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Derivations with values in noncommutative symmetric spaces

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Abstract. Let $E = E(0,\infty)$ be a symmetric function space and $E(\mathcal{M},\tau)$ be the noncommutative symmetric space corresponding to $E(0,\infty)$ associated with a von Neumann algebra with a faithful normal semifinite trace. Our main result identifies the class of spaces E for which every derivation $\delta : \mathcal{A} \to E(\mathcal{M},\tau)$ is necessarily inner for each C^*-subalgebra $\mathcal{A}$ in the class of all semifinite von Neumann algebras $\mathcal{M}$ as those with the Levi property.

Keywords. derivation, noncommutative symmetric space, semifinite von Neumann algebra.

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1. Historical Background and Motivations

Let $\mathcal{A}$ be a C^*-algebra and let J be an $\mathcal{A}$-bimodule [41]. A derivation $\delta : \mathcal{A} \to J$ is a linear mapping satisfying $\delta(xy) = \delta(x)y + x\delta(y)$, $x, y \in \mathcal{A}$. In particular, if $a \in J$, then $\delta_a(x) := xa - ax$ is a derivation. Such derivations implemented by elements in J are said to be *inner*. One of the classical problems in operator algebra theory is the question whether every derivation from $\mathcal{A}$ into J is automatically inner.

At a conference held in 1953, Kaplansky asked Singer if he had an idea of what the derivations of $C(X)$ (the algebra of continuous functions on a compact Hausdorff space X) might be. A day later, Singer gave Kaplansky a short, clever argument that such derivations must map all of $C(X)$ to 0 [28]. Kaplansky's paper [31] and the strong interest in derivations of operator algebras grew out of Singer's result. Kaplansky showed that each derivation of a type I von Neumann algebra (for example, $B(\mathcal{H})$, the algebra of all bounded operators on a Hilbert space $\mathcal{H}$) into itself is inner. In the course of his argument, Kaplansky proved that each such derivation is (norm-)continuous and conjectured that automatic continuity is true for all C^*-algebras. This conjecture was proved a few years later by Sakai [38] and extended by Ringrose [37] to derivations of a C^*-algebra into

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a Banach bimodule. These were among the earliest automatic-continuity results. In [27, 39], it was proved that each derivation of a C^*-algebra acting on $\mathscr{H}$ extends to a derivation of the strong-operator closure of that algebra, a von Neumann algebra, and that each derivation of a von Neumann algebra is inner. However, there exist C^*-algebras $\mathscr{A}$ with non-inner derivations $\delta : \mathscr{A} \to \mathscr{A}$, e.g., if $\mathscr{A}$ is the C^*-algebra $K(\mathscr{H})$ of all compact linear operators on $\mathscr{H}$, then for any $a \notin K(\mathscr{H}) + \mathbb{C}\mathbb{1}$, δ_a is a non-inner derivation on $\mathscr{A}$ [40, Example 4.1.8]. Therefore, it would be desirable to identify those $\mathscr{A}$-bimodules $\mathscr{A}$ such that all derivations from $\mathscr{A}$ into $\mathscr{J}$ are necessarily inner, see e.g. [24, Section 10.11], [22] and [41, p. 60]. Recall several important results in this direction:

(1) Derivations from a C^*-algebra $\mathscr{A}$ into any reflexive $\mathscr{A}$-bimodule are inner [24];
(2) Derivations from a hyperfinite von Neumann algebra $\mathscr{A}$ into any dual normal[1] $\mathscr{A}$-bimodule are inner [25];
(3) Derivations from a nuclear C^*-algebra $\mathscr{A}$ into a dual Banach $\mathscr{A}$-module are inner [21].

Let $\mathscr{M}$ be a semifinite von Neumann algebra acting on a Hilbert space $\mathscr{H}$ equipped with a semifinite faithful normal trace τ. Let $\mathbb{1}$ be the identity of $\mathscr{M}$. Let $\mathscr{P}(\mathscr{M})$ be the collection of all projections in $\mathscr{M}$. A densely-defined closed operator x affiliated with $\mathscr{M}$ is τ-measurable (see [19]) if and only if

$$\tau\left(e^{|x|}(n,\infty)\right) \to 0, \quad n \to \infty,$$

where $e^{|x|}(n,\infty)$ is the spectral projection of $|x|$ corresponding to the interval (n,∞). The collection of all τ-measurable operators with respect to $\mathscr{M}$ is denoted by $S(\mathscr{M},\tau)$. Let $x \in S(\mathscr{M},\tau)$. The generalised singular value function $\mu(x) : t \to \mu(t;x)$, $t > 0$, of the operator x is defined by setting

