INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Rémi Buffe and Takéo Takahashi

Controllability of a fluid-structure interaction system coupling the Navier–Stokes system and a damped beam equation

Volume 361 (2023), p. 1541-1576

https://doi.org/10.5802/crmath.509

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2023, Vol. 361, p. 1541-1576
https://doi.org/10.5802/crmath.509

Control theory / *Théorie du contrôle*

Controllability of a fluid-structure interaction system coupling the Navier–Stokes system and a damped beam equation

Rémi Buffe [a] **and Takéo Takahashi** [*, a]

[a] Université de Lorraine, CNRS, Inria, IECL, 54000 Nancy, France
E-mails: remi.buffe@univ-lorraine.fr, takeo.takahashi@inria.fr

Abstract. We show the local null-controllability of a fluid-structure interaction system coupling a viscous incompressible fluid with a damped beam located on a part of its boundary. The controls act on arbitrary small parts of the fluid domain and of the beam domain. In order to show the result, we first use a change of variables and a linearization to reduce the problem to the null-controllability of a Stokes-beam system in a cylindrical domain. We obtain this property by combining Carleman inequalities for the heat equation, for the damped beam equation and for the Laplace equation with high-frequency estimates. Then, the result on the nonlinear system is obtained by a fixed-point argument.

Keywords. Null controllability, Navier–Stokes systems, Carleman estimates, fluid-structure interaction systems.

2020 Mathematics Subject Classification. 76D05, 35Q30, 74F10, 76D55, 76D27, 93B05, 93B07, 93C10.

Funding. The second author was partially supported by the Agence Nationale de la Recherche, Project TRECOS, ANR-20-CE40-0009.

Manuscript received 29 November 2022, revised 20 April 2023, accepted 21 April 2023.

1. Introduction

We consider a fluid-structure interaction system composed by a viscous incompressible fluid, modeled by the Navier–Stokes system, and by an elastic structure located at a part of the boundary of the fluid domain. We assume that the structure displacement is governed by a damped beam equation. The corresponding model has been introduced in [47] as a first model to study the blood flow in vessels. To simplify our work, we consider here a particular geometry in dimension 2 of space (see Figure 1). The fluid domain is confined into an infinite strip where the bottom boundary is fixed and where the top boundary corresponds to the beam. We also assume periodic condition in the x_1 variables. To be more precise, we set

$$\mathscr{I} := \mathbb{R}/(2\pi\mathbb{Z}),$$

* Corresponding author.

ISSN (electronic) : 1778-3569 https://comptes-rendus.academie-sciences.fr/mathematique/

and for any deformation $\zeta : \mathscr{I} \to (-1,\infty)$, we consider the fluid domain associated with this deformation:

$$\Omega_\zeta = \{(x_1,x_2) \in \mathscr{I} \times \mathbb{R}\ ;\ x_2 \in (0, 1+\zeta(x_1))\}. \tag{1}$$

Then the fluid-structure interaction system writes

$$\begin{cases} \partial_t w + (w\cdot\nabla)w - \operatorname{div}\mathbb{T}(w,\pi) = 1_\omega f & t>0,\ x\in\Omega_{\zeta(t)},\\ \operatorname{div} w = 0 & t>0,\ x\in\Omega_{\zeta(t)},\\ w(t,x_1,1+\zeta(t,x_1)) = (\partial_t\zeta)(t,x_1)e_2, & t>0,\ x_1\in\mathscr{I},\\ w=0 & t>0,\ x\in\Gamma_0,\\ \partial_{tt}\zeta + \alpha_1\partial_{x_1}^4\zeta - \alpha_2\partial_{x_1}^2\zeta - \alpha_3\partial_t\partial_{x_1}^2\zeta = -\widetilde{\mathbb{H}}_\zeta(w,\pi) + 1_{\mathscr{J}} g & t>0,\ x_1\in\mathscr{I},\\ w(0,\cdot) = w^0 \quad \text{in } \Omega_{\zeta_1^0},\quad \zeta(0,\cdot)=\zeta_1^0, & \partial_t\zeta(0,\cdot) = \zeta_2^0 \quad \text{in } \mathscr{I}, \end{cases} \tag{2}$$

where

$$\alpha_1>0,\quad \alpha_2\geqslant 0,\quad \alpha_3>0,$$

and where

$$\Gamma_0 = \mathscr{I}\times\{0\}.$$

In the above system, we have used the following notations: (e_1,e_2) is the canonical basis of $\mathbb{R}^2$ and

$$\mathbb{T}(w,\pi) = 2\mathbb{D}(w) - \pi I_2,\quad \mathbb{D}(w) = \frac{1}{2}\left(\nabla w + (\nabla w)^*\right), \tag{3}$$

$$\widetilde{\mathbb{H}}_\zeta(w,\pi)(t,x_1) = \left[(1+|\partial_{x_1}\zeta|^2)^{1/2}\left[\mathbb{T}(w,\pi)n\right](t,x_1,1+\zeta(t,x_1))\cdot e_2\right]. \tag{4}$$

We have also denoted by n the unit exterior normal to $\Omega_{\zeta(t)}$. In (2), w and π are respectively the velocity and the pressure of the fluid and they satisfy the Navier–Stokes system (two first lines), with no-slip boundary conditions (third and forth equations). The elastic displacement satisfies the damped beam equation written in the fifth line of (2). Finally, our aim is to control (2) by using two distributed controls f and g respectively localized in an arbitrary small nonempty open subset ω of Ω and in an arbitrary small nonempty open subset $\mathscr{J}$ of $\mathscr{I}$.

Let us remark that the well-posedness and the stabilization of system (2) have been already studied in the literature. Let us quote some of the corresponding articles: [13] (existence of weak solutions), [5,21,35,40] (existence of strong solutions), [48] (stabilization of strong solutions), [1] (stabilization of weak solutions around a stationary state). We can also mention some works devoted to the case $\delta = 0$ (undamped beam equation/wave equation): [12,20,43] (weak solutions), [2–4,22] (strong solutions). Some authors have tackled the study of more complex models: [33,34] (linear elastic Koiter shell), [44] (dynamic pressure boundary conditions), [45,46] (3D cylindrical domain with nonlinear elastic cylindrical Koiter shell), [51] and [52] (nonlinear elastic and thermoelastic plate equations), [38,39] (compressible fluids), etc.

A standard strategy to study this kind of systems consists in using a change of variables to write the fluid system into a cylindrical domain, and then in linearizing the system after this transformation. A large part of the work is thus devoted to the corresponding linear system, the results for the nonlinear system are deduced by estimating the coefficients coming from the change of variables and by using a fixed-point argument. We follow here this approach and after a change of variable and a linearization (see Section 6 for the details), we are reduced to work on the spatial domain

$$\Omega := \Omega_0 = \mathscr{I}\times(0,1)$$

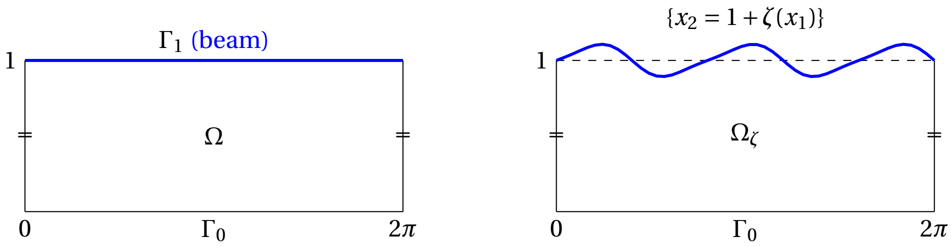

Figure 1. Our geometry

(see Figure 1) and to show the null controllability of the following linear system

$$\begin{cases} \partial_t w - \Delta w + \nabla \pi = 1_\omega f & \text{in } (0,T)\times\Omega,\\ \operatorname{div} w = 0 & \text{in } (0,T)\times\Omega,\\ w = 0 & \text{on } (0,T)\times\Gamma_0,\\ w = (\partial_t \zeta) e_2 & \text{on } (0,T)\times\Gamma_1,\\ \partial_t^2\zeta + \alpha_1\partial_{x_1}^4\zeta - \alpha_2\partial_{x_1}^2\zeta - \alpha_3\partial_t\partial_{x_1}^2\zeta = -\mathbb{T}(w,\pi)n\cdot e_2 + 1_{\mathscr{I}} g & \text{in } (0,T)\times\mathscr{I},\\ w(0,\cdot) = w^0 & \text{in } \Omega,\\ \zeta(0,\cdot) = \zeta_1^0, \quad \partial_t\zeta(0,\cdot) = \zeta_2^0 & \text{in } \mathscr{I}, \end{cases} \tag{5}$$

where

$$\Gamma_1 = \mathscr{I}\times\{1\}.$$

In what follows, to simplify the notation, we take

$$\alpha_1 = \alpha_2 = \alpha_3 = 1.$$

The values of these constants do not play any role in our study. As it is standard (see, for instance, [53, Theorem 11.2.1, p. 357]), the controllability of (5) is equivalent to an observability inequality for the adjoint system

$$\begin{cases} \partial_t u - \Delta u + \nabla p_0 = 0 & \text{in } (0,T)\times\Omega,\\ \operatorname{div} u = 0 & \text{in } (0,T)\times\Omega,\\ u = 0 & \text{on } (0,T)\times\Gamma_0,\\ u = \partial_t\eta e_2 & \text{on } (0,T)\times\Gamma_1,\\ \partial_t^2\eta + \partial_{x_1}^4\eta - \partial_{x_1}^2\eta - \partial_t\partial_{x_1}^2\eta = -\mathbb{T}(u,p_0)n_{|\Gamma_1}\cdot e_2 & \text{in } (0,T)\times\mathscr{I},\\ u(0,\cdot) = u^0 & \text{in } \Omega,\\ \eta(0,\cdot) = \eta_1^0, \quad \partial_t\eta(0,\cdot) = \eta_1^0 & \text{in } \mathscr{I}. \end{cases} \tag{6}$$

Before writing the corresponding observability inequality, let us mention an important remark and introduce some notation. We set

$$L_0^2(\mathscr{I}) := \left\{ f \in L^2(\mathscr{I}) \,;\, \int_0^{2\pi} f(x_1)\,\mathrm{d}x_1 = 0 \right\}.$$

Remark 1. Using the particular geometry considered here, we can simplify the above adjoint system. First on Γ_1, $n = e_2$ and using (3), we deduce

$$-\mathbb{T}(u,p_0)n\cdot e_2 = -2\partial_{x_2}u_2 + p_0 = 2\partial_{x_1}u_1 + p_0 = p_0 \quad \text{on } \Gamma_1, \tag{7}$$

since $u_1(x_1,1) = 0$ for $x_1 \in \mathscr{I}$.

Moreover, using the incompressibility of the fluid and the boundary conditions, we deduce that

$$0 = \int_\Omega \operatorname{div} u \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^{2\pi} \eta \,\mathrm{d}x_1.$$

Assuming that $\eta_1^0 \in L^2_0(\mathscr{I})$ then, we deduce that for all $t \geqslant 0$, $\eta(t,\cdot) \in L^2_0(\mathscr{I})$. Using this condition on the beam equation leads to the following condition on the pressure:

$$\int_0^{2\pi} p_0(t,x_1,1)\,\mathrm{d}x_1 = 0. \tag{8}$$

In particular, in contrast with the standard Stokes system, the pressure is not determined up to a constant.

We define the operators associated with the beam equation:

$$\mathscr{D}(A_1) := H^4(\mathscr{I}) \cap L^2_0(\mathscr{I}), \qquad A_1\eta := \partial^4_{x_1}\eta - \partial^2_{x_1}\eta, \tag{9}$$

$$\mathscr{D}(A_2) := H^2(\mathscr{I}) \cap L^2_0(\mathscr{I}), \qquad A_2\eta := -\partial^2_{x_1}\eta. \tag{10}$$

We also define the Hilbert space of states for our system:

$$\mathscr{H} := \left\{(u,\eta_1,\eta_2) \in L^2(\Omega) \times \mathscr{D}(A_1^{1/2}) \times L^2_0(\mathscr{I})\ ;\ u_2 = \eta_2 \text{ on } \Gamma_1,\ u_2 = 0 \text{ on } \Gamma_0,\ \operatorname{div} u = 0 \text{ in } \Omega\right\}, \tag{11}$$

endowed with the canonical scalar product of $L^2(\Omega) \times \mathscr{D}(A_1^{1/2}) \times L^2(\mathscr{I})$. With the above remark and notation, the adjoint system writes

$$\begin{cases} \partial_t u - \Delta u + \nabla p_0 = 0 & \text{in } (0,T)\times\Omega, \\ \operatorname{div} u = 0 & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\Gamma_0, \\ u = \partial_t\eta e_2 & \text{on } (0,T)\times\Gamma_1, \\ \partial_t^2\eta + A_1\eta + A_2\partial_t\eta = p_{0|_{\Gamma_1}} & \text{in } (0,T)\times\mathscr{I}, \\ u(0,\cdot) = u^0 & \text{in } \Omega, \\ \eta(0,\cdot) = \eta_1^0, \quad \partial_t\eta(0,\cdot) = \eta_2^0 & \text{in } \mathscr{I}, \end{cases} \tag{12}$$

with the condition (8). Our main result stated below is an observability inequality for (12):

Theorem 2. *Assume $T > 0$, $\omega \Subset \Omega$ and $\mathscr{J} \Subset \mathscr{I}$ are nonempty open sets. For any $[u^0,\eta_1^0,\eta_2^0] \in \mathscr{H}$, the solution of* (12) *satisfies*

$$\|u(T,\cdot)\|^2_{L^2(\Omega)} + \left\|\eta(T,\cdot)\right\|^2_{H^2(\mathscr{I})} + \left\|\partial_t\eta(T,\cdot)\right\|^2_{L^2(\mathscr{I})} \leqslant k_T^2 \left(\iint_{(0,T)\times\omega} |u|^2 \,\mathrm{d}x\,\mathrm{d}t + \iint_{(0,T)\times\mathscr{J}} \left|\partial_t\eta\right|^2 \,\mathrm{d}x_1\,\mathrm{d}t \right), \tag{13}$$

and we can choose k_T in the form

$$k_T = Ce^{C/T^2}, \tag{14}$$

with a constant $C > 0$.

The controllability of fluid-structure interaction systems has already been tackled in the case where the structure is a rigid body in [8, 9, 16, 24, 49]. Up to our knowledge, the above theorem is the first result of controllability for the system (5). Let us mention also [42] where the author obtains an observability inequality for the adjoint of a linearized simplified compressible fluid-structure model similar to our system.

Let us point out that due to the structural damping in the beam equation $(-\partial_t\partial^2_{x_1}\zeta)$ the corresponding beam equation becomes a parabolic equation (see, for instance, [14]). In a previous work [11], we have replaced the damped beam equation by a heat equation and we have shown the corresponding controllability result. The proof done here is inspired by our previous work, and in particular, in the proof of the observability, we first apply results on the heat equations to the fluid velocity by considering the pressure as a source term, (in the spirit of [18]). Then, we estimate the pressure by using that it satisfies a Laplace equation. Since the boundary conditions of this Laplace equation are difficult to handle, our estimates on the pressure depend on the boundary value of the pressure and more precisely on the high frequencies of the pressure on the boundary of the fluid domain. To conclude, we apply some energy inequalities combined with a high frequency argument in the horizontal direction to estimate these high frequencies. Using the microlocal analysis near boundaries and interfaces to derive Carleman estimates and to show the controllability of coupled parabolic systems is quite standard and one can quote for instance [6, 7, 10, 25, 28–30] and the recent books [26, 27] for elliptic counterparts).

One of the main differences with [11] is that we work here directly with the time variable whereas in the previous work we show a spectral inequality and then use an abstract method ([31, 32]) to deduce the corresponding observability inequality. Here we do not follow the same approach since it uses that the main operator of our system is self-adjoint, and here our main operator is not self-adjoint or even a perturbation of a self-adjoint operator as in the framework considered in [31]. A consequence of working directly with the time variable is that the separation between low and high frequencies is done through a pseudo-differential operator, which symbol depends on time, and in particular we need some standard commutator estimates from these operators in order to handle the high frequencies.

Remark 3. With respect to [11] or to the stabilization result [1], one should expect to obtain the controllability of (2) or of (5) without any control on the beam equation ($g \equiv 0$). However, with our present approach, it seems difficult to handle the elastic displacement without any observation on the beam equation. Even with the presence of two controls, a particular treatment of the coupling between the pressure and the elastic displacement in the proof of the observability is needed. Concerning the particular geometry, we are using it several times in order to simplify several proofs but the corresponding result in a general geometry should hold even if it is not a direct consequence of our work.

We deduce from Theorem 2 the local controllability of (2):

Theorem 4. *Assume $T > 0$ and that $\omega \Subset \Omega$ and $\mathscr{J} \Subset \mathscr{I}$ are nonempty open sets. There exists $R_0 > 0$ such that for any $\zeta_1^0 \in \mathscr{D}(A_1^{3/4})$, $\zeta_2^0 \in \mathscr{D}(A_1^{1/4})$, $w^0 \in H^1(\Omega_{\zeta_1^0})$ satisfying*

$$\operatorname{div} w^0 = 0 \text{ in } \Omega, \quad w^0 = 0 \text{ on } \Gamma_0, \quad w^0(x_1, 1 + \zeta_1^0(x_1)) = \zeta_2^0(x_1) e_2 \quad (x_1 \in \mathscr{I}), \tag{15}$$

and

$$\|\zeta_1^0\|_{H^3(\mathscr{I})} + \|\zeta_2^0\|_{H^1(\mathscr{I})} + \|w^0\|_{H^1\left(\Omega_{\zeta_1^0}\right)} \leqslant R_0, \tag{16}$$

there exists a control

$$(f, g) \in L^2(0, T; L^2(\omega)) \times L^2(0, T; L^2(\mathscr{J}))$$

such that the solution of (2) *satisfies*

$$\zeta(T, \cdot) = 0, \quad \partial_t \zeta(T, \cdot) = 0 \quad \text{in } \mathscr{I}, \quad w(T, \cdot) = 0 \quad \text{in } \Omega.$$

The proof of Theorem 4 is quite standard from Theorem 2: we need to estimate the coefficients of the change of variables and use a fixed point argument. Similar procedure is done to show the well-posedness or the stabilization of the system. We only sketch the proof of Theorem 4, the details can be found for instance in [1, 48].

The outline of the article is as follows: in the next section, we complete the functional setting needed in this article, introduce the Carleman weights and some classical results on pseudodifferential operators. Section 3 is devoted to Carleman estimates: a Carleman estimate for the heat equation, a Carleman estimate for the damped beam equation and a Carleman estimate for the pressure. Gathering them yields an estimate of the fluid velocity and pressure and of the elastic displacement by terms localized in ω or in $\mathscr{J}$ and by high frequencies of the pressure on the boundary. To get rid of these last terms, we show in Section 4 high frequency estimates using the Stokes system. This allows us to show the observability inequality in Section 5. We give the sketch of the proof of Theorem 4 in Section 6. Finally, in Appendix A, we recall some technical results concerning the Carleman estimates of Section 3.

Notation 5. In the whole paper, we use C as a generic positive constant that does not depend on the other terms of the inequality. The value of the constant C may change from one appearance to another. We also use the notation $X \lesssim Y$ if there exists a constant $C > 0$ such that we have the inequality $X \leqslant CY$.

2. Notation and preliminaries

2.1. *Functional setting*

We complete the notation introduced in the introduction: we consider the control operator for the beam equation:

$$B_{\mathscr{J}} g := P_{L^2_0(\mathscr{I})} \left(1_{\mathscr{J}} g\right),$$

where $P_{L^2_0(\mathscr{I})} : L^2(\mathscr{I}) \to L^2_0(\mathscr{I})$ is the orthogonal projection. With the above notation and (9), (10), the beam equation in (5) writes

$$\partial_t^2 \zeta + A_1 \zeta + A_2 \partial_t \zeta = P_{L^2_0(\mathscr{I})} \pi + B_{\mathscr{J}} g.$$

We also consider the orthogonal projection on the space $\mathscr{H}$ defined by (11):

$$\mathscr{P} : L^2(\Omega) \times \mathscr{D}(A_1^{1/2}) \times L^2_0(\mathscr{I}) \to \mathscr{H}.$$

We recall (see, for instance, [1, Proposition 3.1]) that the orthogonal of $\mathscr{H}$ in $L^2(\Omega) \times \mathscr{D}(A_1^{1/2}) \times L^2_0(\mathscr{I})$ is given by

$$\mathscr{H}^\perp = \left\{ (\nabla p, 0, P_{L^2_0(\mathscr{I})} p_{|\Gamma_1}) \; ; \; p \in H^1(\Omega) \right\}. \tag{17}$$

Then we define the space

$$\mathscr{V} := \left\{ (u, \eta_1, \eta_2) \in H^1(\Omega) \times \mathscr{D}(A_1^{3/4}) \times \mathscr{D}(A_1^{1/4}) \; ; \; u = \eta_2 e_2 \text{ on } \Gamma_1, \quad u = 0 \text{ on } \Gamma_0, \quad \operatorname{div} u = 0 \text{ in } \Omega \right\},$$

and the unbounded operator $\mathscr{A}$ associated with (5):

$$\mathscr{D}(\mathscr{A}) := \mathscr{V} \cap \left[H^2(\Omega) \times \mathscr{D}(A_1) \times \mathscr{D}(A_1^{1/2}) \right], \quad \mathscr{A} \begin{bmatrix} u \\ \eta_1 \\ \eta_2 \end{bmatrix} := \mathscr{P} \begin{bmatrix} \Delta u \\ \eta_2 \\ -A_1 \eta_1 - A_2 \eta_2 \end{bmatrix}.$$

It is shown (see, for instance, [1, Proposition 3.11]) that $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup on $\mathscr{H}$. We have in particular that if $F \in L^2(0,T;\mathscr{H})$, $\Phi^0 \in \mathscr{V}$, then there exists a unique solution

$$\Phi \in L^2(0,T;\mathscr{D}(\mathscr{A})) \cap C^0([0,T];\mathscr{V}) \cap H^1(0,T;\mathscr{H})$$

to

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \mathscr{A}\Phi + F \quad \text{in } (0,T), \quad \Phi(0) = \Phi^0 \tag{18}$$

and we have the estimate

$$\|\Phi\|_{L^2(0,T;H^2(\Omega) \times \mathscr{D}(A_1) \times \mathscr{D}(A_1^{1/2}))} + \|\Phi\|_{H^1(0,T;L^2(\Omega) \times \mathscr{D}(A_1^{1/2}) \times L^2(\mathscr{I}))} \lesssim \|F\|_{L^2(0,T;L^2(\Omega) \times \mathscr{D}(A_1^{1/2}) \times L^2(\mathscr{I}))} + \left\|\Phi^0\right\|_{\mathscr{V}}. \tag{19}$$

Finally, we consider the control operator:

$$\mathscr{B}\begin{bmatrix} f \\ g \end{bmatrix} := \mathscr{P}\begin{bmatrix} 1_\omega f \\ 0 \\ B_{\mathscr{I}} g \end{bmatrix}.$$

Using the above notation and (17), we can write (5) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} w \\ \zeta \\ \partial_t\zeta \end{bmatrix} = \mathscr{A}\begin{bmatrix} w \\ \zeta \\ \partial_t\zeta \end{bmatrix} + \mathscr{B}\begin{bmatrix} f \\ g \end{bmatrix} \quad \text{in } (0,T), \qquad \begin{bmatrix} w \\ \zeta \\ \partial_t\zeta \end{bmatrix}(0) = \begin{bmatrix} w^0 \\ \zeta_1^0 \\ \zeta_2^0 \end{bmatrix}. \tag{20}$$

We say that the above system is null-controllable in time $T > 0$ if for any $\left[w^0,\ \zeta_1^0,\ \zeta_2^0\right] \in \mathscr{H}$, there exists a control $\left[f,\ g\right] \in L^2(0,T; L^2(\omega) \times L^2(\mathscr{I}))$ such that the solution of the above system satisfies

$$\begin{bmatrix} w \\ \zeta \\ \partial_t\zeta \end{bmatrix}(T) = 0.$$

A classical result (see, for instance, [53, Theorem 11.2.1, p. 357]) states that the null-controllability is equivalent to the final-state observability of the adjoint system: there exists $k_T > 0$ such that for any $\begin{bmatrix} u^0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix} \in \mathscr{H}$, the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ \eta_1 \\ \eta_2 \end{bmatrix} = \mathscr{A}^*\begin{bmatrix} u \\ \eta_1 \\ \eta_2 \end{bmatrix} \quad \text{in } (0,T), \qquad \begin{bmatrix} u \\ \eta_1 \\ \eta_2 \end{bmatrix}(0) = \begin{bmatrix} u^0 \\ -\eta_1^0 \\ \eta_2^0 \end{bmatrix} \tag{21}$$

satisfies

$$\left\| \begin{bmatrix} u \\ \eta_1 \\ \eta_2 \end{bmatrix}(T) \right\|_{\mathscr{H}}^2 \leqslant k_T^2 \int_0^T \left\| \mathscr{B}^*\begin{bmatrix} u \\ \eta_1 \\ \eta_2 \end{bmatrix}(t) \right\|_{L^2(\omega)\times L^2(\mathscr{I})}^2 \mathrm{d}t. \tag{22}$$

One can show that

$$\mathscr{D}(\mathscr{A}^*) = \mathscr{D}(\mathscr{A}), \quad \mathscr{A}^*\begin{bmatrix} u \\ \eta_1 \\ \eta_2 \end{bmatrix} = \mathscr{P}\begin{bmatrix} \Delta u \\ -\eta_2 \\ A_1\eta_1 - A_2\eta_2 \end{bmatrix}$$

and

$$\mathscr{B}^*\begin{bmatrix} u \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} u_{|\omega} \\ \eta_{2|\mathscr{I}} \end{bmatrix}.$$

Setting $\eta = -\eta_1$ we see that (21) writes as (12) or in the following abstract form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ \eta \\ \partial_t\eta \end{bmatrix} = \mathscr{A}\begin{bmatrix} u \\ \eta \\ \partial_t\eta \end{bmatrix} \quad \text{in } (0,T), \qquad \begin{bmatrix} u \\ \eta \\ \partial_t\eta \end{bmatrix}(0) = \begin{bmatrix} u^0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix}. \tag{23}$$

The observability inequality (22) writes as (13).

