b, $c > 0$ and b and c are lower and upper bounds of intervals I_j. After scaling with α, the bounds are αb and αc for the corresponding intervals αI_j. Consequently, $|f^{(\alpha)}|^2_{H^s(0,\bar{t}^{(\alpha)})}$ can be written as the same sum with the terms $a(b-c)^{1-2s}$ replaced by $a(\alpha b-\alpha c)^{1-2s}$. This yields $|f^{(\alpha)}|_{H^s(0,\bar{t}^{(\alpha)})} = \alpha^{\frac{1-2s}{2}}|f|_{H^s(0,\bar{t})}$, which proves the third claim. □

Theorem 8. *Let $\underline{t} > 0$ and $\bar{t} \le T$. Let $p \in (0,1)$. Then there exists a sequence $(\alpha^k)_{k\in\mathbb{N}} \subset (0,1]$ such that the functions $f^{(\alpha^k)} = \sum_{i\in J}\chi_{\alpha^k I_i}$ have pairwise disjoint supports and satisfy the estimate*

$$\left|\sum_{k=1}^{\infty} f^{(\alpha^k)}\right|_{H^s(0,T)} \le \left(|f|_{H^s(0,\bar{t})} + \left(\frac{1}{s(1-2s)}\bar{t}^{1-2s}\right)^{\frac{1}{2}}\right)\frac{1}{1-p}.$$

Proof. We proceed inductively and construct the sequence $(\alpha^k)_{k\in\mathbb{N}}$ such that the estimates

$$|f^{(\alpha^k)}|_{H^s(0,\bar{t}^{(\alpha^k)})} \le p^{k-1}|f|_{H^s(0,\bar{t})} \text{ and } \left(\frac{1}{s(1-2s)}(\bar{t}^{(\alpha^k)})^{1-2s}\right)^{\frac{1}{2}} \le p^{k-1}\left(\frac{1}{s(1-2s)}\bar{t}^{1-2s}\right)^{\frac{1}{2}}$$

and the fact that the functions $f^{(\alpha^1)}, \ldots, f^{(\alpha^k)}$ satisfy

$$0 < \min\operatorname{supp} f^{(\alpha^k)} < \max\operatorname{supp} f^{(\alpha^k)} < \cdots < \min\operatorname{supp} f^{(\alpha^1)} < \max\operatorname{supp} f^{(\alpha^1)}$$

hold for all $k \in \mathbb{N}$. This implies that the functions $f^{(\alpha^k)}$ have pairwise disjoint supports. Then the claim follows from

$$\begin{aligned}\left|\sum_{k=1}^{\infty} f^{(\alpha^k)}\right|_{H^s(0,T)} &\le \sum_{k=1}^{\infty}\left|f^{(\alpha^k)}\right|_{H^s(0,T)} \le \sum_{k=1}^{\infty}|f^{(\alpha)}|_{H^s(0,\bar{t}^{(\alpha^k)})} + \left(\frac{1}{s(1-2s)}(\bar{t}^{(\alpha^k)})^{1-2s}\right)^{\frac{1}{2}} \\ &\le \left(|f|_{H^s(0,\bar{t})} + \left(\frac{1}{s(1-2s)}\bar{t}^{1-2s}\right)^{\frac{1}{2}}\right)\sum_{k=1}^{\infty}p^{k-1} = \left(|f|_{H^s(0,\bar{t})} + \left(\frac{1}{s(1-2s)}\bar{t}^{1-2s}\right)^{\frac{1}{2}}\right)\frac{1}{1-p},\end{aligned}$$

where the second inequality follows from Lemma 6 and the inequality $(a+b)^{\frac{1}{2}} \le a^{\frac{1}{2}} + b^{\frac{1}{2}}$ for positive scalars a and b.