$$\mu(t;x) = \inf\left\{\|xp\|_\infty : p \in \mathscr{P}(\mathscr{M}), \tau(\mathbb{1}-p) \le t\right\},$$

where $\|\cdot\|_\infty$ denotes the usual operator norm. If $x \in S(\mathscr{M},\tau)$ satisfies that $\mu(\infty;x) = 0$, then x is said to be a τ-compact operator. The collection of all τ-compact operators in $S(\mathscr{M},\tau)$ is denoted by $S_0(\mathscr{M},\tau)$. Let $\mathscr{E}$ be a linear subset in $S(\mathscr{M},\tau)$ equipped with a complete norm $\|\cdot\|_\mathscr{E}$. We say that $\mathscr{E}$ is a *symmetric space* if for $x \in \mathscr{E}$, $y \in S(\mathscr{M},\tau)$ and $\mu(y) \le \mu(x)$ imply that $y \in \mathscr{E}$ and $\|y\|_\mathscr{E} \le \|x\|_\mathscr{E}$. If $\mathscr{E}$ is a symmetric space, then the carrier projection $c_\mathscr{E} \in \mathscr{P}(\mathscr{M})$ is defined by setting

$$c_\mathscr{E} = \bigvee\{p : p \in \mathscr{P}(\mathscr{M}),\ p \in \mathscr{E}\}.$$

We remark that, replacing the von Neumann algebra $\mathscr{M}$ by the reduced von Neumann algebra $\mathscr{M}_{c_\mathscr{E}}$, it is often assumed that the carrier projection of $\mathscr{E}$ is equal to $\mathbb{1}$, see e.g. [17].

If $x, y \in S(\mathscr{M},\tau)$, then x is said to be *submajorized* by y, denoted by $x \prec\prec y$, if

$$\int_0^t \mu(s;x)\,ds \le \int_0^t \mu(s;y)\,ds \quad \text{for all } t \ge 0.$$

A symmetric space $\mathscr{E} \subset S(\mathscr{M},\tau)$ is called *strongly symmetric* if its norm $\|\cdot\|_\mathscr{E}$ has the additional property that $\|x\|_\mathscr{E} \le \|y\|_\mathscr{E}$ whenever $x, y \in \mathscr{E}$ satisfy $x \prec\prec y$. In addition, if $x \in S(\mathscr{M},\tau)$, $y \in \mathscr{E}$ and $x \prec\prec y$ imply that $x \in \mathscr{E}$ and $\|x\|_\mathscr{E} \le \|y\|_\mathscr{E}$, then $\mathscr{E}$ is called *fully symmetric space* (of τ-measurable operators). If $\mathscr{E}$ is a strongly symmetric space with $c_\mathscr{E} = \mathbb{1}$ (or a symmetric space affiliated with a semifinite von Neumann algebra which is either atomless or atomic with all minimal projections having equal trace), then we have [17, 32]

$$(L_1 \cap L_\infty)(\mathscr{M},\tau) \subset \mathscr{E} \subset (L_1 + L_\infty)(\mathscr{M},\tau). \tag{1}$$

If $\mathscr{E}$ is a symmetric space, then the norm $\|\cdot\|_\mathscr{E}$ is called *order continuous* if $\|x_\alpha\|_\mathscr{E} \to 0$ whenever $\{x_\alpha\}$ is a downwards directed net in $\mathscr{E}^+$ satisfying $x_\alpha \downarrow 0$. A symmetric space $\mathscr{E}$ is said to have the*Levi property* [1, Definition 7][2], if for every upwards directed net $\{x_\beta\}$ in $\mathscr{E}^+$, satisfying

[1] Let $\mathscr{M}$ be a von Neumann algebra. An $\mathscr{M}$-bimodule X is said to be a dual normal X-bimodule if X is a dual space and the maps $m \mapsto mx$ and $m \mapsto xm$ are both ultraweak-weak* continuous from $\mathscr{M}$ into X for each fixed element $x \in X$.

[2] The Soviet school on Banach lattices used the term monotone complete norm or property (B), see also [1, p. 89].

$\sup_\beta \|x_\beta\|_{\mathscr{E}} < \infty$, there exists an element $x \in \mathscr{E}^+$ such that $x_\beta \uparrow x$ in $\mathscr{E}$. It is well known that if a norm is Levi, then necessarily it is also weak Fatou, i.e., there exists a constant $K \geq 1$ such that

$$0 \leq x_\beta \uparrow x \Longrightarrow \|x\|_{\mathscr{E}} \leq K \lim_\beta \|x_\beta\|_{\mathscr{E}}.$$

If the constant K is 1, then $\mathscr{E}$ is said to have the *Fatou property*. If $\mathscr{E}$ has the Fatou property and order continuous norm, then it is said to be a *KB*-space (or Kantorovich–Banach space) [17].

