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Congruences modulo 4 for the number of 3-regular partitions

Cristina Ballantine [*,a] **and Mircea Merca** [b,c]

[a] Department of Mathematics and Computer Science, College of The Holy Cross, Worcester, MA 01610, USA

[b] Department of Mathematical Methods and Models, Fundamental Sciences Applied in Engineering Research Center, University Politehnica of Bucharest, RO-060042 Bucharest, Romania

[c] Academy of Romanian Scientists, RO-050044, Bucharest, Romania

E-mails: cballant@holycross.edu, mircea.merca@profinfo.edu.ro

Abstract. The last decade has seen an abundance of congruences for $b_\ell(n)$, the number of ℓ-regular partitions of n. Notably absent are congruences modulo 4 for $b_3(n)$. In this paper, we introduce Ramanujan type congruences modulo 4 for $b_3(2n)$ involving some primes p congruent to $11, 13, 17, 19, 23$ modulo 24.

Keywords. partitions, regular partitions, congruences.

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1. Introduction

A partition of a positive integer n is a weakly decreasing sequence of positive integers whose sum is n. The positive integers in the sequence are called parts. For more on the theory of partitions, we refer the reader to [1].

For an integer $\ell > 1$, a partition is called ℓ-regular if none of its parts is divisible by ℓ. The number of the ℓ-regular partitions of n is usually denoted by $b_\ell(n)$ and its arithmetic properties were investigated extensively. See, for example, [3–7, 10–12, 17–20, 22]. The generating function for $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n)\, q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}.$$

Here and throughout q is a complex number with $|q|<1$, and the symbol $(a;q)_\infty$ denotes the infinite product

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-a\,q^n).$$

In a recent paper, W. J. Keith and F. Zanello [9] discovered infinite families of Ramanujan type congruences modulo 2 for $b_3(2n)$ involving every prime p with $p \equiv 13, 17, 19, 23 \pmod{24}$.

* Corresponding author.

Theorem 1 (Keith–Zanello). *The sequence $b_3(2n)$ is lacunary modulo 2. If $p \equiv 13, 17, 19, 23 \pmod{24}$ is prime, then*

$$b_3\big(2(p^2 n + pk - 24^{-1})\big) \equiv 0 \pmod{2}$$

for $1 \leqslant k \leqslant p-1$, where 24^{-1} is taken modulo p^2.

Motivated by the Keith–Zanello result, O. X. M. Yao [23] provided new infinite families of Ramanujan type congruences modulo 2 for $b_3(2n)$ involving every prime $p \geqslant 5$.

Theorem 2 (Yao). *Let $p \geqslant 5$ be a prime.*

(1) *If $b_3\left(\frac{p^2-1}{12}\right) \equiv 1 \pmod{2}$, then for $n, k \geqslant 0$*

$$b_3\left(2p^{4k+4} n + 2p^{4k+3} j + \frac{p^{4k+4}-1}{12}\right) \equiv 0 \pmod{2}$$

where $1 \leqslant j \leqslant p-1$ and for $n, k \geqslant 0$

$$b_3\left(\frac{p^{4k}-1}{12}\right) \equiv 1 \pmod{2}.$$

(2) *If $b_3\left(\frac{p^2-1}{12}\right) \equiv 0 \pmod{2}$, then for $n, k \geqslant 0$ with $p \nmid (24n+1)$*

$$b_3\left(2p^{6k+2} n + \frac{p^{6k+2}-1}{12}\right) \equiv 0 \pmod{2}$$

and for $n, k \geqslant 0$

$$b_3\left(\frac{p^{6k}-1}{12}\right) \equiv 1 \pmod{2}.$$

Very recently, Ballantine, Merca and Radu [2] introduced new infinite Ramanujan type congruences modulo 2 for $b_3(2n)$. They complement naturally the results of Keith–Zanello and Yao and involve primes in the set

$$\mathscr{P} = \{p \text{ prime} : \exists\, j \in \{1,4,8\},\ x, y \in \mathbb{Z},\ \gcd(x,y) = 1 \text{ with } x^2 + 216y^2 = jp\}$$

whose Dirichlet density is $1/6$.

Theorem 3. *For every $p \in \mathscr{P}$ and $n \geqslant 0$, we have*

$$b_3\big(2(p^2 n + p\alpha - 24_p^{-1})\big) \equiv 0 \pmod{2},$$

where $0 \leqslant \alpha < p$, $\alpha \neq \lfloor p/24 \rfloor$, and 24_p^{-1} is the inverse of 24 modulo p taken such that $1 \leqslant -24_p^{-1} \leqslant p-1$.