2.2. *Weight functions for the Carleman estimates*

We consider nonempty open subsets $\mathscr{I}_0 \Subset \mathscr{I}$ and $\omega_0 \Subset \omega$ and (see, for instance, [19, Lemma 1.1], [53, Theorem 9.4.3]) two smooth functions $\psi_{\mathscr{I}}$ and ψ_Ω satisfying

$$\psi_{\mathscr{I}} > 0 \text{ in } \mathscr{I}, \quad \psi'_{\mathscr{I}}(x_1) = 0 \Rightarrow x_1 \in \mathscr{I}_0, \tag{24}$$

$$\psi_\Omega > 0 \text{ in } \Omega, \quad \psi_\Omega = 0 \text{ and } \partial_n\psi_\Omega = -1 \text{ on } \partial\Omega, \quad \nabla\psi_\Omega(x) = 0 \Rightarrow x \in \omega_0, \tag{25}$$

with

$$\mathscr{I}_0 \Subset \mathscr{I}, \quad \omega_0 \Subset \omega. \tag{26}$$

In fact, using our particular geometry, one can show directly the existence of such functions $\psi_{\mathscr{I}}$ and ψ_Ω. We set $\psi_1(x_1) := 2 + \sin(x_1)$ and we consider $\psi_2 \in C^\infty([0,1])$, $\psi_2(x_2) = x_2$ in a neighborhood of 0, $\psi_2(x_2) = 1 - x_2$ in a neighborhood of 1 and $\psi_2'(x_2) = 0 \Leftrightarrow x_2 = 1/2$. We also consider $\theta \in C^\infty(\mathbb{R})$ with compact support in $(0,1)$ and such that $\theta \equiv 1$ in a neighborhood of $1/2$. Then for $\varepsilon > 0$ small enough,

$$\widetilde{\psi_\Omega}(x_1, x_2) = \psi_2(x_2) + \varepsilon\theta(x_2)\psi_1(x_1)$$

satisfies $\widetilde{\psi_\Omega} > 0$ in Ω, $\widetilde{\psi_\Omega} = 0$ and $\partial_n \widetilde{\psi_\Omega} = -1$ on $\partial\Omega$ and it has only two critical points: $(\pi/2, 1/2)$ and $(-\pi/2, 1/2)$. By a change of variables on ψ_1 and on $\widetilde{\psi_\Omega}$ (see, for instance, [53, Proposition 14.3.1]), we obtain functions $\psi_{\mathscr{I}}$ and ψ_Ω satisfying (24) and (25).

We also denote by ℓ the function defined by

$$\ell(t) := t(T - t). \tag{27}$$

Let us consider $\Psi := \|\psi_{\mathscr{I}}\|_{L^\infty(\mathscr{I})} + \|\psi_\Omega\|_{L^\infty(\Omega)}$ and for $\lambda \geqslant \mu > 0$, let us define the following functions

$$\varphi(t, x_1, x_2) := \frac{1}{\ell(t)^2}\left(e^{\lambda\psi_\Omega(x_1,x_2)+\mu\psi_{\mathscr{I}}(x_1)+8\lambda\Psi} - e^{10\lambda\Psi}\right), \quad \xi(t, x_1, x_2) := \frac{1}{\ell(t)^2} e^{\lambda\psi_\Omega(x_1,x_2)+\mu\psi_{\mathscr{I}}(x_1)+8\lambda\Psi}, \tag{28}$$

$$\varphi_0(t, x_1) := \frac{1}{\ell(t)^2}\left(e^{\mu\psi_{\mathscr{I}}(x_1)+8\lambda\Psi} - e^{10\lambda\Psi}\right), \qquad \xi_0(t, x_1) := \frac{1}{\ell(t)^2} e^{\mu\psi_{\mathscr{I}}(x_1)+8\lambda\Psi}. \tag{29}$$

We also define for $\lambda \geqslant \mu > 0$ the function

$$\psi(x_1, x_2) := \frac{\mu}{\lambda}\psi_{\mathscr{I}}(x_1) + \psi_\Omega(x_1, x_2). \tag{30}$$

2.3. *Spatial truncation*

In order to use pseudodifferential operators in the x_1 variables, we consider that our functions are 2π-periodic functions defined in the domains

$$\Omega^\infty := \mathbb{R} \times (0,1), \quad \Gamma_0^\infty := \mathbb{R} \times \{0\}, \quad \Gamma_1^\infty := \mathbb{R} \times \{1\}.$$

In the adjoint system (12), we also replace the pressure p_0 that satisfies (8) by a pressure p satisfying another condition. More precisely, we consider ω_1 an open set such that $\omega_0 \Subset \omega_1 \Subset \omega$ and we define

$$c_p(t) := -\int_{\omega_1} p_0(t, x)\, \mathrm{d}x$$

and

$$p := p_0 + c_p. \tag{31}$$

Then the pressure p verifies the condition

$$\int_{\omega_1} p(t, x)\, \mathrm{d}x = 0 \quad \text{in } (0, T). \tag{32}$$

We consider $\chi^\infty \in C^\infty(\mathbb{R}; [0,1])$ with compact support and such that $\chi^\infty \equiv 1$ in $[0, 2\pi]$. We set

$$u^\infty := \chi^\infty u, \quad p^\infty := \chi^\infty p, \quad \eta^\infty := \chi^\infty \eta. \tag{33}$$

Then we deduce from (12) that

$$\begin{cases} \partial_t u^\infty - \Delta u^\infty + \nabla p^\infty = f^{(1)} & \text{in } (0, T) \times \Omega^\infty, \\ \operatorname{div} u^\infty = f^{(2)} & \text{in } (0, T) \times \Omega^\infty, \\ u^\infty = 0 & \text{on } (0, T) \times \Gamma_0^\infty, \\ u^\infty = \partial_t \eta^\infty e_2 & \text{on } (0, T) \times \Gamma_1^\infty, \end{cases} \tag{34}$$

and

$$\Delta p^\infty = f^{(3)} \quad \text{in } (0,T)\times\Omega^\infty, \tag{35}$$

where

$$f^{(1)} := -\left(\chi^\infty\right)'' u - 2\left(\chi^\infty\right)' \partial_{x_1} u + \left(\chi^\infty\right)' p e_1, \quad f^{(2)} := \left(\chi^\infty\right)' u_1, \quad f^{(3)} = (\chi^\infty)'' p + 2(\chi^\infty)' \partial_{x_1} p. \tag{36}$$

2.4. *Pseudodifferential operators*

We consider a parameter $\tau \geqslant 1$ and an order function

$$\Lambda_\tau(k) := \sqrt{\tau^2 + k^2} \quad (k \in \mathbb{R}), \tag{37}$$

where k corresponds to the Fourier variable associated with x_1. For $m \in \mathbb{R}$, we denote by $\mathbf{S}^m_\tau$ the space of complex smooth functions $a = a(x_1, k, \tau)$ defined on $\mathbb{R}\times\mathbb{R}\times[1,\infty)$ and such that for all $\alpha, \beta \in \mathbb{N}$ there exists $C_{\alpha,\beta} > 0$

$$\left|\partial^\alpha_{x_1}\partial^\beta_k a(x_1,k,\tau)\right| \leqslant C_{\alpha,\beta}\Lambda^{m-\beta}_\tau(k) \quad ((x_1,k,\tau)\in\mathbb{R}\times\mathbb{R}\times[1,\infty)). \tag{38}$$

For instance, we have $\Lambda^m_\tau \in \mathbf{S}^m_\tau$ and for any $C \in \mathbb{R}$, the function

$$(k,\tau) \mapsto \frac{\tau^2 - Ck^2}{\tau^2 + k^2}$$

is in $\mathbf{S}^0_\tau$. We also recall the following classical lemma (see, for instance, [26, Proposition 2.3] or [23, p. 73, Lemma 18.1.10] in the classical setting)

Lemma 6. *If $a \in \mathbf{S}^0_\tau$ and $\chi_0 \in C^\infty(\mathbb{R})$. Then $\chi_0(a) \in \mathbf{S}^0_\tau$.*

From $a \in \mathbf{S}^m_\tau$, we can define the following operator on the Schwartz space on $\mathbb{R}$:

$$\left[\mathrm{Op}(a)u\right](x_1) := \frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{ik(x_1-y_1)} a(x_1,k,\tau) u(y_1)\,\mathrm{d}y_1\mathrm{d}k.$$

We can also extend this operator to the Schwartz space on $[0,T]\times\mathbb{R}\times[0,1]$ by a similar formula:

$$\left[\mathrm{Op}(a)u\right](t,x_1,x_2) := \frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{ik(x_1-y_1)} a(x_1,k,\tau) u(t,y_1,x_2)\,\mathrm{d}y_1\mathrm{d}k.$$

From symbolic calculus, we have the following results (see, for instance, [26, pp. 27-28, Theorem 2.22 and Corollary 2.23])

Theorem 7. *Let $m, m' \in \mathbb{R}$ and let $a \in \mathbf{S}^m_\tau$, $b \in \mathbf{S}^{m'}_\tau$. Then there exist $c \in \mathbf{S}^{m+m'}_\tau$ and $d \in \mathbf{S}^{m+m'-1}_\tau$ such that*

$$\mathrm{Op}(a)\circ\mathrm{Op}(b) = \mathrm{Op}(c), \quad [\mathrm{Op}(a),\mathrm{Op}(b)] = \mathrm{Op}(d).$$

We can extend the operator associated with a symbol of order m to Sobolev spaces. For instance we have the following result (see [26, p. 29, Theorem 2.26])

Theorem 8. *Let $m, m' \in \mathbb{R}$, and let $a \in \mathbf{S}^m_\tau$. Then, $\mathrm{Op}(a) : H^{m+m'}(\mathbb{R}) \to H^{m'}(\mathbb{R})$ and if $m, m' \in \mathbb{N}$, we have*

$$\sum_{i+j\leqslant m'} \tau^{2i}\left\|\partial^j_{x_1}\mathrm{Op}(a)u\right\|^2_{L^2(\mathbb{R})} \lesssim \sum_{i+j\leqslant m+m'} \tau^{2i}\left\|\partial^j_{x_1} u\right\|^2_{L^2(\mathbb{R})}.$$

In what follows, we assume that the parameter τ is related to functions defined in Section 2.2 through the formula

$$\tau := \tau(t) = \frac{s\lambda e^{8\lambda\Psi}}{\ell^2(t)}. \tag{39}$$

In particular, τ is a function of time and there exist $s_0 > 0$ and $\lambda_0 > 0$ such that if $s \geqslant s_0 T^4$ and $\lambda \geqslant \lambda_0$, then

$$\tau \geqslant \frac{\tau}{\lambda} \geqslant 1. \tag{40}$$

Remark 9. Due to (39), the symbols in $\mathbf{S}_\tau^m$ depends on the time variable through the parameter τ. The continuity estimates of Theorem 8 are uniform with respect to τ, and thus with respect to the time variable if it only appears in the parameter τ. In what follows, some symbols may depend on time, but not as a function of τ, this occurs for instance when considering $\partial_t \tau$. In that case, we always decompose such symbols in terms of the form $b(t)a(x_1,k,\tau)$ where b is a bounded function of time, and $a \in \mathbf{S}_\tau^m$.

An important example of symbol used in what follows is a function of the form

$$\chi(\tau,k) := \chi_0\left(\frac{\tau^2 - Ck^2}{\tau^2 + k^2}\right),$$

where C is a constant and $\chi_0 \in C^\infty(\mathbb{R})$. From Lemma 6, we have that $\chi \in \mathbf{S}_\tau^0$ and one can check that

$$[\partial_{x_1}, \operatorname{Op}(\chi)] = [\partial_{x_2}, \operatorname{Op}(\chi)] = 0.$$

Moreover, we have the following result on the time derivative of χ:

Lemma 10. *Let χ be defined as above. Then*

$$\partial_t \chi \in \frac{\ell'}{(\lambda s)^{1/2}} \tau^{\frac{5}{2}} \mathbf{S}_\tau^{-2}.$$

Proof. By standard computations and (39),

$$\partial_t \chi(\tau,k) = \chi_0'\left(\frac{\tau^2 - Ck^2}{\tau^2 + k^2}\right) \frac{2(C+1)k^2 \tau \partial_t \tau}{\left(\tau^2 + k^2\right)^2}, \quad \partial_t \tau = -2\ell' \frac{s\lambda e^{8\lambda\Psi}}{\ell^3}. \tag{41}$$

We have

$$\frac{s\lambda e^{8\lambda\Psi}}{\ell^3} \leqslant \frac{\tau^{3/2}}{(s\lambda)^{1/2}}$$

so that using Lemma 6 and Theorem 7, we deduce the result. □

3. Carleman estimates

In this section, we show a Carleman estimate for the solutions of (6). Using the weights introduced in Section 2.2, we define the following weighted integrals:

$$\begin{aligned}
I_1(s,\lambda,\eta) := \lambda \iint_{(0,T)\times\mathscr{I}} & e^{2s\varphi_0}\left(s^{10}\xi_0^{10}|\eta|^2 + s^8\xi_0^8|\partial_{x_1}\eta|^2 + s^6\xi_0^6\left(|\partial_{x_1}^2\eta|^2 + |\partial_t\eta|^2\right)\right) \mathrm{d}t\,\mathrm{d}x_1 \\
& + \lambda \iint_{(0,T)\times\mathscr{I}} e^{2s\varphi_0} s^4\xi_0^4\left(|\partial_{x_1}^3\eta|^2 + |\partial_{x_1}\partial_t\eta|^2\right) \mathrm{d}t\,\mathrm{d}x_1 \\
& + \lambda \iint_{(0,T)\times\mathscr{I}} e^{2s\varphi_0} s^2\xi_0^2\left(|\partial_{x_1}^4\eta|^2 + |\partial_t\partial_{x_1}^2\eta|^2 + |\partial_t^2\eta|^2\right) \mathrm{d}t\,\mathrm{d}x_1 \\
& + \lambda \iint_{(0,T)\times\mathscr{I}} e^{2s\varphi_0}\left(\left|\partial_{x_1}^5\eta\right|^2 + |\partial_t^2\partial_{x_1}\eta|^2 + |\partial_t\partial_{x_1}^3\eta|^2\right) \mathrm{d}t\,\mathrm{d}x_1,
\end{aligned} \tag{42}$$

$$\begin{aligned}
I_2(s,\lambda,u) := \iint_{(0,T)\times\Omega} \lambda^2\left(|\nabla^2 u|^2 + (\partial_t u)^2\right) e^{2s\varphi}\,\mathrm{d}t\,\mathrm{d}x + \iint_{(0,T)\times\Omega} s^2\lambda^4\xi^2 e^{2s\varphi}|\nabla u|^2\,\mathrm{d}t\,\mathrm{d}x \\
+ \iint_{(0,T)\times\Omega} s^4\lambda^6\xi^4 e^{2s\varphi}|u|^2\,\mathrm{d}t\,\mathrm{d}x,
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
I_3(s,\lambda,p^\infty) := \iint_{(0,T)\times\Omega^\infty} s^3\lambda^4\xi^3 e^{2s\varphi}|p^\infty|^2\,\mathrm{d}t\,\mathrm{d}x + \iint_{(0,T)\times\Omega^\infty} s\lambda^2\xi e^{2s\varphi}|\nabla p^\infty|^2\,\mathrm{d}t\,\mathrm{d}x \\
+ \iint_{(0,T)\times\partial\Omega^\infty} s^3\lambda^3\xi_0^3 e^{2s\varphi_0}\left|p^\infty\right|^2\,\mathrm{d}t\,\mathrm{d}x_1 + \iint_{(0,T)\times\partial\Omega^\infty} s\lambda\xi_0 e^{2s\varphi_0}\left|\partial_{x_1} p^\infty\right|^2\,\mathrm{d}t\,\mathrm{d}x_1.
\end{aligned} \tag{44}$$

Remark 11. The above quantities depend also on μ but since we will fix the value of $\mu = \mu_0$ after Section 3.1, we suppress reference to it in the notation.

For $\mu_0 > 1$, we set

$$K_+ := e^{\mu_0 \max \psi_{\mathscr{I}}}, \quad K_- := e^{\mu_0 \min \psi_{\mathscr{I}}}. \tag{45}$$

In particular, with the definition (29) of ξ_0 and the definition (39) of τ, we have

$$K_- \tau \leqslant s\lambda\xi_0 \leqslant K_+ \tau. \tag{46}$$

Using Lemma 6, we can define the following symbol of order 0:

$$\chi(\tau, k) := \chi_0 \left(\frac{\tau^2 - \frac{4K_+}{K_-^3} k^2}{\tau^2 + k^2} \right) \in \mathbf{S}^0_\tau, \quad \text{with} \quad \chi_0 \in C^\infty(\mathbb{R}; [0,1]) \text{ such that } \chi_0 = \begin{cases} 1 & \text{in } [3/4, \infty) \\ 0 & \text{in } (-\infty, 1/2] \end{cases}. \tag{47}$$

The main result of this section is stated below:

Proposition 12. *Assume $\mathscr{J}_0 \Subset \mathscr{J}_1 \Subset \mathscr{J}$ and $\omega_0 \Subset \omega_1 \Subset \omega$. There exist $\mu_0 > 0$, $\lambda_0 > 0$ and s_0 such that for $\mu = \mu_0$, $\lambda \geqslant \lambda_0$ and $s \geqslant s_0(T^2 + T^4)$, any smooth solution $[u, p_0, \eta]$ of* (6) *satisfies*

$$\begin{aligned} I_1(s,\lambda,\eta) + I_2(s,\lambda,u) + I_3(s,\lambda,p^\infty) & \\ \lesssim \lambda \iint_{(0,T)\times \mathscr{J}_1} & e^{2s\varphi_0} \left(s^{10}\xi_0^{10} |\eta|^2 + s^2\xi_0^2 |\partial_t^2 \eta|^2 \right) \,\mathrm{d}t\,\mathrm{d}x_1 \\ & + \iint_{(0,T)\times\omega_1} e^{2s\varphi} \left(s^4\lambda^6\xi^4 |u|^2 \,\mathrm{d}t\,\mathrm{d}x + s^3\lambda^4\xi^3 |p^\infty|^2 \right) \,\mathrm{d}t\,\mathrm{d}x \\ & + \iint_{(0,T)\times\partial\Omega^\infty} \tau \left| \partial_{x_1} \operatorname{Op}(1-\chi) \left[e^{s\varphi_0} p^\infty \right] \right|^2 \,\mathrm{d}t\,\mathrm{d}x_1, \end{aligned} \tag{48}$$

where p^∞ is given by (31) *and* (33).

In order to prove Proposition 12, we first combine a Carleman estimate for the fluid velocity and a Carleman estimate for the elastic deformation (see Section 3.1 and Section 3.2). Both estimates contain pressure terms in the right-hand side and to estimates them, we perform a Carleman estimate for the pressure in Section 3.3. In this last estimate, we need to put in the right-hand side the trace of the pressure at the boundary, microlocalized in the high frequency regime.

3.1. *A Carleman estimate for the elastic deformation*

In this section, we obtain a Carleman estimate for the elastic deformation, mainly based on the results in [41]. This is the only part of the work where $\mu \geqslant \mu_0$, after this, we will fix $\mu = \mu_0$ in the weights $\varphi, \xi, \varphi_0, \xi_0$. To avoid introducing many notations, we keep the same notation μ_0, s_0, λ_0 during the proofs, but their values may change from one appearance to another.

First, we deduce from the definitions (29), the existence of μ_0 such that for $\lambda \geqslant \mu \geqslant \mu_0$, $t \in [0,T]$ and $x_1 \in \mathscr{I}$, and $\alpha \geqslant 0$,

$$\begin{aligned} &\left|\partial_{x_1}^\alpha \varphi_0\right| + \left|\partial_{x_1}^\alpha \xi_0\right| \lesssim \mu^\alpha \xi_0 \quad (k \geqslant 1), \\ &\qquad \left|\partial_t \partial_{x_1}^\alpha \varphi_0\right| + \left|\partial_t \partial_{x_1}^\alpha \xi_0\right| \lesssim T\mu^\alpha \xi_0^{3/2}, \quad \left|\partial_t^2 \partial_{x_1}^\alpha \varphi_0\right| + \left|\partial_t^2 \partial_{x_1}^\alpha \xi_0\right| \lesssim T^2 \mu^\alpha \xi_0^2. \end{aligned} \tag{49}$$

Moreover, there exists μ_0 such that for $\lambda \geqslant \mu \geqslant \mu_0$, for $t \in [0,T]$ and for $x_1 \in \mathscr{I} \setminus \mathscr{J}_0$,

$$\mu\xi_0 \lesssim \left|\partial_{x_1}\varphi_0\right|, \quad \mu^2\xi_0 \lesssim \partial_{x_1}^2 \varphi_0. \tag{50}$$

With these properties, we can obtain the following result which is proven in [41]. More precisely, the Carleman estimate below is obtained in [41] with slightly different weights but the author only uses the above properties in his proof. For sake of completeness, we give in Appendix A.1 a sketch of the corresponding proof.

