Jeffrey S. Case

A simple construction of the Rumin algebra

Volume 361 (2023), p. 1375-1382

Published online: 31 October 2023

https://doi.org/10.5802/crmath.510

This article is licensed under the
Creative Commons Attribution 4.0 International License.
http://creativecommons.org/licenses/by/4.0/
A simple construction of the Rumin algebra

Jeffrey S. Case

\textsuperscript{a} Department of Mathematics, Penn State University, University Park, PA 16802, USA
\textit{E-mail: jscase@psu.edu}

\textbf{Abstract.} The Rumin algebra of a contact manifold is a contact invariant $C_\infty$-algebra of differential forms which computes the de Rham cohomology algebra. We recover this fact by giving a simple and explicit construction of the Rumin algebra via Markl's formulation of the Homotopy Transfer Theorem.

\textbf{Keywords.} Rumin complex, Rumin algebra, contact invariant.

\textbf{2020 Mathematics Subject Classification.} 53D10, 58A10, 58J10.

\textbf{Funding.} This work was partially supported by the Simons Foundation (Grant #524601).

\textit{Manuscript received 5 July 2022, revised 6 April 2023, accepted 25 April 2023.}

\section{Introduction}

A contact manifold is a pair $(M^{2n+1}, \xi)$ of a real $(2n+1)$-dimensional manifold and a subbundle $\xi \subset TM$ of rank $2n$ with the property that if $\theta$ is a local one-form such that $\ker \theta = \xi$, then $\theta \wedge d\theta^n \neq 0$.

The Rumin complex is a cochain complex which is adapted to the contact structure and computes the de Rham cohomology groups. There are many constructions of the Rumin complex, including Rumin’s original construction via subquotients of differential forms [14], constructions via a spectral sequence [1, 8, 16], and constructions including a homotopy equivalence with the de Rham complex [3, 5, 17]. Importantly, the latter constructions yield, as a consequence of the Homotopy Transfer Theorem [9, 12, 13], a $C_\infty$-structure on the Rumin complex which recovers the de Rham cohomology algebra.

The Rumin algebra is a specific contact invariant $C_\infty$-algebra of differential forms which computes the de Rham cohomology algebra; indeed, its higher products $m_k$, $k \geq 4$, all vanish. Its existence follows from the latter constructions above. Calderbank and Diemer showed [3] that a curved Bernstein–Gelfand–Gelfand sequence [4] for a parabolic geometry carries a curved $A_\infty$-structure. Bryant, Eastwood, Gover and Neusser pointed out [1] how to recover the Rumin complex from the BGG sequence for the trivial representation, and their observation extends to the Rumin algebra. Rumin’s work [17] on the Rumin complex of a Carnot–Carathéodory space includes a realization of the Rumin complex as a deformation retract of the de Rham complex. While none of these authors explicitly computed the $C_\infty$-structure, Case gave [5] ad hoc formulas for $m_k$ in a manner which resembles a construction of Tsai, Tseng and Yau [20] on symplectic manifolds.
In this short note we present a simple and explicit construction of the Rumin algebra. The key ingredients are a natural contact invariant deformation retract of the de Rham algebra and an explicit formula [13] for the transfer of the Homotopy Transfer Theorem. Our deformation retract coincides with those of Calderbank–Diemer [3] and Rumin [17], and our application of the Homotopy Transfer Principle explains the ad hoc formulas of Case [5].

By construction, the Rumin algebra is homotopy equivalent, as a $C_\infty$-algebra, to the de Rham algebra. It is known that two simply connected manifolds of finite type are rationally homotopy equivalent if and only if their spaces of polynomial differential forms are homotopy equivalent as $C_\infty$-algebras [6, 7, 10, 18]; and that the homotopy type, as a $C_\infty$-algebra, of the de Rham algebra has many implications for the differential forms because of their contact distribution.

For these reasons, we expect the Rumin algebra to be a fundamental invariant of contact manifolds.

We define the Rumin algebra as a $C_\infty$-algebra on a subspace of differential forms because of the many applications of the latter (e.g. [18]). This choice means that elements of the Rumin algebra are not sections of a vector bundle and the differential forms are homotopy equivalent as $C_\infty$-algebras [6, 7, 10, 18]; and that the homotopy type, as a $C_\infty$-algebra, of the de Rham algebra has many implications for the differential forms because of the diffeomorphism type of a closed, simply connected, smooth manifold [19]. For these reasons, we expect the Rumin algebra to be a fundamental invariant of contact manifolds.

