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A simple construction of the Rumin algebra

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Abstract. The Rumin algebra of a contact manifold is a contact invariant C_∞-algebra of differential forms which computes the de Rham cohomology algebra. We recover this fact by giving a simple and explicit construction of the Rumin algebra via Markl's formulation of the Homotopy Transfer Theorem.

Keywords. Rumin complex, Rumin algebra, contact invariant.

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1. Introduction

A *contact manifold* is a pair (M^{2n+1},ξ) of a real $(2n+1)$-dimensional manifold and a subbundle $\xi\subset TM$ of rank $2n$ with the property that if θ is a local one-form such that $\ker\theta=\xi$, then $\theta\wedge\mathrm{d}\theta^n\neq 0$. The *Rumin complex* is a cochain complex which is adapted to the contact structure and computes the de Rham cohomology *groups*. There are many constructions of the Rumin complex, including Rumin's original construction via subquotients of differential forms [14], constructions via a spectral sequence [1, 8, 16], and constructions including a homotopy equivalence with the de Rham complex [3, 5, 17]. Importantly, the latter constructions yield, as a consequence of the Homotopy Transfer Theorem [9, 12, 13], a C_∞-structure on the Rumin complex which recovers the de Rham cohomology *algebra*.

The *Rumin algebra* is a specific contact invariant C_∞-algebra of differential forms which computes the de Rham cohomology algebra; indeed, its higher products m_k, $k\geq 4$, all vanish. Its existence follows from the latter constructions above. Calderbank and Diemer showed [3] that a curved Bernstein–Gelfand–Gelfand sequence [4] for a parabolic geometry carries a curved A_∞-structure. Bryant, Eastwood, Gover and Neusser pointed out [1] how to recover the Rumin complex from the BGG sequence for the trivial representation, and their observation extends to the Rumin algebra. Rumin's work [17] on the Rumin complex of a Carnot–Carathéodory space includes a realization of the Rumin complex as a deformation retract of the de Rham complex. While none of these authors explicitly computed the C_∞-structure, Case gave [5] ad hoc formulas for m_k in a manner which resembles a construction of Tsai, Tseng and Yau [20] on symplectic manifolds.

In this short note we present a simple and explicit construction of the Rumin algebra. The key ingredients are a natural contact invariant deformation retract of the de Rham algebra and an explicit formula [13] for the transfer of the Homotopy Transfer Theorem. Our deformation retract coincides with those of Calderbank–Diemer [3] and Rumin [17], and our application of the Homotopy Transfer Principle explains the ad hoc formulas of Case [5].

By construction, the Rumin algebra is homotopy equivalent, as a C_∞-algebra, to the de Rham algebra. It is known that two simply connected manifolds of finite type are rationally homotopy equivalent if and only if their spaces of polynomial differential forms are homotopy equivalent as C_∞-algebras [6, 7, 10, 18]; and that the homotopy type, as a C_∞-algebra, of the de Rham algebra has many implications for the diffeomorphism type of a closed, simply connected, smooth manifold [19]. For these reasons, we expect the Rumin algebra to be a fundamental invariant of contact manifolds.

We define the Rumin algebra as a C_∞-algebra on a subspace of differential forms because of the many applications of the latter (e.g. [18]). This choice means that elements of the Rumin algebra are not sections of a vector bundle and the C_∞-structure is not defined in terms of differential operators. However, the construction is local, and can be formulated in the language of sheaves (cf. [5]). Indeed, the Rumin algebra can be equivalently described in terms of sections of vector bundles [1, 3, 5, 14, 15]. This perspective is used, for example, to prove Hodge theorems [5, 15].

This note is organized as follows.

In Section 2 we recall some basic facts about C_∞-algebras, including Markl's version [13] of the Homotopy Transfer Theorem.

In Section 3 we recall some basic facts about contact manifolds and the symplectic structure on their contact distribution.

In Section 4 we construct the Rumin algebra.

2. C_∞-algebras

In this section we discuss some basic facts about C_∞-algebras, which are the commutative versions of A_∞-algebras. We follow the conventions of Keller's survey article on the latter [11]. We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ the set of nonnegative integers.