The so-called Köthe dual is identified with an important part of the dual space. If $\mathscr{E}$ is a symmetric space, then the Köthe dual $\mathscr{E}^\times$ of $\mathscr{E}$ is defined by

$$\mathscr{E}^\times = \left\{ x \in S(\mathscr{M},\tau) : \sup_{\|y\|_{\mathscr{E}} \leq 1, y \in \mathscr{E}} \tau(|xy|) < \infty \right\},$$

and for every $x \in \mathscr{E}^\times$, we set $\|x\|_{\mathscr{E}^\times} = \sup\{\tau(|yx|) : y \in \mathscr{E}, \|y\|_{\mathscr{E}} \leq 1\}$ [17].

Reciprocally, a wide class of symmetric spaces of τ-measurable operators is representable as $E(\mathscr{M},\tau)$, the noncommutative symmetric space associated with a given symmetric function space $E(0,\infty)$ and a given von Neumann algebra $\mathscr{M}$ equipped with a semifinite faithful normal trace τ: let $(E(0,\infty), \|\cdot\|_{E(0,\infty)})$ be a symmetric function space on the semi-axis $(0,\infty)$. The space

$$E(\mathscr{M},\tau) = \{x \in S(\mathscr{M},\tau) : \mu(x) \in E(0,\infty)\}$$

equipped with the norm $\|x\|_{E(\mathscr{M},\tau)} := \|\mu(x)\|_{E(0,\infty)}$ is a symmetric operator space affiliated with $\mathscr{M}$ with $c_{\mathscr{E}} = \mathbb{1}$, see e.g. [30], [17, Proposition 28]. For convenience, we denote $\|\cdot\|_{E(\mathscr{M},\tau)}$ by $\|\cdot\|_E$.

Due to the rapid development of noncommutative analysis and motivated by questions due to Johnson et al., there are a number of papers concerning various versions of the following question [3, 4, 10]:

Question 1. *Assume that $\mathscr{M}$ is a von Neumann algebra equipped with a faithful normal semifinite trace τ. Let $\mathscr{E}$ be a symmetric space of τ-compact operators affiliated with $\mathscr{M}$. How can one identity those $\mathscr{E}$ such that derivations from an arbitrary C^*-subalgebra $\mathscr{A}$ of $\mathscr{M}$ into $\mathscr{E}$ are necessarily inner?*

Experts in the operator theory are probably more familiar with symmetrically normed ideals in $B(\mathscr{H})$, which are a special case of noncommutative symmetric spaces. Various versions of Question 1 for derivations with values in ideals of a von Neumann algebra were asked and discussed in [5, 6, 22, 26, 29, 35, 36].

The Johnson–Parrott–Popa Theorem and Its Semifinite Versions

Johnson and Parrott [26] initiated the study of derivations with values in ideals of a von Neumann algebra by showing that derivations from an abelian/properly infinite von Neumann subalgebra of $B(\mathscr{H})$ into the algebra $K(\mathscr{H})$ of all compact operators on $\mathscr{H}$ are inner. However, they failed to resolve the case when $\mathscr{A}$ is a type II_1 von Neumann algebra, which remained open until Popa's penetrating work [35] in 1987. This result is now known as the so-called the Johnson–Parrott–Popa theorem: *every derivation from an arbitrary von Neumann subalgebra $\mathscr{A}$ of $B(\mathscr{H})$ into $K(\mathscr{H})$ is inner.* Note that the condition that $\mathscr{A}$ is a von Neumann algebra can not be relaxed to the setting of a C^*-subalgebra of $B(\mathscr{H})$ as mentioned above.

A natural development of the Johnson–Parrott–Popa theorem is to establish a suitable semifinite version of the result. In 1985, Kaftal and Weiss [29] proved that if $\mathscr{A}$ is an abelian (or properly infinite) von Neumann subalgebra of $\mathscr{M}$ containing the center $\mathscr{Z}(\mathscr{M})$ of $\mathscr{M}$, then any derivation $\delta : \mathscr{A} \to \mathscr{J}(\mathscr{M})$ is inner, where $\mathscr{J}(\mathscr{M})$ is an ideal of $\mathscr{M}$ generated by all finite projections in a semifinite von Neumann algebra $\mathscr{M}$. This result was later extended to the setting of more general von Neumann subalgebras by Popa and Rădulescu [36]. However, Popa and Rădulescu established the existence of non-inner derivations $\delta : \mathscr{A} \to \mathscr{J}(\mathscr{M})$ for a specific semifinite von Neumann algebra $\mathscr{M}$ and an abelian von Neumann subalgebra $\mathscr{A}$ of $\mathscr{M}$, which is the first non-vanishing 1-cohomological result in the theory of von Neumann algebras.