We denote by $A^k$ the set of homogeneous elements of degree $k$, and by $A = \bigoplus_{k \in \mathbb{Z}} A^k$ be a $\mathbb{Z}$-graded vector space. A homogeneous element of $A$ is an element $\omega \in A^k$ for some $k \in \mathbb{Z}$. In this case we call $k$ the degree of $\omega$ and we set $|\omega| := k$. A map $f: A^\otimes k \to A$ is homogeneous of degree $\ell$ if
\[ f(A^{i_1} \otimes \cdots \otimes A^{i_k}) \subset A^{i_1 + \cdots + i_k + \ell} \]
for all $i_1, \ldots, i_k \in \mathbb{Z}$. In this case we set $|f| := \ell$.

An $A_\infty$-algebra is a pair $(A, m)$ of a $\mathbb{Z}$-graded vector space $A$ and a collection $m = \{m_k\}_{k \in \mathbb{N}}$ of homogeneous operators $m_k: A^\otimes k \to A$ of degree $2 - k$ such that
\[ \sum_{r+s+t=n} (-1)^{r+s} m_{r+s+1} (1^\otimes r \otimes m_s \otimes 1^\otimes t) = 0 \]
for all $n \in \mathbb{N}$, where $1: A \to A$ is the identity map and we use the Koszul sign convention
\[ (f \otimes g)(\alpha \otimes \beta) := (-1)^{|g||\alpha|} f(\alpha) \otimes g(\beta) \]
for homogeneous operators $f, g: A \to A$ and homogeneous elements $\alpha, \beta \in A$. The first three cases of (1) are
\[ m_1 m_1 = 0, \]
\[ m_1 m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1), \]
\[ m_1 m_3 = m_2(1 \otimes m_2 - m_2 \otimes 1) - m_3(m_1 \otimes 1^\otimes 2 + 1 \otimes m_1 \otimes 1 + 1^\otimes 2 \otimes m_1). \]
In particular, if \((A, m)\) is an \(A_\infty\)-algebra, then \((A, m_1)\) is a cochain complex. \textit{Graded (associative) algebras} are \(A_\infty\)-algebras with \(m_k = 0\) for all \(k \neq 2\). \textit{Differential graded algebras} are \(A_\infty\)-algebras with \(m_k = 0\) for all \(k \geq 3\). The \textit{cohomology ring} \((HA, [m_2])\) of \((A, m)\) is the graded algebra

\[
HA^k := \frac{\text{ker} m_1 \cap A^k}{\text{im} m_1 \cap A^k},
\]

\([m_2](\omega_1 \otimes \omega_2) := [m_2(\omega_1 \otimes \omega_2)].\]

Given \(A_\infty\)-algebras \((A, m)\) and \((B, \overline{m})\), an \(A_\infty\)-\textit{morphism} \(f\): \((A, m) \to (B, \overline{m})\) is a collection of homogeneous operators \(f_k: A^\otimes k \to B\) of degree \(1 - k\) such that

\[
\sum_{r+s+t=n} (-1)^{r+s+t} f_{r+t+1} (1^\otimes r \otimes m_s \otimes 1^\otimes t) = \sum_{1 \leq r \leq n} (-1)^{\ell} \overline{m}_r(f_{i_1} \otimes \cdots \otimes f_{i_\ell}),
\]

\[
\ell := \sum_{j=1}^r (r - j)(i_j - 1),
\]

for all \(n \in \mathbb{N}\). Specializing to the cases \(n \in \{1, 2\}\) yields

\[
f_1 m_1 = \overline{m}_1 f_1,
\]

\[
f_1 m_2 = \overline{m}_2 (f_1 \otimes f_1) + \overline{m}_1 f_2 + f_2(m_1 \otimes 1 + 1 \otimes m_1).
\]

In particular, \(f_1: (A, m_1) \to (B, \overline{m}_1)\) is a \textit{cochain map}. We call \(f\) an \(A_\infty\)-\textit{quasi-isomorphism} if \([f_1]: HA \to HB, [f_1](\alpha) := [f_1(\alpha)]\), is an isomorphism of graded algebras.