We set $\alpha^1 := 1$, which gives $f^{(\alpha^1)} = f$ and asserts the base case of our induction. Assume that the claims holds for some $k \in \mathbb{N}$. We apply Lemma 7 to deduce that there is α^{k+1} such that

$$0 < \min\operatorname{supp} f^{(\alpha^{k+1})} < \max\operatorname{supp} f^{(\alpha^{k+1})} < \min\operatorname{supp} f^{(\alpha^k)} < \max\operatorname{supp} f^{(\alpha^k)}$$

holds as well as the estimates

$$|f^{(\alpha^{k+1})}|_{H^s(0,\bar{t}^{(\alpha^{k+1})})} \le p|f^{(\alpha^k)}|_{H^s(0,\bar{t}^{(\alpha^k)})} \underset{\text{induction}}{\le} p^k|f|_{H^s(0,\bar{t})},$$

$$\left(\frac{1}{s(1-2s)}(\bar{t}^{(\alpha^{k+1})})^{1-2s}\right)^{\frac{1}{2}} \le p\left(\frac{1}{s(1-2s)}(\bar{t}^{(\alpha^k)})^{1-2s}\right)^{\frac{1}{2}} \underset{\text{induction}}{\le} p^k\left(\frac{1}{s(1-2s)}\bar{t}^{1-2s}\right)^{\frac{1}{2}},$$

which closes the induction and the proof. □

Corollary 9. *Let $s < 1/2$. There exists a binary-valued function $F \in H^s(0,T)$ such that there are countably many pairwise different $t \in [0,T]$ with associated $\varepsilon > 0$ such that F has countably many discontinuities in $B_\varepsilon(t) \cap (0,T)$.*

Proof. The function f defined in Example 4, has countably many discontinuities in any ball around T intersected with $(0,T)$. Let $F := \sum_{k=1}^{\infty} f^{(\alpha^k)}$ for the sequence $(\alpha^k)_{k\in\mathbb{N}}$ from Theorem 8 for f and an arbitrary $p \in (0,1)$. Then the functions $f^{(\alpha^k)}$ have pairwise disjoint supports and each of them has countably many discontinuities in any ball around $\max\operatorname{supp} f^{(\alpha^k)}$ Moreover, Theorem 8 yields boundedness of $|F|_{H^s(0,T)}$. □

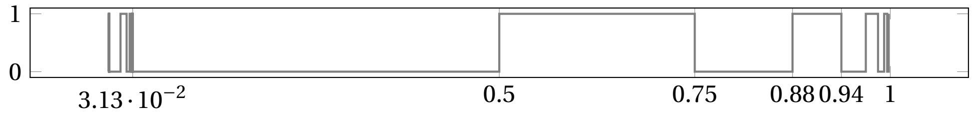

Figure 2. Visualization of F as constructed in Corollary 9 for the choices $q = 1/2$ and $s = 1/4$ based on f defined in Example 4.

We visualize the construction in Corollary 9 in Figure 2 for $q = 1/2$ and $s = 1/4$ with α^k constructed iteratively by the choice $\alpha^k = \alpha^{k-1}2^{-i}$ with the smallest possible $i \in \mathbb{N}$ such that the three conditions of Lemma 7 are met, which yields of $\alpha^k = 0.03125^{k-1}$ for all $k \in \mathbb{N}$.

5. Controls with isolated small jump discontinuities are generally non-optimal

In the space $BV(0,T)$, a switch from zero to one and back (or vice versa) cannot be optimal if the width of the switch is too small, because the induced increase of the total variation is always equal to two. More specifically, if the objective has the form $J(f) + \mathrm{TV}(f)$, where J is continuous with respect to some L^p-norm, $p \in [1,\infty)$, then a small perturbation $\widetilde{f}$ of f with such an isolated switch will always yield $\mathrm{TV}(\widetilde{f}) = \mathrm{TV}(f) + 2$, whereas $J(\widetilde{f})$ can be made arbitrarily close to $J(f)$.

We show that a similar, albeit weaker, result holds true for $H^s(0,T)$. Specifically, if $\widetilde{f} \in H^s(0,T)$ has an interval in which it switches (potentially several times) from zero to one and back that is isolated in the sense that to its left and right there is a wide enough interval on which $\widetilde{f}$ is zero, then $\widetilde{f}$ cannot be optimal if the width of the interval is too small.