In this work, motivated by the results on the parity of $b_3(2n)$, we investigate Ramanujan type congruences modulo 4 for $b_3(2n)$. We note that congruences modulo 4 for ℓ-regular partitions, have been studied in [8] for $\ell = 4, 5, 9$, and in [15] for $\ell = 2$. However, congruences modulo 4 for 3-regular partitions are missing from the literature.

Theorem 4. *For every $p \in \{43, 47, 59, 61, 67, 89, 137, 139, 157\}$ and $n \geqslant 0$ we have*

$$b_3\big(2(p^2 n + p\alpha - 24_p^{-1})\big) \equiv 0 \pmod{4}, \tag{1}$$

where $0 \leqslant \alpha < p$, $\alpha \neq \lfloor p/24 \rfloor$, 24_p^{-1} is the inverse of 24 modulo p taken such that $1 \leqslant -24_p^{-1} \leqslant p-1$.

We conjecture that there are infinitely many primes for which (1) holds. For example, we verified numerically that, in addition to the primes in Theorem 4, the statement of the theorem holds for

$$\begin{aligned} p \in \{&167,181,229,233,277,331,359,379,401,419,421,431,443,479,499,541,\\ &569,593,599,613,643,647,691,709,719,757,761,787,809,827,829,853,\\ &859,863,877,911,929,953,977,983,1021,1031\}. \end{aligned}$$

We were unable to finish the proof of Theorem 4 for these values of p due to computing time limitations.

2. Proof of Theorem 4

2.1. *Modular forms*

As is the case with many proofs of congruences in the literature, we use [13, Lemma 4.5]. For the conveniece of the reader, we first introduce all necessary notation and the statement of [13, Lemma 4.5]. This exposition is nearly identical to that in [2].

Let $\Gamma := SL(2,\mathbb{Z})$, and define

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma \right\}.$$

For a positive integer N, we define the congruence subgroup

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}.$$

If M is a positive integer, we write $R(M)$ for the set of finite integer sequences $r = (r_{\delta_1}, r_{\delta_2}, \ldots, r_{\delta_k})$, where $1 = \delta_1 < \delta_2 < \cdots < \delta_k = M$ are the positive divisors of M. We note for the remainder of this section we only consider positive divisors of a given integer. Given a positive integer m, we denote by S_{24m} the set of invertible quadratic residues modulo $24m$ and, for fixed $0 \leqslant t \leqslant m-1$, we define

$$P_{m,r}(t) := \left\{ ts + \frac{s-1}{24} \sum_{\delta|M} \delta r_\delta \pmod{m} : s \in S_{24m} \right\}.$$

Let m, M and N be positive integers. Moreover, let t be an integer such that $0 \leqslant t \leqslant m-1$ and let $r = (r_\delta) \in R(M)$. We set $\kappa := \gcd(1-m^2, 24)$ and write $\prod_{\delta|M} \delta^{|r_\delta|} =: 2^s v$, where s is a nonnegative integer and v is odd. Then, we say that the tuple $(n, M, N, (r_\delta), t) \in \Delta^*$ if and only if all of the following six conditions are satisfied.

(1) $p \mid m$, p prime, implies $p \mid N$;
(2) $\delta \mid M$, $\delta \geqslant 1$ such that $r_\delta \neq 0$ implies $\delta \mid mN$;
(3) $\kappa N \sum_{\delta|M} r_\delta \frac{mN}{\delta} \equiv 0 \pmod{24}$;
(4) $\kappa N \sum_{\delta|M} r_\delta \equiv 0 \pmod{8}$;
(5) $\left.\frac{24m}{\gcd(\kappa(-24t - \sum_{\delta|M} \delta r_\delta), 24m)} \right| N$;
(6) If $2 \mid m$, then $(4 \mid \kappa N$ and $8 \mid Ns)$ or $(2 \mid s$ and $8 \mid N(1-v))$.

