



INSTITUT DE FRANCE  
Académie des sciences

# *Comptes Rendus*

---


## *Mathématique*

Faouaz Saadi and Radouan Daher

**Equivalence of K-functionals and modulus of smoothness generated by a Dunkl type operator on the interval  $(-1, 1)$**

Volume 361 (2023), p. 1625-1633

<https://doi.org/10.5802/crmath.517>

 This article is licensed under the  
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.  
<http://creativecommons.org/licenses/by/4.0/>



*Les Comptes Rendus. Mathématique* sont membres du  
Centre Mersenne pour l'édition scientifique ouverte  
[www.centre-mersenne.org](http://www.centre-mersenne.org)  
e-ISSN : 1778-3569



Harmonic analysis / *Analyse harmonique*

# Equivalence of $K$ -functionals and modulus of smoothness generated by a Dunkl type operator on the interval $(-1, 1)$

Faouaz Saadi <sup>\*,a</sup> and Radouan Daher <sup>a</sup>

<sup>a</sup> Department of Mathematics, Laboratory of Topology, Algebra, Geometry, and Discrete Mathematics, Faculty of Sciences Aïn Chock University Hassan II, Casablanca, Morocco

*E-mails:* sadfaouaz@gmail.com, rjdaher024@gmail.com

**Abstract.** Our aim in this paper is to show that the modulus of smoothness and the  $K$ -functionals constructed from the Sobolev-type space corresponding to the Dunkl operator are equivalent on the interval  $(-1, 1)$ .

**Keywords.** Fourier–Dunkl series, Dunkl transform, generalized translation operator,  $K$ -functionals, modulus of smoothness.

**2020 Mathematics Subject Classification.** 43A30, 46E35, 33D60.

*Manuscript received 17 October 2022, revised 24 March 2023, accepted 25 May 2023.*

## 1. Introduction

As it is well-known, the modulus of smoothness generated by the standard translation is equivalent with the Peetre’s  $K$ -functional, see e.g. [6, p. 171]. This property is extended to Dunkl translation by E. S. Belkina and S. S. Platonov (see [2]) and Bessel translation in [12]. In this paper, we prove the counterparts of results obtained in [2], i.e., we establish the equivalence between  $K$ -functionals and modulus of smoothness in the Dunkl context (on  $(-1, 1)$ ) by using Fourier–Dunkl expansions introduced in [4], instead of Dunkl transform. The orthonormal system associated with this kind of series is a generalization of the trigonometric one (in particular, the periodicity is lost).

Hereinafter the symbol  $\alpha$  stands for a real value such that  $\alpha > -1$ . We consider the Dunkl operator  $\Lambda_\alpha$  associated with the reflection group  $\mathbb{Z}_2$  on  $\mathbb{R}$  given by

$$\Lambda_\alpha f(x) = \frac{d}{dx} f(x) + \frac{2\alpha + 1}{x} \left( \frac{f(x) - f(-x)}{2} \right).$$

The initial value problem

$$\begin{cases} \Lambda_\alpha f(x) = i\lambda f(x), & \lambda \in \mathbb{R} \\ f(0) = 1 \end{cases}$$

\* Corresponding author.

has a unique solution  $E_\alpha(i\lambda.)$  (called the *Dunkl kernel*) given by:

$$E_\alpha(i\lambda x) = j_\alpha(\lambda x) + \frac{i\lambda x}{2(\alpha + 1)} j_{\alpha+1}(\lambda x), \quad x \in \mathbb{R}, \tag{1}$$

where  $j_\alpha$  is the normalized Bessel function of the first kind defined by

$$j_\alpha(x) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(x)}{x^\alpha} \tag{2}$$

and  $J_\alpha$  is the Bessel functions of the first kind of order  $\alpha$

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^\infty \frac{\left(\frac{ix}{2}\right)^{2n}}{n! \Gamma(n + \alpha + 1)}.$$