Theorem 13. *Assume $r \in \mathbb{R}$ and $\mathscr{J}_0 \Subset \mathscr{J}_1 \Subset \mathscr{J}$. There exist constants $s_0 > 0$ and $\mu_0 > 0$ such that for any smooth function η, for any $s \geqslant s_0(T^2+T^4)$, and for any $\lambda \geqslant \mu \geqslant \mu_0$, we have*

$$\begin{aligned}
&\iint_{(0,T)\times\mathscr{J}} e^{2s\varphi_0}\left(s^{2r+7}\mu^{2r+8}\xi_0^{2r+7}|\eta|^2 + s^{2r+5}\mu^{2r+6}\xi_0^{2r+5}|\partial_{x_1}\eta|^2\right)\,\mathrm{d}t\,\mathrm{d}x_1\\
&\qquad + \iint_{(0,T)\times\mathscr{J}} e^{2s\varphi_0}s^{2r+3}\mu^{2r+4}\xi_0^{2r+3}\left(|\partial_{x_1}^2\eta|^2+|\partial_t\eta|^2\right)\,\mathrm{d}t\,\mathrm{d}x_1\\
&\qquad + \iint_{(0,T)\times\mathscr{J}} e^{2s\varphi_0}s^{2r+1}\mu^{2r+2}\xi_0^{2r+1}\left(|\partial_{x_1}^3\eta|^2+|\partial_t\partial_{x_1}\eta|^2\right)\,\mathrm{d}t\,\mathrm{d}x_1\\
&\qquad + \iint_{(0,T)\times\mathscr{J}} e^{2s\varphi_0}s^{2r-1}\mu^{2r}\xi_0^{2r-1}\left(\left|\partial_{x_1}^4\eta\right|^2+|\partial_t^2\eta|^2+|\partial_t\partial_{x_1}^2\eta|^2\right)\,\mathrm{d}t\,\mathrm{d}x_1\\
&\lesssim \iint_{(0,T)\times\mathscr{J}} e^{2s\varphi_0}s^{2r}\mu^{2r}\xi_0^{2r}\left|(\partial_t^2+\partial_{x_1}^4-\partial_{x_1}^2-\partial_t\partial_{x_1}^2)\eta\right|^2\,\mathrm{d}t\,\mathrm{d}x_1\\
&\qquad + \iint_{(0,T)\times\mathscr{J}_1} s^{2r+7}\mu^{2r+8}\xi_0^{2r+7}e^{2s\varphi_0}|\eta|^2\,\mathrm{d}t\,\mathrm{d}x_1. \qquad (51)
\end{aligned}$$

As a corollary, we have the following result

Corollary 14. *Assume $\mathscr{J}_0 \Subset \mathscr{J}_1 \Subset \mathscr{J}$. There exist constants $s_0 > 0$ and $\mu_0 > 0$ such that for any smooth function η, for any $s \geqslant s_0(T^2+T^4)$, and for any $\lambda \geqslant \mu \geqslant \mu_0$, we have*

$$\begin{aligned}
&\iint_{(0,T)\times\mathscr{J}} e^{2s\varphi_0}\left(s^{10}\mu^{11}\xi_0^{10}|\eta|^2 + s^8\mu^9\xi_0^8|\partial_{x_1}\eta|^2 + s^6\mu^7\xi_0^6\left(|\partial_{x_1}^2\eta|^2+|\partial_t\eta|^2\right)\right)\,\mathrm{d}t\,\mathrm{d}x_1\\
&+ \iint_{(0,T)\times\mathscr{J}} e^{2s\varphi_0}s^4\mu^5\xi_0^4\left(|\partial_{x_1}^3\eta|^2+|\partial_{x_1}\partial_t\eta|^2\right)\mathrm{d}t\,\mathrm{d}x_1\\
&+ \iint_{(0,T)\times\mathscr{J}} e^{2s\varphi_0}\left(s^2\mu^3\xi_0^2\left(|\partial_{x_1}^4\eta|^2+|\partial_t\partial_{x_1}^2\eta|^2+|\partial_t^2\eta|^2\right)+\mu\left(\left|\partial_{x_1}^5\eta\right|^2+|\partial_t^2\partial_{x_1}\eta|^2+|\partial_t\partial_{x_1}^3\eta|^2\right)\right)\mathrm{d}t\,\mathrm{d}x_1\\
&\qquad\lesssim \iint_{(0,T)\times\mathscr{J}} e^{2s\varphi_0}s\mu\xi_0|\partial_{x_1}(\partial_t^2-\partial_{x_1}^2+\partial_{x_1}^4-\partial_t\partial_{x_1}^2)\eta|^2\,\mathrm{d}t\,\mathrm{d}x_1\\
&\qquad\qquad + \iint_{(0,T)\times\mathscr{J}_1} e^{2s\varphi_0}\left(s^{10}\mu^{11}\xi_0^{10}|\eta|^2 + s^2\mu^3\xi_0^2|\partial_t^2\eta|^2\right)\,\mathrm{d}t\,\mathrm{d}x_1. \qquad (52)
\end{aligned}$$

Proof. We first apply Theorem 13 to $\partial_{x_1}\eta$ with $r = 1/2$ and with an open set $\mathscr{J}_2$ such that $\mathscr{J}_0 \Subset \mathscr{J}_2 \Subset \mathscr{J}_1$:

$$\begin{aligned}
&\iint_{(0,T)\times\mathscr{J}} e^{2s\varphi_0}\left(s^8\mu^9\xi_0^8|\partial_{x_1}\eta|^2 + s^6\mu^7\xi_0^6|\partial_{x_1}^2\eta|^2 + s^4\mu^5\xi_0^4\left(|\partial_{x_1}^3\eta|^2+|\partial_t\partial_{x_1}\eta|^2\right)\right)\,\mathrm{d}t\,\mathrm{d}x_1\\
&\quad + \iint_{(0,T)\times\mathscr{J}} e^{2s\varphi_0}\left(s^2\mu^3\xi_0^2\left(|\partial_{x_1}^4\eta|^2+|\partial_t\partial_{x_1}^2\eta|^2\right)+\mu\left(\left|\partial_{x_1}^5\eta\right|^2+|\partial_t^2\partial_{x_1}\eta|^2+|\partial_t\partial_{x_1}^3\eta|^2\right)\right)\,\mathrm{d}t\,\mathrm{d}x_1\\
&\qquad\lesssim \iint_{(0,T)\times\mathscr{J}} e^{2s\varphi_0}s\mu\xi_0\left|\partial_{x_1}(\partial_t^2+\partial_{x_1}^4-\partial_{x_1}^2-\partial_t\partial_{x_1}^2)\eta\right|^2\,\mathrm{d}t\,\mathrm{d}x_1\\
&\qquad\qquad + \iint_{(0,T)\times\mathscr{J}_2} s^8\mu^9\xi_0^8e^{2s\varphi_0}|\partial_{x_1}\eta|^2\,\mathrm{d}t\,\mathrm{d}x_1. \qquad (53)
\end{aligned}$$

Then, we use a Carleman estimate for the gradient operator (see, for instance, [15, Lemma 3]): there exists $s_0 > 0$ such that for any smooth function ζ, and for any $s \geqslant s_0T^4$,

$$\begin{aligned}
&\iint_{(0,T)\times\mathscr{J}} s^{r+2}\mu^{r+3}\xi_0^{r+2}e^{2s\varphi_0}\zeta^2\,\mathrm{d}t\,\mathrm{d}x_1\\
&\qquad\lesssim \iint_{(0,T)\times\mathscr{J}_2} s^{r+2}\mu^{r+3}\xi_0^{r+2}e^{2s\varphi_0}\zeta^2\,\mathrm{d}t\,\mathrm{d}x_1 + \iint_{(0,T)\times\mathscr{J}} s^r\mu^{r+1}\xi_0^re^{2s\varphi_0}(\partial_{x_1}\zeta)^2\,\mathrm{d}t\,\mathrm{d}x_1.
\end{aligned}$$

This Carleman estimate, combined with (53), yields that for $s \geqslant s_0(T^2+T^4)$,

$$\begin{aligned}
&\iint_{(0,T)\times\mathscr{I}} e^{2s\varphi_0}\left(s^{10}\mu^{11}\xi_0^{10}|\eta|^2+s^8\mu^9\xi_0^8|\partial_{x_1}\eta|^2+s^6\mu^7\xi_0^6\left(|\partial_{x_1}^2\eta|^2+|\partial_t\eta|^2\right)\right)\,\mathrm{d}t\,\mathrm{d}x_1\\
&+\iint_{(0,T)\times\mathscr{I}} e^{2s\varphi_0}\left(s^4\mu^5\xi_0^4\left(|\partial_{x_1}^3\eta|^2+|\partial_{x_1}\partial_t\eta|^2\right)\right)\,\mathrm{d}t\,\mathrm{d}x_1\\
&+\iint_{(0,T)\times\mathscr{I}} e^{2s\varphi_0}\left(s^2\mu^3\xi_0^2\left(|\partial_{x_1}^4\eta|^2+|\partial_t\partial_{x_1}^2\eta|^2+|\partial_t^2\eta|^2\right)+\mu\left(\left|\partial_{x_1}^5\eta\right|^2+|\partial_t^2\partial_{x_1}\eta|^2+|\partial_t\partial_{x_1}^3\eta|^2\right)\right)\mathrm{d}t\,\mathrm{d}x_1\\
&\lesssim \iint_{(0,T)\times\mathscr{I}} e^{2s\varphi_0}s\mu\xi_0|\partial_{x_1}(\partial_t^2-\partial_{x_1}^2+\partial_{x_1}^4-\partial_t\partial_{x_1}^2)\eta|^2\,\mathrm{d}t\,\mathrm{d}x_1\\
&\qquad+\iint_{(0,T)\times\mathscr{I}_2} e^{2s\varphi_0}\left(s^{10}\mu^{11}\xi_0^{10}|\eta|^2+s^8\mu^9\xi_0^8|\partial_{x_1}\eta|^2+s^6\mu^7\xi_0^6|\partial_t\eta|^2+s^2\mu^3\xi_0^2|\partial_t^2\eta|^2\right)\mathrm{d}t\,\mathrm{d}x_1. \quad (54)
\end{aligned}$$

Then proceeding as in [41], one can absorb the local terms in $\partial_{x_1}\eta$ and in $\partial_t\eta$ by using a cut-off function and integrations by parts and we deduce the result. □

3.2. *A Carleman estimate for the velocity*

From now on, we take $\mu=\mu_0$ as in Theorem 13 and take $\lambda \geqslant \mu_0$. The constants that follow in the article may depend on μ_0. We have the following standard Carleman estimate for the heat equation (see, for instance, [17] or [19]). For sake of completeness, we also give a sketch of the proof of the following result in Appendix A.2.

Theorem 15. *Assume $\mu=\mu_0$ and $\omega_0 \Subset \omega_1 \Subset \omega$. There exist $s_0>0$ and $\lambda_0>0$ such that for any $\lambda \geqslant \lambda_0$ $s \geqslant s_0(T^2+T^4)$, and for any smooth function u such that*

$$u=0 \quad \text{on } (0,T)\times\Gamma_0, \quad u_1=0 \quad \text{on } (0,T)\times\Gamma_1, \quad \frac{\partial u_2}{\partial n}=0 \quad \text{on } (0,T)\times\Gamma_1,$$

we have

$$\begin{aligned}
&\iint_{(0,T)\times\Omega}\left(|\nabla^2 u|^2+(\partial_t u)^2\right)e^{2s\varphi}\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad+\iint_{(0,T)\times\Omega} s^2\lambda^2\xi^2 e^{2s\varphi}|\nabla u|^2\,\mathrm{d}t\,\mathrm{d}x+\iint_{(0,T)\times\Omega} s^4\lambda^4\xi^4 e^{2s\varphi}|u|^2\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad\qquad\lesssim \iint_{(0,T)\times\Omega} s\xi e^{2s\varphi}|(\partial_t-\Delta)u|^2\,\mathrm{d}t\,\mathrm{d}x+\iint_{(0,T)\times\omega_1} s^4\lambda^4\xi^4 e^{2s\varphi}|u|^2\,\mathrm{d}t\,\mathrm{d}x. \quad (55)
\end{aligned}$$

3.3. *A Carleman estimate for the pressure*

In order to obtain a Carleman estimate for the pressure, we use that from (12), the pressure p_0 is harmonic in Ω. We recall that p^∞ is defined from p_0 by (31) and (33). In particular, it satisfies the Laplace equation(34) but without any explicit boundary condition. Thus in our Carleman estimate, we keep in the right-hand side a boundary term microlocalized in a high frequency regime (represented by $\operatorname{supp}(1-\chi)$, with χ defined by (47)). We recall that τ is defined in (39).

Proposition 16. *Assume $\mu = \mu_0$ and $\omega_0 \Subset \omega_1 \Subset \omega$. There exist $s_0 > 0$ $\lambda_0 > 0$ and $C > 0$ such that for any $s \geqslant s_0(T^2 + T^4)$, $\lambda \geqslant \lambda_0$ and for any smooth function p, the function $p^\infty := p\chi^\infty$ satisfies*

$$\begin{aligned}\iint_{(0,T)\times\Omega^\infty} s^3\lambda^4\xi^3 e^{2s\varphi}|p^\infty|^2 \,\mathrm{d}t\,\mathrm{d}x + \iint_{(0,T)\times\Omega^\infty} s\lambda^2\xi e^{2s\varphi}|\nabla p^\infty|^2 \,\mathrm{d}t\,\mathrm{d}x \\ + \iint_{(0,T)\times\partial\Omega^\infty} s^3\lambda^3\xi_0^3 e^{2s\varphi_0}\left|p^\infty\right|^2 \,\mathrm{d}t\,\mathrm{d}x_1 + \iint_{(0,T)\times\partial\Omega^\infty} s\lambda\xi_0 e^{2s\varphi_0}\left|\partial_{x_1} p^\infty\right|^2 \,\mathrm{d}t\,\mathrm{d}x_1 \\ \leqslant C\left(\iint_{(0,T)\times\Omega^\infty} e^{2s\varphi}|\Delta p^\infty|^2 \,\mathrm{d}t\,\mathrm{d}x + \iint_{(0,T)\times\omega_1} s^3\lambda^4\xi^3 e^{2s\varphi}|p^\infty|^2 \,\mathrm{d}t\,\mathrm{d}x \right.\\ \left. + \iint_{(0,T)\times\partial\Omega^\infty} \tau\left|\partial_{x_1} \operatorname{Op}(1-\chi)\left[e^{s\varphi_0}p^\infty\right]\right|^2 \,\mathrm{d}t\,\mathrm{d}x_1\right). \quad (56)\end{aligned}$$

Proof. We start by a standard Carleman estimate for p^∞ in Ω^∞, using that χ^∞ has a compact support. First, we set

$$q = e^{s\varphi}p^\infty$$

and we perform standard computations (see, for instance, [17,32], [26, pp. 106–117]), to obtain the existence of positive constants c, C, s_0 such that for $s \geqslant s_0(T^2 + T^4)$,

$$\begin{aligned} c\iint_{(0,T)\times\Omega^\infty}\left(s^3\lambda^4\xi^3 q^2 + s\lambda^2\xi\left|\nabla q\right|^2 + \frac{1}{s\xi}\left|\Delta q\right|^2\right)\mathrm{d}t\,\mathrm{d}x \\ + \iint_{(0,T)\times\partial\Omega^\infty}\left(-s^3\lambda^3\xi^3\left|\nabla\psi\right|^2\frac{\partial\psi}{\partial n}q^2 - 2s\lambda^2\xi\left|\nabla\psi\right|^2\frac{\partial q}{\partial n}q - 2s\lambda\xi\nabla\psi\cdot\nabla q\frac{\partial q}{\partial n} + s\lambda\xi\frac{\partial\psi}{\partial n}\left|\nabla q\right|^2\right)\mathrm{d}t\,\mathrm{d}x_1 \\ \leqslant C\left(\iint_{(0,T)\times\Omega^\infty}(-\Delta p)^2 e^{2s\varphi}\,\mathrm{d}t\,\mathrm{d}x + \iint_{(0,T)\times\omega_1} s^3\lambda^4\xi^3 q^2\,\mathrm{d}x\right). \quad (57)\end{aligned}$$

From (30) and (25), we have

$$\frac{\partial\psi}{\partial x_1} = \frac{\mu_0}{\lambda}\psi'_{\mathscr{I}}, \quad \frac{\partial\psi}{\partial n} = -1 \quad \text{on } \partial\Omega^\infty.$$

Thus there exist $\lambda_0 > 0$ and $s_0 > 0$ such that for $\lambda \geqslant \lambda_0$, $s \geqslant s_0(T^2 + T^4)$, we have on $(0,T)\times\partial\Omega$,

$$\begin{aligned} -s^3\lambda^3\xi^3\left|\nabla\psi\right|^2\frac{\partial\psi}{\partial n}q^2 - 2s\lambda^2\xi\left|\nabla\psi\right|^2\frac{\partial q}{\partial n}q - 2s\lambda\xi\nabla\psi\cdot\nabla q\frac{\partial q}{\partial n} + s\lambda\xi\frac{\partial\psi}{\partial n}\left|\nabla q\right|^2 \\ = s^3\lambda^3\xi^3\left|\nabla\psi\right|^2 q^2 + s\lambda\xi\left(\frac{\partial q}{\partial n}\right)^2 - s\lambda\xi\left(\frac{\partial q}{\partial x_1}\right)^2 - 2s\lambda^2\xi\left|\nabla\psi\right|^2\frac{\partial q}{\partial n}q - 2s\mu_0\xi\psi'_{\mathscr{I}}\frac{\partial q}{\partial x_1}\frac{\partial q}{\partial n} \\ \geqslant \frac{1}{2}s^3\lambda^3\xi_0^3 q^2 + \frac{1}{2}s\lambda\xi_0\left(\frac{\partial q}{\partial n}\right)^2 - 2s\lambda\xi_0\left(\frac{\partial q}{\partial x_1}\right)^2. \quad (58)\end{aligned}$$

Let us denote by $\widehat{q}$ the Fourier transform of q in the x_1 direction. Then by using the Plancherel theorem, there exists $c > 0$ such that

$$\iint_{(0,T)\times\partial\Omega^\infty}\left(\frac{1}{2}s^3\lambda^3\xi_0^3 q^2 - 2s\lambda\xi_0\left(\frac{\partial q}{\partial x_1}\right)^2\right)\mathrm{d}x_1\,\mathrm{d}t \geqslant c\iint_{(0,T)\times\partial\Omega^\infty}\tau\left(K_-^3\tau^2 - 4K_+k^2\right)\left|\widehat{q}\right|^2\,dk\,\mathrm{d}t$$

and thus, there exist two constant $c, C > 0$ such that

$$\begin{aligned}\iint_{(0,T)\times\partial\Omega^\infty}\left(\frac{1}{2}s^3\lambda^3\xi_0^3 q^2 - 2s\lambda\xi_0\left(\frac{\partial q}{\partial x_1}\right)^2\right)\mathrm{d}x_1\,\mathrm{d}t + C\iint_{(0,T)\times\partial\Omega^\infty}(1-\chi)^2\tau k^2\left|\widehat{q}\right|^2\,dk\,\mathrm{d}t \\ \geqslant c\iint_{(0,T)\times\partial\Omega^\infty}\tau\left(\tau^2 + k^2\right)\left|\widehat{q}\right|^2\,dk\,\mathrm{d}t. \quad (59)\end{aligned}$$

Using again the Plancherel theorem, and combining the above relation with (58) and with (57), we deduce the result. ☐

3.4. *Gathering the Carleman estimates*

We are now in a position to prove Proposition 12

Proof. Assume that (u, p_0, η) is the solution of (12). We consider p defined from p_0 by (31) and p^∞ defined from p by (33). We apply Corollary 14, Theorem 15 and Proposition 16 and using that $\nabla p_0 = \nabla p$, we obtain the following relations for I_1, I_2 and I_3 (defined by (42)–(44)):

$$I_1(s,\lambda,\eta) \lesssim \iint_{(0,T)\times\mathscr{I}} e^{2s\varphi_0} s\lambda\xi_0 |\partial_{x_1} p|^2 \,\mathrm{d}t\,\mathrm{d}x_1 + \lambda \iint_{(0,T)\times\mathscr{I}_1} e^{2s\varphi_0} \left(s^{10}\xi_0^{10}|\eta|^2 + s^2\xi_0^2|\partial_t^2\eta|^2\right) \mathrm{d}t\,\mathrm{d}x_1, \tag{60}$$

$$I_2(s,\lambda,u) \lesssim \iint_{(0,T)\times\Omega} s\lambda^2\xi e^{2s\varphi} \left|\nabla p\right|^2 \,\mathrm{d}t\,\mathrm{d}x + \iint_{(0,T)\times\omega_1} s^4\lambda^6\xi^4 e^{2s\varphi}|u|^2 \,\mathrm{d}t\,\mathrm{d}x, \tag{61}$$

and

$$I_3(s,\lambda,p^\infty) \lesssim \iint_{(0,T)\times\Omega^\infty} e^{2s\varphi}|f^{(3)}|^2 \,\mathrm{d}t\,\mathrm{d}x + \iint_{(0,T)\times\omega_1} s^3\lambda^4\xi^3 e^{2s\varphi}|p^\infty|^2 \,\mathrm{d}t\,\mathrm{d}x \\ + \iint_{(0,T)\times\partial\Omega^\infty} \tau \left|\partial_{x_1} \mathrm{Op}(1-\chi)\left[e^{s\varphi_0} p^\infty\right]\right|^2 \mathrm{d}t\,\mathrm{d}x_1. \tag{62}$$

Then, we can estimate $f^{(3)}$ by using (36) and we deduce that

$$\iint_{(0,T)\times\Omega^\infty} e^{2s\varphi}|f^{(3)}|^2 \,\mathrm{d}t\,\mathrm{d}x \leqslant C\lambda^{-2} I_3(s,\lambda,p^\infty).$$

Using that $\chi^\infty \equiv 1$ in $(0,2\pi)\times(0,1)$, and taking $\lambda \geqslant \lambda_0$ with $\lambda_0 > 0$ sufficiently large, we can combine the three Carleman estimates (60)–(62) and the above relation to obtain (48). □

4. High frequency estimates

In this section, we eliminate the last term in (48) by showing high frequency estimates for u and p. The method used here is the same as the one used in [11]. We conjugate the system (34) with $e^{s\varphi_0}$, using that the spatial derivatives of φ_0 involve only powers of μ_0 that is fixed, instead of powers of λ for the spatial derivatives of φ. This allows us to perform energy estimates of the Stokes system, by considering all the terms coming from the conjugaison as lower order terms in the high frequency regime.

4.1. *Estimates from the Stokes system*

We recall that u^∞, p^∞ and η^∞ are defined in (33) by using the function χ^∞. We introduce

$$\widetilde{\chi}^\infty \in C_0^\infty(\mathbb{R},[0,1]), \quad \widetilde{\chi}^\infty \equiv 1 \quad \text{in } \operatorname{supp}\chi^\infty. \tag{63}$$

Then, we set

$$\check{u} := e^{s\varphi_0} u^\infty, \qquad \check{p} := e^{s\varphi_0} p^\infty, \qquad \check{\eta} := e^{s\varphi_0}\eta^\infty, \tag{64}$$

$$\widetilde{u} := \mathrm{Op}(1-\chi)\check{u}, \qquad \widetilde{p} := \mathrm{Op}(1-\chi)\check{p}, \qquad \widetilde{\eta} := \mathrm{Op}(1-\chi)\check{\eta}, \tag{65}$$

$$\widetilde{u}^\infty := \widetilde{\chi}^\infty \mathrm{Op}(1-\chi)\check{u}, \qquad \widetilde{p}^\infty := \widetilde{\chi}^\infty \mathrm{Op}(1-\chi)\check{p}, \qquad \widetilde{\eta}^\infty := \widetilde{\chi}^\infty \mathrm{Op}(1-\chi)\check{\eta}. \tag{66}$$

Our aim is to estimate $\widetilde{p}$ (see (48)) but we need to use $\widetilde{\chi}^\infty$ to work on a bounded domain and to apply the elliptic regularity of the Stokes system. In order to estimate $\widetilde{p}$, we use that, with our choice of truncation functions, we have the relations

$$\widetilde{p} = \widetilde{p}^\infty + [1-\widetilde{\chi}^\infty, \mathrm{Op}(1-\chi)]\check{p}.$$

Then, using the commutator property in Theorem 7, we can estimate $\widetilde{p}$ from $\widetilde{p}^\infty$ and $\check{p}$.