Denote by \(S_n\) the group of permutations of \(\{1, 2, \ldots, n\}\). We require the left action of \(S_n\) on \(A_\otimes n\) defined as follows: Given \(\sigma \in S_n\), let \(\rho_\sigma: A_\otimes n \to A_\otimes n\) be the linear map defined on tensor products of homogeneous elements \(x_1, \ldots, x_n \in A\) by

\[
\rho_\sigma(x_1 \otimes \cdots \otimes x_n) = \text{sgn}(\sigma) \epsilon(\sigma; x_1, \ldots, x_n)x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)},
\]

where \(\epsilon(\sigma; x_1, \ldots, x_n)\) is the Koszul sign determined by

\[
x_1 \wedge \cdots \wedge x_{p+q} = \sum_{\sigma \in S_{p+q}} \epsilon(\sigma; x_1, \ldots, x_{p+q}) x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(p+q)}.
\]

Given \(p, q \in \mathbb{N}_0\), a \((p, q)\)-\textit{shuffle} is an element \(\sigma \in S_{p+q}\) such that

\[
\sigma(1) < \cdots < \sigma(p), \quad \sigma(p+1) < \cdots < \sigma(p+q).
\]

Note that a \((p, q)\)-shuffle \(\sigma\) is uniquely determined by the set \(\{\sigma(1), \ldots, \sigma(p)\}\). Denote by \(\text{Sh}_{p,q} \subset S_{p+q}\) the subset of \((p, q)\)-shuffles. Given \(s, t \in \mathbb{N}\), the sets

\[
\text{Sh}_{p,q}(s, t) := \left\{ \sigma \in \text{Sh}_{p,q}: \#(\{\sigma^{-1}(p+q-j)\}_{j=0}^{s+t-1} \cap \{1, \ldots, p\}) = s \right\}
\]

of \((p, q)\)-shuffles such that \(\{\sigma^{-1}(p+q-j)\}_{j=0}^{s+t-1}\) contains exactly \(s\) elements of \(\{1, \ldots, p\}\) give rise to a partition

\[
\text{Sh}_{p,q} = \bigsqcup_{s+t=r} \text{Sh}_{p,q}(s, t) \tag{2}
\]

of \(\text{Sh}_{p,q}\). Note that if \(\sigma \in \text{Sh}_{p,q}(s, t)\), then there are unique \(\alpha \in \text{Sh}_{p-s,q-t}\) and \(\beta \in \text{Sh}_{s,t}\) such that

\[
\sigma = (\alpha \times \beta) \varphi_{s,t}, \tag{3}
\]

where \(\alpha\) and \(\beta\) act on \(\{1, \ldots, p+q-s-t\}\) and \(\{p+q-s-t+1, \ldots, p+q\}\), respectively, and \(\varphi_{s,t} \in \text{Sh}_{p,q}\) is the \((p, q)\)-shuffle

\[
\varphi_{s,t}(j) = \begin{cases} 
  j, & \text{if } 1 \leq j \leq p-s, \\
  j+q-t, & \text{if } p-s+1 \leq j \leq p, \\
  j-s, & \text{if } p+1 \leq j \leq p+q-t, \\
  j, & \text{if } p+q-t+1 \leq j \leq p+q,
\end{cases}
\]

which fixes \(\{1, \ldots, p\}\) and maps \(\{p-s+1, \ldots, p\}\) to \(\{p+q-s-t+1, \ldots, p+q-t\}\).
The \((p,q)\)-shuffle product \(v_{p,q}: A^{\oplus(p+q)} \to A^{\oplus(p+q)}\) is
\[
v_{p,q} := \sum_{\sigma \in S_{p,q}} \rho_\sigma.
\]
For example, if \(\omega, \tau, \eta \in A\) are homogeneous, then
\[
v_{1,1}(\omega \otimes \tau) = \omega \otimes \tau - (-1)^{[\omega][\tau]} \tau \otimes \omega,
\]
\[
v_{1,2}(\omega \otimes \tau \otimes \eta) = \omega \otimes \tau \otimes \eta - (-1)^{[\omega][\tau]} \tau \otimes \eta \otimes \omega + (-1)^{[\omega][\tau][\eta]} \tau \otimes \eta \otimes \omega,
\]
\[
v_{2,1}(\omega \otimes \tau \otimes \eta) = \omega \otimes \tau \otimes \eta - (-1)^{[\tau][\eta]} \omega \otimes \eta \otimes \tau + (-1)^{[\eta][\tau][\eta]} \eta \otimes \omega \otimes \tau.
\]
Equations (2) and (3) imply that if \(p, q, r \in \mathbb{N}\) are such that \(r \leq p + q\), then
\[
v_{p,q} = \sum_{s, t \in \mathbb{N}_0 \atop s + t = r} (v_{p-s,q-t} \otimes v_{s,t}) \rho_{q,i}.
\]
A \(C_\infty\)-algebra is an \(A_\infty\)-algebra \((A, m)\) such that
\[
m_{p+q} \circ v_{p,q} = 0
\]
for all \(p, q \in \mathbb{N}\). Graded commutative (associative) algebras are \(C_\infty\)-algebras with \(m_k = 0\) for all \(k \neq 2\). Commutative differential graded algebras are \(C_\infty\)-algebras with \(m_k = 0\) for all \(k \geq 3\).