Let $A = \bigoplus_{k\in\mathbb{Z}} A^k$ be a $\mathbb{Z}$-graded vector space. A *homogeneous element of* A is an element $\omega \in A^k$ for some $k \in \mathbb{Z}$. In this case we call k the *degree* of ω and we set $|\omega| := k$. A map $f\colon A^{\otimes k} \to A$ is *homogeneous of degree* ℓ if

$$f(A^{i_1} \otimes \cdots \otimes A^{i_k}) \subset A^{i_1+\cdots+i_k+\ell}$$

for all $i_1, \ldots, i_k \in \mathbb{Z}$. In this case we set $|f| := \ell$.

An A_∞*-algebra* is a pair (A, m) of a $\mathbb{Z}$-graded vector space A and a collection $m = \{m_k\}_{k\in\mathbb{N}}$ of homogeneous operators $m_k\colon A^{\otimes k} \to A$ of degree $2-k$ such that

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0 \tag{1}$$

for all $n \in \mathbb{N}$, where $1\colon A \to A$ is the identity map and we use the Koszul sign convention

$$(f \otimes g)(\alpha \otimes \beta) := (-1)^{|g||\alpha|} f(\alpha) \otimes g(\beta)$$

for homogeneous operators $f, g\colon A \to A$ and homogeneous elements $\alpha, \beta \in A$. The first three cases of (1) are

$$\begin{aligned}
&m_1 m_1 = 0,\\
&m_1 m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1),\\
&m_1 m_3 = m_2(1 \otimes m_2 - m_2 \otimes 1) - m_3(m_1 \otimes 1^{\otimes 2} + 1 \otimes m_1 \otimes 1 + 1^{\otimes 2} \otimes m_1).
\end{aligned}$$

In particular, if (A, m) is an A_∞-algebra, then (A, m_1) is a cochain complex. *Graded (associative) algebras* are A_∞-algebras with $m_k = 0$ for all $k \neq 2$. *Differential graded algebras* are A_∞-algebras with $m_k = 0$ for all $k \geq 3$. The *cohomology ring* $(HA, [m_2])$ of (A, m) is the graded algebra

$$HA^k := \frac{\ker m_1 \cap A^k}{\operatorname{im} m_1 \cap A^k},$$
$$[m_2]([\omega_1] \otimes [\omega_2]) := [m_2(\omega_1 \otimes \omega_2)].$$

Given A_∞-algebras (A, m) and $(B, \overline{m})$, an A_∞*-morphism* $f\colon (A, m) \to (B, \overline{m})$ is a collection of homogeneous operators $f_k\colon A^{\otimes k} \to B$ of degree $1 - k$ such that

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = \sum_{\substack{1 \leq r \leq n \\ i_1+\cdots+i_r=n}} (-1)^{\ell} \overline{m}_r(f_{i_1} \otimes \cdots \otimes f_{i_r}),$$
$$\ell := \sum_{j=1}^{r} (r-j)(i_j - 1),$$

for all $n \in \mathbb{N}$. Specializing to the cases $n \in \{1, 2\}$ yields

$$f_1 m_1 = \overline{m}_1 f_1,$$
$$f_1 m_2 = \overline{m}_2(f_1 \otimes f_1) + \overline{m}_1 f_2 + f_2(m_1 \otimes 1 + 1 \otimes m_1).$$

In particular, $f_1\colon (A, m_1) \to (B, \overline{m}_1)$ is a *cochain map*. We call f an A_∞*-quasi-isomorphism* if $[f_1]\colon HA \to HB$, $[f_1]([\alpha]) := [f_1(\alpha)]$, is an isomorphism of graded algebras.