Using (46), (28) and (29), we have

$$\tau \lesssim s\lambda\xi_0, \quad \tau \lesssim s\lambda\xi.$$

This leads us to define (see (42)–(44))

$$\begin{aligned} I_4(s,\lambda,\check{\eta}) := \iint_{(0,T)\times\mathscr{I}} & \left(\lambda^{-9}\tau^{10}|\check{\eta}|^2 + \lambda^{-7}\tau^8|\partial_{x_1}\check{\eta}|^2 + \lambda^{-5}\tau^6\left(|\partial^2_{x_1}\check{\eta}|^2 + |\partial_t\check{\eta}|^2\right)\right) \mathrm{d}t\,\mathrm{d}x_1 \\ & + \iint_{(0,T)\times\mathscr{I}} \left(\lambda^{-3}\tau^4\left(|\partial^3_{x_1}\check{\eta}|^2 + |\partial_{x_1}\partial_t\check{\eta}|^2\right)\right) \mathrm{d}t\,\mathrm{d}x_1 \\ & + \iint_{(0,T)\times\mathscr{I}} \Big(\lambda^{-1}\tau^2\Big(|\partial^4_{x_1}\check{\eta}|^2 + |\partial_t\partial^2_{x_1}\check{\eta}|^2 + |\partial^2_t\check{\eta}|^2\Big) \\ & \qquad + \lambda\Big(\left|\partial^5_{x_1}\check{\eta}\right|^2 + |\partial^2_t\partial_{x_1}\check{\eta}|^2 + |\partial_t\partial^3_{x_1}\check{\eta}|^2\Big)\Big) \mathrm{d}t\,\mathrm{d}x_1, \end{aligned} \tag{67}$$

$$I_5(s,\lambda,\check{u}) := \lambda^2 \iint_{(0,T)\times\Omega} \left(|\nabla^2\check{u}|^2 + (\partial_t\check{u})^2 + \tau^2|\nabla\check{u}|^2 + \tau^4|\check{u}|^2\right) \mathrm{d}t\,\mathrm{d}x, \tag{68}$$

and

$$I_6(s,\lambda,\check{p}) := \lambda \iint_{(0,T)\times\Omega^\infty} \left(\tau^3|\check{p}|^2 + \tau|\nabla\check{p}|^2\right) \mathrm{d}t\,\mathrm{d}x + \iint_{(0,T)\times\partial\Omega^\infty} \left(\tau^3\left|\check{p}\right|^2 + \tau\left|\partial_{x_1}\check{p}\right|^2\right) \mathrm{d}t\,\mathrm{d}x_1. \tag{69}$$

Noting that

$$I_4(s,\lambda,\check{\eta}) \lesssim I_1(s,\lambda,\eta), \quad I_5(s,\lambda,\check{u}) \lesssim I_2(s,\lambda,u), \quad I_6(s,\lambda,\check{p}) \lesssim I_3(s,\lambda,p^\infty),$$

we deduce from (48) that

$$\begin{aligned} I_4(s,\lambda,\check{\eta}) + I_5(s,\lambda,\check{u}) + I_6(s,\lambda,\check{p}) \lesssim \lambda \iint_{(0,T)\times\mathscr{I}_1} & e^{2s\varphi_0}\left(s^{10}\xi_0^{10}|\eta|^2 + s^2\xi_0^2|\partial^2_t\eta|^2\right) \mathrm{d}t\,\mathrm{d}x_1 \\ & + \iint_{(0,T)\times\omega_1} e^{2s\varphi}\left(s^4\lambda^6\xi^4|u|^2\,\mathrm{d}t\,\mathrm{d}x + s^3\lambda^4\xi^3|p^\infty|^2\right) \mathrm{d}t\,\mathrm{d}x \\ & + \iint_{(0,T)\times\partial\Omega^\infty} \tau\left|\partial_{x_1}\widetilde{p}\right|^2 \mathrm{d}t\,\mathrm{d}x_1. \end{aligned} \tag{70}$$

The aim of this section is to show the following result:

Proposition 17. *There exist $\lambda_0 > 0$ and $s_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $s \geqslant s_0(T^2 + T^4)$, any smooth solutions $[u, p_0, \eta]$ of* (6) *satisfies*

$$\begin{aligned} I_4(s,\lambda,\check{\eta}) + I_5(s,\lambda,\check{u}) \lesssim \lambda \iint_{(0,T)\times\mathscr{I}_1} & e^{2s\varphi_0}\left(s^{10}\xi_0^{10}|\eta|^2 + s^2\xi_0^2|\partial^2_t\eta|^2\right) \mathrm{d}t\,\mathrm{d}x_1 \\ & + \iint_{(0,T)\times\omega_1} e^{2s\varphi}\left(s^4\lambda^6\xi^4|u|^2\,\mathrm{d}t\,\mathrm{d}x + s^3\lambda^4\xi^3|p|^2\right) \mathrm{d}t\,\mathrm{d}x, \end{aligned} \tag{71}$$

where $\check{\eta}$ and $\check{u}$ are defined by (64) *and p is defined by* (31).

Before proving Proposition 17, let us first introduce some preliminary results and notation. Recalling that χ is defined in (47), we deduce that if $\chi \neq 1$ then

$$\tau \lesssim |k|. \tag{72}$$

This yields the following semi-classical trace inequality:

Lemma 18. *There exists $s_0 > 0$ such that for any $s \geqslant s_0 T^4$ and for any $f \in H^1(\Omega^\infty)$,*

$$\tau^{1/2}\left\|\mathrm{Op}(1-\chi)f_{|\partial\Omega^\infty}\right\|_{L^2(\partial\Omega^\infty)} \lesssim \left\|\nabla\,\mathrm{Op}(1-\chi)f\right\|_{L^2(\Omega^\infty)}.$$

Proof. We write $g := \text{Op}(1-\chi)f$ and

$$g^2(x_1,1) = g^2(x_1,x_2) + 2\int_{x_2}^{1} g(x_1,y_2)\partial_{x_2} g(x_1,y_2)\, dy_2$$

so that

$$\tau\int_{\Gamma_1^\infty} g(x_1,1)^2\, \mathrm{d}x_1 \leqslant \tau\left\|g\right\|^2_{L^2(\Omega^\infty)} + 2\tau\left\|g\right\|_{L^2(\Omega^\infty)}\left\|\partial_{x_2} g\right\|_{L^2(\Omega^\infty)} \leqslant (\tau+4\tau^2)\left\|g\right\|^2_{L^2(\Omega^\infty)} + \left\|\partial_{x_2} g\right\|^2_{L^2(\Omega^\infty)}$$

and we conclude by using (72). □

In particular, using Lemma 18, we can estimate the last term of (70) as follows:

$$\iint_{(0,T)\times\partial\Omega^\infty} \tau\left|\partial_{x_1}\widetilde{p}_{|\partial\Omega^\infty}\right|^2\, \mathrm{d}t\,\mathrm{d}x_1 \lesssim \left\|\nabla\partial_{x_1}\widetilde{p}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}. \tag{73}$$

We have also the following result that will allows us to estimate boundary terms:

Lemma 19. *There exists $s_0 > 0$ such that for any $s \geqslant s_0 T^4$ and for any $f \in H^2(\mathbb{R})$,*

$$\left\|\text{Op}(1-\chi)f\right\|_{H^{3/2}(\mathbb{R})} \lesssim \tau^{-1/2}\left\|\text{Op}(1-\chi)\partial_{x_1}^2 f\right\|_{L^2(\mathbb{R})}.$$

Proof. Denoting by $\widehat{f}$ the Fourier transform of f, we have

$$\left\|\text{Op}(1-\chi)f\right\|^2_{H^{3/2}(\mathbb{R})} = \int_{\mathbb{R}} (1+k^2)^{3/2}(1-\chi(\tau,k))^2\left|\widehat{f}(k)\right|^2\, dk$$

and by using (72), we deduce the result. □

In order to estimate $\widetilde{p}$ (and prove Proposition 17), we consider the system verified by $\widetilde{u}^\infty$ and $\widetilde{p}^\infty$: from (34), we have

$$\begin{cases} \partial_t\widetilde{u}^\infty - \Delta\widetilde{u}^\infty + \nabla\widetilde{p}^\infty = \widetilde{f}^{(1)} & \text{in } (0,T)\times\Omega^\infty, \\ \operatorname{div}\widetilde{u}^\infty = \widetilde{f}^{(2)} & \text{in } (0,T)\times\Omega^\infty, \\ \widetilde{u}^\infty = 0 & \text{on } (0,T)\times\Gamma_0^\infty, \\ \widetilde{u}^\infty = \widetilde{h}e_2 & \text{on } (0,T)\times\Gamma_1^\infty, \\ \widetilde{u}^\infty(0,\cdot) = \widetilde{u}^\infty(T,\cdot) = 0 & \text{in } \Omega^\infty, \end{cases} \tag{74}$$

where

$$\begin{aligned} \widetilde{f}^{(1)} = {} & \widetilde{\chi}^\infty\text{Op}(1-\chi)e^{s\varphi_0}f^{(1)} + (s\partial_t\varphi_0)\widetilde{u}^\infty - s\left(\partial_{x_1}^2\varphi_0\right)\widetilde{u}^\infty + s^2\left(\partial_{x_1}\varphi_0\right)^2\widetilde{u}^\infty - 2s\partial_{x_1}\varphi_0\partial_{x_1}\widetilde{u}^\infty + s\nabla\varphi_0\widetilde{p}^\infty \\ & + \left[-(s\partial_t\varphi_0) + s\left(\partial_{x_1}^2\varphi_0\right) - s^2\left(\partial_{x_1}\varphi_0\right)^2 + 2s\partial_{x_1}\varphi_0\partial_{x_1}, \widetilde{\chi}^\infty\text{Op}(1-\chi)\right]\check{u} \\ & + \left[-s\nabla\varphi_0, \widetilde{\chi}^\infty\text{Op}(1-\chi)\right]\check{p} - \widetilde{\chi}^\infty\text{Op}(\partial_t\chi)\check{u} - \left(\widetilde{\chi}^\infty\right)''\widetilde{u} - 2\left(\widetilde{\chi}^\infty\right)'\partial_{x_1}\widetilde{u} + \left(\widetilde{\chi}^\infty\right)'\widetilde{p}e_1, \end{aligned} \tag{75}$$

$$\widetilde{f}^{(2)} = \widetilde{\chi}^\infty\text{Op}(1-\chi)(e^{s\varphi_0}f^{(2)}) + s\partial_{x_1}\varphi_0\widetilde{u}_1^\infty - [s\partial_{x_1}\varphi_0, \widetilde{\chi}^\infty\text{Op}(1-\chi)]\check{u}_1 + \left(\widetilde{\chi}^\infty\right)'\widetilde{u}_1, \tag{76}$$

$$\widetilde{h} := \widetilde{\chi}^\infty\text{Op}(1-\chi)(\partial_t\check{\eta} - s\left(\partial_t\varphi_0\right)\check{\eta}). \tag{77}$$

We also define

$$\begin{aligned} \widetilde{I}(\widetilde{u},\widetilde{p}) := {} & \left\|\partial_t\partial_{x_1}\widetilde{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\partial_{x_1}\widetilde{u}\right\|^2_{L^2(0,T;H^2(\Omega^\infty))} + \left\|\nabla\partial_{x_1}\widetilde{p}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} \\ & + \left\|\tau\partial_t\widetilde{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\tau^3\widetilde{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\tau^2\partial_{x_1}\widetilde{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} \\ & + \left\|\tau\partial_{x_1}^2\widetilde{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\tau\partial_{x_1}\partial_{x_2}\widetilde{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\tau\partial_{x_2}^2\widetilde{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} \\ & + \left\|\tau^2\partial_{x_2}\widetilde{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\tau^2\widetilde{p}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\tau\nabla\widetilde{p}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}. \end{aligned} \tag{78}$$

From relation (72), we have

$$\widetilde{I}(\widetilde{u},\widetilde{p}) \lesssim \left\|\partial_t\partial_{x_1}\widetilde{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\partial_{x_1}\widetilde{u}\right\|^2_{L^2(0,T;H^2(\Omega^\infty))} + \left\|\nabla\partial_{x_1}\widetilde{p}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}. \tag{79}$$

We have the following a priori estimate on (74).

Proposition 20. *There exist $\lambda_0 > 0$ and $s_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $s \geqslant s_0(T^2+T^4)$, any smooth solutions $[u, p_0, \eta]$ of* (6) *satisfies*

$$\widetilde{I}(\widetilde{u},\widetilde{p}) \lesssim \left\|\partial_{x_1}\widetilde{f}^{(1)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\partial_{x_1}\widetilde{f}^{(2)}\right\|^2_{L^2(0,T;H^1(\Omega^\infty))} + \left\|\partial_t\widetilde{f}^{(2)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\partial_{x_1}\widetilde{h}\right\|^2_{L^2(0,T;H^{3/2}(\Gamma_1^\infty))} + \left\|\tau^{-1/2}\partial_t\partial_{x_1}\widetilde{h}\right\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))} + \lambda^{-1}\left(I_5(s,\lambda,\check{u}) + I_6(s,\lambda,\check{p})\right). \quad (80)$$

Proof. First, we differentiate (74) with respect to x_1:

$$\begin{cases} \partial_t\partial_{x_1}\widetilde{u}^\infty - \Delta\partial_{x_1}\widetilde{u}^\infty + \nabla\partial_{x_1}\widetilde{p}^\infty = \partial_{x_1}\widetilde{f}^{(1)} & \text{in } (0,T)\times\Omega^\infty, \\ \operatorname{div}\partial_{x_1}\widetilde{u}^\infty = \partial_{x_1}\widetilde{f}^{(2)} & \text{in } (0,T)\times\Omega^\infty, \\ \partial_{x_1}\widetilde{u}^\infty = 0 & \text{on } (0,T)\times\Gamma_0^\infty, \\ \partial_{x_1}\widetilde{u}^\infty = \partial_{x_1}\widetilde{h}e_2 & \text{on } (0,T)\times\Gamma_1^\infty, \\ \partial_{x_1}\widetilde{u}^\infty(0,\cdot) = \partial_{x_1}\widetilde{u}^\infty(T,\cdot) = 0 & \text{in } \Omega^\infty. \end{cases} \quad (81)$$

Let us consider a bounded smooth domain $\Omega^\natural \subset \Omega^\infty$ containing $\overline{\operatorname{supp}\widetilde{\chi}^\infty} \times (0,1)$. Let us also write

$$h^\natural = \begin{cases} \partial_{x_1}\widetilde{u}^\infty = 0 & \text{on } (0,T)\times\left(\partial\Omega^\natural \setminus \Gamma_1^\infty\right), \\ \partial_{x_1}\widetilde{u}^\infty = \partial_{x_1}\widetilde{h}e_2 & \text{on } (0,T)\times\left(\partial\Omega^\natural \cap \Gamma_1^\infty\right). \end{cases}$$

Using [36, p. 33, Theorem 7.5], there exists $H \in H^2(\{(x_1,x_2)\in\mathbb{R}^2\ ;\ x_2<1\})$ such that $H = \partial_{x_1}\widetilde{h}$ on Γ_1^∞. Multiplying H by an adequate cut-off function we deduce the existence of $H^\natural \in H^2(\Omega^\natural)$ such that $H^\natural = h^\natural$ on $\partial\Omega^\natural$. Therefore $h^\natural \in H^{3/2}(\partial\Omega^\natural)$ and we have the estimate

$$\left\|h^\natural\right\|_{H^{3/2}(\partial\Omega^\natural)} \lesssim \left\|\partial_{x_1}\widetilde{h}\right\|_{H^{3/2}(\Gamma_1^\infty)}.$$

With the above notation, we deduce from (81) that $\left(\partial_{x_1}\widetilde{u}^\infty, \partial_{x_1}\widetilde{p}^\infty\right)$ satisfies a Stokes system in $\Omega^\natural$:

$$\begin{cases} -\Delta\partial_{x_1}\widetilde{u}^\infty + \nabla\partial_{x_1}\widetilde{p}^\infty = \partial_{x_1}\widetilde{f}^{(1)} - \partial_t\partial_{x_1}\widetilde{u}^\infty & \text{in } (0,T)\times\Omega^\natural, \\ \operatorname{div}\partial_{x_1}\widetilde{u}^\infty = \partial_{x_1}\widetilde{f}^{(2)} & \text{in } (0,T)\times\Omega^\natural, \\ \partial_{x_1}\widetilde{u}^\infty = h^\natural & \text{on } (0,T)\times\partial\Omega^\natural. \end{cases} \quad (82)$$

Using the elliptic regularity of the Stokes system (see, for instance, [50, Proposition 2.2 p. 33]) we obtain

$$\left\|\partial_{x_1}\widetilde{u}^\infty\right\|^2_{L^2(0,T;H^2(\Omega^\infty))} + \left\|\nabla\partial_{x_1}\widetilde{p}^\infty\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} \lesssim \left\|\partial_{x_1}\widetilde{f}^{(1)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\partial_t\partial_{x_1}\widetilde{u}^\infty\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\partial_{x_1}\widetilde{f}^{(2)}\right\|^2_{L^2(0,T;H^1(\Omega^\infty))} + \left\|\partial_{x_1}\widetilde{h}\right\|^2_{L^2(0,T;H^{3/2}(\Gamma_1^\infty))}. \quad (83)$$

On the other hand, by multiplying the first equation of (81) by $\partial_t\partial_{x_1}\widetilde{u}$ and integrating by parts, we deduce

$$\int_0^T \left\|\partial_t\partial_{x_1}\widetilde{u}^\infty\right\|^2_{L^2(\Omega^\infty)}\,\mathrm{d}t + \iint_{(0,T)\times\Gamma_1^\infty} \partial_{x_1}\widetilde{p}^\infty_{|\Gamma_1^\infty}\overline{\partial_t\partial_{x_1}\widetilde{h}}\,\mathrm{d}x_1\mathrm{d}t + \iint_{(0,T)\times\Omega^\infty} \partial^2_{x_1}\widetilde{p}^\infty\overline{\partial_t\widetilde{f}^{(2)}}\,\mathrm{d}t\mathrm{d}x = \iint_{(0,T)\times\Omega^\infty} \partial_{x_1}\widetilde{f}^{(1)}\cdot\overline{\partial_t\partial_{x_1}\widetilde{u}^\infty}\,\mathrm{d}x\mathrm{d}t.$$

The above relation yields that for any $\varepsilon > 0$,

$$\left\|\partial_t\partial_{x_1}\widetilde{u}^\infty\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} \lesssim \varepsilon\left\|\nabla\partial_{x_1}\widetilde{p}^\infty\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \varepsilon\left\|\tau^{1/2}\partial_{x_1}\widetilde{p}^\infty\right\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))} + \frac{1}{\varepsilon}\left\|\partial_t\widetilde{f}^{(2)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \frac{1}{\varepsilon}\left\|\tau^{-1/2}\partial_t\partial_{x_1}\widetilde{h}\right\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))} + \left\|\partial_{x_1}\widetilde{f}^{(1)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}.$$

We deduce from the above relation, from (66) and from Lemma 18 that

$$\begin{aligned}\left\|\partial_t\partial_{x_1}\widetilde{u}^{\infty}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}\\ \lesssim \varepsilon\left\|\nabla\partial_{x_1}\widetilde{p}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}+\varepsilon\left\|\widetilde{p}\right\|^2_{L^2(0,T;H^1(\Omega^\infty))}+\frac{1}{\varepsilon}\left\|\partial_t\widetilde{f}^{(2)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}\\ +\frac{1}{\varepsilon}\left\|\tau^{-1/2}\partial_t\partial_{x_1}\widetilde{h}\right\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))}+\left\|\partial_{x_1}\widetilde{f}^{(1)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}. \quad (84)\end{aligned}$$

Now using (63) and (33), we have

$$\nabla^2\partial_{x_1}\widetilde{u}=\nabla^2\partial_{x_1}\widetilde{u}^{\infty}+\nabla^2\partial_{x_1}\left[(1-\widetilde{\chi}^{\infty}),\mathrm{Op}(1-\chi)\right]\check{u}.$$

Since, $(1-\widetilde{\chi}^{\infty}), 1-\chi\in\mathbf{S}^0_\tau$, we deduce from Theorem 7 and Theorem 8 that

$$\begin{aligned}\left\|\nabla^2\partial_{x_1}\widetilde{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}\lesssim\left\|\nabla^2\partial_{x_1}\widetilde{u}^{\infty}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}+\left\|\tau^2\check{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}\\ +\left\|\tau\nabla\check{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}+\left\|\nabla^2\check{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}.\end{aligned}$$

Using the compact support of χ^∞ and the periodicity of $ue^{s\varphi_0}$, we deduce from the above relation and from (68) that

$$\left\|\nabla^2\partial_{x_1}\widetilde{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}\lesssim\left\|\nabla^2\partial_{x_1}\widetilde{u}^{\infty}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}+\lambda^{-2}I_5(s,\lambda,\check{u}). \quad (85)$$

Similarly, with (69), we have

$$\left\|\nabla\partial_{x_1}\widetilde{p}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}\lesssim\left\|\nabla\partial_{x_1}\widetilde{p}^{\infty}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}+\lambda^{-1}I_6(s,\lambda,\check{p}). \quad (86)$$

Finally,

$$\partial_t\partial_{x_1}\widetilde{u}=\partial_t\partial_{x_1}\widetilde{u}^{\infty}-\partial_{x_1}[1-\widetilde{\chi}^{\infty},\mathrm{Op}(\partial_t\chi)]\check{u}+\partial_{x_1}[1-\widetilde{\chi}^{\infty},\mathrm{Op}(1-\chi)]\partial_t\check{u},$$

and from Lemma 10,

$$\partial_t\chi\in\frac{\ell'}{(\lambda s)^{1/2}}\tau^{\frac{1}{2}}\mathbf{S}^0_\tau.$$

In particular, if $s\geqslant T^2$, we can combine the two previous relations with Theorem 7 and Theorem 8 to obtain

$$\left\|\partial_t\partial_{x_1}\widetilde{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}\lesssim\left\|\partial_t\partial_{x_1}\widetilde{u}^{\infty}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}+\lambda^{-2}I_5(s,\lambda,\check{u}). \quad (87)$$

Combining (79), (83), (84), (85), (86) and (87), we deduce the result by taking $\varepsilon>0$ small enough. □

Combining (80), (73) and (70), we deduce the existence of $\lambda_0>0$ and $s_0>0$ such that for $\lambda\geqslant\lambda_0$ and $s\geqslant s_0(T^2+T^4)$,

$$\begin{aligned}I_4(s,\lambda,\check{\eta})+I_5(s,\lambda,\check{u})+I_6(s,\lambda,\check{p})+\widetilde{I}(\widetilde{u},\widetilde{p})\\ \lesssim\lambda\iint_{(0,T)\times\mathscr{I}_1}e^{2s\varphi_0}\left(s^{10}\xi_0^{10}|\eta|^2+s^2\xi_0^2|\partial_t^2\eta|^2\right)\,\mathrm{d}t\,\mathrm{d}x_1\\ +\iint_{(0,T)\times\omega_1}e^{2s\varphi}\left(s^4\lambda^6\xi^4|u|^2\,\mathrm{d}t\,\mathrm{d}x+s^3\lambda^4\xi^3|p^\infty|^2\right)\,\mathrm{d}t\,\mathrm{d}x\\ +\left\|\partial_{x_1}\widetilde{f}^{(1)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}+\left\|\partial_{x_1}\widetilde{f}^{(2)}\right\|^2_{L^2(0,T;H^1(\Omega^\infty))}+\left\|\partial_t\widetilde{f}^{(2)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}\\ +\left\|\tau^{-1/2}\partial_t\partial_{x_1}\widetilde{h}\right\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))}+\left\|\partial_{x_1}\widetilde{h}\right\|^2_{L^2(0,T;H^{3/2}(\Gamma_1^\infty))}. \quad (88)\end{aligned}$$

4.2. *Estimates of* $\widetilde{f}^{(1)}$, $\widetilde{f}^{(2)}$ *and* $\widetilde{h}$

To obtain Proposition 17, it remains to estimate the right-hand side of (88). We recall that $\widetilde{f}^{(1)}$, $\widetilde{f}^{(2)}$ and $\widetilde{h}$ are given by (75), (76) and (77).