Given \(C_\infty\)-algebras \((A, m)\) and \((B, \overline{m})\), a \(C_\infty\)-morphism \(f: (A, m) \to (B, \overline{m})\) is an \(A_\infty\)-morphism such that
\[
f_{p+q} \circ v_{p,q} = 0
\]
for all \(p, q \in \mathbb{N}\). We call \(f\) a \(C_\infty\)-quasi-isomorphism if \([f_1]: HA \to HB\) is an isomorphism of graded commutative algebras.

The Homotopy Transfer Theorem [12, Theorem 10.3.1] constructs a \(C_\infty\)-structure on a deformation retract of a commutative differential graded algebra and an extension of the inclusion to a \(C_\infty\)-quasi-isomorphism. We require explicit formulas for the transferred structure and induced quasi-isomorphism (cf. [12, Theorem 13.1.7]).

**Theorem 1.** Let \((A, d, \mu)\) be a commutative differential graded algebra and let \((B, d)\) be a subcomplex of \((A, d)\). Suppose that
\[
h \subseteq_i (A, d) \xrightarrow{\pi} (B, d)
\]
is a deformation retract; i.e. \(\pi: (A, d) \to (B, d)\) and \(i: (B, d) \to (A, d)\) are cochain maps, \(h: A \to A\) is homogeneous of degree \(-1\), and \(i \pi = 1_A - dh - hd\) and \(i = 1_B\). Recursively define \(\psi_n: A^\otimes n \to A, n \geq 2, by
\[
\psi_n := \sum_{s+t=n} (-1)^{s+1} \mu(h \psi_s \otimes h \psi_t),
\]
with the convention \(h \psi_1 = -1_A\). Set \(m_1 := d\) and \(m_k := \pi \psi_k i^\otimes k, k \geq 2, and f_k := -h \psi_k i^\otimes k, k \in \mathbb{N}\). Then \((B, m)\) is a \(C_\infty\)-algebra and \(f: (B, m) \to (A, d, \mu)\) is a \(C_\infty\)-quasi-isomorphism with \(f_1 = i\).

**Proof.** Markl computed [13, Theorem 5] that \((B, m)\) is an \(A_\infty\)-algebra and that \(f: (B, m) \to (A, d, \mu)\) is an \(A_\infty\)-quasi-morphism. We prove the final conclusion by proving that \(\psi_n \circ v_{p,q} = 0\) for all \(p, q \in \mathbb{N}\) such that \(p + q = n\). The proof is by strong induction in \(n\). Since \(\psi_2 = \mu\), we have that \(\psi_2 \circ v_{1,1} = 0\). Suppose now that \(\psi_k \circ v_{p,q} = 0\) for all \(2 \leq k \leq n\) and all \(p, q \in \mathbb{N}\) such that \(p + q = k\). Let \(p, q \in \mathbb{N}\) be such that \(p + q = n + 1\). On the one hand, Equation (4) implies that
\[
\psi_{n+1} \circ v_{p,q} = \sum_{s+t=p+q, a+b=t} (a,b \in \mathbb{N}_0} \sum (a,b) (-1)^{s+1} \mu(h \psi_s \otimes h \psi_t) (v_{p-a,q-b} \otimes v_{a,b}) \rho_{a,b}.\]
On the other hand, the inductive hypothesis implies that the only nonzero summands are those for which \((a, b) = (0, q)\) or \((a, b) = (p, 0)\). Combining these facts with the observations \(\nu_{p, 0} = 1\) and \(\nu_{0, q} = 1\) and \(p \nu_{0, q} = 1\) yields
\[
\psi_{n+1} \circ \nu_{p, q} = (-1)^{p+1} \mu(h \psi_p \otimes h \psi_q) + (-1)^{p+1} \mu(h \psi_q \otimes h \psi_p) \nu_{p, q}.
\]
Combining this with the fact \(\mu \circ \nu_{1, 1} = 0\) yields \(\psi_{n+1} \circ \nu_{p, q} = 0\), as desired. \(\square\)