From [15], for all  $x \in \mathbb{R}$ , we have

$$|E_\alpha(ix)| \leq 1 \quad \text{and} \quad |E'_\alpha(ix)| \leq 1. \tag{3}$$

Let  $L^p((-1, 1), d\mu_\alpha)$ ,  $p \geq 1$ , denote the Lebesgue spaces on the interval  $(-1, 1)$  endowed with the norm

$$\|f\|_{\alpha,p} = \left( \int_{-1}^1 |f(t)|^p d\mu_\alpha(x) \right)^{\frac{1}{p}}.$$

where  $d\mu_\alpha(x) = (2^{\alpha+1} \Gamma(\alpha + 1))^{-1} |x|^{2\alpha+1} dx$ . The Dunkl transform is a generalization of the Fourier transform. It is defined for  $f \in L^1((-1, 1), d\mu_\alpha)$  by the identity (see [7, 9])

$$\mathcal{F}_\alpha f(y) = \int_{\mathbb{R}} f(x) E_\alpha(-iyx) d\mu_\alpha(x), \quad y \in \mathbb{R}.$$

The Fourier transform corresponds with the case  $\alpha = -1/2$  because  $E_{-1/2}(ix) = e^{ix}$  and  $d\mu_{-1/2}$  is, up to a multiplicative factor, the Lebesgue measure on  $\mathbb{R}$ .

## 2. Equivalence of K-functionals and modulus of smoothness generated by a Dunkl type operator on the interval $(-1, 1)$ .

Let  $\{\lambda_n := \lambda_{\alpha+1,n}, n \in \mathbb{N}\}$  be the increasing sequence of positive zeros of  $J_{\alpha+1}$ . It is proved in [10] that

$$\lambda_n \leq n\pi + \alpha\pi/2 + \pi/4 \quad \text{for } \alpha > -1/2. \tag{4}$$

In [8] we find the following inequality

$$\lambda_n > \alpha + n\pi - \frac{\pi}{2} + \frac{3}{2}, \quad \alpha > -1, \quad n = 1, 2, \dots$$

then

$$\lambda_n > n, \quad \alpha > -1, \quad n = 1, 2, \dots \tag{5}$$

When the range of  $\alpha$  is fixed, like  $-1 < \alpha \leq -\frac{1}{2}$  (see Schafheitlin in [18, p. 490]) and no (essential) restriction on  $n$  :

$$n\pi + \alpha\pi/2 + \pi/4 < \lambda_n < n\pi \quad \left(-1 < \alpha \leq -\frac{1}{2}\right). \tag{6}$$

Let  $c_\alpha = \pi + \max\{0, \alpha\pi/2 + \pi/4\}$ . Combining (4), (5), and (6) gives

$$n < \lambda_n < c_\alpha n \quad \text{for all } \alpha > -1. \tag{7}$$

The real-valued function  $\Im E_\alpha(ix) = \frac{x}{2(\alpha+1)} j_{\alpha+1}(x)$  is odd and its zeros are  $\{\lambda_n, n \in \mathbb{Z}\}$  where  $\lambda_{-n} = -\lambda_n$  and  $\lambda_0 = 0$ .

Theorem 1 in [4] establishes that  $\{E_\alpha(i\lambda_n x)\}_{n \in \mathbb{Z}}$  is a complete orthogonal system in  $L^2((-1, 1), d\mu_\alpha)$ . That is to say

$$\int_{-1}^1 E_\alpha(i\lambda_n x) \overline{E_\alpha(i\lambda_m x)} d\mu_\alpha(x) = \|E_\alpha(i\lambda_n \cdot)\|_{2,\alpha}^2 \delta_{nm}.$$

For each appropriate function  $f$  on  $(-1, 1)$ , we define its Fourier series related to the system  $\{E_\alpha(i\lambda_n x)\}_{n \in \mathbb{Z}}$ , which are called Fourier–Dunkl series, as

$$f \sim \sum_{n \in \mathbb{Z}} c_n(f) E_\alpha(i\lambda_n x) \theta_n, \quad c_n(f) = \int_{-1}^1 f(y) \overline{E_\alpha(i\lambda_n y)} d\mu_\alpha(y).$$

and

$$\theta_n = \|E_\alpha(i\lambda_n \cdot)\|_{2,\alpha}^{-2}.$$

We notice that Ó. Ciaurri and his collaborators have studied in [3] the weighted norm convergence of the Fourier–Dunkl series and proved in [5] an uncertainty inequality associated to this system. From [4, Lemma 1] we have

$$\theta_n = \frac{2^\alpha \Gamma(\alpha + 1)}{|j_\alpha(\lambda_n)|^2}, \quad n \in \mathbb{Z} \setminus \{0\} \quad (\text{we recall that } \lambda_n := \lambda_{\alpha+1,n}) \tag{8}$$

and  $\theta_0 = 2^{\alpha+1} \Gamma(\alpha + 2)$ . The following asymptotic formulas hold for the Bessel function  $J_\alpha(u)$  ([18, p. 490]):

$$J_\alpha(u) = \sqrt{\frac{2}{\pi u}} \left[ \cos\left(u - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{u}\right) \right], \quad u \rightarrow \infty. \tag{9}$$