To be clear, we consider a binary-valued function $f \in H^s(0,T)$ and positive scalars $0 \le t_L < t_\ell < t_r < t_R \le T$ with $t_\ell - t_L = \varepsilon$, $t_R - t_r = \varepsilon$, and $t_r - t_\ell = \delta$ for $\varepsilon > 0$ and $\delta > 0$ such that $f(t) = 0$ for a.e. $t \in (t_L, t_R)$. Moreover, we define $\widetilde{f} := f + \chi_{I^h}$ for a measurable set $I^h \subset (t_\ell, t_r)$ and $h = |I^h| > 0$, in particular $0 < h \le \delta$. Thus $\widetilde{f}$ is one on a set I^h of measure h that is surrounded by intervals of length of at least ε, where it is zero.

The objective of an H^s-regularized (integer) optimal control problem reads

$$\mathscr{J}(f) := J(f) + \frac{\alpha}{2}\|f\|^2_{L^2(0,T)} + \frac{\alpha}{2}|f|^2_{H^s(0,T)}$$

for some $\alpha > 0$. We obtain the following certificate of non-optimality for $\widetilde{f}$.

Theorem 10. *Let $J : L^1(0,T) \to \mathbb{R}$ be Lipschitz continuous. Let $f, \widetilde{f} \in H^s(0,T)$ be as above with arbitrary but fixed $\varepsilon > 0$. There is $\delta_0 > 0$ such that if $\delta \le \delta_0$, then $\mathscr{J}(f) < \mathscr{J}(\widetilde{f})$ holds regardless of the specific positions of t_L, t_R, t_ℓ, and t_r in $[0,T]$.*

Proof. We define $I^m := \{t \in (0,t_L) \cup (t_R,T) \,|\, f(t) = 1\}$, $O^m := \{t \in (0,t_\ell) \cup (t_r,T) \,|\, f(t) = 0\}$, and $O^h := (t_\ell, t_r) \setminus I^h$. We deduce

$$\begin{aligned}
\frac{1}{2}|\widetilde{f}|^2_{H^s(0,T)} - \frac{1}{2}|f|^2_{H^s(0,T)} &= \int_{t\in I^m\cup I^h}\int_{\tau\in O^m\cup O^h} \frac{1}{|t-\tau|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}t - \int_{t\in I^m}\int_{\tau\in O^m\cup(t_\ell,t_r)} \frac{1}{|t-\tau|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_{t\in I^h}\int_{\tau\in O^m\cup O^h} \frac{1}{|t-\tau|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}t - \int_{t\in I^m}\int_{\tau\in I^h} \frac{1}{|t-\tau|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}t \\
&\ge a(h) + b(h) - c(h) - d(h)
\end{aligned}$$

with

$$a(h) := \int_{t\in(t_r-h,t_r)}\int_{\tau\in(t_L,t_\ell)} \frac{1}{|t-\tau|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}t, \qquad b(h) := \int_{t\in(t_\ell,t_\ell+h)}\int_{\tau\in(t_r,t_R)} \frac{1}{|t-\tau|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}t,$$

$$c(h) := \int_{t\in(0,t_L)}\int_{\tau\in(t_\ell,t_\ell+h)} \frac{1}{|t-\tau|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}t, \qquad d(h) := \int_{t\in(t_R,T)}\int_{\tau\in(t_r-h,t_r)} \frac{1}{|t-\tau|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}t,$$

where

$$\int_{t\in I^h}\int_{\tau\in O^m\cup O^h}\frac{1}{|t-\tau|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}t \ge \int_{t\in I^h}\int_{\tau\in(t_L,t_\ell)\cup(t_r,t_R)}\frac{1}{|t-\tau|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}t \ge a(h)+b(h),$$