In 1987, Christensen [11] introduced the notion of *generalized compacts* associated with a von Neumann algebra and showed that derivations from a properly infinite von Neumann algebra into the generalized compacts associated with this von Neumann algebra are inner. However, the question whether derivations from a type II_1 von Neumann algebra into the generalized compacts associated with this von Neumann algebra are inner was left open, which was recently answered in the affirmative in [20]. We note that the ideals considered in [11, 20, 29, 36] are not necessarily symmetrically normed ideals in a semifinite von Neumann algebra (see e.g. [6, Section 2.3]) and these results lie outside of the scope covered by Question 1.

In our recent joint paper [6] with Ber and Levitina, we established the Johnson–Parrott–Popa theorem for another type of semifinite version of the ideal $K(\mathscr{H})$, namely the ideal $C_0(\mathscr{M},\tau)$ of τ-compact operators in a semifinite von Neumann algebra $\mathscr{M}$: *every derivation from a von Neumann subalgebra of $\mathscr{M}$ into $C_0(\mathscr{M},\tau)$ is necessarily inner.* Even though $C_0(\mathscr{M},\tau)$ and $\mathscr{J}(\mathscr{M})$ are similar in many respects (see [6]), our result is in strong contrast with the result in [36] and our result seems to be spiritually closer to the original Johnson–Parrott–Popa theorem, since we do not impose any additional condition on the von Neumann subalgebra $\mathscr{A}$.

Derivations into an Ideal of a von Neumann Algebra

An important class of ideals in a von Neumann algebra is given by the Schatten–von Neumann p-classes. In [22], Hoover used the Ryll-Nardzewski fixed point theorem (as suggested by Johnson [24, 26]) and the reflexivity of the Schatten–von Neumann p-class $C_p(\mathscr{H})$, $1 < p < \infty$, to show that every derivation from a C^*-subalgebra of $B(\mathscr{H})$ into $C_p(\mathscr{H})$ is inner. Actually, the Ryll-Nardzewski fixed point theorem is applicable to all reflexive $\mathscr{A}$-bimodules (in particular, noncommutative L_p-spaces when $1 < p < \infty$), see e.g. [24, Theorem 3.4]. Hoover also resolved the special case for the trace class $C_1(\mathscr{H})$ by a p-convexification technique [22].

Let $\mathscr{M}$ be a von Neumann algebra equipped with a semifinite faithful normal trace τ. Denote by $L_p(\mathscr{M},\tau)$, $p \ge 1$, the noncommutative L_p-space affiliated with $\mathscr{M}$, and denote $C_p(\mathscr{M},\tau) := L_p(\mathscr{M},\tau) \cap \mathscr{M}$. In general, $C_p(\mathscr{M},\tau)$ is not reflexive even for $1 < p < \infty$ and therefore the Ryll-Nardzewski fixed point theorem can not be applied directly (the method used in [3] or [34] is not applicable, either). In 1985, using Johnson and Parrott's trick [26], Kaftal and Weiss [29] showed that every derivation from an abelian (or properly infinite) von Neumann subalgebra of $\mathscr{M}$ into the C_p ideal of $\mathscr{M}$ is inner when $1 \le p < \infty$. However, the case for general von Neumann subalgebras of $\mathscr{M}$ was left unanswered. Using noncommutative integration techniques, it was proved in [5] that derivations from a C^*-subalgebra of $\mathscr{M}$ into any symmetric *KB*-ideal of $\mathscr{M}$ are inner, which, in particular, fully resolves the untreated cases for derivations with values in $C_p(\mathscr{M},\tau)$ in the paper [29] by Kaftal and Weiss. For the special case when $\mathscr{A}$ is a von Neumann subalgebra of $\mathscr{M}$, it is shown in [6] that derivations from $\mathscr{A}$ into any ideal of τ-compact operators, generated by a noncommutative strongly symmetric space $\mathscr{E}$ having the Fatou property, are inner; it was shown in [7, 8] that derivations from a (not necessarily semifinite) von Neumann algebra $\mathscr{M}$ into any ideal of $\mathscr{M}$ are inner.