Combining (8) and (9) gives

$$\theta_n \sim \pi |\lambda_n|^{2\alpha+1}, \quad |n| \rightarrow \infty. \tag{10}$$

The sequence  $\{c_n(f), n \in \mathbb{Z}\}$  is called the discrete Fourier–Dunkl transform of  $f$ . We define the weighted spaces  $l^p(\mathbb{Z}, (\theta_n)_{n \in \mathbb{Z}})$  by

$$l^p(\mathbb{Z}, (\theta_n)_{n \in \mathbb{Z}}) = \left\{ (x_n)_{n \in \mathbb{Z}} : \mathbb{Z} \rightarrow \mathbb{C} : \left( \sum_{n \in \mathbb{Z}} |x_n|^p \theta_n \right)^{1/p} < +\infty \right\}.$$

If  $f \in L^2((-1, 1), d\mu_\alpha)$ , then the sequence  $\{c_n(f), n \in \mathbb{Z}\}$  belongs to  $l^2(\mathbb{Z}, (\theta_n)_{n \in \mathbb{Z}})$  and we have

$$\|f\|_{2,\alpha} = \sqrt{\sum_{n=-\infty}^{\infty} |c_n(f)|^2 \theta_n}. \tag{11}$$

The Dunkl translation operator of a function  $f$  is defined for all  $h \in \mathbb{R}$  by (see [1, 13])

$$\tau_y^\alpha f(x) = \sum_{n=0}^{\infty} \Lambda_\alpha^n f(x) \frac{y^n}{\gamma_{n,\alpha}}, \quad \alpha > -1 \tag{12}$$

where  $\Lambda_\alpha^0$  is the identity operator,  $\Lambda_\alpha^{n+1} = \Lambda_\alpha(\Lambda_\alpha^n)$ , and

$$\gamma_{n,\alpha} = \begin{cases} 2^{2k} k!(\alpha + 1)_k & \text{if } n = 2k, \\ 2^{2k+1} k!(\alpha + 1)_{k+1} & \text{if } n = 2k + 1. \end{cases}$$

The definition (12) is valid only for  $C^\infty$ -functions, and assuming also that the series on the right is convergent. In particular, this can be guaranteed when  $f$  is a polynomial, because the operator  $\Lambda_\alpha$  applied to a polynomial of degree  $k$  generates a polynomial of degree  $k - 1$ , so the series (12) has only a finite number of nonzero summands. In the case  $\alpha = -1/2$ , the translation  $\tau_y^\alpha f$  is just the Taylor expansion of a function  $f$  around a fixed point  $x$ , that is,

$$f(x + y) = \sum_{n=0}^{\infty} f^{(n)}(x) \frac{y^n}{n!}.$$

Some properties of the translation operator, including an integral expression, can be found in [11, 14, 16, 17]. For our purposes, we only need the identity [13, formula (4.2.2)]

$$\tau_h^\alpha(E_\alpha(i\lambda x)) = E_\alpha(i\lambda h) E_\alpha(i\lambda x) \text{ for all } x, h \in \mathbb{R}. \tag{13}$$

That resembles the classical

$$e^{\lambda(h+x)} = e^{\lambda h} e^{\lambda x}.$$

The scalar product in the Hilbert space  $L^2((-1, 1), d\mu_\alpha)$  obeys the formula

$$(f, g) = \int_{-1}^1 f(x)\overline{g(x)}d\mu_\alpha(x).$$

We denote by  $\mathcal{E} = \mathcal{E}((-1, 1))$ , the set of all infinitely differentiable with compact support included in the interval  $(-1, 1)$ . By the partial integration one can verify the relation

$$\int_{-1}^1 \Lambda_\alpha f(x)\overline{g(x)}d\mu_\alpha(x) = \frac{f(1)\overline{g(1)} - f(-1)\overline{g(-1)}}{2^{\alpha+1}\Gamma(\alpha+1)} - \int_{-1}^1 f(x)\overline{\Lambda_\alpha g(x)}d\mu_\alpha(x). \tag{14}$$

