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Académie des sciences

Comptes Rendus
Mathématique

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Volume 361 (2023), p. 1761-1766

Published online: 21 December 2023

https://doi.org/10.5802/crmath.515

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2023, Vol. 361, p. 1761-1766
https://doi.org/10.5802/crmath.515

Mathematical physics, Spectral theory / *Physique mathématique, Théorie spectrale*

Asymptotic formula for large eigenvalues of the two-photon quantum Rabi model

Formule asymptotique pour les grandes valeurs propres du modèle quantique de Rabi à deux photons

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Abstract. We prove that the spectrum of the two-photon quantum Rabi Hamiltonian consists of two eigenvalue sequences $(E^+_m)^\infty_{m=0}$, $(E^-_m)^\infty_{m=0}$ satisfying a three-term asymptotic formula with the remainder estimate $O(m^{-1}\ln m)$ when m tends to infinity. By analogy to the one-photon quantum Rabi model, the leading three terms of this asymptotic formula, describe a generalized rotating-wave approximation for large eigenvalues of the two-photon quantum Rabi model.

Résumé. Nous démontrons que le spectre de l'hamiltonien du modèle quantique de Rabi à deux photons est constitué de deux suites de valeurs propres $(E^+_m)^\infty_{m=0}$, $(E^-_m)^\infty_{m=0}$ vérifiant une formule asymptotique à trois termes avec l'estimation de l'erreur $O(m^{-1}\ln m)$ quand m tend vers l'infini. Par analogie avec le modèle quantique de Rabi à un photon, les trois termes dominants dans cette formule asymptotique décrivent l'approximation de l'onde tournante pour les grandes valeurs propres du modèle quantique de Rabi à deux photons.

Manuscript received 11 May 2023, accepted 16 May 2023.

1. Introduction

The quantum Rabi model describes the interactions between a two-level system and a single-mode quantum field. It is considered as a fundamental model in various domains of theoretical physics, e.g. cavity optics, a theory of nanostructured semiconductors, superconducting circuits, trapped ions and quantum information. We refer to [15] for an exhaustive overview of theoretical and experimental works in the domain.

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The papers Boutet de Monvel, Naboko, Silva [7, 8] and Yanovich [14] (see also [13]) investigate the behaviour of large eigenvalues of operators related to the quantum Rabi model. The paper [9] gives the three-term asymptotic formula for large eigenvalues of the one-photon Rabi model (see Definition 1 (1)). The two-term asymptotic formula for the two-photon Rabi model (see Definition 1(2)) is given in [10] and this note gives the three-term asymptotic formula for this model. Concerning the two-photon Rabi model, we refer to [2, 5, 15].

In Section 2 we give the definition of the basic quantum Rabi model in the one-photon and two-photon case. Our main result is Theorem 2, stated in Section 3. We describe main ingredients of the proof in Sections 4, 5 and 6.

We remark that our result is closely related to the method of the generalized rotating-wave approximation (GRWA), used by a great number of physicists working with the quantum Rabi model. The method takes its name from the famous paper of Irish [6], but the same idea was also used in the paper Feranchuk, Komarov, Ulyanenkov [4] under the name of the zeroth order approximation of the operator method (see also [3]). It appears that in the case of the one-photon quantum Rabi model, the GRWA given by the formula (25) in [4], coincides with the three-term asymptotic formula proved in [9]. In Section 6 we describe the relation between the assertion of Theorem 2 and the GRWA for large eigenvalues of the two-photon quantum Rabi model.

2. Definition of the quantum Rabi model

In what follows, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N} = \{n \in \mathbb{Z} : n \geq 0\}$ and $\ell^2(\mathbb{N})$ is the Hilbert space of square summable sequences $x : \mathbb{N} \to \mathbb{C}$. For $s \geq 0$ we denote

$$\ell^{2,s}(\mathbb{N}) = \left\{ x \in \ell^2(\mathbb{N}) : \sum_{k\in\mathbb{N}} (1+k^2)^s |x(k)|^2 < \infty \right\}.$$

We define the photon annihilation and creation operators $\widehat{a}$ and $\widehat{a}^\dagger$, as linear maps $\ell^{2,1/2}(\mathbb{N}) \to \ell^2(\mathbb{N})$ satisfying

$$\widehat{a}^\dagger e_m = \sqrt{m+1}\, e_{m+1} \quad \text{for } m = 0, 1, 2, \dots$$
$$\widehat{a} e_m = \sqrt{m-1}\, e_{m-1} \quad \text{for } m = 1, 2, 3, \dots \text{ and } \widehat{a} e_0 = 0,$$

where $\{e_m\}_{m\in\mathbb{N}}$ is the canonical basis of $\ell^2(\mathbb{N})$.