Denote by S_n the group of permutations of $\{1, 2, \ldots, n\}$. We require the left action of S_n on $A^{\otimes n}$ defined as follows: Given $\sigma \in S_n$, let $\rho_\sigma\colon A^{\otimes n} \to A^{\otimes n}$ be the linear map defined on tensor products of homogeneous elements $x_1, \ldots, x_n \in A$ by

$$\rho_\sigma(x_1 \otimes \cdots \otimes x_n) = \operatorname{sgn}(\sigma)\epsilon(\sigma; x_1, \ldots, x_n) x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)},$$

where $\epsilon(\sigma; x_1, \ldots, x_n)$ is the Koszul sign determined by

$$x_1 \wedge \cdots \wedge x_{p+q} = \sum_{\sigma \in S_{p+q}} \epsilon(\sigma; x_1, \ldots, x_{p+q}) x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(p+q)}.$$

Given $p, q \in \mathbb{N}_0$, a (p, q)*-shuffle* is an element $\sigma \in S_{p+q}$ such that

$$\sigma(1) < \cdots < \sigma(p), \quad \sigma(p+1) < \cdots < \sigma(p+q).$$

Note that a (p, q)-shuffle σ is uniquely determined by the set $\{\sigma(1), \ldots, \sigma(p)\}$. Denote by $Sh_{p,q} \subset S_{p+q}$ the subset of (p, q)-shuffles. Given $s, t \in \mathbb{N}$, the sets

$$Sh_{p,q}(s, t) := \left\{\sigma \in Sh_{p,q} : \#\bigl(\{\sigma^{-1}(p+q-j)\}_{j=0}^{s+t-1} \cap \{1, \ldots, p\}\bigr) = s\right\}$$

of (p, q)-shuffles such that $\{\sigma^{-1}(p+q-j)\}_{j=0}^{s+t-1}$ contains exactly s elements of $\{1, \ldots, p\}$ give rise to a partition

$$Sh_{p,q} = \coprod_{s+t=r} Sh_{p,q}(s, t) \tag{2}$$

of $Sh_{p,q}$. Note that if $\sigma \in Sh_{p,q}(s, t)$, then there are unique $\alpha \in Sh_{p-s,q-t}$ and $\beta \in Sh_{s,t}$ such that

$$\sigma = (\alpha \times \beta)\varphi_{s,t}, \tag{3}$$

where α and β act on $\{1, \ldots, p+q-s-t\}$ and $\{p+q-s-t+1, \ldots, p+q\}$, respectively, and $\varphi_{s,t} \in Sh_{p,q}$ is the (p, q)-shuffle

$$\varphi_{s,t}(j) = \begin{cases} j, & \text{if } 1 \leq j \leq p-s, \\ j+q-t, & \text{if } p-s+1 \leq j \leq p, \\ j-s, & \text{if } p+1 \leq j \leq p+q-t, \\ j, & \text{if } p+q-t+1 \leq j \leq p+q, \end{cases}$$

which fixes $\{1, \ldots, p\}$ and maps $\{p-s+1, \ldots, p\}$ to $\{p+q-s-t+1, \ldots, p+q-t\}$.

The (p,q)-*shuffle product* $\nu_{p,q}\colon A^{\otimes(p+q)} \to A^{\otimes(p+q)}$ is

$$\nu_{p,q} := \sum_{\sigma\in Sh_{p,q}} \rho_\sigma.$$

For example, if $\omega,\tau,\eta\in A$ are homogeneous, then

$$\begin{aligned}
\nu_{1,1}(\omega\otimes\tau) &= \omega\otimes\tau - (-1)^{|\omega||\tau|}\tau\otimes\omega,\\
\nu_{1,2}(\omega\otimes\tau\otimes\eta) &= \omega\otimes\tau\otimes\eta - (-1)^{|\omega||\tau|}\tau\otimes\omega\otimes\eta + (-1)^{|\omega|(|\tau|+|\eta|)}\tau\otimes\eta\otimes\omega,\\
\nu_{2,1}(\omega\otimes\tau\otimes\eta) &= \omega\otimes\tau\otimes\eta - (-1)^{|\tau||\eta|}\omega\otimes\eta\otimes\tau + (-1)^{|\eta|(|\omega|+|\tau|)}\eta\otimes\omega\otimes\tau.
\end{aligned}$$

Equations (2) and (3) imply that if $p,q,r\in\mathbb{N}$ are such that $r\le p+q$, then

$$\nu_{p,q} = \sum_{\substack{s,t\in\mathbb{N}_0\\ s+t=r}} (\nu_{p-s,q-t}\otimes\nu_{s,t})\rho_{\varphi_{s,t}}. \tag{4}$$

A C_∞-*algebra* is an A_∞-algebra (A,m) such that

$$m_{p+q}\circ\nu_{p,q} = 0$$

for all $p,q\in\mathbb{N}$. *Graded commutative (associative) algebras* are C_∞-algebras with $m_k=0$ for all $k\neq 2$. *Commutative differential graded algebras* are C_∞-algebras with $m_k=0$ for all $k\ge 3$.