3. The Lefschetz operator on a contact manifold

In this section we discuss the Lefschetz operator on a contact manifold \((M^{2n+1}, \xi)\). While this material is standard (cf. [15, Section 2], [17, Section 5.3]), we take the unorthodox approach of working with subspaces of the space of differential forms, rather than sections of \(\Lambda^* \xi\). This choice reflects our definition of the Rumin algebra.

Let \((M^{2n+1}, \xi)\) be a contact manifold. Locally there exists a contact form; i.e. a real one-form \(\theta\) with kernel \(\xi\). We say that \((M^{2n+1}, \xi)\) is coorientable if a global contact form exists.

Denote by \(\mathcal{A}^k\) the (real) vector space of differential \(k\)-forms on \(M^{2n+1}\) and denote by \(\mathcal{A}^k_0\) the space of vertical forms; i.e. \(\mathcal{A}^k_0 \subset \mathcal{A}^k\) is the subspace annihilated by taking the exterior product with any local contact form. We require the following simple observation about the exterior derivative on \(\mathcal{A}^k_0\).

**Lemma 2.** Let \(\theta\) be a local contact form on a contact manifold \((M^{2n+1}, \xi)\). If \(\omega \in \mathcal{A}^k_0\), then \(\theta \wedge d\omega = \omega \wedge d\theta\) wherever \(\theta\) is defined.

**Proof.** Since \(\omega \in \mathcal{A}^k_0\), it holds that \(\omega = \theta \wedge \tau\) for some \(\tau \in \mathcal{A}^{k-1}\). Therefore
\[
\theta \wedge d\omega = \theta \wedge d(\theta \wedge \tau) = \theta \wedge \tau \wedge d\theta = \omega \wedge d\theta.
\]

Suppose that \((M^{2n+1}, \xi)\) is coorientable. Given a choice of contact form \(\theta\), the Lefschetz operator \(\mathcal{L}_\theta : \mathcal{A}^k_0 \to \mathcal{A}^{k+2}_0\) is
\[
\mathcal{L}_\theta \omega := \omega \wedge d\theta.
\]
The restriction to vertical forms ensures that if \(u \in C^\infty(M)\) and \(\omega \in \mathcal{A}^k_0\), then
\[
\mathcal{L}_{\theta^k} \omega = e^u \mathcal{L}_\theta \omega.
\]

The Lefschetz operator inherits many properties from the symplectic form \(d\theta\big|_\xi\). For example, its powers are isomorphisms when suitably restricted.

**Lemma 3.** Let \((M^{2n+1}, \xi)\) be a coorientable contact manifold with global contact form \(\theta\). If \(k \leq n\), then \(\mathcal{L}_\theta^k : \mathcal{A}^{n-k+1}_0 \to \mathcal{A}^{n+k+1}_0\) is an isomorphism.

**Proof.** Denote by \(\mathcal{W}^k\) the vector space of smooth sections of \(\Lambda^k \xi^*\). Given \(\omega \in \mathcal{W}^k\), denote by \(\tilde{\omega}\big|_\xi\) the unique element of \(\mathcal{A}^k\) such that \(\tilde{\omega}\big|_\xi = \omega\) and \(\tilde{\omega}(T_\theta, \cdot) = 0\), where \(T_\theta\) is the Reeb vector field determined by \(\theta(T) = 0\) and \(d\theta(T, \cdot) = 0\). Then the map
\[
\mathcal{W}^k \ni \omega \mapsto \omega \wedge \tilde{\omega} \in \mathcal{A}^{k+1}_0
\]
is an isomorphism. Since \(d\theta|_\xi\) is a symplectic form, \((d\theta|_\xi)^k : \mathcal{W}^{n-k} \to \mathcal{W}^{n+k}\) is an isomorphism [2, Proposition 1.1]. The conclusion readily follows. \(\square\)