Definition 1. *We fix two real parameters: the energy spacing of the two-level system Δ and the coupling constant g.*

(1) *The Hamiltonian of the one-photon quantum Rabi model is given by the linear map $\mathbb{C}^2 \otimes \ell^{2,1}(\mathbb{N}) \to \mathbb{C}^2 \otimes \ell^2(\mathbb{N})$ of the form*

$$H_1 = \begin{pmatrix} \frac{\Delta}{2} & 0 \\ 0 & -\frac{\Delta}{2} \end{pmatrix} \otimes I_{\ell^2(\mathbb{N})} + I_{\mathbb{C}^2} \otimes \widehat{a}^\dagger \widehat{a} + \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix} \otimes \left(\widehat{a}^\dagger + \widehat{a} \right). \tag{1}$$

(2) *The Hamiltonian of the two-photon quantum Rabi model is given by the linear map $\mathbb{C}^2 \otimes \ell^{2,1}(\mathbb{N}) \to \mathbb{C}^2 \otimes \ell^2(\mathbb{N})$ of the form*

$$H_2 = \begin{pmatrix} \frac{\Delta}{2} & 0 \\ 0 & -\frac{\Delta}{2} \end{pmatrix} \otimes I_{\ell^2(\mathbb{N})} + I_{\mathbb{C}^2} \otimes \widehat{a}^\dagger \widehat{a} + \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix} \otimes \left((\widehat{a}^\dagger)^2 + \widehat{a}^2 \right). \tag{2}$$

3. Main result

In what follows, we assume $0 < g < 1/2$ and introduce

$$\beta := \sqrt{1 - 4g^2}. \tag{3}$$

Let H_2^0 denote the operator given by (2) with $\Delta = 0$. If $0 < g < 1/2$, then the spectrum of H_2^0 is explicitly known (see [2]): it is composed of the sequence of eigenvalues

$$E_m^0 = m\beta + (\beta - 1)/2, \quad m = 0, 1, 2, \ldots \tag{4}$$

and each eigenvalue E_m^0 is of multiplicity 2. Thus $0 < g < 1/2$ ensures the fact that H_2^0 is self-adjoint and has compact resolvent. Since $H_2 - H_2^0$ is bounded, the operator H_2 is self-adjoint and has compact resolvent if $0 < g < 1/2$. The explicit values of eigenvalues of H_2 are not known when $\Delta \neq 0$, but we can described their asymptotic behavior in

Theorem 2. *Assume that* $0 < g < 1/2$. *Then one can find* $\{v_m^+\}_{m\in\mathbb{N}} \cup \{v_m^-\}_{m\in\mathbb{N}}$, *an orthonormal basis of* $\mathbb{C}^2 \otimes \ell^2(\mathbb{N})$, *such that*

$$H_2 v_m^\pm = E_m^\pm v_m^\pm, \quad m = 0, 1, 2, \ldots$$

and the eigenvalue sequences $(E_m^+)_{m\in\mathbb{N}}$, $(E_m^-)_{m\in\mathbb{N}}$, *satisfy the large* m *estimates*

$$E_m^\pm = m\beta + (\beta - 1)/2 \pm r_m + O(m^{-1}\ln m) \tag{5}$$

with r_m *given by the formula*

$$r_m = \begin{cases} \dfrac{\Delta}{2}\sqrt{\dfrac{\beta}{\pi g m}}\,\cos((2m+1)\alpha) & \text{if } m \text{ is even} \\[2ex] \dfrac{\Delta}{2}\sqrt{\dfrac{\beta}{\pi g m}}\,\sin((2m+1)\alpha) & \text{if } m \text{ is odd} \end{cases} \tag{6}$$

where we have denoted

$$\alpha := \arctan\left(\sqrt{\frac{1-2g}{1+2g}}\right). \tag{7}$$

Remarks.

(a) One has $E_m^\pm - E_m^0 = O(m^{-1/2})$ in spite of the fact that $H_2 - H_2^0$ is not compact. A similar fact was established in [14] for the one-photon Rabi model.

(b) Following [12], one can prove that the spectrum of H_2 is not discrete if $g \geq 1/2$.