Given C_∞-algebras (A,m) and $(B,\overline{m})$, a C_∞-*morphism* $f\colon (A,m)\to(B,\overline{m})$ is an A_∞-morphism such that

$$f_{p+q}\circ\nu_{p,q} = 0$$

for all $p,q\in\mathbb{N}$. We call f a C_∞-*quasi-isomorphism* if $[f_1]\colon HA\to HB$ is an isomorphism of graded commutative algebras.

The Homotopy Transfer Theorem [12, Theorem 10.3.1] constructs a C_∞-structure on a deformation retract of a commutative differential graded algebra and an extension of the inclusion to a C_∞-quasi-isomorphism. We require explicit formulas for the transferred structure and induced quasi-isomorphism (cf. [12, Theorem 13.1.7]).

Theorem 1. *Let (A,d,μ) be a commutative differential graded algebra and let (B,d) be a subcomplex of (A,d). Suppose that*

$$h \circlearrowright (A,d) \underset{i}{\overset{\pi}{\rightleftarrows}} (B,d) \tag{5}$$

is a deformation retract*; i.e. $\pi\colon (A,d)\to(B,d)$ and $i\colon (B,d)\to(A,d)$ are cochain maps, $h\colon A\to A$ is homogeneous of degree -1, and $i\pi = 1_A - dh - hd$ and $\pi i = 1_B$. Recursively define $\psi_n\colon A^{\otimes n}\to A$, $n\ge 2$, by*

$$\psi_n := \sum_{s+t=n} (-1)^{s+1}\mu(h\psi_s\otimes h\psi_t), \tag{6}$$

with the convention $h\psi_1 = -1_A$. Set $m_1 := d$ and $m_k := \pi\psi_k i^{\otimes k}$, $k\ge 2$, and $f_k := -h\psi_k i^{\otimes k}$, $k\in\mathbb{N}$. Then (B,m) is a C_∞-algebra and $f\colon (B,m)\to(A,d,\mu)$ is a C_∞-quasi-isomorphism with $f_1 = i$.

Proof. Markl computed [13, Theorem 5] that (B,m) is an A_∞-algebra and that $f\colon (B,m)\to(A,d,\mu)$ is an A_∞-quasi-morphism. We prove the final conclusion by proving that $\psi_n\circ\nu_{p,q}=0$ for all $p,q\in\mathbb{N}$ such that $p+q=n$. The proof is by strong induction in n. Since $\psi_2=\mu$, we have that $\psi_2\circ\nu_{1,1}=0$. Suppose now that $\psi_k\circ\nu_{p,q}=0$ for all $2\le k\le n$ and all $p,q\in\mathbb{N}$ such that $p+q=k$. Let $p,q\in\mathbb{N}$ be such that $p+q=n+1$. On the one hand, Equation (4) implies that

$$\psi_{n+1}\circ\nu_{p,q} = \sum_{s+t=p+q}\sum_{\substack{a,b\in\mathbb{N}_0\\ a+b=t}} (-1)^{s+1}\mu(h\psi_s\otimes h\psi_t)(\nu_{p-a,q-b}\otimes\nu_{a,b})\rho_{\varphi_{a,b}}.$$

On the other hand, the inductive hypothesis implies that the only nonzero summands are those for which $(a,b)=(0,q)$ or $(a,b)=(p,0)$. Combining these facts with the observations $\nu_{p,0}=1$ and $\nu_{0,q}=1$ and $\rho_{\varphi_{0,q}}=1$ yields

$$\psi_{n+1}\circ\nu_{p,q}=(-1)^{p+1}\mu(h\psi_p\otimes h\psi_q)+(-1)^{q+1}\mu(h\psi_q\otimes h\psi_p)\rho_{\varphi_{p,0}}.$$

Combining this with the fact $\mu\circ\nu_{1,1}=0$ yields $\psi_{n+1}\circ\nu_{p,q}=0$, as desired. □

3. The Lefschetz operator on a contact manifold

In this section we discuss the Lefschetz operator on a contact manifold (M^{2n+1},ξ). While this material is standard (cf. [15, Section 2], [17, Section 5.3]), we take the unorthodox approach of working with subpaces of the space of differential forms, rather than sections of $\Lambda\xi^*$. This choice reflects our definition of the Rumin algebra.