Finally, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and m and $r = (r_\delta) \in R(M)$ as above, we define

$$p_{m,r}(\gamma) := \min_{d \in \{0,\ldots,m-1\}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd^2(\delta(a + \kappa dc), mc)}{\delta m}$$

and for $a = (a_\delta) \in R(N)$, we define

$$p_a^*(\gamma) := \frac{1}{24} \sum_{\delta|N} a_\delta \frac{\gcd^2(\delta, c)}{\delta}.$$

We use the notation

$$\sum_{n=0}^{\infty} c(n)\, q^n \equiv \sum_{n=0}^{\infty} d(n)\, q^n \pmod{u}$$

to mean that $c(n) \equiv d(n) \pmod{u}$ for all $n \geqslant 0$. Similarly,

$$\sum_{n=0}^{\infty} c(n)\, q^n \equiv 0 \pmod{u}$$

means $c(n) \equiv 0 \pmod{u}$ for all $n \geqslant 0$.

Lemma 5 ([13, Lemma 4.5]). *Let u be a positive integer, $(m, M, N, t, r = (r_\delta)) \in \Delta^*$, $a = (a_\delta) \in R(N)$. Let $\{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$ be a complete set of representatives of the double cosets in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Assume that $p_{m,r}(\gamma_i) + p_a^*(\gamma_i) \geqslant 0$ for all $0 \leqslant i \leqslant n$. Let $t_{\min} := \min_{t' \in P_{m,r}(t)} t'$ and*

$$\nu := \frac{1}{24}\left(\left(\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta a_\delta\right) - \frac{1}{24m}\sum_{\delta|M} \delta r_\delta - \frac{t_{\min}}{m}.$$

Suppose

$$\prod_{\delta|M} \prod_{n=1}^{\infty} (1 - q^{\delta n})^{r_\delta} = \sum_{n=0}^{\infty} c_r(n) q^n.$$

If

$$\sum_{n=0}^{\lfloor \nu \rfloor} c_r(mn + t') q^n \equiv 0 \pmod{u}, \quad \textit{for all } t' \in P_{m,r}(t),$$

then

$$\sum_{n=0}^{\infty} c_r(mn + t') q^n \equiv 0 \pmod{u}, \quad \textit{for all } t' \in P_{m,r}(t).$$

2.2. *Proof of Theorem 4*

As customary, we use the notation

$$f_i := \prod_{k=1}^{\infty} (1 - q^{ki}).$$

From [21], identity (2.18), we have

$$\sum_{n=0}^{\infty} b_3(2n)\, q^n = \frac{f_2 f_3 f_8 f_{12}^2}{f_1^2 f_4 f_6 f_{24}}. \tag{2}$$

Moreover, since for $i \geq 1$, $f_{2i}^2 \equiv f_i^4 \pmod{4}$, we have

$$\frac{f_{2i}}{f_i} = \frac{f_{2i}^2}{f_i^4} \frac{f_i^3}{f_{2i}} \equiv \frac{f_i^3}{f_{2i}} \pmod{4}. \tag{3}$$

Using (2) and (3), we obtain

$$\sum_{n=0}^{\infty} b_3(2n)\, q^n \equiv \frac{f_1^2 f_3 f_4^3 f_6^3}{f_2 f_8 f_{24}} \pmod{4}.$$

To use the Lemma 5, we write

$$\sum_{n=0}^{\infty} c(n) q^n := \frac{f_1^2 f_3 f_4^3 f_6^3}{f_2 f_8 f_{24}} = \prod_{\delta|M} \prod_{n=1}^{\infty} (1 - q^{\delta n})^{r_\delta}.$$

Thus, with the notation of Lemma 5, we take $u = 4$, $m = p^2$, $M = 24$, and

$$(r_1, r_2, r_3, r_4, r_6, r_8, r_{12}, r_{24}) = (2, -1, 1, 3, 3, -1, 0, -1).$$

We have $\kappa = 24$ and we calculate

$$\sum_{\delta|M} \frac{r_\delta}{\delta} = \frac{35}{12}, \qquad \sum_{\delta|M} r_\delta = 6, \qquad \sum_{\delta|M} \delta r_\delta = 1.$$

We take $N = 24p$. It is straightforward to verify that for any $t = p\alpha - 24_p^{-1}$ conditions (1)–(6) are satisfied.