Then for any functions  $f, g \in \mathcal{E}$ , we have

$$(\Lambda_\alpha f, g) = -(f, \Lambda_\alpha g). \tag{15}$$

As usual, we endow the space  $\mathcal{E}$  with a topology; this turns it into a topological vector space. Let  $\mathcal{E}'$  stand for the set of generalized functions, i.e., linear continuous functionals on the space  $\mathcal{E}$ . We denote the value of a functional  $f \in \mathcal{E}'$  on a function  $\varphi \in \mathcal{E}$  by  $\langle f, \varphi \rangle$ . The space  $L^2((-1, 1), d\mu_\alpha)$  is embedded into  $\mathcal{E}'$ , provided that for  $f \in L^2((-1, 1), d\mu_\alpha)$  and  $\varphi \in \mathcal{E}'$  we put

$$\langle f, \varphi \rangle = \int_{-1}^1 f(x)\varphi(x)d\mu_\alpha(x)$$

We can extend the action of Dunkl operator  $\Lambda_\alpha$  onto the space of generalized functions  $\mathcal{E}'$ , putting

$$\langle \Lambda_\alpha f, \varphi \rangle = -\langle f, \Lambda_\alpha \varphi \rangle, f \in \mathcal{E}', \varphi \in \mathcal{E}.$$

In particular, the action of the operator  $\Lambda_\alpha f$  is defined for any function  $f \in L^2((-1, 1), d\mu_\alpha)$ , but, generally speaking,  $\Lambda_\alpha f$  is a generalized function.

Analogously to  $\Lambda_\alpha$ , we can extend the operator  $\tau_h^\alpha$  by continuity on the whole space  $L^2((-1, 1), d\mu_\alpha)$ . Indeed, Let  $P$  be a vector space generated by the system  $\{E_\alpha(i\lambda_n x)\}_{n \in \mathbb{Z}}$  and  $f \in P$ , then  $f$  can be written as  $f(x) = \sum_{n=-m}^m c_n E_\alpha(i\lambda_n x)\theta_n$ . Using (3) and (13) we check easily that

$$\|\tau_h^\alpha f\|_{2,\alpha} \leq \|f\|_{2,\alpha}. \tag{16}$$

As  $P$  is a dense subspace of  $L^2((-1, 1), d\mu_\alpha)$ , it follows from (16) that  $\tau_h^\alpha$  can be extended by continuity to a bounded operator in  $L^2((-1, 1), d\mu_\alpha)$ . The extended operator is also denoted by  $\tau_h^\alpha$ ; inequality (16) remains valid for it.

For every function  $f \in L^2((-1, 1), d\mu_\alpha)$  we define the differences  $\Delta_h^m f$  of order,  $m \in \mathbb{N} = \{1, 2, 3, \dots\}$ , with step  $h \in \mathbb{R}$  by the formula  $\Delta_h^1 f(t) = \Delta_h f(t) = (\tau_h^\alpha - I)f(t)$ , where  $I$  is the identity operator in  $L^2((-1, 1), d\mu_\alpha)$  and for  $m > 1$

$$\Delta_h^m f(t) = \Delta_h(\Delta_h^{m-1} f(t)) = (\tau_h^\alpha - I)^m f(t) = \sum_{i=0}^m (-1)^{m-1} \binom{m}{i} (\tau_h^\alpha)^i f(t),$$

where

$$(\tau_h^\alpha)^0 f(t) = f(t), \quad (\tau_h^\alpha)^i f(t) = \tau_h^\alpha((\tau_h^\alpha)^{i-1} f(t)), \quad i = 1, 2, \dots, m.$$

The moduli of smoothness generated by general translations are defined as follows.

$$\omega_m(f, \delta)_{2,\alpha} := \sup_{0 < h \leq \delta} \|\Delta_h^m f\|_{2,\alpha}, \quad \delta > 0, \quad f \in L^2((-1, 1), d\mu_\alpha).$$

Let  $W_{2,\alpha}^m$  be the Sobolev space constructed by the operator  $\Lambda_\alpha$ , i.e.,

$$W_{2,\alpha}^m := \left\{ f \in L^2((-1, 1), d\mu_\alpha) : \Lambda_\alpha^j f \in L^2((-1, 1), d\mu_\alpha), j = 1, 2, \dots, m \right\}.$$