2. Derivations into Symmetric Spaces of Possibly Unbounded Operators

An important class of $\mathscr{A}$-bimodules is given by the noncommutative L_p-spaces. Since a noncommutative L_p-space is reflexive if $1 < p < \infty$, it follows immediately from Johnson's result that derivations into a noncommutative L_p-space, $1 < p < \infty$, are inner. Let $\mathscr{U}(\mathscr{A})$ be the unitary group of a unital C^*-algebra $\mathscr{A}$. Consider a derivation δ from $\mathscr{A}$ into a reflexive $\mathscr{A}$-bimodule $\mathscr{J}$. Since the unit ball of a reflexive Banach space is weakly compact, it follows that the set

$$K_\delta := \overline{\mathrm{conv}\{\delta(u)u^* \mid u \in \mathscr{U}(A)\}}^{\|\cdot\|_{\mathscr{J}}}$$[3]

is weakly compact in $\mathscr{J}$. Therefore, one can easily apply the Ryll-Nardzewski fixed point theorem to prove the innerness of δ. Recall, also, that every derivation from a hyperfinite von Neumann algebra $\mathscr{A}$ into a dual normal $\mathscr{A}$-bimodule is inner. However, the predual (i.e., a noncommutative L_1-space) of a semifinite von Neumann algebra $\mathscr{M}$ is not reflexive unless the underlying von Neumann algebra is finite-dimensional, which is the main obstacle to resolve derivation problem for noncommutative L_1-spaces; moreover, in general, a noncommutative L_1-space does not even have a predual (i.e., it does not have the “dual normal” property), which makes this problem even harder.

Hoover [22] resolved the special case for the trace class. The case for derivations from a C^*-subalgebra of a finite von Neumann algebra $\mathscr{M}$ into the predual of $\mathscr{M}$ was resolved by Bunce and Paschke in [10], where they also proved the derivations from a semifinite von Neumann algebra $\mathscr{M}$ into its predual $\mathscr{M}_*$ are automatically inner (see also [21] for the case of type *III* von Neumann algebras). However, it was a long-standing open question whether every derivation a C^*-subalgebra of $\mathscr{M}$ into $\mathscr{M}_*$ must be inner [10, p. 247], which was resolved completely by Bader, Gelander and Monod in 2012 [3] (see also [34] for a slightly different proof due to Pfitzner). Bader, Gelander and Monod considered the so-called Chebyshev center (which is weakly compact) in the so-called L-embedded Banach spaces[4] (e.g. the predual of a von Neumann algebra), where the Ryll-Nardzewski fixed point theorem is applicable. Hereby, they provided a beautiful and short resolution to the derivation problem for noncommutative L_1-spaces. In fixed point theory, it is natural to claim only convex compactness instead of compactness [34, Remark 8]. Therefore, the idea used by Pfitzner in [34] seems to be more natural, where he introduced a new topology for general L-embedded spaces (in particular, for the predual of von Neumann algebras) and apply a variant of the Ryll-Nardzewski fixed point theorem to a weakly compact convex set. This direction of thought has been completed in [23], which shows that derivations into L-embedded symmetric spaces are inner. It is worth mentioning that derivations from a semifinite von Neumann algebra into a symmetric space affiliated with itself are necesarily inner [4].

3. Main Results and Methods

L-embedded Banach spaces are very special in the field of Banach spaces, and the method used in [3] (or [34]) does not have any chance to deliver the full answer on Question 1. This fact was emphaiszed in [3, Section 3], where the following points were raised:

(a) In marked contrast to the classical fixed point theorems, there is no hope to find a fixed point inside a general bounded closed convex subset of L^1 $\cdots$ the weak compactness $\cdots$ seems almost unavoidable $\cdots$

(b) $\cdots$ a canonical norm one projection $V^{**} \to V$ is not enough.

(c) It would be interesting to find a purely geometric version of the proposition $\cdots$

The fact that the “fixed point” obtained in [3] is not inside a general bounded closed convex subset of L^1 leads to extra difficulties in the general case. Our principal result provides a complete answer to Question 1 above.

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[3]This convex set K_δ was first considered by Kadison, in connection with innerness of derivations on finite von Neumann algebras, and using a fixed point method.

[4]That is, a Banach space V whose bidual can be decomposed as $V^{**} = V \oplus_1 V_0$ for some $V_0 \subset V^{**}$ (and $\oplus_1$ indicates that the norm is the sum of the norms on V and V_0).

Recall the noncommutative version of Grothendieck's theorem established in [2] stating that an arbitrary bounded linear map from a C^*-algebra into a weakly sequentially complete Banach space is weakly compact. Moreover, a symmetric KB-function space $F(0,\infty)$ generates a weakly sequentailly complete noncommutative symmetric space $F(\mathcal{M},\tau)$ [17]. We obtain that derivations into $F(\mathcal{M},\tau)$ must be weakly compact. However, it is not enough to show that the convex set K_δ is weakly compact in $F(\mathcal{M},\tau)$, see e.g., the case when $F(0,\infty) = L_1(0,\infty)$ and $\mathcal{M}$ is a non-finite semifinite von Neumann algebra [42, Theorem III.5.4]. We need the following result to guarantee the weak compactness of K_δ, whose proof relies on the weak compactness criteria in symmetric spaces obtained in [13, 18].