Let (M^{2n+1},ξ) be a contact manifold. Locally there exists a *contact form*; i.e. a real one-form θ with kernel ξ. We say that (M^{2n+1},ξ) is *coorientable* if a global contact form exists.

Denote by $\mathscr{A}^k$ the (real) vector space of differential k-forms on M^{2n+1} and denote by $\mathscr{A}_0^k$ the space of *vertical forms*; i.e. $\mathscr{A}_0^k\subset\mathscr{A}^k$ is the subspace annihilated by taking the exterior product with any local contact form. We require the following simple observation about the exterior derivative on $\mathscr{A}_0^k$.

Lemma 2. *Let θ be a local contact form on a contact manifold (M^{2n+1},ξ). If $\omega\in\mathscr{A}_0^k$, then $\theta\wedge\mathrm{d}\omega=\omega\wedge\mathrm{d}\theta$ wherever θ is defined.*

Proof. Since $\omega\in\mathscr{A}_0^k$, it holds that $\omega=\theta\wedge\tau$ for some $\tau\in\mathscr{A}^{k-1}$. Therefore

$$\theta\wedge\mathrm{d}\omega=\theta\wedge d(\theta\wedge\tau)=\theta\wedge\tau\wedge\mathrm{d}\theta=\omega\wedge\mathrm{d}\theta.$$ □

Suppose that (M^{2n+1},ξ) is coorientable. Given a choice of contact form θ, the *Lefschetz operator* $\mathscr{L}_\theta\colon\mathscr{A}_0^k\to\mathscr{A}_0^{k+2}$ is

$$\mathscr{L}_\theta\omega:=\omega\wedge\mathrm{d}\theta.$$

The restriction to vertical forms ensures that if $u\in C^\infty(M)$ and $\omega\in\mathscr{A}_0^k$, then

$$\mathscr{L}_{e^u\theta}\omega=e^u\mathscr{L}_\theta\omega. \tag{7}$$

The Lefschetz operator inherits many properties from the symplectic form $\mathrm{d}\theta|_\xi$. For example, its powers are isomorphisms when suitably restricted.

Lemma 3. *Let (M^{2n+1},ξ) be a coorientable contact manifold with global contact form θ. If $k\le n$, then $\mathscr{L}_\theta^k\colon\mathscr{A}_0^{n-k+1}\to\mathscr{A}_0^{n+k+1}$ is an isomorphism.*

Proof. Denote by $\mathscr{W}^k$ the vector space of smooth sections of $\Lambda^k\xi^*$. Given $\omega\in\mathscr{W}^k$, denote by $\widetilde{\omega}_\theta$ the unique element of $\mathscr{A}^k$ such that $\widetilde{\omega}_\theta|_\xi=\omega$ and $\widetilde{\omega}_\theta(T_\theta,\cdot)=0$, where T_θ is the Reeb vector field determined by $\theta(T)=0$ and $\mathrm{d}\theta(T,\cdot)=0$. Then the map

$$\mathscr{W}^k\ni\omega\mapsto\theta\wedge\widetilde{\omega}_\theta\in\mathscr{A}_0^{k+1}$$

is an isomorphism. Since $\mathrm{d}\theta|_\xi$ is a symplectic form, $(\mathrm{d}\theta|_\xi)^k\colon\mathscr{W}^{n-k}\to\mathscr{W}^{n+k}$ is an isomorphism [2, Proposition 1.1]. The conclusion readily follows. □

We say that $\omega\in\mathscr{A}^k$, $k\le n$, is *primitive* if $\mathscr{L}_\theta^{n+1-k}(\theta\wedge\omega)=0$ for any choice of local contact form. The following proposition identifies, in a contact invariant way, the non-primitive part of an arbitrary differential form.