Then the corresponding K-functional is

$$K(f, t, W_{2,\alpha}^m) := \inf \left\{ \|f - g\|_{2,\alpha} + t \|\Lambda_\alpha^m g\|_{2,\alpha} : g \in W_{2,\alpha}^m \right\}$$

where  $f \in L^2((-1, 1), d\mu_\alpha)$  and  $t > 0$ . For brevity, we denote

$$K_m(f, t)_{2,\alpha} := K(f, t, W_{2,\alpha}^m).$$

The following theorem establishes an equivalence between the modulus of smoothness and the K-functional. It is analogous to the theorem on the equivalence between the modulus of smoothness and the K-functional in classical approximation theory.

**Theorem 1.** *One can find positive numbers  $C_1 = C_1(m, \alpha)$  and  $C_2 = C_2(m, \alpha)$  which satisfy the inequality*

$$C_1\omega_m(f, \delta)_{2,\alpha} \leq K_m(f, \delta^m)_{2,\alpha} \leq C_2\omega_m(f, \delta)_{2,\alpha}$$

where  $f \in L^2((-1, 1), d\mu_\alpha)$  and  $\delta > 0$ .

Using that  $E_{-1/2}(ix) = e^{ix}$ , it is easy to check that Theorem 1 reduces to an equivalence between the modulus of smoothness and the K-functional on the basis of Fourier series and usual translation.

When we consider real even and odd functions the Fourier–Dunkl series can be seen as Fourier–Dini and Fourier–Bessel series respectively. From this fact, applying Theorem 1 to even or odd functions, we can deduce analogs of Theorem 1 for these kinds of series.

### 3. Proof

Some technical facts are needed to prove our result. They are included in the next lemmas.

**Lemma 2.**

(i) *For  $x \in \mathbb{R}$  the following inequality is fulfilled*

$$|1 - E_\alpha(ix)| \leq |x|.$$

(ii) *For  $|x| \geq 1$ , there exists a certain constant  $c > 0$  which depends only on  $\alpha$  such that*

$$|1 - E_\alpha(ix)| \geq c.$$

**Proof. (i).** It follows by the estimates provided in (3) together with standard application of Lagrange’s mean value theorem.

**(ii).** The asymptotic formulas (9) imply that  $j_\alpha(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Consequently, a number  $\eta > 0$  exists such that with  $|x| \geq \eta$  the inequality  $|j_\alpha(x)| \leq 1/2$  is true. Let

$$m = \min_{1 \leq |x| \leq \eta} |1 - j_\alpha(x)|.$$

With  $|x| \geq 1$  we get the inequality  $|1 - j_\alpha(x)| \geq c$ , where  $c = \min\{m, 1/2\}$ . Taking only the reel part of  $1 - E_\alpha(i\lambda_n x)$  gives

$$c \leq |1 - j_\alpha(x)| \leq |1 - E_\alpha(i\lambda_n x)|.$$

□

**Proposition 3.** *Let  $x \in (-1, 1)$  and  $h \in \mathbb{R}$ . If  $f \in L^2((-1, 1), d\mu_\alpha)$  with*

$$f(x) = \sum_{n \in \mathbb{Z}} c_n(f) E_\alpha(i\lambda_n x) \theta_n,$$

then

$$\tau_h^\alpha f(x) = \sum_{n \in \mathbb{Z}} c_n(f) E_\alpha(i\lambda_n h) E_\alpha(i\lambda_n x) \theta_n. \tag{17}$$

**Proof.** By product formula (13) of  $\tau_h^\alpha$ , we have

$$\tau_h^\alpha E_\alpha(i\lambda_n x) = E_\alpha(i\lambda_n h) E_\alpha(i\lambda_n x).$$

So, for any Dunkl-polynomial function

$$\mathcal{Q}_N(x) = \sum_{n=-N}^N c_n(f) E_\alpha(i\lambda_n x) \theta_n,$$

(this can be considered as a generalization of the trigonometric polynomial in the classical case  $\alpha = -1/2$ ) since  $\tau_h^\alpha$  is linear, we have

$$\tau_h^\alpha \mathcal{Q}_N(x) = \sum_{n=-N}^N c_n(f) E_\alpha(i\lambda_n h) E_\alpha(i\lambda_n x) \theta_n. \tag{18}$$