4. Initial reformulations

It is easy to check that the operator H_2 has four closed invariant subspaces:

$$\begin{aligned}
&\mathscr{H}_0^- \text{ spanned by } \{(1,0)\otimes e_{4k} : k \in \mathbb{N}\} \cup \{(0,1)\otimes e_{4k+2} : k \in \mathbb{N}\} \\
&\mathscr{H}_0^+ \text{ spanned by } \{(0,1)\otimes e_{4k} : k \in \mathbb{N}\} \cup \{(1,0)\otimes e_{4k+2} : k \in \mathbb{N}\} \\
&\mathscr{H}_1^- \text{ spanned by } \{(1,0)\otimes e_{4k+1} : k \in \mathbb{N}\} \cup \{(0,1)\otimes e_{4k+3} : k \in \mathbb{N}\} \\
&\mathscr{H}_1^+ \text{ spanned by } \{(0,1)\otimes e_{4k+1} : k \in \mathbb{N}\} \cup \{(1,0)\otimes e_{4k+3} : k \in \mathbb{N}\}
\end{aligned}$$

and the matrix of H_2 in a suitable basis of $\mathscr{H}_\mu^\pm$ $\mu = 0, 1$, can be written in the form

$$J_\mu^\pm = \begin{pmatrix} d_\mu^\pm(0) & b_\mu(0) & 0 & 0 & \cdots \\ b_\mu(0) & d_\mu^\pm(1) & b_\mu(1) & 0 & \cdots \\ 0 & b_\mu(1) & d_\mu^\pm(2) & b_\mu(2) & \cdots \\ 0 & 0 & b_\mu(2) & d_\mu^\pm(3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{8}$$

where

$$\begin{aligned}
d_\mu^\pm(m) &:= 2m + \mu \pm (-1)^m \Delta/2, \\
b_\mu(m) &:= g\sqrt{(2m+1+\mu)(2m+2+\mu)}.
\end{aligned}$$

It therefore remains to investigate the asymptotic behavior of eigenvalues of operators defined by J_0^-, J_0^+, J_1^- and J_1^+. For this purpose, we remark that

$$b_\mu(m) = 2g(m+\gamma) + O(m^{-1}) \text{ holds with } \gamma := \frac{\mu}{2} + \frac{3}{4}. \tag{9}$$

Using the result of Rozenblum stated in Theorem 1.1 of [11], we find that modulo $O(m^{-1})$, the asymptotic behaviour of the m-th eigenvalue remains the same if the entries $\{b_\mu(m)\}_{m\in\mathbb{N}}$ are replaced by $\{2g(m+\gamma)\}_{m\in\mathbb{N}}$ with $\gamma := \frac{\mu}{2} + \frac{3}{4}$. This fact allows us to deduce the assertion of Theorem 2 from

Theorem 3. *Assume that* $0 < g < 1/2$. *Let* δ, γ *be some real numbers and let* $\widehat{J}_\gamma^\delta$ *be the linear map* $\ell^{2,1}(\mathbb{N}) \to \ell^2(\mathbb{N})$ *defined by the formula*

$$\widehat{J}_\gamma^\delta e_m = \big(m + (-1)^m\delta\big)e_m + g(m+\gamma)e_{m+1} + g(m-1+\gamma)e_{m-1}, \quad m \in \mathbb{N}, \tag{10}$$

where $\{e_m\}_{m\in\mathbb{N}}$ *is the canonical basis in* $\ell^2(\mathbb{N})$ *and by convention* $g(m-1+\gamma)e_{m-1} = 0$ *if* $m = 0$. *Then* $\widehat{J}_\gamma^\delta$ *has discrete spectrum and its non-decreasing eigenvalue sequence satisfies the large* n *asymptotic formula*

$$\lambda_n(\widehat{J}_\gamma^\delta) = \beta n + (\gamma - 1/2)(\beta - 1) + \delta r_{\gamma,n} + O(n^{-1}\ln n) \tag{11}$$

with

$$r_{\gamma,n} = \sqrt{\frac{\beta}{2\pi g n}}\cos\big(4\alpha n + \widehat{\theta}_\gamma\big), \tag{12}$$

where β *is given by* (3), α *by* (7) *and*

$$\widehat{\theta}_\gamma = (\gamma - 1/2)(4\alpha - \pi) + \pi/4. \tag{13}$$

In order to prove Theorem 3, we move from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{Z})$, the Hilbert space of square summable sequences $x : \mathbb{Z} \to \mathbb{C}$. For $s > 0$ we denote