For a Banach space X, we denote by B_X the unit ball of X.

Proposition 2. *Let $\mathcal{M}$ be a von Neumann algebra equipped with a semifinite faithful normal trace τ. Assume that $\mathcal{E}$ is a strongly symmetric KB-space such that $\mathcal{E}^\times \subset S_0(\mathcal{M},\tau)$. Let $\mathcal{A}$ be a C^*-subalgebra $\mathcal{M}$ and let T be a bounded linear operator from $\mathcal{A}$ into $\mathcal{E}$. Then, the set*

$$B_{\mathcal{M}} T(B_{\mathcal{A}}) B_{\mathcal{M}} := \{aT(x)b : a, b \in B_{\mathcal{M}}, x \in B_{\mathcal{A}}\}$$

is relatively weakly compact in $\mathcal{E}$.

For derivations into noncommutative symmetric spaces, we have the following consequence.

Proposition 3. *Let $\mathcal{M}$ be a von Neumann algebra equipped with a semifinite faithful normal trace τ and let $\mathcal{A}$ be a unital C^*-subalgebra $\mathcal{M}$. Assume that $\mathcal{E}$ is a strongly symmetric KB-space such that $\mathcal{E}^\times \subset S_0(\mathcal{M},\tau)$. Let $\delta : \mathcal{A} \to \mathcal{E}$ be a derivation. Then,*

$$\{\delta(u)u^* \mid u \in \mathcal{U}(\mathcal{A})\}$$

is relatively weakly compact in $\mathcal{E}$. Consequently, the closure $\overline{\mathrm{conv}\{\delta(u)u^* \mid u \in \mathcal{U}(\mathcal{A})\}}^{\|\cdot\|_{\mathcal{E}}}$ of the convex hull is weakly compact.

Assume that $\mathcal{E}$ is a strongly symmetric space with $c_{\mathcal{E}} = \mathbb{1}$. Note that, if $\tau(\mathbb{1}) < \infty$, then the condition $\mathcal{E}^\times \subset S_0(\mathcal{M},\tau)$ holds for any symmetric space $\mathcal{E}$. If $\tau(\mathbb{1}) = \infty$, then $\mathcal{E}^\times \subset S_0(\mathcal{M},\tau)$ if and only if $\mathcal{E} \cap \mathcal{M} \supsetneqq L_1(\mathcal{M},\tau) \cap \mathcal{M}$. Indeed, assume that $\mathcal{E} \cap \mathcal{M} \supsetneqq L_1(\mathcal{M},\tau) \cap \mathcal{M}$. Then, there exists an element $0 \le z \in \mathcal{E} \cap \mathcal{M}$ but $z \notin L_1(\mathcal{M},\tau)$. In particular, we have $\tau(z) = \infty$. By the definition of Köthe duals, we infer that $\mathbb{1} \notin \mathcal{E}^\times$, which, in turn, implies that $\mathcal{E}^\times \subset S_0(\mathcal{M},\tau)$. On the other hand, assume by contradiction that $L_1(\mathcal{M},\tau) \cap \mathcal{M}$ is not a proper subspace of $\mathcal{E} \cap \mathcal{M}$. By (1), we have $L_1(\mathcal{M},\tau) \cap \mathcal{M} \subset \mathcal{E} \cap \mathcal{M}$. Therefore, we obtain that $\mathcal{E} \cap \mathcal{M} = L_1(\mathcal{M},\tau) \cap \mathcal{M}$. By the fact that $\mathcal{E} \overset{(1)}{\subset} (L_1 + L_\infty)(\mathcal{M},\tau)$, we obtain that for any element $x \in \mathcal{E}$, $\mu(x)\chi_{(0,1)} \in L_1(0,1)$. Hence, all elements in $\mathcal{E}$ belong to $L_1(\mathcal{M},\tau)$. Therefore, $\mathcal{E}^\times \supset L_1(\mathcal{M},\tau)^\times = \mathcal{M}$ [14, Definition 5.1]. That is, $\mathcal{E}^\times \not\subset S_0(\mathcal{M},\tau)$, which completes the proof.

Now, one may apply the Ryll-Nardzewski fixed point theorem to obtain the following result for a wide class of noncommutative symmetric spaces.