Since

$$[\Gamma : \Gamma_0(N)] = N \prod_{x|N} (1 + x^{-1}),$$

where the product is taken after all prime divisors of N, we have

$$[\Gamma : \Gamma_0(N)] = 48(p+1).$$

In general, it is nontrivial to find a complete set of representatives for the double cosets in $\Gamma_0(N) \setminus \Gamma/\Gamma_\infty$. If N is square free, it is shown in [14, Lemma 2.6] that a complete set of representatives for $\Gamma_0(N)\setminus\Gamma/\Gamma_\infty$ is given by

$$\mathscr{A}_N = \left\{ \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} : \delta \mid N, \delta \geqslant 1 \right\}.$$

This result has been extended to N such that $N/2$ is square free in [16, Lemma 4.3]. While for $N = 24p$, neither N nor $N/2$ is square free, when $m = p^2, \kappa = 24$, and $(r_1, r_2, r_3, r_4, r_6, r_8, r_{12}, r_{24}) = (2, -1, 1, 3, 3, -1, 0, -1)$ we can avoid finding a complete set of representatives all together.

Since for any integers $i, j \geq 1$ we have

$$\gcd(j(a + \kappa dc), mc) \leqslant \gcd(ij(a + \kappa dc), mc) \leqslant i \gcd(j(a + \kappa dc), mc),$$

an easy calculation shows that for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we have

$$\sum_{\delta|M} r_\delta \frac{\gcd^2(\delta(a + \kappa dc), mc)}{\delta m} \geq 0,$$

and thus $p_{m,r}(\gamma) \geq 0$. Hence, we can use $a := (a_\delta)_{\delta|N}$ with $a_\delta = 0$ for each $\delta \mid N$ to calculate $\lfloor \nu \rfloor$. It is clear from the definition of ν in Lemma 5 that

$$\lfloor \nu \rfloor = 12\,(p+1) - 1.$$

Let

$$\mathscr{R}_p := \{p\alpha - 24_p^{-1} : 0 \leqslant \alpha < p,\ \alpha \neq \lfloor p/24 \rfloor\}.$$

For each p, we used Mathematica™ to write the set $\mathscr{R}_p$ as

$$\mathscr{R}_p = P_{m,r}(-24_p^{-1}) \cup P_{m,r}(Ap - 24_p^{-1})$$

for a minimal A. For example, when $p = 43$, we have $A = 2$ and the Mathematica™ calculation gives

$$\begin{aligned} P_{m,r}(-24_p^{-1}) = \{&34, 163, 206, 292, 378, 421, 593, 851, 894, 937, 1023, 1195, \\ &1238, 1281, 1324, 1367, 1453, 1496, 1539, 1668, 1754\} \end{aligned}$$

and

$$\begin{aligned} P_{m,r}(2 \cdot 43 - 24_p^{-1}) = \{&120, 249, 335, 464, 507, 550, 636, 679, 722, 765, 808, 980, \\ &1066, 1109, 1152, 1410, 1582, 1625, 1711, 1797, 1840\}. \end{aligned}$$

If $\alpha = \lfloor 43/24 \rfloor = 1$ we have $P_{m,r}(p - 24_p^{-1}) = P_{m,r}(77) = \{77\}$.

For $p \in \{43, 47, 139, 157\}$ we obtained $A = 2$ and for $p \in \{59, 61, 67, 89, 137\}$ we obtained $A = 1$. Moreover, if $t^* = p\lfloor p/24 \rfloor - 24_p^{-1}$, then $P_{m,r}(t^*) = \{t^*\}$.

Finally, in each case, we verified that for each $t \in \mathscr{R}_p$ we have

$$c(p^2 n + t) \equiv 0 \pmod 4 \quad \text{for } 0 \leqslant n \leqslant 12(p+1) - 1.$$

Then, [13, Lemma 4.5] implies that, for each

$$p \in \mathscr{U} := \{43, 47, 59, 61, 67, 89, 137, 139, 157\}$$

we have

$$c(p^2 n + t) \equiv 0 \pmod 4 \quad \text{for all } n \geqslant 0 \text{ and } t \in \mathscr{R}_p.$$

Our calculations show that for each prime $p \in \mathscr{U}$, we have that

$$c(p\lfloor p/24\rfloor - 24_p^{-1}) \not\equiv 0 \pmod 4$$

and thus the requirement that $\alpha \neq \lfloor p/24 \rfloor$ in the statement of Theorem 4 is necessary.

3. Concluding remarks

Several Ramanujan type congruences modulo 4 for $b_3(2n)$ involving some primes p with $p \equiv 11, 13, 17, 19, 23 \pmod{24}$ have been proved in this paper using modular forms and [13, Lemma 4.5].