By using the fact that  $\tau_h^\alpha$  is extended to a continuous linear operator in  $L^2((-1, 1), d\mu_\alpha)$  and the set of all polynomials  $\mathcal{Q}_N(x)$  is everywhere dense in  $L^2((-1, 1), d\mu_\alpha)$ , passage to the limit in (18) gives the desired equality.  $\square$

**Corollary 4.** *Let  $f \in L^2((-1, 1), d\mu_\alpha)$  and  $h \in \mathbb{R}$ , then*

$$\|\Delta_h^m f\|_{2,\alpha} \leq 2^m \|f\|_{2,\alpha}. \tag{19}$$

**Proof.** Let  $h \in \mathbb{R}$ . According to the formula (17), we obtain

$$\begin{aligned} c_n(\Delta_h^1 f) &= c_n(\tau_h^\alpha f) - c_n(f) \\ &= (E_\alpha(i\lambda_n h) - 1) c_n(f). \end{aligned}$$

Using induction with respect to  $m$ , we have

$$c_n(\Delta_h^m f) = (E_\alpha(i\lambda_n h) - 1)^m c_n(f). \tag{20}$$

Then

$$c_n(\Delta_h^m f) \leq 2^m c_n(f). \tag{21}$$

$\square$

**Lemma 5.** *If  $f \in \mathcal{E}$ , we get*

$$c_n(\Lambda_\alpha f) = i\lambda_n c_n(f) \tag{21}$$

for all  $n \in \mathbb{Z}$ .

**Proof.** Let  $f \in \mathcal{E}$ , we put  $c_n(f) = \int_{-1}^1 f(y) \overline{E_\alpha(i\lambda_n y)} d\mu_\alpha(y)$ . It follows from (15) that

$$\begin{aligned} c_n(\Lambda_\alpha f) &= \int_{-1}^1 \Lambda_\alpha f(y) E_\alpha(-i\lambda_n y) d\mu_\alpha(y), \\ &= - \int_{-1}^1 f(y) \Lambda_\alpha E_\alpha(-i\lambda_n y) d\mu_\alpha(y), \\ &= i\lambda_n \int_{-1}^1 f(y) E_\alpha(-i\lambda_n y) d\mu_\alpha(y), \\ &= i\lambda_n c_n(f). \end{aligned}$$

Then the equality (21) is valid in  $\mathcal{E}$ .  $\square$

**Remark 6.** Using induction with respect to  $m$  and Lemma 5, we can see that for all  $f \in W_{2,\alpha}^m$

$$c_n(\Lambda_\alpha^m f) = (i\lambda_n)^m c_n(f) \tag{22}$$

for all  $n \in \mathbb{Z}$  and  $m = 0, 1, 2, \dots$

**Lemma 7.** *Assume that  $\delta > 0$  and  $f \in W_{2,\alpha}^m$ . The following inequality is true:*

$$\omega_m(f, \delta)_{2,\alpha} \leq \delta^m \|\Lambda^m f\|_{2,\alpha}.$$

**Proof.** Let  $h \in (0, \delta]$ . According to the formula (20), we have

$$c_n(\Delta_h^m f) = (E_\alpha(i\lambda_n h) - 1)^m c_n(f).$$

It follows from the Parseval identity (11) and Lemma 2 that

$$\begin{aligned} \|\Delta_h^m f\|_{2,\alpha}^2 &= \sum_{n \in \mathbb{Z}} (1 - E_\alpha(i\lambda_n h))^{2m} |c_n(f)|^2 \theta_n \leq h^{2m} \sum_{n \in \mathbb{Z}} \left( \frac{1 - E_\alpha(i\lambda_n h)}{\lambda_n h} \right)^{2m} |\lambda_n^m c_n(f)|^2 \theta_n \\ &\leq h^{2m} \|\Lambda_\alpha^m f\|_{2,\alpha}^2 \leq \delta^{2m} \|\Lambda_\alpha^m f\|_{2,\alpha}^2. \end{aligned}$$

Calculating the supremum with respect to all  $h \in (0, \delta]$ , we obtain  $\omega_m(f, \delta)_{2,\alpha} \leq \delta^m \|\Lambda_\alpha^m f\|_{2,\alpha}$ .  $\square$