in which the first inequality is due to leaving out parts of the integration domain of the inner integral and the second inequality is due to splitting the inner integral and replacing I^h with sets of the same Lebesgue measure h, which lead to smaller values of $\frac{1}{|t-\tau|^{1+2s}}$ in the splitted integrals. Similarly, by adding parts of the integration domain, splitting the inner integral and replacing I^h therein so that larger values of $\frac{1}{|t-\tau|^{1+2s}}$ are attained, we obtain

$$-\int_{t\in I^m}\int_{\tau\in I^h}\frac{1}{|t-\tau|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}t \ge -\int_{t\in(0,t_L)\cup(t_R,T)}\int_{\tau\in I_h}\frac{1}{|t-\tau|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}t \ge -c(h)-d(h),$$

Let $q=1-2s$. Then we obtain with the same computations as in Lemma 3

$$\begin{aligned}
2sq\,a(h) &= -(\delta-h)^q+(\varepsilon+\delta-h)^q+\delta^q-(\varepsilon+\delta)^q,\\
2sq\,b(h) &= -(\delta-h)^q+\delta^q+(\varepsilon+\delta-h)^q-(\varepsilon+\delta)^q,\\
2sq\,c(h) &= -\varepsilon^q+(\varepsilon+t_L)^q+(\varepsilon+h)^q-(t_L+\varepsilon+h)^q,\\
2sq\,d(h) &= -\varepsilon^q+(\varepsilon+h)^q+(\varepsilon+T-t_R)^q-(\varepsilon+T-t_R+h)^q.
\end{aligned}$$

Clearly, $\|\widetilde{f}-f\|_{L^1}=h$ and we obtain

$$\begin{aligned}
2sq\,c(h)/h &\to q\varepsilon^{-2s}-q(t_L+\varepsilon)^{-2s}\le q\varepsilon^{-2s},\\
2sq\,d(h)/h &\to q\varepsilon^{-2s}-q(T-t_R+\varepsilon)^{-2s}\le q\varepsilon^{-2s}
\end{aligned}$$

for $h\searrow 0$, where the inequalities ensure that our arguments and the construction of δ_0 will be independent of the specific choice of $I^h\subset(t_\ell,t_r)$ (and also t_L, t_ℓ, t_r, t_R inside $(0,T)$).

We consider $\delta\searrow 0$ and $\delta\ge h$. The functions $t\mapsto t^q$ and its derivatives are Lipschitz continuous in compact intervals around ε that are bounded away from zero and thus the mean value theorem yields

$$\frac{(\varepsilon+\delta-h)^q-(\varepsilon+\delta)^q}{h}=-q(\varepsilon+\xi)^{-2s}\to -q\varepsilon^{-2s}$$

with $\xi\in[\delta-h,h]$ for $\delta\ge h>0$ and $\delta\searrow 0$. Moreover, the mean value theorem yields

$$\frac{\delta^q-(\delta-h)^q}{h}=q\xi^{-2s}\to\infty$$

with $\xi\in[\delta-h,\delta]$ for $\delta\ge h>0$ and $\delta\searrow 0$.

Because $\|g\|_{L^2}^2=|g|_{L^1}$ holds for binary-valued functions g, there is $L>0$ so that $J+\frac{\alpha}{2}\|\cdot\|_{L^2}^2$ is L-Lipschitz from $L^1(0,T)$ to $\mathbb{R}$ on the set of $\{0,1\}$-valued measurable functions for some $L>0$. We combine the estimates above to obtain

$$\liminf_{\delta\searrow 0,\delta\ge h}\frac{\mathscr{J}(\widetilde{f})-\mathscr{J}(f)}{\|\widetilde{f}-f\|_{L^1}}\ge -L+\liminf_{\delta\searrow 0,\delta\ge h}\frac{2\alpha}{sq}\left(\frac{\delta^q-(\delta-h)^q}{h}\right)-\frac{4\alpha}{s}\varepsilon^{-2s}\to\infty,$$

which implies that there is $\delta_0>0$ such that for all $0<h\le\delta\le\delta_0$ we have $\mathscr{J}(f)<\mathscr{J}(\widetilde{f})$. □

In the interest of a clear presentation, we restrict to the case that f is zero on (t_L,t_ℓ) and (t_r,t_R) and $\widetilde{f}$ is one on I^h here but note that the same arguments hold true with interchanged roles of zero and one.