Combining (49), (46), (29) and (38), we deduce that for $\alpha \geqslant 0$,

$$\left|\partial_{x_1}^{\alpha}\varphi_0\right| \lesssim \frac{\tau}{\lambda s} \quad (k\geqslant 1), \quad \left|\partial_t\partial_{x_1}^{\alpha}\varphi_0\right| \lesssim T\left(\frac{\tau}{\lambda s}\right)^{3/2}, \quad \left|\partial_t^2\partial_{x_1}^{\alpha}\varphi_0\right| \lesssim T^2\left(\frac{\tau}{\lambda s}\right)^{2} \tag{89}$$

$$\partial_{x_1}^{\alpha}\varphi_0 \in \frac{\tau}{\lambda s}\mathbf{S}_\tau^0 \quad (k\geqslant 1), \quad \partial_t\partial_{x_1}^{\alpha}\varphi_0 \in \ell'\left(\frac{\tau}{\lambda s}\right)^{3/2}(e^{-4\lambda\Psi}+e^{-2\lambda\Psi})\mathbf{S}_\tau^0,$$
$$\partial_t^2\partial_{x_1}^{\alpha}\varphi_0 \in \left(2\ell+3(\ell')^2\right)\left(\frac{\tau}{\lambda s}\right)^{2}(e^{-8\lambda\Psi}+e^{-6\lambda\Psi})\mathbf{S}_\tau^0. \tag{90}$$

Proposition 21. *There exist* $s_0>0$ *and* $\lambda_0>0$ *such that the function* $\widetilde{f}^{(1)}$ *defined by* (75) *satisfies for* $s\geqslant s_0(T^2+T^4)$ *and for* $\lambda\geqslant\lambda_0$,

$$\left\|\partial_{x_1}\widetilde{f}^{(1)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} \leqslant \lambda^{-1}\left(I_4(s,\lambda,\check{\eta})+I_5(s,\lambda,\check{u})+I_6(s,\lambda,\check{p})+\widetilde{I}(\widetilde{u},\widetilde{p})\right).$$

Proof. Differentiating (75) yields,

$$\partial_{x_1}\widetilde{f}^{(1)} = \sum_{i=1}^{6} F^{(i)} \tag{91}$$

where

$$F^{(1)} := \partial_{x_1}\left(\widetilde{\chi}^\infty \operatorname{Op}(1-\chi)e^{s\varphi_0}f^{(1)}\right), \tag{92}$$
$$F^{(2)} := \partial_{x_1}\left((s\partial_t\varphi_0)\widetilde{u}^\infty - s\left(\partial_{x_1}^2\varphi_0\right)\widetilde{u}^\infty + s^2\left(\partial_{x_1}\varphi_0\right)^2\widetilde{u}^\infty - 2s\partial_{x_1}\varphi_0\partial_{x_1}\widetilde{u}^\infty + s\nabla\varphi_0\widetilde{p}^\infty\right), \tag{93}$$
$$F^{(3)} := \partial_{x_1}\left[-(s\partial_t\varphi_0) + s\left(\partial_{x_1}^2\varphi_0\right) - s^2\left(\partial_{x_1}\varphi_0\right)^2 + 2s\partial_{x_1}\varphi_0\partial_{x_1}, \widetilde{\chi}^\infty\operatorname{Op}(1-\chi)\right]\check{u}, \tag{94}$$
$$F^{(4)} := \partial_{x_1}[-s\nabla\varphi_0, \widetilde{\chi}^\infty\operatorname{Op}(1-\chi)]\check{p}, \quad F^{(5)} := -\partial_{x_1}\left(\widetilde{\chi}^\infty\operatorname{Op}(\partial_t\chi)\check{u}\right), \tag{95}$$
$$F^{(6)} := \partial_{x_1}\left(-\left(\widetilde{\chi}^\infty\right)''\widetilde{u} - 2\left(\widetilde{\chi}^\infty\right)'\partial_{x_1}\widetilde{u} + \left(\widetilde{\chi}^\infty\right)'\widetilde{p}e_1\right). \tag{96}$$

From (92), we have

$$F^{(1)} = \widetilde{\chi}^\infty\operatorname{Op}(1-\chi)\left(s\partial_{x_1}\varphi_0 e^{s\varphi_0}f^{(1)} + e^{s\varphi_0}\partial_{x_1}f^{(1)}\right) + (\widetilde{\chi}^\infty)'\operatorname{Op}(1-\chi)e^{s\varphi_0}f^{(1)}.$$

From (36), (89), the properties of χ^∞ and the periodicity of u and p in the x_1 variable,

$$\begin{aligned}\left\|s\partial_{x_1}\varphi_0 e^{s\varphi_0}f^{(1)} + e^{s\varphi_0}\partial_{x_1}f^{(1)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|e^{s\varphi_0}f^{(1)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} \\ \lesssim \iint_{(0,T)\times(0,2\pi)\times(0,1)} e^{2s\varphi_0}\left(\frac{\tau^2}{\lambda^2}\left(|u|^2+\left|\partial_{x_1}u\right|^2+|p|^2\right)+\left|\partial_{x_1}^2u\right|^2+\left|\partial_{x_1}p\right|^2\right)\mathrm{d}x\,\mathrm{d}t \\ \lesssim \iint_{(0,T)\times\Omega^\infty}\left(\left|\partial_{x_1}^2\check{u}\right|^2+\left(\frac{\tau}{\lambda}\right)^2\left|\partial_{x_1}\check{u}\right|^2+\left(\frac{\tau}{\lambda}\right)^4|\check{u}|^2+\left|\partial_{x_1}\check{p}\right|^2+\left(\frac{\tau}{\lambda}\right)^2|\check{p}|^2\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $1-\chi\in\mathbf{S}_\tau^0$ (see Lemma 6), we deduce from the above estimate, from Theorem 8, and from (68)-(69), that

$$\left\|F^{(1)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} \leqslant \lambda^{-1}\left(I_5(s,\lambda,\check{u})+I_6(s,\lambda,\check{p})\right). \tag{97}$$

From (40), (93) and (89), we have for $s\geqslant s_0\left(T^2+T^4\right)$ and $\lambda\geqslant\lambda_0$,

$$\left|F^{(2)}\right| \lesssim \frac{\tau^2}{\lambda^2}\left(|\widetilde{u}|+\left|\partial_{x_1}\widetilde{u}\right|\right)+\frac{\tau}{\lambda}\left(\left|\partial_{x_1}^2\widetilde{u}\right|+|\widetilde{p}|+\left|\partial_{x_1}\widetilde{p}\right|\right), \tag{98}$$

and thus with (78),

$$\left\|F^{(2)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} \leqslant \lambda^{-2}\widetilde{I}(\widetilde{u},\widetilde{p}). \tag{99}$$

On the other hand, from (94), (95),

$$F^{(3)} = \left[-(s\partial_t\varphi_0) + s\left(\partial_{x_1}^2\varphi_0\right) - s^2\left(\partial_{x_1}\varphi_0\right)^2 + 2s\partial_{x_1}\varphi_0\partial_{x_1}, \widetilde{\chi}^\infty \operatorname{Op}(1-\chi)\right]\partial_{x_1}\check{u}$$
$$+\left[-(s\partial_t\partial_{x_1}\varphi_0) + s\left(\partial_{x_1}^3\varphi_0\right) - 2s^2\partial_{x_1}\varphi_0\partial_{x_1}^2\varphi_0 + 2s\partial_{x_1}^2\varphi_0\partial_{x_1} + 2s\partial_{x_1}\varphi_0\partial_{x_1}^2, \widetilde{\chi}^\infty \operatorname{Op}(1-\chi)\right]\check{u}, \quad (100)$$

$$F^{(4)} = [-s\nabla\varphi_0, \widetilde{\chi}^\infty \operatorname{Op}(1-\chi)]\partial_{x_1}\check{p} + [-s\nabla\partial_{x_1}\varphi_0, \widetilde{\chi}^\infty \operatorname{Op}(1-\chi)]\check{p}. \quad (101)$$

From (90),

$$s\left(\partial_{x_1}^2\varphi_0\right) - s^2\left(\partial_{x_1}\varphi_0\right)^2 - 2s\partial_{x_1}\varphi_0 ik \in \frac{1}{\lambda}\mathbf{S}_\tau^2, \quad s\partial_t\varphi_0 \in \frac{\ell'}{s^{1/2}\lambda}\mathbf{S}_\tau^{3/2}$$
$$s\left(\partial_{x_1}^3\varphi_0\right) - 2s^2\partial_{x_1}\varphi_0\partial_{x_1}^2\varphi_0 - 2s\partial_{x_1}^2\varphi_0 ik - 2s\partial_{x_1}\varphi_0 k^2 \in \frac{1}{\lambda}\mathbf{S}_\tau^3, \quad s\partial_t\partial_{x_1}\varphi_0 \in \frac{\ell'}{s^{1/2}\lambda}\mathbf{S}_\tau^{3/2},$$
$$-s\nabla\varphi_0, -s\nabla\partial_{x_1}\varphi_0 \in \frac{1}{\lambda}\mathbf{S}_\tau^1,$$

so that, from Theorem 7, Theorem 8 and (78),

$$\left\|F^{(3)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|F^{(4)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}$$
$$\lesssim \frac{1}{\lambda^2}\left(\left\|\tau^2\check{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\tau\partial_{x_1}\check{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\partial_{x_1}^2\check{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}\right)$$
$$+\frac{1}{\lambda^2}\left(\left\|\check{p}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\partial_{x_1}\check{p}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}\right) \lesssim \frac{1}{\lambda^2}\left(I_5(s,\lambda,\check{u}) + I_6(s,\lambda,\check{p})\right). \quad (102)$$

From (95),

$$F^{(5)} = -\widetilde{\chi}^\infty \operatorname{Op}(\partial_t\chi)\partial_{x_1}\check{u} - \left(\widetilde{\chi}^\infty\right)' \operatorname{Op}(\partial_t\chi)\check{u}. \quad (103)$$

From Lemma 10,

$$\partial_t\chi \in \frac{\ell'}{(\lambda s)^{1/2}}\mathbf{S}_\tau^{1/2}$$

so that from Theorem 8 and (68),

$$\left\|F^{(5)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} \lesssim \lambda^{-1}\left(\left\|\tau\partial_{x_1}\check{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\tau\check{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} + \left\|\partial_{x_1}^2\check{u}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}\right)$$
$$\lesssim \lambda^{-2} I_5(s,\lambda,\check{u}). \quad (104)$$

Finally, from (96), (78) and (40),

$$\left\|F^{(6)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} \lesssim \lambda^{-2}\widetilde{I}(\widetilde{u},\widetilde{p}).$$

Gathering (91), (97), (99), (102), (104) and the above estimate, we deduce the result. □

Proposition 22. *There exist $s_0 > 0$ and $\lambda_0 > 0$ such that the function $\widetilde{f}^{(2)}$ defined by* (76) *satisfies for $s \geqslant s_0(T^2+T^4)$ and for $\lambda \geqslant \lambda_0$,*

$$\left\|\partial_{x_1}\widetilde{f}^{(2)}\right\|^2_{L^2(0,T;H^1(\Omega^\infty))} + \left\|\partial_t\widetilde{f}^{(2)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))} \leqslant \lambda^{-1}\left(I_4(s,\lambda,\check{\eta}) + I_5(s,\lambda,\check{u}) + I_6(s,\lambda,\check{p}) + \widetilde{I}(\widetilde{u},\widetilde{p})\right).$$

Proof. From (76)

$$\partial_{x_1}\widetilde{f}^{(2)} = G^{(1)} + G^{(2)} + G^{(3)} + G^{(4)}, \quad (105)$$

with

$$G^{(1)} := \partial_{x_1}\left(\widetilde{\chi}^\infty \operatorname{Op}(1-\chi)\left(e^{s\varphi_0}f^{(2)}\right)\right), \qquad G^{(2)} := \partial_{x_1}\left(s\partial_{x_1}\varphi_0\widetilde{u}_1^\infty\right), \quad (106)$$

$$G^{(3)} := -\partial_{x_1}[s\partial_{x_1}\varphi_0, \widetilde{\chi}^\infty \operatorname{Op}(1-\chi)]\check{u}_1, \qquad G^{(4)} := \partial_{x_1}\left(\left(\widetilde{\chi}^\infty\right)'\widetilde{u}_1\right). \quad (107)$$

From (106), we have

$$G^{(1)} = \left(\widetilde{\chi}^\infty\right)' \operatorname{Op}(1-\chi)\left(e^{s\varphi_0}f^{(2)}\right) + \widetilde{\chi}^\infty \operatorname{Op}(1-\chi)\partial_{x_1}\left(e^{s\varphi_0}f^{(2)}\right),$$
$$\partial_{x_1}G^{(1)} = \left(\widetilde{\chi}^\infty\right)'' \operatorname{Op}(1-\chi)\left(e^{s\varphi_0}f^{(2)}\right) + 2\left(\widetilde{\chi}^\infty\right)' \operatorname{Op}(1-\chi)\partial_{x_1}\left(e^{s\varphi_0}f^{(2)}\right) + \widetilde{\chi}^\infty \operatorname{Op}(1-\chi)\partial_{x_1}^2\left(e^{s\varphi_0}f^{(2)}\right),$$
$$\partial_{x_2}G^{(1)} = \left(\widetilde{\chi}^\infty\right)' \operatorname{Op}(1-\chi)\left(e^{s\varphi_0}\partial_{x_2}f^{(2)}\right) + \widetilde{\chi}^\infty \operatorname{Op}(1-\chi)\partial_{x_1}\left(e^{s\varphi_0}\partial_{x_2}f^{(2)}\right),$$

with

$$\partial_{x_1}\left(e^{s\varphi_0}f^{(2)}\right)=s\partial_{x_1}\varphi_0 e^{s\varphi_0}f^{(2)}+e^{s\varphi_0}\partial_{x_1}f^{(2)},$$
$$\partial^2_{x_1}\left(e^{s\varphi_0}f^{(2)}\right)=s\partial^2_{x_1}\varphi_0 e^{s\varphi_0}f^{(2)}+s^2\left(\partial_{x_1}\varphi_0\right)^2e^{s\varphi_0}f^{(2)}+2s\partial_{x_1}\varphi_0 e^{s\varphi_0}\partial_{x_1}f^{(2)}+e^{s\varphi_0}\partial^2_{x_1}f^{(2)},$$
$$\partial_{x_1}\left(e^{s\varphi_0}\partial_{x_2}f^{(2)}\right)=s\partial_{x_1}\varphi_0 e^{s\varphi_0}\partial_{x_2}f^{(2)}+e^{s\varphi_0}\partial_{x_1}\partial_{x_2}f^{(2)}.$$

The above relations, combined with (36), (89), with the properties of χ^∞ and with the periodicity of u and p in the x_1 variable, imply

$$\left\|e^{s\varphi_0}f^{(2)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}+\left\|\nabla\left(e^{s\varphi_0}f^{(2)}\right)\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}+\left\|\nabla\partial_{x_1}\left(e^{s\varphi_0}f^{(2)}\right)\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}$$
$$\lesssim\iint_{(0,T)\times(0,2\pi)\times(0,1)}e^{2s\varphi_0}\left(\left(\frac{\tau}{\lambda}\right)^4|u|^2+\left(\frac{\tau}{\lambda}\right)^2|\nabla u|^2+\left|\partial^2_{x_1}u\right|^2+\left|\partial_{x_1}\partial_{x_2}u\right|^2\right)\,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim\iint_{(0,T)\times\Omega^\infty}\left(\left|\nabla^2\check{u}\right|^2+\left(\frac{\tau}{\lambda}\right)^2|\nabla\check{u}|^2+\left(\frac{\tau}{\lambda}\right)^4|\check{u}|^2\right)\,\mathrm{d}x\,\mathrm{d}t.$$

Thus, using $1-\chi\in\mathbf{S}^0_\tau$ (Lemma 6) along with Theorem 8, the property of χ^∞ and (68), we deduce that

$$\left\|G^{(1)}\right\|^2_{L^2(0,T;H^1(\Omega^\infty))}\leqslant\lambda^{-1}I_5(s,\lambda,\check{u}). \tag{108}$$

Moreover, using (89), we deduce

$$\left|G^{(2)}\right|+\left|\nabla G^{(2)}\right|\lesssim\frac{\tau}{\lambda}\left(|\widetilde{u}|+\left|\partial_{x_1}\widetilde{u}\right|+\left|\partial^2_{x_1}\widetilde{u}\right|+\left|\partial_{x_2}\widetilde{u}\right|+\left|\partial_{x_1}\partial_{x_2}\widetilde{u}\right|\right). \tag{109}$$

We deduce from the above relation and (78) that

$$\left\|G^{(2)}\right\|^2_{L^2(0,T;H^1(\Omega^\infty))}\leqslant\lambda^{-2}\widetilde{I}(\widetilde{u},\widetilde{p}). \tag{110}$$

From (107),

$$\partial_{x_1}G^{(3)}=-\partial^2_{x_1}[s\partial_{x_1}\varphi_0,\widetilde{\chi}^\infty\operatorname{Op}(1-\chi)]\check{u}_1,\quad\partial_{x_2}G^{(3)}=-\partial_{x_1}[s\partial_{x_1}\varphi_0,\widetilde{\chi}^\infty\operatorname{Op}(1-\chi)]\partial_{x_2}\check{u}_1.$$

Thus from (90), (68), Theorem 7 and Theorem 8,

$$\left\|G^{(3)}\right\|^2_{L^2(0,T;H^1(\Omega^\infty))}\lesssim\lambda^{-4}I_5(s,\lambda,\check{u}). \tag{111}$$

From (107),

$$\partial_{x_1}G^{(4)}=\partial^2_{x_1}\left(\left(\widetilde{\chi}^\infty\right)'\widetilde{u}_1\right),\quad\partial_{x_2}G^{(4)}=\partial_{x_1}\left(\left(\widetilde{\chi}^\infty\right)'\partial_{x_2}\widetilde{u}_1\right),$$

and thus, from (78) and (40),

$$\left\|G^{(4)}\right\|^2_{L^2(0,T;H^1(\Omega^\infty))}\lesssim\lambda^{-2}\widetilde{I}(\widetilde{u},\widetilde{p}).$$

Gathering (105), (108), (110), (111) and the above relation, we deduce the estimate for $\partial_{x_1}\widetilde{f}^{(2)}$. To estimate $\partial_t\widetilde{f}^{(2)}$, we derive (76) with respect to time:

$$\partial_t\widetilde{f}^{(2)}=H^{(1)}+H^{(2)}+H^{(3)}, \tag{112}$$

with

$$H^{(1)}:=-\widetilde{\chi}^\infty\operatorname{Op}(-\partial_t\chi)\left(e^{s\varphi_0}f^{(2)}\right)+\widetilde{\chi}^\infty\operatorname{Op}(1-\chi)\left(\left(s\partial_t\varphi_0 f^{(2)}+\partial_t f^{(2)}\right)e^{s\varphi_0}\right), \tag{113}$$
$$H^{(2)}:=\partial_t\left(s\partial_{x_1}\varphi_0\widetilde{u}_1^\infty+\left(\widetilde{\chi}^\infty\right)'\widetilde{u}_1\right), \tag{114}$$
$$H^{(3)}:=-\partial_t[s\partial_{x_1}\varphi_0,\widetilde{\chi}^\infty\operatorname{Op}(1-\chi)]\check{u}_1. \tag{115}$$

Combining (36), Lemma 10, (89), Theorem 8, the property of χ^∞ and (68), we deduce that

$$\left\|H^{(1)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}\lesssim\lambda^{-1}I_5(s,\lambda,\check{u}). \tag{116}$$

Using (36), (89) and (78), we also find

$$\left\|H^{(2)}\right\|^2_{L^2(0,T;L^2(\Omega^\infty))}\leqslant\lambda^{-2}\widetilde{I}(\widetilde{u},\widetilde{p}). \tag{117}$$

For the last term, we write

$$H^{(3)} = -[s\partial_{x_1}\partial_t\varphi_0, \widetilde{\chi}^\infty \operatorname{Op}(1-\chi)]\check{u}_1 + [s\partial_{x_1}\varphi_0, \widetilde{\chi}^\infty \operatorname{Op}(\partial_t\chi)]\check{u}_1 - [s\partial_{x_1}\varphi_0, \widetilde{\chi}^\infty \operatorname{Op}(1-\chi)]\partial_t\check{u}_1.$$

Combining (90), Lemma 10, Theorem 7 and Theorem 8, we deduce

$$\|H^{(3)}\|^2_{L^2(0,T;L^2(\Omega^\infty))} \lesssim \lambda^{-1} I_5(s,\lambda,\check{u}).$$

Gathering the above estimate, (116) and (117) yields the result. □

Proposition 23. *There exist $s_0>0$ and $\lambda_0>0$ such that the function $\widetilde{h}$ defined by* (77) *satisfies for $s \geqslant s_0(T^2+T^4)$ and for $\lambda \geqslant \lambda_0$,*

$$\|\tau^{-1/2}\partial_t\partial_{x_1}\widetilde{h}\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))} + \|\partial_{x_1}\widetilde{h}\|^2_{L^2(0,T;H^{3/2}(\Gamma_1^\infty))} \lesssim \lambda^{-1} I_4(s,\lambda,\check{\eta}).$$

Proof. From (77)

$$\partial_{x_1}\widetilde{h} = (\widetilde{\chi}^\infty)' \operatorname{Op}(1-\chi)\widetilde{h}^{(1)} + \widetilde{\chi}^\infty \operatorname{Op}(1-\chi)\widetilde{h}^{(2)}, \tag{118}$$

with

$$\widetilde{h}^{(1)} := \partial_t\check{\eta} - s\partial_t\varphi_0\check{\eta}, \quad \widetilde{h}^{(2)} := \partial_{x_1}\widetilde{h}^{(1)} = \partial_t\partial_{x_1}\check{\eta} - s\left(\partial_t\partial_{x_1}\varphi_0\right)\check{\eta} - s\left(\partial_t\varphi_0\right)\partial_{x_1}\check{\eta}.$$

Applying Lemma 19 and using (118), we deduce that

$$\begin{aligned}\|\partial_{x_1}\widetilde{h}\|^2_{L^2(0,T;H^{3/2}(\Gamma_1^\infty))} &\lesssim \|\operatorname{Op}(1-\chi)\widetilde{h}^{(1)}\|^2_{L^2(0,T;H^{3/2}(\Gamma_1^\infty))} + \|\operatorname{Op}(1-\chi)\widetilde{h}^{(2)}\|^2_{L^2(0,T;H^{3/2}(\Gamma_1^\infty))}\\ &\lesssim \|\tau^{-1/2}\operatorname{Op}(1-\chi)\partial^2_{x_1}\widetilde{h}^{(1)}\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))} + \|\tau^{-1/2}\operatorname{Op}(1-\chi)\partial^2_{x_1}\widetilde{h}^{(2)}\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))}.\end{aligned}$$

Then, using that $1-\chi \in \mathbf{S}^0_\tau$, (89) and Theorem 8, we find

$$\begin{aligned}\|\partial_{x_1}\widetilde{h}\|^2_{L^2(0,T;H^{3/2}(\Gamma_1^\infty))} &\lesssim \|\tau^{-1/2}\partial_t\partial^3_{x_1}\check{\eta}\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))}\\ &\quad + \|\tau^{-1/2}\partial_t\partial^2_{x_1}\check{\eta}\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))} + \sum_{j=0}^3 \left\|\frac{\tau^{3/2}}{\lambda^{3/2}}\partial^j_{x_1}\check{\eta}\right\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))}.\end{aligned}$$

From (67), we deduce from the above relation that

$$\|\partial_{x_1}\widetilde{h}\|^2_{L^2(0,T;H^{3/2}(\Gamma_1^\infty))} \lesssim \lambda^{-1} I_4(s,\lambda,\check{\eta}). \tag{119}$$

By differentiating (118) with respect to t, we obtain

$$\partial_t\partial_{x_1}\widetilde{h} = -(\widetilde{\chi}^\infty)'\operatorname{Op}(\partial_t\chi)\widetilde{h}^{(1)} + (\widetilde{\chi}^\infty)'\operatorname{Op}(1-\chi)\partial_t\widetilde{h}^{(1)} - \widetilde{\chi}^\infty\operatorname{Op}(\partial_t\chi)\widetilde{h}^{(2)} + \widetilde{\chi}^\infty\operatorname{Op}(1-\chi)\partial_t\widetilde{h}^{(2)}. \tag{120}$$

Applying Lemma 10, Theorem 8 and (89), we have for $s \geqslant s_0(T^2+T^4)$ and $\lambda \geqslant \lambda_0$,

$$\begin{aligned}\|\tau^{-1/2}\partial_t\partial_{x_1}\widetilde{h}\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))} &\lesssim \left\|\frac{\tau^{3/2}}{\lambda^{3/2}}\partial_{x_1}\check{\eta}\right\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))} + \left\|\frac{\tau^{3/2}}{\lambda^{3/2}}\check{\eta}\right\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))}\\ &\quad + \|\tau^{-1/2}\partial_t^2\check{\eta}\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))} + \|\tau^{-1/2}\partial_t^2\partial_{x_1}\check{\eta}\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))}\\ &\quad + \left\|\frac{\tau}{\lambda^{3/2}}\partial_{x_1}\partial_t\check{\eta}\right\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))} + \left\|\frac{\tau}{\lambda^{3/2}}\partial_t\check{\eta}\right\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))}.\end{aligned} \tag{121}$$

From (67), we deduce from the above relation that

$$\|\tau^{-1/2}\partial_t\partial_{x_1}\widetilde{h}\|^2_{L^2(0,T;L^2(\Gamma_1^\infty))} \lesssim \lambda^{-1} I_4(s,\lambda,\check{\eta}). \tag{122}$$

□

The proof of Proposition 17 consists now in combining (88) with Proposition 21, Proposition 22 and Proposition 23. In the next section, we show the observability result from the above relation.