We say that \(\omega \in \mathcal{A}^k\), \(k \leq n\), is primitive if \(\mathcal{L}_\theta^{n-1-k}(\theta \wedge \omega) = 0\) for any choice of local contact form. The following proposition identifies, in a contact invariant way, the non-primitive part of an arbitrary differential form.
Proposition 4. Let \((M^{2n+1}, \xi)\) be a contact manifold. There is a unique contact invariant linear map \(\Gamma : \mathfrak{A}^k \to \mathfrak{A}^{k-1}\) such that given a local contact form \(\theta\) and an \(\omega \in \mathfrak{A}^k\), it holds that
\[
\theta \wedge \omega \wedge d\theta^{n+1-k} = \Gamma \omega \wedge d\theta^{n+2-k}, \quad \text{if } k \leq n,
\]
\[
\theta \wedge \omega = \Gamma \omega \wedge d\theta, \quad \Gamma \omega \in \text{im} \mathcal{L}_\theta^{k-1}, \quad \text{if } k \geq n+1.
\]
Moreover,
\[\begin{align}
(1) \quad & \Gamma(\mathfrak{A}_0^k) = \{0\}; \\
(2) \quad & \Gamma d\Gamma = \Gamma; \quad \text{and} \\
(3) \quad & \Gamma d = 1 \text{ on } \mathfrak{A}_0^k, \quad k \leq n.
\end{align}\]

Proof. Let \(\theta\) be a local contact form. Let \(\omega \in \mathfrak{A}^k\). If \(k \leq n\), then Lemma 3 yields a unique \(\zeta_\theta \in \mathfrak{A}^{k-1}\) such that
\[
\theta \wedge \omega \wedge d\theta^{n+1-k} = \zeta_\theta \wedge d\theta^{n+2-k}.
\]
Equation (7) implies that \(\Gamma_\theta \omega := \zeta_\theta \wedge d\theta^{k-n-1}\) is independent of the choice of \(\theta\). The existence, uniqueness, and contact invariance of \(\Gamma\) readily follow. It follows immediately from (8) that \(\Gamma(\mathfrak{A}_0^k) = \{0\}\). Let \(\omega \in \mathfrak{A}^k\) and let \(\theta\) be a local contact form. Suppose first that \(k \leq n\). Let \(\zeta \in \mathfrak{A}_0^{k-1}\) be as in (9). Lemma 2 implies that
\[
\theta \wedge \omega = \zeta \wedge d\theta^{k-n}.
\]
Equation (7) implies that \(\Gamma_\theta \omega := \zeta_\theta \wedge d\theta^{k-n-1}\) is independent of the choice of \(\theta\). The existence, uniqueness, and contact invariance of \(\Gamma\) readily follow. It follows immediately from (8) that \(\Gamma(\mathfrak{A}_0^k) = \{0\}\). Let \(\omega \in \mathfrak{A}^k\). Denote \(\zeta \in \mathfrak{A}_0^{k-1}\) be as in (9). Lemma 2 implies that
\[
\theta \wedge d\omega \wedge d\theta^{n+1-k} = \omega \wedge d\theta^{n+2-k}.
\]
Hence \(\Gamma d\Gamma \omega = \Gamma \omega\) if \(k \leq n\). Suppose now that \(k \geq n+1\). Let \(\zeta \in \mathfrak{A}_0^{2n-k+1}\) be as in (10). Lemma 2 implies that
\[
\theta \wedge d(\zeta \wedge d\theta^{k-n-1}) = \zeta \wedge d\theta^{k-n}.
\]
Hence \(\Gamma d\Gamma \omega = \Gamma \omega\) if \(k \geq n+1\). Finally, let \(\omega \in \mathfrak{A}_0^k, \quad k \leq n\). Lemma 2 implies that
\[
\theta \wedge d\omega \wedge d\theta^{n-k} = \omega \wedge d\theta^{n+1-k}.
\]
Therefore \(\Gamma d\omega = \omega\).

4. The Rumin algebra

Let \((M^{2n+1}, \xi)\) be a contact manifold. Set
\[
\mathcal{R}^k := \left\{ \omega \in \mathfrak{A}^k : \Gamma \omega = 0, \Gamma d\omega = 0 \right\},
\]
where \(\Gamma\) is as in Proposition 4. Note that \(d(\mathcal{R}^k) \subseteq \mathcal{R}^{k+1}\) for all \(k \in \mathbb{N}_0\). Denote \(\mathcal{R} := \bigoplus_k \mathcal{R}^k\) and \(\mathfrak{A} := \bigoplus_k \mathfrak{A}^k\).