$$\ell^{2,s}(\mathbb{Z}) = \left\{x \in \ell^2(\mathbb{Z}) : \sum_{k\in\mathbb{Z}} (1+k^2)^s |x(k)|^2 < \infty\right\}$$

and define $\widetilde{J}_\gamma^\delta$ as the linear map $\ell^{2,1}(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ given by

$$\widetilde{J}_\gamma^\delta e_k = \big(k + (-1)^k\delta\big)e_k + g(k+\gamma)e_{k+1} + g(k-1+\gamma)e_{k-1}, \quad k \in \mathbb{Z}, \tag{14}$$

where $\{e_k\}_{k\in\mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$. We can identify $\ell^2(\mathbb{N})$ with

$$\{x \in \ell^2(\mathbb{Z}) : x(k) = 0 \text{ for } k \in \mathbb{Z} \setminus \mathbb{N}\}$$

and consider $\widetilde{J}_\gamma^\delta$ as an extension of $\widehat{J}_\gamma^\delta$. Using Theorem 1.1 in [11], we find that the spectrum of $\widetilde{J}_\gamma^\delta$ is composed of a non-decreasing sequence of eigenvalues $\{\lambda_j(\widetilde{J}_\gamma^\delta)\}_{j\in\mathbb{Z}}$, which can be labeled so that for any $N > 0$ one has the estimate

$$\lambda_n(\widetilde{J}_\gamma^\delta) = \lambda_n(\widehat{J}_\gamma^\delta) + O(n^{-N}) \text{ when } n \to \infty \tag{15}$$

5. Diagonalisation of $\widetilde{J}_\gamma^\delta$ when $\delta = 0$

The operator $\widetilde{J}_\gamma^\delta$ with $\delta = 0$ was investigated in [1]. Let S be the shift $Se_j = e_{j+1}$ in $\ell^2(\mathbb{Z})$ and denote $\Lambda := \operatorname{diag}(j)_{j\in\mathbb{Z}}$. Then (14) gives

$$\widetilde{J}_\gamma^0 = \Lambda + g\big(S(\Lambda+\gamma) + (\Lambda+\gamma)S^{-1}\big) = \Lambda + g\big(S(\Lambda+\gamma) + \text{h.c.}\big)$$

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and let $\mathscr{F}_\mathbb{T}$ be the isometric isomorphism $\mathrm{L}^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ given by

$$(\mathscr{F}_\mathbb{T} f)(j) = \int_{-\pi}^{\pi} f(\theta)\,\mathrm{e}^{-\mathrm{i}j\theta}\,\frac{\mathrm{d}\theta}{2\pi}. \tag{16}$$

Then $L_\gamma := \mathscr{F}_{\mathbb{T}}^{-1}\widetilde{J}_\gamma^0\mathscr{F}_{\mathbb{T}}$ is the first order differential operator

$$L_\gamma = \frac{1}{2}\left(\left(1+2g\cos\theta\right)\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)+\text{h.c.}\right)+(1+2\gamma)\cos\theta$$

and following [1], we define the diffeomorphism of $]-\pi,\pi[$ given by

$$\Phi(\theta) := \int_0^\theta \frac{\beta\,\mathrm{d}\theta'}{1+2g\cos\theta'} = 2\arctan\left(\sqrt{\frac{1-2g}{1+2g}}\tan\left(\frac{\theta}{2}\right)\right). \tag{17}$$

The change of variable $\eta = \Phi(\theta)$ defines the unitary operator acting in $\mathrm{L}^2(\mathbb{T})$ according to the formula

$$(U_\Phi f)(\theta) = \Phi'(\theta)^{1/2} f(\Phi(\theta)) \tag{18}$$

and the direct computation gives

$$U_\Phi^{-1} L_\gamma U_\Phi = \beta\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\eta}+q_\gamma(\eta)\right)$$

with

$$q_\gamma(\eta) := \beta^{-1}(1+2\gamma)\cos(\Phi^{-1}(\eta)).$$

Let $\widetilde{q}_\gamma$ be a primitive of q_γ. We compute

$$\langle q_\gamma\rangle := (\widetilde{q}_\gamma(\pi)-\widetilde{q}_\gamma(-\pi))/(2\pi) = (\gamma-1/2)(1-1/\beta)$$

and remark that $\eta \to \langle q_\gamma\rangle\eta - \widetilde{q}_\gamma(\eta)$ is 2π-periodic, hence we can define $(f_{\gamma,j})_{j\in\mathbb{Z}}$, the orthonormal basis in $\mathrm{L}^2(\mathbb{T})$ given by

$$f_{\gamma,j}(\eta) = \mathrm{e}^{\mathrm{i}j\eta}\,\mathrm{e}^{\mathrm{i}(\langle q_\gamma\rangle\eta-\widetilde{q}_\gamma(\eta))}. \tag{19}$$