6. The squared $H^s(0,T)$-seminorm is not differentiable for canonical variations

In [8], the fact that a feasible control in $BV(0,T)$ has only finitely many switches gives rise to a stationarity condition. Specifically, it arises from the Fermat principle $\frac{\mathrm{d}}{\mathrm{d}h}J(f^h)+\alpha\,\mathrm{TV}(f^h)\big|_{h=0}=0$ where f^h is identical to a given function f except that one of the switching points t of f is perturbed from t to $t+h$. In this section we show that such a stationarity condition cannot hold in $H^s(0,T)$ because the function $h\mapsto\frac{1}{2}|f^h|^2_{H^s}$ is not differentiable for any isolated switching point in $(0,T)$.

We only show the result for points, where the function switches from 0 to 1 but the similar arguments hold for switches from 1 to 0 as well. Before stating the result, we define our base function. Let $f\in H^s(0,T)$ be binary-valued and not constant. Let $t\in(0,T)$ satisfy $\lim_{\tilde{t}\uparrow t}f(\tilde{t})=0<1=\lim_{\tilde{t}\downarrow t}f(\tilde{t})$. We can write f and $1-f$ (a.e.) as

$$f=\chi_I+\chi_{J_t}\quad\text{and}\quad 1-f=\chi_O+\chi_{R_t}$$

where $J_t=(t,\bar{t})$ for some $\bar{t}\in(t,T]$ and $R_t=(\underline{t},t)$ for some $\underline{t}\in[0,t)$ and further disjoint sets I and O. We also define

$$f_h:=\chi_I+\chi_{J_{t+h}}$$

for all $h\in(\underline{t}-t,\bar{t}-t)$ as well as $J_{t+h}:=(t+h,\bar{t})$ and $R_{t+h}=(\underline{t},t+h)$.

Proposition 11. *Let f, t, f_h for $h\in(\underline{t}-t,\bar{t}-t)$ be as introduced above. Then the function $h\mapsto\frac{1}{2}|f_h|^2_{H^s}$ is not differentiable at zero. Specifically,*

$$\limsup_{h\to 0}\frac{|f_h|^2_{H^s(0,T)}-|f_0|^2_{H^s(0,T)}}{2h}=-\infty.$$

Proof. We observe

$$\frac{1}{2}|f_h|^2_{H^s(0,T)}-\frac{1}{2}|f_0|^2_{H^s(0,T)}=\int_{\sigma\in I\cup J_{t+h}}\int_{\tau\in O\cup R_{t+h}}\frac{1}{|\tau-\sigma|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}\sigma-\int_{\sigma\in I\cup J_t}\int_{\tau\in O\cup R_t}\frac{1}{|\tau-\sigma|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}\sigma,$$

which gives

$$\begin{aligned}\frac{1}{2}|f_h|^2_{H^s(0,T)}-\frac{1}{2}|f_0|^2_{H^s(0,T)}&=\underbrace{\int_{\sigma\in J_{t+h}}\int_{\tau\in O}\frac{1}{|\tau-\sigma|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}\sigma-\int_{\sigma\in J_t}\int_{\tau\in O}\frac{1}{|\tau-\sigma|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}\sigma}_{=:A(h)}\\&\quad+\underbrace{\int_{\sigma\in I}\int_{\tau\in R_{t+h}}\frac{1}{|\tau-\sigma|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}\sigma-\int_{\sigma\in I}\int_{\tau\in R_t}\frac{1}{|\tau-\sigma|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}\sigma}_{=:B(h)}\\&\quad+\underbrace{\int_{\sigma\in J_{t+h}}\int_{\tau\in R_{t+h}}\frac{1}{|\tau-\sigma|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}\sigma-\int_{J_t}\int_{R_t}\frac{1}{|\tau-\sigma|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}\sigma}_{=:C(h)}.\end{aligned}$$