Our main result is that \((\mathcal{R}, d)\) is a deformation retract of the de Rham complex.

Theorem 5. Let \((M^{2n+1}, \xi)\) be a contact manifold. Then
\[
\pi \omega := \omega - d\Gamma \omega - \Gamma d\omega
\]
is a homogeneous projection \(\pi : \mathfrak{A} \to \mathcal{R}\) of degree zero. In particular,
\[
\Gamma \prec (\mathfrak{A}, d) \xrightarrow{\pi} (\mathcal{R}, d)
\]
is a deformation retract, where \(i : \mathcal{R} \to \mathfrak{A}\) is the inclusion.

Proof. Equation (11) implies that \(d\pi = \pi d\). On the one hand, the definition of \(\mathcal{R}\) implies that \(\pi|_{\mathfrak{A}} = 1_{\mathfrak{A}}\). On the other hand, Proposition 4 implies that \(\Gamma \circ \Gamma = 0\) and \(\Gamma d\Gamma = \Gamma\). Therefore \(\Gamma \pi = 0\) and \(\Gamma d\pi = 0\), and hence \(\pi(\mathfrak{A}) \subseteq \mathcal{R}\).
Applying Theorem 1 to (12) yields an explicit $C_\infty$-structure on the Rumin complex which is $C_\infty$-quasi-isomorphic to the de Rham algebra.

**Corollary 6.** Let $(M^{2n+1},\xi)$ be a contact manifold. Define $m_k: \mathcal{R}^k \to \mathcal{R}$ and $f_k: \mathcal{R}^k \to \mathcal{A}$ by

- $m_1 = di$,
- $m_2 = \pi \mu i^2$,
- $m_3 = \pi \mu (\Gamma \mu \odot 1 - 1 \odot \Gamma \mu) i^3$,
- $m_k = 0,$ \hspace{1cm} if $k \geq 4$.

- $f_1 = i$,
- $f_2 = -\Gamma \mu i^2$,
- $f_3 = 0$,
- $f_k = 0,$ \hspace{1cm} if $k \geq 4$.

Then $(\mathcal{R}, m)$ is a $C_\infty$-algebra and $f: \mathcal{R} \to \mathcal{A}$ is a $C_\infty$-quasi-isomorphism.

**Proof.** It suffices to compute $m_k$ and $f_k$ in Theorem 1. Since $\Gamma(\mathcal{A}) \subseteq \mathcal{A}_0$, it holds that $\mu(\Gamma \odot \Gamma) = 0$ and $\Gamma(\mu \odot 1) = 0 = \Gamma(1 \odot \Gamma)$. We deduce from Equation (6) that

$$\psi_2 = \mu,$$

$$\psi_3 = \mu(\Gamma \mu \odot 1 - 1 \odot \Gamma).$$

Therefore $\Gamma \psi_3 = 0$. Using Equation (6) again yields

$$\psi_4 = -\mu(\Gamma \psi_2 \odot \Gamma \psi_2) = 0.$$

Finally let $k \geq 5$. If $s, t \in \mathbb{N}$ satisfy $s + t = k$, then $\max\{s, t\} \geq 3$. A straightforward induction argument then yields $\psi_k = 0$. The conclusion readily follows.

The $C_\infty$-algebra $(\mathcal{R}, m)$ of Corollary 6 is the *Rumin algebra*. Note that if $\omega, \tau, \eta \in \mathcal{R}$ are homogeneous, then

$$m_2 (\omega \odot \tau) = \pi (\omega \wedge \tau),$$

$$m_3 (\omega \odot \tau \odot \eta) = \pi \left( \Gamma(\omega \wedge \tau) \wedge \eta - (-1)^{|\omega|} \omega \wedge \Gamma(\tau \wedge \eta) \right),$$

recovering the formulas of Case [5, Definitions 8.4 and 8.6, and Theorem 8.8].

**Acknowledgements**

I thank Andreas Čap for directing me to the work of Calderbank and Diemer, the University of Washington for providing a productive research environment, and the referee for helpful suggestions for improving the clarity of this article.

**References**