Proposition 4. *Let* (M^{2n+1},ξ) *be a contact manifold. There is a unique contact invariant linear map* $\Gamma\colon \mathscr{A}^k \to \mathscr{A}_0^{k-1}$ *such that given a local contact form* θ *and an* $\omega\in\mathscr{A}^k$*, it holds that*

$$\begin{aligned} \theta\wedge\omega\wedge\mathrm{d}\theta^{n+1-k} &= \Gamma\omega\wedge\mathrm{d}\theta^{n+2-k}, && && \text{if } k\leq n,\\ \theta\wedge\omega &= \Gamma\omega\wedge\mathrm{d}\theta, && \Gamma\omega\in\operatorname{im}\mathscr{L}_\theta^{k-n-1}, && \text{if } k\geq n+1. \end{aligned} \tag{8}$$

Moreover,

(1) $\Gamma(\mathscr{A}_0^k)=\{0\}$*;*
(2) $\Gamma\mathrm{d}\Gamma=\Gamma$*; and*
(3) $\Gamma\mathrm{d}=1$ *on* $\mathscr{A}_0^k$*,* $k\leq n$*.*

Proof. Let θ be a local contact form. Let $\omega\in\mathscr{A}^k$. If $k\leq n$, then Lemma 3 yields a unique $\zeta_\theta\in\mathscr{A}_0^{k-1}$ such that

$$\theta\wedge\omega\wedge\mathrm{d}\theta^{n+1-k}=\zeta_\theta\wedge\mathrm{d}\theta^{n+2-k}. \tag{9}$$

Equation (7) implies that $\Gamma_\theta\omega:=\zeta_\theta$ is independent of the choice of θ. If $k\geq n+1$, then Lemma 3 yields a unique $\zeta_\theta\in\mathscr{A}_0^{2n-k+1}$ such that

$$\theta\wedge\omega=\zeta_\theta\wedge\mathrm{d}\theta^{k-n}. \tag{10}$$

Equation (7) implies that $\Gamma_\theta\omega:=\zeta_\theta\wedge\mathrm{d}\theta^{k-n-1}$ is independent of the choice of θ. The existence, uniqueness, and contact invariance of Γ readily follow. It follows immediately from (8) that $\Gamma(\mathscr{A}_0^k)=\{0\}$. Let $\omega\in\mathscr{A}^k$ and let θ be a local contact form. Suppose first that $k\leq n$. Let $\zeta\in\mathscr{A}_0^{k-1}$ be as in (9). Lemma 2 implies that

$$\theta\wedge\mathrm{d}\zeta\wedge\mathrm{d}\theta^{n+1-k}=\zeta\wedge\mathrm{d}\theta^{n+2-k}.$$

Hence $\Gamma\mathrm{d}\Gamma\omega=\Gamma\omega$ if $k\leq n$. Suppose now that $k\geq n+1$. Let $\zeta\in\mathscr{A}_0^{2n-k+1}$ be as in (10). Lemma 2 implies that

$$\theta\wedge\mathrm{d}(\zeta\wedge\mathrm{d}\theta^{k-n-1})=\zeta\wedge\mathrm{d}\theta^{k-n}.$$

Hence $\Gamma\mathrm{d}\Gamma\omega=\Gamma\omega$ if $k\geq n+1$. Finally, let $\omega\in\mathscr{A}_0^k$, $k\leq n$. Lemma 2 implies that

$$\theta\wedge\mathrm{d}\omega\wedge\mathrm{d}\theta^{n-k}=\omega\wedge\mathrm{d}\theta^{n+1-k}.$$

Therefore $\Gamma\mathrm{d}\omega=\omega$. □

4. The Rumin algebra

Let (M^{2n+1},ξ) be a contact manifold. Set

$$\mathscr{R}^k:=\left\{\omega\in\mathscr{A}^k : \Gamma\omega=0, \Gamma\mathrm{d}\omega=0\right\},$$

where Γ is as in Proposition 4. Note that $d(\mathscr{R}^k)\subseteq\mathscr{R}^{k+1}$ for all $k\in\mathbb{N}_0$. Denote $\mathscr{R}:=\bigoplus_k\mathscr{R}^k$ and $\mathscr{A}:=\bigoplus_k\mathscr{A}^k$.