Lemma 4. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal semifinite trace τ and let $\mathcal{A}$ be a unital C^*-subalgebra of $\mathcal{M}$. Let $\mathcal{E}$ a strongly symmetric KB-space such that $\mathcal{E}^\times \subset S_0(\mathcal{M},\tau)$. For every derivation $\delta : \mathcal{A} \to \mathcal{E}$, there exists an element $a \in \overline{\mathrm{conv}\{\delta(u)u^* \mid u \in \mathcal{U}(A)\}}^{\|\cdot\|_{\mathcal{E}}}$ such that $\delta = \delta_a$ on $\mathcal{A}$. In particular, $\|a\|_{\mathcal{E}} \le \|\delta\|_{\mathcal{A}\to\mathcal{E}}$.*

For the case of general noncommutative symmetric spaces (in particular, $L_1(\mathcal{M},\tau)$), we need the following noncommutative version of the "reflexive gate type" result. Here, we recall the reflexive gate type result [33, Corollary 3.2.3] since it seems not to be well-known but plays a significant role in our approach: if $E(0,1) \neq L_1(0,1)$ and $E(0,1) \neq L_\infty(0,1)$, then there exist two reflexive symmetric spaces $F_1(0,1)$ and $F_2(0,1)$ such that $F_2(0,1) \subset E(0,1) \subset F_1(0,1)$. The main ingredients of the proof of [33, Corollary 3.2.3] are the construction of a Lorentz space which embeds into $E(0,1)$ continuously, and a p-convexification technique. For the case when the measure/trace is infinite, we need the famous Davis–Figiel–Johnson–Pełczyński construction for reflexive spaces[12].

Theorem 5. *Let $\mathcal{M}$ be a von Neumann algebra equipped with a semifinite faithful normal trace τ. Let $\mathcal{E}$ be a strongly symmetric space such that $\mathcal{E}^{\times\times} \subset S_0(\mathcal{M},\tau)$. Then, there exists a symmetric KB-function space $F(0,\infty)$ such that*

$$\mathcal{E} \subset F(\mathcal{M},\tau).$$

It is important to emphasize that the Fatou/Levi property was hatched in the theory of Banach lattices [1,9], and was even included into the original definition of Banach function spaces over σ-finite measure spaces. The property is somewhat analogous to the so-called "dual normal" property. The importance of the Fatou/Levi property in the theory of Banach function spaces and symmetric operator spaces is hard to overestimate [14–16]. It seems appropriate to recall here that every derivation from a hyperfinite von Neumann algebra $\mathcal{A}$ into a dual normal $\mathcal{A}$-bimodule is inner. Recall also, that derivations from a nuclear C^*-algebra $\mathcal{A}$ into a dual Banach $\mathcal{A}$-module are inner. However, Theorem 6 below holds for arbitrary C^*-subalgebras $\mathcal{A}$ of $\mathcal{M}$ and for symmetric spaces which may not have a predual.

Theorem 6. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal semifinite trace τ and let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{M}$. If $\mathcal{E}$ is a strongly symmetric space of τ-compact operators (i.e., $\mathcal{E} \subset S_0(\mathcal{M},\tau)$[5]) having the Fatou property (resp., the Levi property), then every derivation $\delta : \mathcal{A} \to \mathcal{E}$ is inner. That is, there exists an element $a \in \overline{\mathrm{conv}\{\delta(u)u^* \mid u \in \mathcal{U}(\mathcal{A})\}}^{t_\tau} \subset \mathcal{E}$ with $\|a\|_{\mathcal{E}} \le \|\delta\|_{\mathcal{A}\to\mathcal{E}}$ (resp., $\|a\|_{\mathcal{E}} \le c\|\delta\|_{\mathcal{A}\to\mathcal{E}}$ for some constant c depending on $\mathcal{E}$ only) such that $\delta = \delta_a$ on $\mathcal{A}$.*

Proof. Without loss of generality, we may assume that the carrier projection $c_{\mathcal{E}} = \mathbb{1}$, $\mathcal{A}$ is unital and $\mathcal{E}$ has the Fatou property.

Since $\mathcal{E}$ has the Fatou property and $\mathcal{E} \subset S_0(\mathcal{M},\tau)$, it follows that $\mathcal{E}^{\times\times} \subset S_0(\mathcal{M},\tau)$. By Theorem 5, there exists a symmetric KB-function space $F(0,\infty) \subsetneqq (L_1 + C_0)(0,\infty)$ such that

$$\mathcal{E} \subset F(\mathcal{M},\tau) \subset S_0(\mathcal{M},\tau).$$

Without loss of generality, we may assume that $L_2(0,\infty) \subset F(0,\infty)$ by replacing $F(0,\infty)$ with $L_2(0,\infty)+F(0,\infty)$. In particular, $F(0,\infty)\cap L_\infty(0,\infty) \supsetneqq L_1(0,\infty)\cap L_\infty(0,\infty)$. In particular, $F(0,\infty)^\times \subset S_0(0,\infty)$, and therefore, $F(\mathcal{M},\tau)^\times \subset S_0(\mathcal{M},\tau)$.