**Definition 8.** For any function  $f \in L^2((-1, 1), d\mu_\alpha)$  and any number  $\sigma > 0$ , we define the function

$$\mathcal{P}_\sigma(f)(t) := \sum_{n \in \mathbb{Z}} \mathcal{X}_\sigma(\lambda_n) c_n(f) E_\alpha(i\lambda_n t) \theta_n$$

where  $\mathcal{X}_\sigma(n)$  is the characteristic function defined by

$$\mathcal{X}_\sigma(\lambda_n) := \begin{cases} 1 & \text{if } |\lambda_n| \leq \sigma, \\ 0 & \text{if } |\lambda_n| > \sigma. \end{cases}$$

**Proposition 9.** Let  $\sigma > 0$ . For any function  $f \in L^2((-1, 1), d\mu_\alpha)$  the following inequality is true:

$$\|f - \mathcal{P}_\sigma(f)\|_{2,\alpha} \leq C \|\Delta_{\frac{1}{\sigma}}^m f\|_{2,\alpha}.$$

**Proof.** Using the Parseval equality, we obtain

$$\|f - \mathcal{P}_\sigma(f)\|_{2,\alpha}^2 = \sum_{n \in \mathbb{Z}} (1 - \mathcal{X}_\sigma(\lambda_n)) |c_n(f)|^2 \theta_n, \tag{23}$$

$$= \sum_{n \in \mathbb{Z}} \frac{(1 - \mathcal{X}_\sigma(\lambda_n))}{(1 - E_\alpha(i\lambda_n \sigma^{-1}))^{2m}} (1 - E_\alpha(i\lambda_n \sigma^{-1}))^{2m} |c_n(f)|^2 \theta_n. \tag{24}$$

Note that  $C_1 \leq |1 - E_\alpha(ix)|$  with  $|x| \geq 1$  (see Lemma 2). Hence

$$\sup_{n \in \mathbb{Z}} \frac{1 - \mathcal{X}_\sigma(\lambda_n)}{1 - E_\alpha(i\lambda_n \sigma^{-1})} \leq \sup_{|x| \geq 1} \frac{1}{1 - E_\alpha(ix)} \leq \frac{1}{C_1}. \tag{25}$$

Relations (24) and (25) give

$$\|f - \mathcal{P}_\sigma(f)\|_{2,\alpha} \leq C \|\Delta_{\frac{1}{\sigma}}^m f\|_{2,\alpha}$$

where  $C = \frac{1}{C_1^m}$ .  $\square$

**Corollary 10.** For any function  $f \in L^2((-1, 1), d\mu_\alpha)$  the following inequality is true:

$$\|f - \mathcal{P}_\sigma(f)\|_{2,\alpha} \leq C \omega_m\left(f, \frac{1}{\sigma}\right)_{2,\alpha}.$$

**Proposition 11.** Suppose that  $f \in L^2((-1, 1), d\mu_\alpha)$ ,  $m \in \mathbb{N}$ , and  $\sigma > 0$ . Then we have

$$\|\Lambda_\alpha^m \mathcal{P}_\sigma(f)\|_{2,\alpha} \leq C_3 \sigma^m \|\Delta_{\frac{1}{\sigma}}^m f\|_{2,\alpha}.$$

**Proof.** Using the Parseval equality, we obtain

$$\|\Lambda_\alpha^m \mathcal{P}_\sigma(f)\|_{2,\alpha}^2 = \sum_{n \in \mathbb{Z}} \lambda_n^{2m} \mathcal{X}_\sigma(\lambda_n) |c_n(f)|^2 \theta_n, \tag{26}$$

$$= \sigma^{2m} \sum_{n \in \mathbb{Z}} \frac{\mathcal{X}_\sigma(\lambda_n) (\lambda_n \sigma^{-1})^{2m}}{(1 - E_\alpha(i\lambda_n \sigma^{-1}))^{2m}} (1 - E_\alpha(i\lambda_n \sigma^{-1}))^{2m} |c_n(f)|^2 \theta_n. \tag{27}$$

Note that

$$\sup_{n \in \mathbb{Z}} \left| \frac{\mathcal{X}_\sigma(\lambda_n) \lambda_n \sigma^{-1}}{1 - E_\alpha(i\lambda_n \sigma^{-1})} \right| \leq \sup_{|x| \leq 1} \left| \frac{x}{1 - E_\alpha(ix)} \right| = C_2. \tag{28}$$