5. Proof of the observability

This section is devoted to the proof of Theorem 2. We first remove in (71) the local terms in p and in $\partial_t^2\eta$. Setting

$$\varphi_1(t) := \frac{1}{\ell(t)^2}\left(e^{8\lambda\Psi} - e^{10\lambda\Psi}\right), \qquad \xi_1(t) := \frac{1}{\ell(t)^2} e^{8\lambda\Psi}, \tag{123}$$

$$\varphi_2(t) := \frac{1}{\ell(t)^2}\left(e^{(9\lambda+\mu_0)\Psi} - e^{10\lambda\Psi}\right), \qquad \xi_2(t) := \frac{1}{\ell(t)^2} e^{(9\lambda+\mu_0)\Psi}, \tag{124}$$

we have (see (28))

$$\varphi_1(t) \leqslant \varphi(t,\cdot) \leqslant \varphi_2(t), \quad \xi_1(t) \leqslant \xi(t,\cdot) \leqslant \xi_2(t) \quad (t \in (0,T)).$$

Let us set

$$\rho_0 := \lambda\tau^2 e^{s\varphi_1} = \lambda^3 e^{16\lambda\Psi}\frac{s^2}{\ell^4} e^{s\varphi_1}, \tag{125}$$

$$\rho_1 := s^{11/2}\lambda^{-7}\xi_2^{11/2} e^{4s\varphi_2 - 3s\varphi_1}, \quad \rho_2 := s^9\lambda^{-1}\xi_2^9 e^{4s\varphi_2 - 3s\varphi_1}. \tag{126}$$

Then we have the following result.

Proposition 24. *There exist $s_0 > 0$ and $\lambda_0 > 0$ such that for any $s \geqslant s_0(T^2 + T^4)$ and for any $\lambda \geqslant \lambda_0$, any smooth solution of* (6) *satisfies*

$$\int_0^T \rho_0^2 \left\|[u,\eta,\partial_t\eta]\right\|_{\mathscr{H}}^2 \,\mathrm{d}t \lesssim \iint_{(0,T)\times\mathscr{I}} \rho_1^2 |\partial_t\eta|^2 \,\mathrm{d}t\,\mathrm{d}x_1 + \iint_{(0,T)\times\omega} \rho_2^2 |u|^2 \,\mathrm{d}t\,\mathrm{d}x. \tag{127}$$

Proof. Using (32) and applying the Poincaré–Wirtinger inequality, we deduce that

$$\iint_{(0,T)\times\omega_1} s^3\lambda^4\xi^3 e^{2s\varphi}|p|^2 \,\mathrm{d}t\,\mathrm{d}x \lesssim \iint_{(0,T)\times\omega_1} s^3\lambda^4\xi_2^3 e^{2s\varphi_2}|\nabla p|^2 \,\mathrm{d}t\,\mathrm{d}x$$

and with (12),

$$\iint_{(0,T)\times\omega_1} s^3\lambda^4\xi^3 e^{2s\varphi}|p|^2 \,\mathrm{d}t\,\mathrm{d}x \lesssim \iint_{(0,T)\times\omega_1} s^3\lambda^4\xi_2^3 e^{2s\varphi_2}\left(|\partial_t u|^2 + |\Delta u|^2\right) \mathrm{d}t\,\mathrm{d}x. \tag{128}$$

From (125), (67) and (68), we have

$$\int_0^T \rho_0(t)^2 \left(\|u(t)\|_{L^2(\Omega)}^2 + \left\|\eta(t)\right\|_{H^2(\mathscr{I})}^2 + \left\|\partial_t\eta(t)\right\|_{L^2(\mathscr{I})}^2\right) \mathrm{d}t \lesssim I_4(s,\lambda,\check{\eta}) + I_5(s,\lambda,\check{u}). \tag{129}$$

Combining (71), (129) and (128), we deduce

$$\begin{aligned}
&\left\|\rho_0 u\right\|_{L^2(0,T;L^2(\Omega))}^2 + \left\|\rho_0\partial_t\eta\right\|_{L^2(0,T;L^2(\Gamma_1))}^2 + \left\|\rho_0\eta\right\|_{L^2(0,T;H^2(\Gamma_1))}^2 \\
&\quad\lesssim \lambda \iint_{(0,T)\times\mathscr{I}_1} e^{2s\varphi_0}\left(s^{10}\xi_0^{10}|\eta|^2 + s^2\xi_0^2|\partial_t^2\eta|^2\right) \mathrm{d}t\,\mathrm{d}x_1 \\
&\qquad + \iint_{(0,T)\times\omega_1} s^4\lambda^6\xi^4 e^{2s\varphi}|u|^2 \,\mathrm{d}t\,\mathrm{d}x + \iint_{(0,T)\times\omega_1} s^3\lambda^4\xi_2^3 e^{2s\varphi_2}\left(|\partial_t u|^2 + |\Delta u|^2\right) \mathrm{d}t\,\mathrm{d}x.
\end{aligned} \tag{130}$$

Now we set

$$\rho_3(t) := \lambda^3 e^{6\lambda\Psi}\frac{s^{1/2}}{\ell(t)} e^{s\varphi_1(t)}, \quad \rho_4(t) := \lambda^3 e^{-4\lambda\Psi} s^{-1}\ell(t)^2 e^{s\varphi_1(t)}. \tag{131}$$

We have $\rho_3, \rho_4 \in C^1([0,T])$, $\rho_3(0) = \rho_4(0) = 0$ and

$$\left|\rho_3'\right| \lesssim \rho_0 \quad \text{and} \quad \left|\rho_4'\right| \lesssim \rho_3. \tag{132}$$

We recall that (12) can be written as (23). Then, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_3\begin{bmatrix} u \\ \eta \\ \partial_t\eta \end{bmatrix} = \mathscr{A}\rho_3\begin{bmatrix} u \\ \eta \\ \partial_t\eta \end{bmatrix} + \rho_3'\begin{bmatrix} u \\ \eta \\ \partial_t\eta \end{bmatrix} \quad \text{in } (0,T), \quad \begin{bmatrix} \rho_3 u \\ \rho_3\eta \\ \rho_3\partial_t\eta \end{bmatrix}(0) = 0. \tag{133}$$

From (132), (129) and (19),

$$\begin{aligned}\left\|\rho_3\partial_t u\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\rho_3 u\right\|_{L^2(0,T;H^2(\Omega))} \\ + \left\|\rho_3\partial_t^2\eta\right\|_{L^2(0,T;L^2(\Gamma_1))} + \left\|\rho_3\partial_t\eta\right\|_{L^2(0,T;H^2(\Gamma_1))} + \left\|\rho_3\eta\right\|_{L^2(0,T;H^4(\Gamma_1))} \\ \lesssim \left\|\rho_0 u\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|\rho_0\partial_t\eta\right\|^2_{L^2(0,T;L^2(\Gamma_1))} + \left\|\rho_0\eta\right\|^2_{L^2(0,T;H^2(\Gamma_1))}. \end{aligned} \tag{134}$$

Then, we deduce from (23) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_4\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ \eta \\ \partial_t\eta\end{bmatrix}\right) = \mathscr{A}\left(\rho_4\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ \eta \\ \partial_t\eta\end{bmatrix}\right) + \rho_4'\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ \eta \\ \partial_t\eta\end{bmatrix} \quad \text{in } (0,T), \quad \left(\rho_4\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ \eta \\ \partial_t\eta\end{bmatrix}\right)(0) = 0. \tag{135}$$

From (132), (129), (134) and (19),

$$\begin{aligned}\left\|\rho_4\partial_t^2 u\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\rho_4\partial_t u\right\|_{L^2(0,T;H^2(\Omega))} \\ + \left\|\rho_4\partial_t^3\eta\right\|_{L^2(0,T;L^2(\Gamma_1))} + \left\|\rho_4\partial_t^2\eta\right\|_{L^2(0,T;H^2(\Gamma_1))} + \left\|\rho_4\partial_t\eta\right\|_{L^2(0,T;H^4(\Gamma_1))} \\ \lesssim \left\|\rho_0 u\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|\rho_0\partial_t\eta\right\|^2_{L^2(0,T;L^2(\Gamma_1))} + \left\|\rho_0\eta\right\|^2_{L^2(0,T;H^2(\Gamma_1))}. \end{aligned} \tag{136}$$

Then, from the standard elliptic regularities for the stationary Stokes system ([50, Proposition 2.2 p. 33]) and for A_1, we have moreover

$$\begin{aligned}\left\|\rho_4 u\right\|_{L^2(0,T;H^4(\Omega))} + \left\|\rho_4\eta\right\|_{L^2(0,T;H^6(\Gamma_1))} \\ \lesssim \left\|\rho_0 u\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|\rho_0\partial_t\eta\right\|^2_{L^2(0,T;L^2(\Gamma_1))} + \left\|\rho_0\eta\right\|^2_{L^2(0,T;H^2(\Gamma_1))}. \end{aligned} \tag{137}$$

By integration by parts, we obtain

$$\begin{aligned}\iint_{(0,T)\times\mathscr{I}_1} s^2\lambda\xi_0^2 e^{2s\varphi_0}|\partial_t^2\eta|^2 \,\mathrm{d}t\,\mathrm{d}x_1 = \frac{1}{2}\iint_{(0,T)\times\mathscr{I}_1} \partial_t^2\left(s^2\lambda\xi_0^2 e^{2s\varphi_0}\right)|\partial_t\eta|^2 \,\mathrm{d}t\,\mathrm{d}x_1 \\ - \iint_{(0,T)\times\mathscr{I}_1} s^2\lambda\xi_0^2 e^{2s\varphi_0}\partial_t^3\eta\partial_t\eta \,\mathrm{d}t\,\mathrm{d}x_1.\end{aligned}$$

Using (49), we deduce that for $s \geqslant s_0(T^2+T^4)$, for any $\varepsilon>0$, there exists $C>0$ such that

$$\begin{aligned}\iint_{(0,T)\times\mathscr{I}_1} s^2\lambda\xi_0^2 e^{2s\varphi_0}|\partial_t^2\eta|^2 \,\mathrm{d}t\,\mathrm{d}x_1 \leqslant C\iint_{(0,T)\times\mathscr{I}_1} s^5\lambda\xi_0^5 e^{2s\varphi_0}|\partial_t\eta|^2 \,\mathrm{d}t\,\mathrm{d}x_1 \\ + \varepsilon\left\|\rho_4\partial_t^3\eta\right\|^2_{L^2(0,T;L^2(\Gamma_1))} + C\iint_{(0,T)\times\mathscr{I}_1} s^6\lambda^{-4}\xi_0^6 e^{4s\varphi_0-2s\varphi_1}|\partial_t\eta|^2 \,\mathrm{d}t\,\mathrm{d}x_1 \\ \leqslant \varepsilon\left\|\rho_4\partial_t^3\eta\right\|^2_{L^2(0,T;L^2(\Gamma_1))} + C\iint_{(0,T)\times\mathscr{I}_1} s^6\lambda^{-4}\xi_0^6 e^{4s\varphi_0-2s\varphi_1}|\partial_t\eta|^2 \,\mathrm{d}t\,\mathrm{d}x_1.\end{aligned} \tag{138}$$

Then, we integrate by parts the last term and we obtain that for any $\varepsilon>0$, there exists $C>0$ such that

$$\begin{aligned}\iint_{(0,T)\times\mathscr{I}_1} s^6\lambda^{-4}\xi_0^6 e^{4s\varphi_0-2s\varphi_1}|\partial_t\eta|^2 \,\mathrm{d}t\,\mathrm{d}x_1 \\ \leqslant \varepsilon\left\|\rho_3\partial_t^2\eta\right\|^2_{L^2(0,T;L^2(\Gamma_1))} + C\iint_{(0,T)\times\mathscr{I}_1} s^{11}\lambda^{-14}\xi_0^{11} e^{8s\varphi_0-6s\varphi_1}|\eta|^2 \,\mathrm{d}t\,\mathrm{d}x_1.\end{aligned} \tag{139}$$

Similarly, for any $\varepsilon>0$, there exists $C>0$ such that

$$\begin{aligned}\iint_{(0,T)\times\omega_1} s^3\lambda^4\xi_2^3 e^{2s\varphi_2}|\partial_t u|^2 \,\mathrm{d}t\,\mathrm{d}x \leqslant C\iint_{(0,T)\times\omega_1} s^6\lambda^4\xi_2^6 e^{2s\varphi_2}|u|^2 \,\mathrm{d}t\,\mathrm{d}x \\ + \varepsilon\left\|\rho_4\partial_t^2 u\right\|^2_{L^2(0,T;L^2(\Omega))} + C\iint_{(0,T)\times\omega_1} s^8\lambda^2\xi_2^8 e^{4s\varphi_2-2s\varphi_1}|u|^2 \,\mathrm{d}t\,\mathrm{d}x \\ \leqslant \varepsilon\left\|\rho_4\partial_t^2 u\right\|^2_{L^2(0,T;L^2(\Omega))} + C\iint_{(0,T)\times\omega_1} s^8\lambda^2\xi_2^8 e^{4s\varphi_2-2s\varphi_1}|u|^2 \,\mathrm{d}t\,\mathrm{d}x.\end{aligned} \tag{140}$$

Finally, we consider a nonnegative smooth function χ_1 with compact support in ω and such that $\chi_1 \equiv 1$ in ω_1. Then by integrating by parts,

$$\begin{aligned}\iint_{(0,T)\times\omega_1} & s^3\lambda^4\xi_2^3 e^{2s\varphi_2}|\Delta u|^2 \,\mathrm{d}t\,\mathrm{d}x \\ &\leqslant \iint_{(0,T)\times\omega} \chi_1 s^3\lambda^4\xi_2^3 e^{2s\varphi_2}|\Delta u|^2 \,\mathrm{d}t\,\mathrm{d}x = \iint_{(0,T)\times\omega} s^3\lambda^4\xi_2^3 e^{2s\varphi_2}\Delta\left(\chi_1\Delta u\right) u \,\mathrm{d}t\,\mathrm{d}x \\ &\qquad\leqslant \varepsilon \left\|\rho_4 u\right\|^2_{L^2(0,T;H^4(\Omega))} + C\iint_{(0,T)\times\omega} s^8\lambda^2\xi_2^8 e^{4s\varphi_2-2s\varphi_1}|u|^2 \,\mathrm{d}t\,\mathrm{d}x. \end{aligned} \tag{141}$$

Gathering (130), (138), (139), (140), and (141), and using (134), (136) and (137) we deduce

$$\begin{aligned}\int_0^T \rho_0(t)^2 & \left(\|u(t)\|^2_{L^2(\Omega)} + \left\|\eta(t)\right\|^2_{H^2(\mathscr{I})} + \left\|\partial_t\eta(t)\right\|^2_{L^2(\mathscr{I})}\right) \mathrm{d}t \\ &\lesssim \iint_{(0,T)\times\mathscr{I}_1} \rho_1^2|\eta|^2 \,\mathrm{d}t\,\mathrm{d}x_1 + \iint_{(0,T)\times\omega} \rho_5^2|u|^2 \,\mathrm{d}t\,\mathrm{d}x,\end{aligned} \tag{142}$$

with ρ_1 defined by (126) and with

$$\rho_5 := s^4\lambda\xi_2^4 e^{2s\varphi_2-s\varphi_1}. \tag{143}$$

To end the proof of Proposition 24, we need to replace in the above estimate the observation by η with an observation by $\partial_t\eta$. This is done by using the smoothing effet of the parabolic system (23). More precisely, we apply (142) to $(\partial_t u, \partial_t\eta, \partial_t^2\eta)$ and we deduce

$$\begin{aligned}\int_0^T \rho_0(t)^2 & \left(\|\partial_t u(t)\|^2_{L^2(\Omega)} + \left\|\partial_t\eta(t)\right\|^2_{H^2(\mathscr{I})} + \left\|\partial_t^2\eta(t)\right\|^2_{L^2(\mathscr{I})}\right) \mathrm{d}t \\ &\lesssim \iint_{(0,T)\times\mathscr{I}} \rho_1^2|\partial_t\eta|^2 \,\mathrm{d}t\,\mathrm{d}x_1 + \iint_{(0,T)\times\omega} \rho_5^2|\partial_t u|^2 \,\mathrm{d}t\,\mathrm{d}x.\end{aligned} \tag{144}$$

On the other hand, using (23) and the fact that $0 \in \rho(\mathscr{A})$ (see, for instance, [1, Proposition 3.5]),

$$\left\|\partial_t[u,\eta,\partial_t\eta]\right\|_{\mathscr{H}} = \left\|\mathscr{A}[u,\eta,\partial_t\eta]\right\|_{\mathscr{H}} \geqslant c\left\|[u,\eta,\partial_t\eta]\right\|_{\mathscr{H}}.$$

Combining the above estimate and (144) implies

$$\int_0^T \rho_0^2\left\|[u,\eta,\partial_t\eta]\right\|^2_{\mathscr{H}} \,\mathrm{d}t \lesssim \iint_{(0,T)\times\mathscr{I}} \rho_1^2|\partial_t\eta|^2 \,\mathrm{d}t\,\mathrm{d}x_1 + \iint_{(0,T)\times\omega} \rho_5^2|\partial_t u|^2 \,\mathrm{d}t\,\mathrm{d}x. \tag{145}$$

We integrate by parts the last term: recalling (143), we obtain that for any $\varepsilon > 0$, there exists $C > 0$ such that

$$\begin{aligned}\iint_{(0,T)\times\omega_1} \rho_5^2|\partial_t u|^2 \,\mathrm{d}t\,\mathrm{d}x \leqslant C\iint_{(0,T)\times\omega_1} s^{11}\lambda^2\xi_2^{11} e^{4s\varphi_2-2s\varphi_1}|u|^2 \,\mathrm{d}t\,\mathrm{d}x + \varepsilon\left\|\rho_4\partial_t^2 u\right\|^2_{L^2(0,T;L^2(\Omega))} \\ + C\iint_{(0,T)\times\omega_1} s^{18}\lambda^{-2}\xi_2^{18} e^{8s\varphi_2-6s\varphi_1}|u|^2 \,\mathrm{d}t\,\mathrm{d}x.\end{aligned} \tag{146}$$

We deduce (127) by combining (145), (146) and (137). □

Using Proposition 24, one can deduce Theorem 2:

Proof of Theorem 2. We fix $\lambda = \lambda_0$ and $s = s_0(T^2 + T^4)$ in (125) and (126). In particular the constants that follows may depend on λ_0 and s_0. Then we deduce from (127) that

$$\int_{T/4}^{3T/4} \rho_0^2\left\|[u,\eta,\partial_t\eta]\right\|^2_{\mathscr{H}} \,\mathrm{d}t \lesssim \iint_{(0,T)\times\mathscr{I}} \rho_1(t)^2|\partial_t\eta|^2 \,\mathrm{d}t\,\mathrm{d}x_1 + \iint_{(0,T)\times\omega} \rho_2(t)^2|u|^2 \,\mathrm{d}t\,\mathrm{d}x. \tag{147}$$

From (125) and (126), there exists $C > 0$ such that

$$\frac{C}{T}e^{-C/T^2} \leqslant \rho_0 \quad \text{in } \left(\frac{T}{4}, \frac{3T}{4}\right),$$

and

$$\rho_1 \lesssim 1, \quad \rho_2 \lesssim 1.$$

Since $\mathscr{A}$ is the generator of a semigroup of contractions (see, for instance, [1, Proposition 3.4]), we deduce the result from the above relations. □

6. Proof of Theorem 4

We give here a sketch of the proof of Theorem 4. First we construct a change of variables to write (2) in a cylindrical domain, then we use the "source term method" and Theorem 2 to show Theorem 4 by a fixed-point argument.

6.1. *Change of variables*

We can assume that for some $\delta > 0$

$$\omega \subset \mathscr{I} \times (0, 1-\delta).$$

Let us consider a smooth function $\theta \in C^\infty([0,1];[0,1])$ with compact support in $(1-\delta, 1]$ and such that $\theta \equiv 1$ in $[1-\delta/2, 1]$. We consider the change of variables

$$X(t,\cdot) : \Omega \to \Omega_{\zeta(t)}, \quad (y_1, y_2) \mapsto (y_1, y_2 + \theta(y_2)\zeta(t, y_1)) \tag{148}$$

that is a diffeomorphism if

$$\|\theta'\|_{L^\infty(0,1)} \|\zeta\|_{L^\infty(0,T;L^\infty(\mathscr{I}))} \leqslant \frac{1}{2}. \tag{149}$$

We denote by $Y(t,\cdot)$ the inverse of $X(t,\cdot)$.

We write

$$W(t,y) := \operatorname{Cof}(\nabla X)^*(t,y) w(t, X(t,y)), \quad \Pi(t,y) := \pi(t, X(t,y))$$
$$X^0 := X(0,\cdot), \quad W^0 := \operatorname{Cof}(\nabla X^0)^* w^0 \circ X^0.$$

We also write

$$a := \operatorname{Cof}(\nabla Y)^*. \tag{150}$$

After some standard calculations (see, for instance, [2]) (2) is transformed into

$$\begin{cases} \partial_t W - \operatorname{div} \mathbb{T}(W,\Pi) = 1_\omega f + \mathbb{F}_\zeta(W,\Pi) & t>0,\ x \in \Omega, \\ \operatorname{div} W = 0 & t>0,\ x\in\Omega, \\ W = \partial_t \zeta e_2 & t>0,\ x \in \Gamma_1, \\ W = 0 & t>0,\ x\in\Gamma_0, \\ \partial_{tt}\zeta + A_1\zeta + A_2\partial_t\zeta = P_{L^2_0(\mathscr{I})}\left(\Pi + 1_{\mathscr{I}} g + \mathbb{G}_\zeta(W)\right) & t>0,\ x_1 \in \mathscr{I}, \end{cases} \tag{151}$$

$$W(0,\cdot) = W^0 \quad \text{in } \Omega, \quad \zeta(0,\cdot) = \zeta_1^0, \quad \partial_t\zeta(0,\cdot) = \zeta_2^0 \quad \text{in } \mathscr{I}, \tag{152}$$

with

$$\begin{aligned} \left[\mathbb{F}_\zeta(W,\Pi)\right]_i :=& -\sum_k \partial_t a_{i,k}(X) W_k + \left(a_{i,k}(X) - \delta_{i,k}\right)\partial_t W_k - \sum_{k,l} a_{i,k}(X) \frac{\partial W_k}{\partial y_l}\frac{\partial Y_l}{\partial t}(X) \\ &+ \sum_{k,j} \frac{\partial^2 a_{i,k}}{\partial x_j^2}(X) W_k + 2\sum_{k,j,l} \frac{\partial a_{i,k}}{\partial x_j}(X)\frac{\partial W_k}{\partial y_l}\frac{\partial Y_l}{\partial x_j}(X) + \sum_{k,j,l} a_{i,k}(X)\frac{\partial W_k}{\partial y_l}\frac{\partial^2 Y_l}{\partial x_j^2}(X) \\ &+ \sum_{k,j,l,m}\left(a_{i,k}(X)\frac{\partial Y_l}{\partial x_j}(X)\frac{\partial Y_m}{\partial x_j}(X) - \delta_{i,k}\delta_{l,j}\delta_{m,j}\right)\frac{\partial W_k}{\partial y_l \partial y_m} - \sum_l \left(\frac{\partial Y_l}{\partial x_i}(X) - \delta_{l,i}\right)\frac{\partial \Pi}{\partial y_l} \\ &- \sum_{j,m,k} a_{j,m}(X)\frac{\partial a_{i,k}}{\partial x_j}(X) W_m W_k - \frac{1}{\det(\nabla X)}\sum_{m,k} a_{i,k}(X) W_m \frac{\partial W_k}{\partial y_m}, \end{aligned} \tag{153}$$

$$\mathbb{G}_\zeta(W) := -\left(\partial_{x_1}\zeta + \left(\partial_{x_1}\zeta\right)^2\right) \left.\frac{\partial W_1}{\partial x_2}\right|_{x_2=1}. \tag{154}$$

Then, we can write Theorem 4 as follows:

Theorem 25. *Assume $T > 0$, $\omega \Subset \Omega$ and $\mathscr{J} \Subset \mathscr{I}$ are nonempty open sets. There exists $R_0 > 0$ such that for any $[W^0, \zeta_1^0, \zeta_2^0] \in \mathcal{V}$ with*

$$\left\| [W^0, \zeta_1^0, \zeta_2^0] \right\|_{\mathcal{V}} \leqslant R_0, \tag{155}$$

there exists a control

$$(f, g) \in L^2(0, T; L^2(\omega)) \times L^2(0, T; L^2(\mathscr{J}))$$

such that the solution of (151), (152), (153) *and* (154) *satisfies*

$$\zeta(T,\cdot) = 0, \quad \partial_t\zeta(T,\cdot) = 0 \quad \text{in } \mathscr{I}, \quad W(T,\cdot) = 0 \quad \text{in } \Omega.$$

6.2. *The fixed point argument*

Using the notation of Section 2.1, the result of Theorem 2 states the existence of k_T satisfying (14) such that for any $\left[u^0, \eta_1^0, \eta_2^0\right]$,

$$\left\| e^{T\mathscr{A}^*} \begin{bmatrix} u^0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix} \right\|_{\mathscr{H}}^2 \leqslant k_T^2 \int_0^T \left\| \mathscr{B}^* e^{t\mathscr{A}^*} \begin{bmatrix} u^0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix} \right\|_{L^2(\omega)\times L^2(\mathscr{J})}^2 \mathrm{d}t. \tag{156}$$

From standard results (see, for instance, [53, Theorem 11.2.1, p. 357]), this yields the null-controllability of (5). Using the "source term method" (see, [37]), one can improve this result. Let us consider the following weight functions

$$\sigma_1(t) := e^{-\frac{C_1}{(T-t)^2}}, \quad \sigma_2(t) := e^{-\frac{C_2}{(T-t)^2}}, \quad \sigma_3(t) := e^{-\frac{C_3}{(T-t)^2}} \tag{157}$$

and the corresponding spaces (for $\sigma = \sigma_1, \sigma_2$ or σ_3)

$$L^p_\sigma(0,T;\mathscr{X}) := \left\{ f/\sigma \in L^p(0,T;\mathscr{X}) \right\},$$
$$C^\alpha_\sigma([0,T];\mathscr{X}) := \left\{ f/\sigma \in C^\alpha([0,T];\mathscr{X}) \right\},$$
$$H^s_\sigma(0,T;\mathscr{X}) := \left\{ f/\sigma \in H^s(0,T;\mathscr{X}) \right\},$$

for $p \geqslant 1$, $k \in \mathbb{N}$, $s \in \mathbb{R}_+$ and $\mathscr{X}$ a Banach space. The abstract result proved in [37] yields the following result:

Proposition 26. *Assume* (156) *with* (14)*. Then there exist $\sigma_1, \sigma_2, \sigma_3$ as in* (157) *and a bounded map*

$$\mathbb{E}_T : \mathcal{V} \times L^2_{\sigma_1}(0,T; L^2(\Omega) \times L^2(\mathscr{I})) \to L^2_{\sigma_2}(0,T; L^2(\omega) \times L^2(\mathscr{J}))$$

such that for any $[W^0, \zeta_1^0, \zeta_2^0] \in \mathcal{V}$ and for any $(F,G) \in L^2_{\sigma_1}(0,T; L^2(\Omega) \times L^2(\mathscr{I}))$, the solution of

$$\begin{cases} \partial_t W - \operatorname{div} \mathbb{T}(W,\Pi) = 1_\omega f + F & t > 0,\ x \in \Omega, \\ \operatorname{div} W = 0 & t > 0,\ x \in \Omega, \\ W = \partial_t \zeta e_2 & t > 0,\ x \in \Gamma_1, \\ W = 0 & t > 0,\ x \in \Gamma_0, \\ \partial_{tt}\zeta + A_1\zeta + A_2\partial_t\zeta = P_{L^2_0(\mathscr{I})}\left(\Pi + 1_{\mathscr{J}} g + G\right) & t > 0,\ x_1 \in \mathscr{I}, \end{cases} \tag{158}$$

$$W(0,\cdot) = W^0 \quad \text{in } \Omega, \quad \zeta(0,\cdot) = \zeta_1^0, \quad \partial_t\zeta(0,\cdot) = \zeta_2^0 \quad \text{in } \mathscr{I}, \tag{159}$$

with the control

$$(f,g) = \mathbb{E}_T([W^0, \zeta_1^0, \zeta_2^0], (F,G))$$

satisfies

$$\begin{aligned}\|W\|_{L^2_{\sigma_3}(0,T;H^2(\Omega))\cap C^0_{\sigma_3}([0,T];H^1(\Omega))\cap H^1_{\sigma_3}(0,T;L^2(\Omega))} + \|\Pi\|_{L^2_{\sigma_3}(0,T;H^1_0(\Omega))} \\ + \|\zeta\|_{L^2_{\sigma_3}(0,T;H^4(\mathscr{I}))} + \|\zeta\|_{C^0_{\sigma_3}([0,T];H^3(\mathscr{I}))} + \|\zeta\|_{H^1_{\sigma_3}(0,T;H^2(\mathscr{I}))} \\ + \|\zeta\|_{C^1_{\sigma_3}([0,T];H^1(\mathscr{I}))} + \|\zeta\|_{H^2_{\sigma_3}([0,T];L^2(\mathscr{I}))} \\ \lesssim \left\|[W^0,\zeta^0_1,\zeta^0_2]\right\|_{\mathcal{V}} + \|(F,G)\|_{L^2_{\sigma_1}(0,T;L^2(\Omega)\times L^2(\mathscr{I}))}. \end{aligned} \tag{160}$$

Moreover, we can assume

$$\sigma_3^2 \lesssim \sigma_1. \tag{161}$$

We are now in a position to prove Theorem 25 and thus Theorem 4.

Proof of Theorem 25. Assume that $[W^0,\zeta^0_1,\zeta^0_2]$ satisfies (155) for some R_0 and let us assume that

$$\|(F,G)\|_{L^2_{\sigma_1}(0,T;L^2(\Omega)\times L^2(\mathscr{I}))} \leqslant R_0.$$

Applying Proposition 26, we deduce the existence of a control $(f,g)\in L^2_{\sigma_2}(0,T;L^2(\omega)\times L^2(\mathscr{I}))$ such that the corresponding solution of (158), (159) satisfies

$$\begin{aligned}\|W\|_{L^2_{\sigma_3}(0,T;H^2(\Omega))\cap C^0_{\sigma_3}([0,T];H^1(\Omega))\cap H^1_{\sigma_3}(0,T;L^2(\Omega))} + \|\Pi\|_{L^2_{\sigma_3}(0,T;H^1_0(\Omega))} \\ + \|\zeta\|_{L^2_{\sigma_3}(0,T;H^4(\mathscr{I}))} + \|\zeta\|_{C^0_{\sigma_3}([0,T];H^3(\mathscr{I}))} + \|\zeta\|_{H^1_{\sigma_3}(0,T;H^2(\mathscr{I}))} \\ + \|\zeta\|_{C^1_{\sigma_3}([0,T];H^1(\mathscr{I}))} + \|\zeta\|_{H^2_{\sigma_3}([0,T];L^2(\mathscr{I}))} \leqslant CR_0 \end{aligned} \tag{162}$$

for some constant $C>0$. Using the Sobolev embeddings, we have in particular that

$$\|\zeta\|_{C^0([0,T];W^{2,\infty}(\mathscr{I}))} \leqslant CR_0 \tag{163}$$

for some constant $C>0$. This yields that for R_0 small enough, (149) holds and we can consider the change of variables of Section 6.1. We thus define X, a, $\mathbb{F}$ and $\mathbb{G}$ by respectively, (148), (150), (153) and (154). Moreover, following the arguments in [1,48] and using (161), one can show that

$$\left\|\mathbb{F}_\zeta(W,\Pi)\right\|_{L^2_{\sigma_1}(0,T;L^2(\Omega))} + \left\|\mathbb{G}_\zeta(W)\right\|_{L^2_{\sigma_1}(0,T;L^2(\mathscr{I}))} \leqslant CR_0^2, \tag{164}$$

and in particular for R_0 small enough, the closed set

$$B_{R_0} := \left\{(F,G)\in L^2_{\sigma_1}(0,T;L^2(\Omega)\times L^2(\mathscr{I}))\ ;\ \|(F,G)\|_{L^2_{\sigma_1}(0,T;L^2(\Omega)\times L^2(\mathscr{I}))} \leqslant R_0\right\}$$

is invariant under the map

$$\mathcal{Z} : (F,G) \to (\mathbb{F}_\zeta(W,\Pi), \mathbb{G}_\zeta(W)).$$

One can also show that for $R_0>0$ small enough, the above map is a strict contraction on B_{R_0}. Using the Banach fixed point we deduce the existence of fixed point (F,G) for $\mathcal{Z}$. One can notice that the corresponding solution (W,Π,ζ) of (158)–(159) verifies the conclusion of Theorem 25. □

Appendix A. Technical results

A.1. *A Carleman estimates for the damped beam equation*

The proof of Theorem 13 follows directly from the proof done in [41]. The differences with respect to this article is the weight in time and the powers of s, μ, and ξ_0. For sake of completeness, we give here a brief sketch of the proof of Theorem 13 by using what is already done in [41].

We recall that φ_0 and ξ_0 are given by (29). We set

$$f_\eta := \partial_t^2\eta + \partial_{x_1}^4\eta - \partial_{x_1}^2\eta - \partial_t\partial_{x_1}^2\eta, \tag{165}$$

$$\zeta := e^{s\varphi_0}\xi_0^r\eta. \tag{166}$$

We say that a function g is l. o. t (lower order term) if it satisfies for some $\varepsilon_1, \varepsilon_2 \geqslant 0$, $\varepsilon_1\varepsilon_2 \neq 0$,

$$|g| \lesssim s^{-\varepsilon_1}\lambda^{-\varepsilon_2}\xi_0^{-\varepsilon_1}\left(s^{7/2}\mu^4\xi_0^{7/2}|\zeta| + s^{5/2}\mu^3\xi_0^{5/2}\left|\partial_{x_1}\zeta\right| + s^{3/2}\mu^2\xi_0^{3/2}\left(\left|\partial_{x_1}^2\zeta\right| + |\partial_t\zeta|\right)\right.$$
$$\left. + s^{1/2}\mu\xi_0^{1/2}\left(\left|\partial_{x_1}^3\zeta\right| + \left|\partial_t\partial_{x_1}\zeta\right|\right) + s^{-1/2}\xi_0^{-1/2}\left(\left|\partial_{x_1}^4\zeta\right| + \left|\partial_t\partial_{x_1}^2\zeta\right| + \left|\partial_t^2\zeta\right|\right)\right).$$

From the Leibniz formula

$$e^{s\varphi_0}\xi_0^r\frac{\partial^4}{\partial x_1^4}\left(e^{-s\varphi_0}\xi_0^{-r}\zeta\right) = \sum_{\alpha=0}^{4}\binom{4}{\alpha}e^{s\varphi_0}\frac{\partial^\alpha}{\partial x_1^\alpha}\left(e^{-s\varphi_0}\zeta\right)\xi_0^r\frac{\partial^{4-\alpha}}{\partial x_1^{4-\alpha}}\left(\xi_0^{-r}\right).$$

From (49), we obtain

$$\left|\xi_0^r\frac{\partial^{4-\alpha}}{\partial x_1^{4-\alpha}}\left(\xi_0^{-r}\right)\right| \lesssim \mu^{4-\alpha}$$

and thus a direct computation and (49) yield that for $s \geqslant s_0(T^2+T^4)$,

$$\sum_{\alpha=0}^{3}\binom{4}{\alpha}e^{s\varphi_0}\frac{\partial^\alpha}{\partial x_1^\alpha}\left(e^{-s\varphi_0}\zeta\right)\xi_0^r\frac{\partial^{4-\alpha}}{\partial x_1^{4-\alpha}}\left(\xi_0^{-r}\right) = \text{l. o. t.}$$

We also deduce from (49) that

$$e^{s\varphi_0}\frac{\partial^4}{\partial x_1^4}\left(e^{-s\varphi_0}\zeta\right) = -4s^3\left(\partial_{x_1}\varphi_0\right)^3\partial_{x_1}\zeta + 6s^2\left(\partial_{x_1}\varphi_0\right)^2\partial_{x_1}^2\zeta - 4s\partial_{x_1}\varphi_0\partial_{x_1}^3\zeta + \partial_{x_1}^4\zeta + s^4\left(\partial_{x_1}\varphi_0\right)^4\zeta$$
$$- 12s^3\left(\partial_{x_1}\varphi_0\right)^2\partial_{x_1}^2\varphi_0\zeta + \text{l. o. t}$$

and thus

$$e^{s\varphi_0}\xi_0^r\frac{\partial^4}{\partial x_1^4}\left(e^{-s\varphi_0}\xi_0^{-r}\zeta\right) = -4s^3\left(\partial_{x_1}\varphi_0\right)^3\partial_{x_1}\zeta + 6s^2\left(\partial_{x_1}\varphi_0\right)^2\partial_{x_1}^2\zeta - 4s\partial_{x_1}\varphi_0\partial_{x_1}^3\zeta + \partial_{x_1}^4\zeta$$
$$+ s^4\left(\partial_{x_1}\varphi_0\right)^4\zeta - 12s^3\left(\partial_{x_1}\varphi_0\right)^2\partial_{x_1}^2\varphi_0\zeta + \text{l. o. t.} \quad (167)$$

Note that

$$-12s^3\left(\partial_{x_1}\varphi_0\right)^2\partial_{x_1}^2\varphi_0\zeta = \text{l. o. t},$$

but we follow the trick of [41] to keep this term in order to show the Carleman estimate.

We can show similarly that

$$e^{s\varphi_0}\xi_0^r\frac{\partial^2}{\partial x_1^2}\left(e^{-s\varphi_0}\xi_0^{-r}\zeta\right) = \text{l. o. t.} \quad (168)$$

We also have

$$e^{s\varphi_0}\xi_0^r\frac{\partial^2}{\partial t^2}\left(e^{-s\varphi_0}\xi_0^{-r}\zeta\right) = e^{s\varphi_0}\frac{\partial^2}{\partial t^2}\left(e^{-s\varphi_0}\zeta\right) + 2e^{s\varphi_0}\frac{\partial}{\partial t}\left(e^{-s\varphi_0}\zeta\right)\xi_0^r\frac{\partial}{\partial t}\left(\xi_0^{-r}\right) + \zeta\xi_0^r\frac{\partial^2}{\partial t^2}\left(\xi_0^{-r}\right).$$

Thus, using (49), for $s \geqslant s_0(T^2+T^4)$,

$$e^{s\varphi_0}\xi_0^r\frac{\partial^2}{\partial t^2}\left(e^{-s\varphi_0}\xi_0^{-r}\zeta\right) = \partial_t^2\zeta + \text{l. o. t.} \quad (169)$$

Finally,

$$e^{s\varphi_0}\xi_0^r\frac{\partial}{\partial_t}\frac{\partial^2}{\partial x_1^2}\left(e^{-s\varphi_0}\xi_0^{-r}\zeta\right) = e^{s\varphi_0}\frac{\partial^2}{\partial x_1^2}\left(e^{-s\varphi_0}\zeta\right)\xi_0^r\frac{\partial}{\partial t}\left(\xi_0^{-r}\right) + e^{s\varphi_0}\frac{\partial}{\partial t}\frac{\partial^2}{\partial x_1^2}\left(e^{-s\varphi_0}\zeta\right)$$
$$+ 2e^{s\varphi_0}\frac{\partial}{\partial x_1}\left(e^{-s\varphi_0}\zeta\right)\xi_0^r\frac{\partial}{\partial t}\frac{\partial}{\partial x_1}\left(\xi_0^{-r}\right) + 2e^{s\varphi_0}\frac{\partial}{\partial t}\frac{\partial}{\partial x_1}\left(e^{-s\varphi_0}\zeta\right)\xi_0^r\frac{\partial}{\partial x_1}\left(\xi_0^{-r}\right)$$
$$+ \zeta\xi_0^r\frac{\partial}{\partial t}\frac{\partial^2}{\partial x_1^2}\left(\xi_0^{-r}\right) + e^{s\varphi_0}\frac{\partial}{\partial t}\left(e^{-s\varphi_0}\zeta\right)\xi_0^r\frac{\partial^2}{\partial x_1^2}\left(\xi_0^{-r}\right).$$

From (49), for $s \geqslant s_0(T^2+T^4)$, and for $p=0,1,2$,

$$\left|\xi_0^r \frac{\partial^p}{\partial t^p}\frac{\partial^\alpha}{\partial x_1^\alpha}\left(\xi_0^{-r}\right)\right| \lesssim \mu^\alpha \xi_0^{p/2}$$

and

$$e^{s\varphi_0}\xi_0^r \frac{\partial}{\partial_t}\frac{\partial^2}{\partial x_1^2}\left(e^{-s\varphi_0}\xi_0^{-r}\zeta\right) = e^{s\varphi_0}\frac{\partial}{\partial t}\frac{\partial^2}{\partial x_1^2}\left(e^{-s\varphi_0}\zeta\right) + \text{l.o.t.}$$

Consequently, using (49), we deduce that for $s \geqslant s_0(T^2+T^4)$,

$$e^{s\varphi_0}\xi_0^r \frac{\partial}{\partial t}\frac{\partial^2}{\partial x_1^2}\left(e^{-s\varphi_0}\xi_0^{-r}\zeta\right) = s^2\left(\partial_{x_1}\varphi_0\right)^2\partial_t\zeta - 2s\partial_{x_1}\varphi_0\partial_t\partial_{x_1}\zeta + \partial_t\partial_{x_1}^2\zeta + \text{l.o.t.}$$

Gathering (167), (168), (169), and the above relation and combining them with (165) and (166), we deduce

$$M_1\zeta + M_2\zeta = e^{s\varphi_0}\xi_0^r f_\eta + \text{l.o.t}, \tag{170}$$

with

$$M_1\zeta := s^4\left(\partial_{x_1}\varphi_0\right)^4\zeta + 6s^2\left(\partial_{x_1}\varphi_0\right)^2\partial_{x_1}^2\zeta + \partial_{x_1}^4\zeta + 2s\left(\partial_{x_1}\varphi_0\right)\partial_t\partial_{x_1}\zeta + \partial_t^2\zeta,$$

and

$$M_2\zeta := -4s^3\left(\partial_{x_1}\varphi_0\right)^3\partial_{x_1}\zeta - 4s\partial_{x_1}\varphi_0\partial_{x_1}^3\zeta - \partial_t\partial_{x_1}^2\zeta - s^2\left(\partial_{x_1}\varphi_0\right)^2\partial_t\zeta - 12s^3\left(\partial_{x_1}\varphi_0\right)^2\partial_{x_1}^2\varphi_0\zeta.$$

In what follows, we say that a term G is a L.O.T (Lower Order Term) if there exist $\varepsilon_1 \geqslant 0$, $\varepsilon_2 \geqslant 0$, $\varepsilon_1\varepsilon_2 \neq 0$, such that

$$\begin{aligned} |G| \lesssim \iint_{(0,T)\times\mathscr{I}} s^{-\varepsilon_1}\lambda^{-\varepsilon_2}\xi_0^{-\varepsilon_1}\Big(s^7\mu^8\xi_0^7|\zeta|^2 + s^5\mu^6\xi_0^5|\partial_{x_1}\zeta|^2 + s^3\mu^4\xi_0^3\left(|\partial_{x_1}^2\zeta|^2 + |\partial_t\zeta|^2\right) \\ + s\mu^2\xi_0\left(|\partial_{x_1}^3\zeta|^2 + |\partial_t\partial_{x_1}\zeta|^2\right) + s^{-1}\xi_0^{-1}\left(\left|\partial_{x_1}^4\zeta\right|^2 + |\partial_t^2\zeta|^2 + |\partial_t\partial_{x_1}^2\zeta|^2\right)\Big)\,\mathrm{d}t\,\mathrm{d}x_1 \end{aligned} \tag{171}$$

Then, we deduce

$$\begin{aligned} \|M_1\zeta\|^2_{L^2(0,T;L^2(\Omega))} + \|M_2\zeta\|^2_{L^2(0,T;L^2(\Omega))} + 2\iint_{(0,T)\times\mathscr{I}} M_1\zeta\cdot M_2\zeta\,\mathrm{d}x_1\,\mathrm{d}t \\ = \left\|e^{s\varphi_0}\xi_0^r f_\eta\zeta\right\|^2_{L^2(0,T;L^2(\Omega))} + \text{L.O.T.} \end{aligned} \tag{172}$$

Writing $I_{i,j}$ for the product term of the i-th term of $M_1\zeta$ with the j-th term if $M_2\zeta$, we have

$$\iint_{(0,T)\times\mathscr{I}} M_1\zeta\cdot M_2\zeta\,\mathrm{d}x_1\,\mathrm{d}t = \sum_{i,j\in\{1,\dots,5\}} I_{i,j}$$

and we have to estimate all the terms $I_{i,j}$. This is done in a precise way in [41] using (49). For instance, by integration by parts,

$$\begin{aligned} I_{1,2} = -4s^5\iint_{(0,T)\times\mathscr{I}}\left(\partial_{x_1}\varphi_0\right)^5\zeta\partial_{x_1}^3\zeta\,\mathrm{d}x_1\,\mathrm{d}t = -30s^5\iint_{(0,T)\times\mathscr{I}}\left(\partial_{x_1}\varphi_0\right)^4\partial_{x_1}^2\varphi_0\left(\partial_{x_1}\zeta\right)^2\,\mathrm{d}x_1\,\mathrm{d}t \\ + 20s^5\iint_{(0,T)\times\mathscr{I}}\left[4\left(\partial_{x_1}\varphi_0\right)^3\left(\partial_{x_1}^2\varphi_0\right)^2 + \left(\partial_{x_1}\varphi_0\right)^4\partial_{x_1}^3\varphi_0\right]\zeta\partial_{x_1}\zeta\,\mathrm{d}x_1\,\mathrm{d}t \end{aligned}$$

and using (49), we deduce, as in [41] that

$$I_{1,2} = -4s^5\iint_{(0,T)\times\mathscr{I}}\left(\partial_{x_1}\varphi_0\right)^5\zeta\partial_{x_1}^3\zeta\,\mathrm{d}x_1\,\mathrm{d}t = -30s^5\iint_{(0,T)\times\mathscr{I}}\left(\partial_{x_1}\varphi_0\right)^4\partial_{x_1}^2\varphi_0\left(\partial_{x_1}\zeta\right)^2\,\mathrm{d}x_1\,\mathrm{d}t + \text{L.O.T.}$$

Then, following the computations in [41], we find

$$\iint_{(0,T)\times\mathscr{I}} M_1\zeta\cdot M_2\zeta \,\mathrm{d}x_1\,\mathrm{d}t \geqslant c\iint_{(0,T)\times\mathscr{I}} \Big(s^7\mu^8\xi_0^7|\zeta|^2 + s^5\mu^6\xi_0^5|\partial_{x_1}\zeta|^2 + s^3\mu^4\xi_0^3\left(|\partial_{x_1}^2\zeta|^2 + |\partial_t\zeta|^2\right)$$
$$+ s\mu^2\xi_0\left(|\partial_{x_1}^3\zeta|^2 + |\partial_t\partial_{x_1}\zeta|^2\right) + s^{-1}\xi_0^{-1}\left(\left|\partial_{x_1}^4\zeta\right|^2 + |\partial_t^2\zeta|^2 + |\partial_t\partial_{x_1}^2\zeta|^2\right)\Big)\,\mathrm{d}t\,\mathrm{d}x_1$$
$$- C\iint_{(0,T)\times\mathscr{I}_0} \Big(s^7\mu^8\xi_0^7|\zeta|^2 + s^5\mu^6\xi_0^5|\partial_{x_1}\zeta|^2 + s^3\mu^4\xi_0^3\left(|\partial_{x_1}^2\zeta|^2 + |\partial_t\zeta|^2\right)$$
$$+ s\mu^2\xi_0\left(|\partial_{x_1}^3\zeta|^2 + |\partial_t\partial_{x_1}\zeta|^2\right) + s^{-1}\xi_0^{-1}\left(\left|\partial_{x_1}^4\zeta\right|^2 + |\partial_t^2\zeta|^2 + |\partial_t\partial_{x_1}^2\zeta|^2\right)\Big)\,\mathrm{d}t\,\mathrm{d}x_1.$$

Then by using standard techniques as in [41], we deduce the result.