Note that $\delta(\mathcal{A}) \subset \mathcal{E} \subset F(\mathcal{M},\tau)$. By Lemma 4, there is an element

$$a \in \overline{\mathrm{conv}\{\delta(u)u^* \mid u \in \mathcal{U}(\mathcal{A})\}}^{\|\cdot\|_F}$$

such that $\delta = \delta_a$ on $\mathcal{A}$. Hence, there exists a sequence

$$(x_n)_{n=1}^\infty \subset \mathrm{conv}\left\{\delta(u)u^* \mid u \in \mathcal{U}(\mathcal{A})\right\}$$

such that $\|x_n - a\|_F \to_n 0$. Since $F(\mathcal{M},\tau)$ is a symmetric space, it follows from [17, Proposition 20] that $x_n \to_{t_\tau} a$ as $n \to \infty$[6].

By Ringrose's theorem [37], we have that $\delta : (\mathcal{A}, \|\cdot\|_\infty) \to (\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a bounded mapping. Since $\mathcal{E}$ has the Fatou property, it follows that the closed ball $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ with radius $\|\delta\|_{\mathcal{A}\to\mathcal{E}}$ is closed in $S(\mathcal{M},\tau)$ with respect to the measure topology [14, 17]. Noticing that every element x_n, $n \ge 1$, belongs to the ball of radius $\|\delta\|_{\mathcal{A}\to\mathcal{E}}$ in $\mathcal{E}$ and $x_n \to a$ in the measure topology, we conclude that $a \in \mathcal{E}$ with $\|a\|_{\mathcal{E}} \le \|\delta\|_{\mathcal{A}\to\mathcal{E}}$. □

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[5] If $\tau(\mathbb{1}) < \infty$, then $\mathcal{E} \subset S_0(\mathcal{M},\tau)$ holds for any symmetric space $\mathcal{E}$ affiliated with $\mathcal{M}$.

[6] For every $\varepsilon, \delta > 0$, we define the set

$$V(\varepsilon,\delta) = \left\{x \in S(\mathcal{M},\tau) : \exists\, p \in \mathcal{P}(\mathcal{M}) \text{ such that } \left\|x(\mathbb{1}-p)\right\|_\infty \le \varepsilon,\ \tau(p) \le \delta\right\}.$$

The topology generated by the sets $V(\varepsilon,\delta)$, $\varepsilon,\delta > 0$, is called the *measure topology* t_τ on $S(\mathcal{M},\tau)$ [19]. It is well known that the algebra $S(\mathcal{M},\tau)$ equipped with the measure topology is a complete metrizable topological algebra.

The new approach devised in this paper answers to points (a), (b) and (c) above raised in [3]. In particular, it provides an alternative proof for the resolution of the question raised by Bunce and Paschke[10], without involving weak compactness of a subset in a L-embedded space as [3] and [34] did.

(1) This enables us to find a "fixed point" (implementing the derivation) from a not necessarily weakly compact closed convex subset of a noncommutative symmetric space.
(2) The Levi property of a symmetric space $\mathscr{E}$ (not necessarily an L-embedded Banach space) is equivalent to the existence of a canonical norm one projection from the bidual $\mathscr{E}^{**}$ onto $\mathscr{E}$ [15, 16][7]), which is enough to prove the existence of a fixed point, see Theorem 6.
(3) On the other hand, the Levi property of the space $\mathscr{E}$ means that $\mathscr{E}$ coincides with its second Köthe dual and this geometrical condition is the only one required in Theorem 6 thus delivering (at least spiritually) an answer to the question suggested in [3, Comment c] above.

We believe that the method developed in this work is of interest in its own right.

We also show that whenever $E(0,\infty)$ does not have the Levi property, there exist non-inner derivations $\delta : \mathscr{A} \to E(\mathscr{M},\tau)$ for some semifinite von Neumann algebra $\mathscr{M}$ and a C^*-subalgebra $\mathscr{A}$ of $\mathscr{M}$.

Corollary 7. *For a given symmetric function space $E(0,\infty) \subset S_0(0,\infty)$, the following two statements are equivalent:*

(1) *for any von Neumann algebra $\mathscr{M}$ equipped with a semifinite faithful normal trace τ and any C^*-subalgebra $\mathscr{A}$ of $\mathscr{M}$, derivations $\delta : \mathscr{A} \to E(\mathscr{M},\tau)$ are necessarily inner;*
(2) *$E(0,\infty)$ has the Levi property.*

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