Our main result is that $(\mathscr{R},d)$ is a deformation retract of the de Rham complex.

Theorem 5. *Let* (M^{2n+1},ξ) *be a contact manifold. Then*

$$\pi\omega:=\omega-\mathrm{d}\Gamma\omega-\Gamma\mathrm{d}\omega \tag{11}$$

is a homogeneous projection $\pi\colon\mathscr{A}\to\mathscr{R}$ *of degree zero. In particular,*

$$\Gamma \circlearrowright (\mathscr{A},\mathrm{d}) \underset{i}{\overset{\pi}{\rightleftarrows}} (\mathscr{R},\mathrm{d}) \tag{12}$$

is a deformation retract, where $i\colon\mathscr{R}\to\mathscr{A}$ *is the inclusion.*

Proof. Equation (11) implies that $\mathrm{d}\pi=\pi\mathrm{d}$. On the one hand, the definition of $\mathscr{R}$ implies that $\pi|_{\mathscr{R}}=1_{\mathscr{R}}$. On the other hand, Proposition 4 implies that $\Gamma\circ\Gamma=0$ and $\Gamma\mathrm{d}\Gamma=\Gamma$. Therefore $\Gamma\pi=0$ and $\Gamma\mathrm{d}\pi=0$, and hence $\pi(\mathscr{A})\subseteq\mathscr{R}$. □

Applying Theorem 1 to (12) yields an explicit C_∞-structure on the Rumin complex which is C_∞-quasi-isomorphic to the de Rham algebra.

Corollary 6. *Let (M^{2n+1},ξ) be a contact manifold. Define $m_k\colon \mathscr{R}^{\otimes k}\to\mathscr{R}$ and $f_k\colon \mathscr{R}^{\otimes k}\to\mathscr{A}$ by*

$$\begin{aligned} m_1 &= \mathrm{d}i, & f_1 &= i, \\ m_2 &= \pi\mu i^{\otimes 2}, & f_2 &= -\Gamma\mu i^{\otimes 2}, \\ m_3 &= \pi\mu(\Gamma\mu\otimes 1 - 1\otimes\Gamma\mu)i^{\otimes 3}, & f_3 &= 0, \\ m_k &= 0, & f_k &= 0, & &\text{if } k\geq 4. \end{aligned}$$

Then $(\mathscr{R},m)$ is a C_∞-algebra and $f\colon\mathscr{R}\to\mathscr{A}$ is a C_∞-quasi-isomorphism.

Proof. It suffices to compute m_k and f_k in Theorem 1. Since $\Gamma(\mathscr{A})\subseteq\mathscr{A}_0$, it holds that $\mu(\Gamma\otimes\Gamma)=0$ and $\Gamma\mu(\Gamma\otimes 1)=0=\Gamma\mu(1\otimes\Gamma)$. We deduce from Equation (6) that

$$\begin{aligned} \psi_2 &= \mu, \\ \psi_3 &= \mu(\Gamma\mu\otimes 1 - 1\otimes\Gamma). \end{aligned}$$

Therefore $\Gamma\psi_3=0$. Using Equation (6) again yields

$$\psi_4 = -\mu(\Gamma\psi_2\otimes\Gamma\psi_2)=0.$$

Finally let $k\geq 5$. If $s,t\in\mathbb{N}$ satisfy $s+t=k$, then $\max\{s,t\}\geq 3$. A straightforward induction argument then yields $\psi_k=0$. The conclusion readily follows. ☐

The C_∞-algebra $(\mathscr{R},m)$ of Corollary 6 is the *Rumin algebra*. Note that if $\omega,\tau,\eta\in\mathscr{R}$ are homogeneous, then

$$\begin{aligned} m_2(\omega\otimes\tau) &= \pi(\omega\wedge\tau), \\ m_3(\omega\otimes\tau\otimes\eta) &= \pi\left(\Gamma(\omega\wedge\tau)\wedge\eta - (-1)^{|\omega|}\omega\wedge\Gamma(\tau\wedge\eta)\right), \end{aligned}$$

recovering the formulas of Case [5, Definitions 8.4 and 8.6, and Theorem 8.8].

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