We analyze the rows on the right hand side one by one. For the first row, we obtain

$$\frac{A(h)}{h}=-\int_{\sigma\in O}\frac{1}{h}\int_{\tau\in[t,t+h]}\frac{1}{|\tau-\sigma|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}\sigma\underset{h\to 0}{\to}-\int_O\frac{1}{|t-\sigma|^{1+2s}}\,\mathrm{d}\sigma$$

by virtue of Lebesgue's dominated convergence theorem. Simiarly, we obtain

$$\frac{B(h)}{h}=\int_{\sigma\in I}\frac{1}{h}\int_{\tau\in[t,t+h]}\frac{1}{|\tau-\sigma|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}\sigma\underset{h\to 0}{\to}\int_I\frac{1}{|t-\sigma|^{1+2s}}\,\mathrm{d}\sigma$$

for the second row. The third row can be rewritten as

$$C(h) = \underbrace{-\int_{\sigma\in[t,t+h]}\int_{\tau\in R_t}\frac{1}{|\tau-\sigma|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}\sigma}_{=:C_1(h)} + \underbrace{\int_{\sigma\in J_t}\int_{\tau\in[t,t+h]}\frac{1}{|\tau-\sigma|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}\sigma}_{=:C_2(h)} \underbrace{-\int_{\sigma\in[t,t+h]}\int_{\tau\in[t,t+h]}\frac{1}{|\tau-\sigma|^{1+2s}}\,\mathrm{d}\tau\,\mathrm{d}\sigma}_{=:C_3(h)}.$$

For the first two of these three terms we obtain

$$\frac{C_1(h)}{h}\to\int_{R_t}\frac{1}{|t-\tau|^{1+2s}}\,\mathrm{d}\tau \quad\text{and}\quad \frac{C_2(h)}{h}\to\int_{J_t}\frac{1}{|t-\sigma|^{1+2s}}\,\mathrm{d}\sigma,$$

where we have used Lebesgue's dominated convergence theorem again. We continue and analyze $C_3(h)$. Because all other terms are bounded (after dividing by h and passing to the limit $h\to 0$), the claim holds if $C_3(h)/h$ is unbounded below.

We consider the case $h>0$. Then elementary computations yield

$$\begin{aligned}
-C_3(h) &= \frac{1}{2s}\int_{[t,t+h]}\left|(\sigma-t)^{-2s}-((t+h)-\sigma)^{-2s}\right|\mathrm{d}\sigma\\
&= \frac{1}{s}\int_{[t,t+0.5h]}(\sigma-t)^{-2s}-((t+h)-\sigma)^{-2s}\,\mathrm{d}\sigma\\
&= \frac{1}{s(1-2s)}\left((\sigma-t)^{1-2s}\Big|_{\sigma=t}^{\sigma=t+0.5h}+((t+h)-\sigma)^{1-2s}\Big|_{\sigma=t}^{\sigma=t+0.5h}\right)\\
&= \frac{1}{s(1-2s)}\left(\frac{h^{1-2s}}{2^{1-2s}}+\frac{h^{1-2s}}{2^{1-2s}}-h^{1-2s}\right).
\end{aligned}$$

We consider the case $h<0$. Then elementary computations similarly yield

$$-C_3(h) = \frac{1}{2s}\int_{[t,t+h]}\left|(t-\sigma)^{-2s}-(\sigma-(t+h))^{-2s}\right|\mathrm{d}\sigma = \frac{-1}{s(1-2s)}\left(\frac{|h|^{1-2s}}{2^{1-2s}}+\frac{|h|^{1-2s}}{2^{1-2s}}-|h|^{1-2s}\right).$$

This implies $\frac{C_3(h)}{h}\to-\infty$ for $h\to 0$. □

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