A.2. *A Carleman estimate for the heat equation*

We give here a sketch of the proof of Theorem 15. We recall that φ and ξ are defined by (28) and we define

$$\begin{aligned}\widetilde{\varphi}(t,x_1,x_2) &:= \frac{1}{\ell(t)^2}\left(e^{-\lambda\psi_\Omega(x_1,x_2)+\mu\psi_{\mathscr{I}}(x_1)+8\lambda\Psi} - e^{10\lambda\Psi}\right),\\ \widetilde{\xi}(t,x_1,x_2) &:= \frac{1}{\ell(t)^2}e^{-\lambda\psi_\Omega(x_1,x_2)+\mu\psi_{\mathscr{I}}(x_1)+8\lambda\Psi}.\end{aligned} \tag{173}$$

We also recall that ψ is defined by (30) and we define

$$\widetilde{\psi}(x_1,x_2) := \frac{\mu}{\lambda}\psi_{\mathscr{I}}(x_1) - \psi_\Omega(x_1,x_2). \tag{174}$$

We have in particular

$$\varphi = \frac{1}{\ell^2}\left(e^{\lambda(\psi+8\Psi)} - e^{10\lambda\Psi}\right),\quad \xi := \frac{1}{\ell^2}e^{\lambda(\psi+8\Psi)},\quad \widetilde{\varphi} = \frac{1}{\ell^2}\left(e^{\lambda(\widetilde{\psi}+8\Psi)} - e^{10\lambda\Psi}\right),\quad \widetilde{\xi} := \frac{1}{\ell^2}e^{\lambda(\widetilde{\psi}+8\Psi)}.$$

We set

$$v := e^{s\varphi}\xi^r u,\quad \widetilde{v} := e^{s\widetilde{\varphi}}\widetilde{\xi}^r u.$$

Using (25), we have

$$\psi = \widetilde{\psi},\quad \varphi = \widetilde{\varphi},\quad \xi = \widetilde{\xi},\quad v = \widetilde{v},\quad \frac{\partial\psi}{\partial x_1} = \frac{\partial\widetilde{\psi}}{\partial x_1},\quad \frac{\partial\psi}{\partial n} = -\frac{\partial\widetilde{\psi}}{\partial n}\quad \text{on } (0,T)\times\partial\Omega, \tag{175}$$

and using that $\frac{\partial u_2}{\partial n} = 0$ on $(0,T)\times\Gamma_1$, we deduce that

$$\frac{\partial v_2}{\partial n} = -\frac{\partial\widetilde{v}_2}{\partial n}\quad \text{on } (0,T)\times\Gamma_1. \tag{176}$$

Since $\mu = \mu_0$, taking $\lambda_0 \geqslant \mu_0$ and $\lambda \geqslant \lambda_0$, we deduce that

$$\left|\psi\right| + \left|\nabla\psi\right| + \left|\nabla^2\psi\right| + \left|\widetilde{\psi}\right| + \left|\widetilde{\nabla}\psi\right| + \left|\widetilde{\nabla}^2\psi\right| \lesssim 1.$$

There exists $s_0 > 0$ such that for $s \geqslant s_0(T^2 + T^4)$,

$$1 \leqslant s\xi,\ \left|\nabla^\alpha\xi\right| + \left|\nabla^\alpha\varphi\right| \lesssim \lambda^\alpha\xi\ (k\geqslant 1),\ \left|\partial_t\nabla^\alpha\xi\right| + \left|\partial_t\nabla^\alpha\varphi\right| \lesssim \lambda^\alpha T\xi^{3/2},\ \left|\partial_t^2\nabla^\alpha\xi\right| + \left|\partial_t^2\nabla^\alpha\varphi\right| \lesssim \lambda^\alpha T^2\xi^2,$$

$$1 \leqslant s\widetilde{\xi},\ \left|\nabla^\alpha\widetilde{\xi}\right| + \left|\nabla^\alpha\widetilde{\varphi}\right| \lesssim \lambda^\alpha\widetilde{\xi}\ (k\geqslant 1),\ \left|\partial_t\nabla^\alpha\widetilde{\xi}\right| + \left|\partial_t\nabla^\alpha\widetilde{\varphi}\right| \lesssim \lambda^\alpha T\widetilde{\xi}^{3/2},\ \left|\partial_t^2\nabla^\alpha\widetilde{\xi}\right| + \left|\partial_t^2\nabla^\alpha\widetilde{\varphi}\right| \lesssim \lambda^\alpha T^2\widetilde{\xi}^2,$$

Using the above relations and following standard calculations (see, for instance, [17]), we obtain the existence of $s_0, c, C, \widetilde{c}, \widetilde{C} > 0$ such that for $s \geqslant s_0(T^2+T^4)$,

$$\begin{aligned}
c \iint_{(0,T)\times\Omega} &\left(s^3\lambda^4\xi^3 |v|^2 + s\lambda^2\xi |\nabla v|^2 + \frac{1}{s\xi} |\Delta v|^2 + \frac{1}{s\xi} |\partial_t v|^2 \right) \mathrm{d}x\mathrm{d}t \\
&- \iint_{(0,T)\times\partial\Omega} 2s^3\lambda^3\xi^3 \left|\nabla\psi\right|^2 \frac{\partial\psi}{\partial n} |v|^2 \, \mathrm{d}x_1\mathrm{d}t - \iint_{(0,T)\times\partial\Omega} 4s\lambda^2\xi \left|\nabla\psi\right|^2 \frac{\partial v}{\partial n} \cdot v \, \mathrm{d}x_1\mathrm{d}t \\
&- \iint_{(0,T)\times\partial\Omega} 4s\lambda\xi \left(\nabla v \nabla\psi\right) \cdot \frac{\partial v}{\partial n} \, \mathrm{d}x_1\mathrm{d}t + \iint_{(0,T)\times\partial\Omega} 2s\lambda\xi \frac{\partial\psi}{\partial n} |\nabla v|^2 \, \mathrm{d}x_1\mathrm{d}t \\
&- \iint_{(0,T)\times\partial\Omega} 2\partial_t v \cdot \frac{\partial v}{\partial n} \, \mathrm{d}x_1\mathrm{d}t - \iint_{(0,T)\times\partial\Omega} 2s^2\lambda\xi \frac{\partial\psi}{\partial n} \partial_t\varphi |v|^2 \, \mathrm{d}x_1\mathrm{d}t \\
&\leqslant C \left(\iint_{(0,T)\times\Omega} |\partial_t u - \Delta u|^2 \xi^{2r} e^{2s\varphi} \, \mathrm{d}x\mathrm{d}t + \iint_{(0,T)\times\omega_1} s^3\lambda^4\xi^3 |v|^2 \, \mathrm{d}x\mathrm{d}t \right) \quad (177)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{c} \iint_{(0,T)\times\Omega} &\left(s^3\lambda^4\widetilde{\xi}^3 |\widetilde{v}|^2 + s\lambda^2\widetilde{\xi} |\nabla \widetilde{v}|^2 + \frac{1}{s\widetilde{\xi}} |\Delta \widetilde{v}|^2 + \frac{1}{s\widetilde{\xi}} |\partial_t \widetilde{v}|^2 \right) \mathrm{d}x\mathrm{d}t \\
&- \iint_{(0,T)\times\partial\Omega} 2s^3\lambda^3\widetilde{\xi}^3 \left|\nabla\widetilde{\psi}\right|^2 \frac{\partial\widetilde{\psi}}{\partial n} |\widetilde{v}|^2 \, \mathrm{d}x_1\mathrm{d}t - \iint_{(0,T)\times\partial\Omega} 4s\lambda^2\widetilde{\xi} \left|\nabla\widetilde{\psi}\right|^2 \frac{\partial \widetilde{v}}{\partial n} \cdot \widetilde{v} \, \mathrm{d}x_1\mathrm{d}t \\
&- \iint_{(0,T)\times\partial\Omega} 4s\lambda\widetilde{\xi} \left(\nabla \widetilde{v} \nabla\widetilde{\psi}\right) \cdot \frac{\partial \widetilde{v}}{\partial n} \, \mathrm{d}x_1\mathrm{d}t + \iint_{(0,T)\times\partial\Omega} 2s\lambda\widetilde{\xi} \frac{\partial\widetilde{\psi}}{\partial n} |\nabla \widetilde{v}|^2 \, \mathrm{d}x_1\mathrm{d}t \\
&- \iint_{(0,T)\times\partial\Omega} 2\partial_t \widetilde{v} \cdot \frac{\partial \widetilde{v}}{\partial n} \, \mathrm{d}x_1\mathrm{d}t - \iint_{(0,T)\times\partial\Omega} 2s^2\lambda\widetilde{\xi} \frac{\partial\widetilde{\psi}}{\partial n} \partial_t\widetilde{\varphi} |\widetilde{v}|^2 \, \mathrm{d}x_1\mathrm{d}t \\
&\leqslant \widetilde{C} \left(\iint_{(0,T)\times\Omega} |\partial_t u - \Delta u|^2 \widetilde{\xi}^{2r} e^{2s\widetilde{\varphi}} \, \mathrm{d}x\mathrm{d}t + \iint_{(0,T)\times\omega_1} s^3\lambda^4\widetilde{\xi}^3 |\widetilde{v}|^2 \, \mathrm{d}x\mathrm{d}t \right). \quad (178)
\end{aligned}$$

Summing (177) and (178) and using (176), (176), we deduce that

$$\begin{aligned}
c \iint_{(0,T)\times\Omega} &\left(s^3\lambda^4\xi^3 |v|^2 + s\lambda^2\xi |\nabla v|^2 + \frac{1}{s\xi} |\Delta v|^2 + \frac{1}{s\xi} |\partial_t v|^2 \right) \mathrm{d}x\mathrm{d}t \\
&+ \widetilde{c} \iint_{(0,T)\times\Omega} \left(s^3\lambda^4\widetilde{\xi}^3 |\widetilde{v}|^2 + s\lambda^2\widetilde{\xi} |\nabla \widetilde{v}|^2 + \frac{1}{s\widetilde{\xi}} |\Delta \widetilde{v}|^2 + \frac{1}{s\widetilde{\xi}} |\partial_t \widetilde{v}|^2 \right) \mathrm{d}x\mathrm{d}t \\
&\leqslant C \left(\iint_{(0,T)\times\Omega} |\partial_t u - \Delta u|^2 \xi^{2r} e^{2s\varphi} \, \mathrm{d}x\mathrm{d}t + \iint_{(0,T)\times\omega_0} s^3\lambda^4\xi^3 |v|^2 \, \mathrm{d}x\mathrm{d}t \right) \\
&+ \widetilde{C} \left(\iint_{(0,T)\times\Omega} |\partial_t u - \Delta u|^2 \widetilde{\xi}^{2r} e^{2s\widetilde{\varphi}} \, \mathrm{d}x\mathrm{d}t + \iint_{(0,T)\times\omega_1} s^3\lambda^4\widetilde{\xi}^3 |\widetilde{v}|^2 \, \mathrm{d}x\mathrm{d}t \right). \quad (179)
\end{aligned}$$

Then, using that $\widetilde{\varphi} \leqslant \varphi$ and $\widetilde{\xi} \leqslant \xi$, we deduce that

$$\begin{aligned}
\iint_{(0,T)\times\Omega} &\left(s^3\lambda^4\xi^3 |v|^2 + s\lambda^2\xi |\nabla v|^2 + \frac{1}{s\xi} |\Delta v|^2 + \frac{1}{s\xi} |\partial_t v|^2 \right) \mathrm{d}x\mathrm{d}t \\
&\lesssim \iint_{(0,T)\times\Omega} |\partial_t u - \Delta u|^2 \xi^{2r} e^{2s\varphi} \, \mathrm{d}x\mathrm{d}t + \iint_{(0,T)\times\omega_1} s^3\lambda^4\xi^3 |v|^2 \, \mathrm{d}x\mathrm{d}t. \quad (180)
\end{aligned}$$

From the elliptical regularity of the Laplace operator, we deduce

$$\begin{aligned}
\iint_{(0,T)\times\Omega} &\left(s^3\lambda^4\xi^3 |v|^2 + s\lambda^2\xi |\nabla v|^2 + \frac{1}{s\xi} \left|\nabla^2 v\right|^2 + \frac{1}{s\xi} |\partial_t v|^2 \right) \mathrm{d}x\mathrm{d}t \\
&\lesssim \iint_{(0,T)\times\Omega} |\partial_t u - \Delta u|^2 \xi^{2r} e^{2s\varphi} \, \mathrm{d}x\mathrm{d}t + \iint_{(0,T)\times\omega_1} s^3\lambda^4\xi^3 |v|^2 \, \mathrm{d}x\mathrm{d}t \quad (181)
\end{aligned}$$

and with standard computations, we can come back to u:

$$\iint_{(0,T)\times\Omega} e^{2s\varphi}\left(s^{2r+3}\lambda^{2r+4}\xi^{2r+3}|u|^2+s^{2r+1}\lambda^{2r+2}\xi^{2r+1}|\nabla u|^2 \right.$$
$$\left. +s^{2r-1}\lambda^{2r}\xi^{2r-1}\left|\nabla^2 u\right|^2+s^{2r-1}\lambda^{2r}\xi^{2r-1}|\partial_t u|^2\right)\,\mathrm{d}x\mathrm{d}t$$
$$\lesssim \iint_{(0,T)\times\Omega}|\partial_t u-\Delta u|^2(s\lambda\xi)^{2r}e^{2s\varphi}\,\mathrm{d}x\mathrm{d}t+\iint_{(0,T)\times\omega_1}s^{2r+3}\lambda^{2r+4}\xi^{2r+3}e^{2s\varphi}|u|^2\,\mathrm{d}x\mathrm{d}t. \quad (182)$$

We deduce Theorem 15 by taking $r=1/2$.

References

[1] M. Badra, T. Takahashi, "Feedback boundary stabilization of 2D fluid-structure interaction systems", *Discrete Contin. Dyn. Syst.* **37** (2017), no. 5, p. 2315-2373.

[2] ———, "Gevrey regularity for a system coupling the Navier–Stokes system with a beam equation", *SIAM J. Math. Anal.* **51** (2019), no. 6, p. 4776-4814.

[3] ———, "Gevrey regularity for a system coupling the Navier–Stokes system with a beam: the non-flat case", *Funkc. Ekvacioj* **65** (2022), no. 1, p. 63-109.

[4] ———, "Maximal regularity for the Stokes system coupled with a wave equation: application to the system of interaction between a viscous incompressible fluid and an elastic wall", *J. Evol. Equ.* **22** (2022), no. 3, article no. 71 (55 pages).

[5] H. Beirão da Veiga, "On the existence of strong solutions to a coupled fluid-structure evolution problem", *J. Math. Fluid Mech.* **6** (2004), no. 1, p. 21-52.

[6] M. Bellassoued, J. Le Rousseau, "Carleman estimates for elliptic operators with complex coefficients. Part I: Boundary value problems", *J. Math. Pures Appl.* **104** (2015), no. 4, p. 657-728.

[7] ———, "Carleman estimates for elliptic operators with complex coefficients. Part II: Transmission problems", *J. Math. Pures Appl.* **115** (2018), p. 127-186.

[8] M. Boulakia, S. Guerrero, "Local null controllability of a fluid-solid interaction problem in dimension 3", *J. Eur. Math. Soc.* **15** (2013), no. 3, p. 825-856.

[9] M. Boulakia, A. Osses, "Local null controllability of a two-dimensional fluid-structure interaction problem", *ESAIM, Control Optim. Calc. Var.* **14** (2008), no. 1, p. 1-42.

[10] R. Buffe, "Stabilization of the wave equation with Ventcel boundary condition", *J. Math. Pures Appl.* **108** (2017), no. 2, p. 207-259.

[11] R. Buffe, T. Takahashi, "Controllability of a Stokes system with a diffusive boundary condition", *ESAIM, Control Optim. Calc. Var.* **28** (2022), article no. 63 (29 pages).

[12] S. Čanić, B. Muha, M. Bukač, "Fluid-structure interaction in hemodynamics: modeling, analysis, and numerical simulation", in *Fluid-structure interaction and biomedical applications*, Advances in Mathematical Fluid Mechanics, Springer, 2014, p. 79-195.

[13] A. Chambolle, B. Desjardins, M. J. Esteban, C. Grandmont, "Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate", *J. Math. Fluid Mech.* **7** (2005), no. 3, p. 368-404.

[14] S. P. Chen, R. Triggiani, "Proof of extensions of two conjectures on structural damping for elastic systems", *Pac. J. Math.* **136** (1989), no. 1, p. 15-55.

[15] J.-M. Coron, S. Guerrero, "Null controllability of the N-dimensional Stokes system with $N-1$ scalar controls", *J. Differ. Equations* **246** (2009), no. 7, p. 2908-2921.

[16] A. Doubova, E. Fernández-Cara, "Some control results for simplified one-dimensional models of fluid-solid interaction", *Math. Models Methods Appl. Sci.* **15** (2005), no. 5, p. 783-824.

[17] E. Fernández-Cara, M. González-Burgos, S. Guerrero, J.-P. Puel, "Null controllability of the heat equation with boundary Fourier conditions: the linear case", *ESAIM, Control Optim. Calc. Var.* **12** (2006), p. 442-465.

[18] E. Fernández-Cara, S. Guerrero, O. Y. Imanuvilov, J.-P. Puel, "Local exact controllability of the Navier–Stokes system", *J. Math. Pures Appl.* **83** (2004), no. 12, p. 1501-1542.

[19] A. Fursikov, O. Imanuvilov, *Controllability of evolution equations*, Lecture Notes Series, Seoul, vol. 34, Seoul National University, 1996, iv+163 pages.

[20] C. Grandmont, "Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate", *SIAM J. Math. Anal.* **40** (2008), no. 2, p. 716-737.

[21] C. Grandmont, M. Hillairet, "Existence of global strong solutions to a beam-fluid interaction system", *Arch. Ration. Mech. Anal.* **220** (2016), no. 3, p. 1283-1333.

[22] C. Grandmont, M. Hillairet, J. Lequeurre, "Existence of local strong solutions to fluid-beam and fluid-rod interaction systems", *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **36** (2019), no. 4, p. 1105-1149.

[23] L. Hörmander, *The analysis of linear partial differential operators. III: Pseudo-differential operators*, reprint of the 1994 ed. ed., Classics in Mathematics, Springer, 2007.
[24] O. Imanuvilov, T. Takahashi, "Exact controllability of a fluid-rigid body system", *J. Math. Pures Appl.* **87** (2007), no. 4, p. 408-437.
[25] J. Le Rousseau, M. Léautaud, L. Robbiano, "Controllability of a parabolic system with a diffuse interface", *J. Eur. Math. Soc.* **15** (2013), no. 4, p. 1485-1574.
[26] J. Le Rousseau, G. Lebeau, L. Robbiano, *Elliptic Carleman estimates and applications to stabilization and controllability. Volume I. Dirichlet boundary conditions on Euclidean space*, Progress in Nonlinear Differential Equations and their Applications, vol. 97, Birkhäuser, 2021.
[27] ———, *Elliptic Carleman estimates and applications to stabilization and controllability. Volume II. General Boundary Conditions on Riemannian Manifolds*, Progress in Nonlinear Differential Equations and their Applications, vol. 97, Birkhäuser, 2022.
[28] J. Le Rousseau, N. Lerner, "Carleman estimates for anisotropic elliptic operators with jumps at an interface", *Anal. PDE* **6** (2013), no. 7, p. 1601-1648.
[29] J. Le Rousseau, L. Robbiano, "Carleman estimate for elliptic operators with coefficients with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations", *Arch. Ration. Mech. Anal.* **195** (2010), no. 3, p. 953-990.
[30] ———, "Local and global Carleman estimates for parabolic operators with coefficients with jumps at interfaces", *Invent. Math.* **183** (2011), no. 2, p. 245-336.
[31] M. Léautaud, "Spectral inequalities for non-selfadjoint elliptic operators and application to the null-controllability of parabolic systems", *J. Funct. Anal.* **258** (2010), no. 8, p. 2739-2778.
[32] G. Lebeau, L. Robbiano, "Contrôle exact de l'équation de la chaleur", *Commun. Partial Differ. Equations* **20** (1995), no. 1-2, p. 335-356.
[33] D. Lengeler, "Weak solutions for an incompressible, generalized Newtonian fluid interacting with a linearly elastic Koiter type shell", *SIAM J. Math. Anal.* **46** (2014), no. 4, p. 2614-2649.
[34] D. Lengeler, M. Růžička, "Weak solutions for an incompressible Newtonian fluid interacting with a Koiter type shell", *Arch. Ration. Mech. Anal.* **211** (2014), no. 1, p. 205-255.
[35] J. Lequeurre, "Existence of strong solutions to a fluid-structure system", *SIAM J. Math. Anal.* **43** (2011), no. 1, p. 389-410.
[36] J.-L. Lions, E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I*, Grundlehren der Mathematischen Wissenschaften, vol. 181, Springer, 1972, Translated from the French by P. Kenneth, xvi+357 pages.
[37] Y. Liu, T. Takahashi, M. Tucsnak, "Single input controllability of a simplified fluid-structure interaction model", *ESAIM, Control Optim. Calc. Var.* **19** (2013), no. 1, p. 20-42.
[38] D. Maity, A. Roy, T. Takahashi, "Existence of strong solutions for a system of interaction between a compressible viscous fluid and a wave equation", *Nonlinearity* **34** (2021), no. 4, p. 2659-2687.
[39] D. Maity, T. Takahashi, "Existence and uniqueness of strong solutions for the system of interaction between a compressible Navier–Stokes–Fourier fluid and a damped plate equation", *Nonlinear Anal., Real World Appl.* **59** (2021), article no. 103267 (34 pages).
[40] ———, "L^p theory for the interaction between the incompressible Navier–Stokes system and a damped plate", *J. Math. Fluid Mech.* **23** (2021), no. 4, article no. 103 (23 pages).
[41] S. Mitra, "Carleman estimate for an adjoint of a damped beam equation and an application to null controllability", *J. Math. Anal. Appl.* **484** (2020), no. 1, article no. 123718 (29 pages).
[42] ———, "Observability and unique continuation of the adjoint of a linearized simplified compressible fluid-structure model in a 2D channel", *ESAIM, Control Optim. Calc. Var.* **27** (2021), article no. S18 (51 pages).
[43] B. Muha, S. Canić, "Existence of a weak solution to a nonlinear fluid-structure interaction problem modeling the flow of an incompressible, viscous fluid in a cylinder with deformable walls", *Arch. Ration. Mech. Anal.* **207** (2013), no. 3, p. 919-968.
[44] ———, "Existence of a weak solution to a nonlinear fluid-structure interaction problem modeling the flow of an incompressible, viscous fluid in a cylinder with deformable walls", *Arch. Ration. Mech. Anal.* **207** (2013), no. 3, p. 919-968.
[45] B. Muha, S. Čanić, "A nonlinear, 3D fluid-structure interaction problem driven by the time-dependent dynamic pressure data: a constructive existence proof", *Commun. Inf. Syst.* **13** (2013), no. 3, p. 357-397.
[46] ———, "Fluid-structure interaction between an incompressible, viscous 3D fluid and an elastic shell with nonlinear Koiter membrane energy", *Interfaces Free Bound.* **17** (2015), no. 4, p. 465-495.
[47] A. Quarteroni, M. Tuveri, A. Veneziani, "Computational vascular fluid dynamics: problems, models and methods", *Comput. Vis. Sci.* **2** (2000), no. 4, p. 163-197.
[48] J.-P. Raymond, "Feedback stabilization of a fluid-structure model", *SIAM J. Control Optim.* **48** (2010), no. 8, p. 5398-5443.

[49] A. Roy, T. Takahashi, "Local null controllability of a rigid body moving into a Boussinesq flow", *Math. Control Relat. Fields* **9** (2019), no. 4, p. 793-836.

[50] R. Temam, *Navier–Stokes equations. Theory and numerical analysis*, revised ed., Studies in Mathematics and its Applications, vol. 2, North-Holland, 1979, with an appendix by F. Thomasset, x+519 pages.

[51] S. Trifunović, Y.-G. Wang, "Existence of a weak solution to the fluid-structure interaction problem in 3D", *J. Differ. Equations* **268** (2020), no. 4, p. 1495-1531.

[52] ———, "Weak solution to the incompressible viscous fluid and a thermoelastic plate interaction problem in 3D", *Acta Math. Sci., Ser. B, Engl. Ed.* **41** (2021), no. 1, p. 19-38.

[53] M. Tucsnak, G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts. Basler Lehrbücher, Birkhäuser, 2009, xii+483 pages.