As mentioned in the introduction, we verified the statement of Theorem 4 numerically up to 10^8 for many more primes equivalent to 11, 13, 17, 19, 23 modulo 24. In fact, our computations suggest that (1) is also true for primes $p = 1033$ and $p = 1153$ which are congruent to 1 modulo 24. We did not encounter any primes congruent to 5 or 7 modulo 24 for which (1) holds. We leave it as an open problem to characterize an infinite family of primes for which the statement of Theorem 4 holds and to prove the theorem for all these primes.

References

[1] G. E. Andrews, *The Theory of Partitions*, Cambridge Mathematical Library, Cambridge University Press, 1998.

[2] C. Ballantine, M. Merca, C.-S. Radu, "Parity of 3-regular partition numbers and Diophantine equations", 2022, https://arxiv.org/abs/2212.09810.

[3] R. Carlson, J. J. Webb, "Infinite families of infinite families of congruences for k-regular partitions", *Ramanujan J.* **33** (2014), no. 3, p. 329-337.

[4] S.-P. Cui, N. S. S. Gu, "Arithmetic properties of ℓ-regular partitions", *Adv. Appl. Math.* **51** (2013), no. 4, p. 507-523.

[5] B. Dandurand, D. Penniston, "l-Divisibility of l-regular partition functions", *Ramanujan J.* **19** (2009), no. 1, p. 63-70.

[6] D. Furcy, D. Penniston, "Congruences for ℓ-regular partition functions modulo 3", *Ramanujan J.* **27** (2012), no. 1, p. 101-108.

[7] M. D. Hirschhorn, J. A. Sellers, "Elementary proofs of parity results for 5-regular partitions", *Bull. Aust. Math. Soc.* **81** (2010), no. 1, p. 58-63.

[8] W. J. Keith, "Congruences for m-regular partitions modulo 4", *Integers* **15A** (2015), article no. A11 (12 pages).

[9] W. J. Keith, F. Zanello, "Parity of the coefficients of certain eta-quotients", *J. Number Theory* **235** (2022), p. 275-304.

[10] J. Lovejoy, D. Penniston, "3-regular partitions and a modular $K3$ surface", in *q-series with applications to combinatorics, number theory, and physics*, Contemporary Mathematics, vol. 291, American Mathematical Society, 2001, p. 177-182.

[11] D. Penniston, "The p^a-regular partition function modulo p^j", *J. Number Theory* **94** (2002), no. 2, p. 320-325.

[12] ———, "Arithmetic of ℓ-regular partition functions", *Int. J. Number Theory* **4** (2008), no. 2, p. 295-302.

[13] S. Radu, "An algorithmic approach to Ramanujan's congruences", *Ramanujan J.* **20** (2009), no. 2, p. 215-251.

[14] S. Radu, J. A. Sellers, "Congruence properties modulo 5 and 7 for the pod function", *Int. J. Number Theory* **7** (2011), no. 8, p. 2249-2259.

[15] Ø. Rødseth, "Congruence properties of the partition functions $q(n)$ and $q_0(n)$", *Årbok Univ. Bergen, Mat.-Naturv. Ser.* **1969**, no. 13.

[16] L. Wang, "Arithmetic properties of (k, ℓ)-regular bipartitions", *Bull. Aust. Math. Soc.* **95** (2017), no. 3, p. 353-364.

[17] ———, "Congruences for 5-regular partitions modulo powers of 5", *Ramanujan J.* **44** (2017), no. 2, p. 343-358.

[18] ———, "Arithmetic properties of 7-regular partitions", *Ramanujan J.* **47** (2018), no. 1, p. 99-115.

[19] J. J. Webb, "Arithmetic of the 13-regular partition function modulo 3", *Ramanujan J.* **25** (2011), p. 49-56.

[20] E. X. W. Xia, "Congruences for some l-regular partitions modulo l", *J. Number Theory* **152** (2015), p. 105-117.

[21] E. X. W. Xia, O. X. M. Yao, "New Ramanujan-like congruences modulo powers of 2 and 3 for overpartitions", *J. Number Theory* **133** (2013), no. 6, p. 1932-1949.

[22] ———, "Parity results for 9-regular partitions", *Ramanujan J.* **34** (2014), no. 1, p. 109-117.

[23] O. X. M. Yao, "New parity results for 3-regular partitions", *Quaest. Math.* **46** (2023), no. 3, p. 465-471.