Then (see [1]), for every $j\in\mathbb{Z}$, one has

$$\beta\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\eta}+q_\gamma\right)f_{\gamma,j} = \beta(j+\langle q_\gamma\rangle)f_{\gamma,j}.$$

Consequently, for every $j\in\mathbb{Z}$,

$$\widetilde{J}_\gamma^0 u_{\gamma,j} = d_{\gamma,j}u_{\gamma,j} \tag{20}$$

holds with

$$u_{\gamma,j} = \mathscr{F}_{\mathbb{T}}U_\Phi f_{\gamma,j}, \tag{21}$$

$$d_{\gamma,j} = \beta(j+\langle q_\gamma\rangle) = \beta j+(\gamma-1/2)(\beta-1). \tag{22}$$

6. Generalized rotating-wave approximation (GRWA)

The idea of the GRWA consists in using the diagonal entries of a perturbation as the first correction for eigenvalues of a perturbed diagonal matrix and we refer to [3] for numerous examples of this approach. In order to apply this idea, we consider $\widetilde{J}_\gamma^\delta$ as a perturbation of $\widetilde{J}_\gamma^0$ and use the diagonalisation (20). We remark that

$$\widetilde{J}_\gamma^\delta = \widetilde{J}_\gamma^0+\delta V \tag{23}$$

holds with $V := \mathrm{diag}((-1)^j)_{j\in\mathbb{Z}}$. Let U_γ be the unitary operator in $\ell^2(\mathbb{Z})$ defined by $U_\gamma e_j = u_{\gamma,j}$, where $\{e_j\}_{j\in\mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$ and $\{u_{\gamma,j}\}_{j\in\mathbb{Z}}$ is the basis given by (21). Then (23) gives

$$U_\gamma^{-1}\widetilde{J}_\gamma^\delta U_\gamma = D_\gamma+\delta V_\gamma, \tag{24}$$

where $D_\gamma = \mathrm{diag}(d_{\gamma,j})_{j\in\mathbb{Z}}$ and $V_\gamma = U_\gamma^{-1}VU_\gamma$. We claim that the asymptotic estimate (11)–(13) follows from

$$\lambda_j(D_\gamma+\delta V_\gamma) = d_{\gamma,j}+\delta V_\gamma(j,j)+O(j^{-1}\ln j)\text{ when } j\to\infty \tag{25}$$

where

$$V_\gamma(j,j) = \langle e_j, V_\gamma e_j\rangle_{\ell^2(\mathbb{Z})} = \langle u_{\gamma,j}, Vu_{\gamma,j}\rangle_{\ell^2(\mathbb{Z})}.$$

Indeed, since $\mathscr{F}_{\mathbb{T}}^{-1} V \mathscr{F}_{\mathbb{T}} = T_\pi$ is the translation $\theta + 2\pi\mathbb{Z} \to \theta + \pi + 2\pi\mathbb{Z}$, we find that

$$V_\gamma(j,j) = \langle U_\Phi f_{\gamma,j}, T_\pi U_\Phi f_{\gamma,j}\rangle_{\mathrm{L}^2(\mathbb{T})} = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}j(\Phi - T_\pi\Phi)(\theta)} p_\gamma(\theta)\, \overline{(T_\pi p_\gamma)(\theta)}\, \frac{\mathrm{d}\theta}{2\pi} \tag{26}$$

holds with

$$p_\gamma(\theta) := \mathrm{e}^{\mathrm{i}\langle q_\gamma\rangle \Phi(\theta) - \mathrm{i}\widetilde{q}_\gamma(\Phi(\theta))} \beta^{1/2} (1 + 2g\cos\theta)^{-1/2}$$

and the stationary phase method gives

$$V_\gamma(j,j) = r_{\gamma,j} + O(j^{-1}) \text{ when } j \to \infty \tag{27}$$

where $r_{\gamma,j}$ is given by (12)–(13). Thus (11)–(13) follow from (25), (27) and (15).

The proof of (25) is based on the approach of Yanovich [14] and the explicit expressions of the entries

$$V_\gamma(j,k) = \langle U_\Phi f_{\gamma,j}, T_\pi U_\Phi f_{\gamma,k}\rangle_{\mathrm{L}^2(\mathbb{T})}. \tag{28}$$

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