Then formula (27) yields

$$\|\Lambda_\alpha^m \mathcal{P}_\sigma(f)\|_{2,\alpha} \leq C_3 \sigma^m \|\Delta_{\frac{1}{\sigma}}^m f\|_{2,\alpha}$$

where  $C_3 = C_2^m$ .  $\square$

**Corollary 12.** For any function  $f \in L^2((-1, 1), d\mu_\alpha)$  the following inequality is true:

$$\|\Lambda_\alpha^m \mathcal{P}_\sigma(f)\|_{2,\alpha} \leq C_3 \sigma^m \omega_m\left(f, \frac{1}{\sigma}\right)_{2,\alpha}.$$



**Proof of Theorem 1.**

**Proof of the inequality**  $2^{-m}\omega_m(f, \delta)_{2,\alpha} \leq K_m(f, \delta^m)_{2,\alpha}$ .

Let  $h \in (0, \delta]$ ,  $g \in W_{2,\alpha}^m$ . Using Lemma 7 and inequality (19), we obtain

$$\begin{aligned} \|\Delta_h^m f\|_{2,\alpha} &\leq \|\Delta_h^m f - g\|_{2,\alpha} + \|\Delta_h^m g\|_{2,\alpha} \leq 2^m \|f - g\|_{2,\alpha} + h^m \|\Lambda_\alpha^m f\|_{2,\alpha} \\ &\leq 2^m \left( \|f - g\|_{2,\alpha} + h^m \|\Lambda_\alpha^m f\|_{2,\alpha} \right). \end{aligned}$$

Calculating the supremum with respect to  $h \in (0, \delta]$  and the infimum with respect to all possible functions  $g \in W_{2,\alpha}^m$  we obtain

$$2^{-m}\omega_m(f, \delta)_{2,\alpha} \leq K_m(f, \delta^m)_{2,\alpha}.$$

**Proof of the inequality**  $K_m(f, \delta^m)_{2,\alpha} \leq C_1\omega_m(f, \delta)_{2,\alpha}$ .

Since  $\mathcal{P}_\sigma(f) \in W_{2,\alpha}^m$  by the definition of a  $K$ -functional we have

$$K_m(f, \delta^m)_{2,\alpha} \leq \|f - \mathcal{P}_\sigma(f)\|_{2,\alpha} + \delta^m \|\Lambda_\alpha^m \mathcal{P}_\sigma(f)\|_{2,\alpha}.$$

Using Corollaries 10 and 12, this gives

$$K_m(f, \delta^m)_{2,\alpha} \leq \omega_m\left(f, \frac{1}{\sigma}\right)_{2,\alpha} + C_3(\delta\sigma)^n \omega_m\left(f, \frac{1}{\sigma}\right)_{2,\alpha}.$$

Since  $\sigma$  is an arbitrary positive value, choosing  $\sigma = 1/\delta$ , we obtain the inequality.  $\square$

**Acknowledgements**

The authors would like to thank the anonymous referee and J. Faraut, who both found misprints as well as suggested valuable improvements of the text.

**References**

- [1] A. G. Asensi, J. L. Varona, "Appell-Dunkl sequences and Hurwitz-Dunkl zeta functions", *J. Math. Anal. Appl.* **520** (2023), no. 1, article no. 126870 (40 pages).
- [2] E. S. Belkina, S. S. Platonov, "Equivalence of  $K$ -functionals and modulus of smoothness constructed by generalized Dunkl translations", *Izv. Vyssh. Uchebn. Zaved., Mat.* **2008** (2008), no. 8, p. 3-15.
- [3] Ó. Ciaurri, M. Pérez, J. M. Reyes, J. L. Varona, "Mean convergence of Fourier–Dunkl series", *J. Math. Anal. Appl.* **372** (2010), no. 2, p. 470-485.
- [4] Ó. Ciaurri, J. L. Varona, "A Whittaker-Shannon-Kotel'nikov sampling theorem related to the Dunkl transform", *Proc. Am. Math. Soc.* **135** (2007), no. 9, p. 2939-2947.
- [5] ———, "An uncertainty inequality for Fourier–Dunkl series", *J. Comput. Appl. Math.* **233** (2010), no. 6, p. 1499-1504.
- [6] R. A. DeVore, G. G. Lorentz, *Constructive Approximation*, Grundlehren der Mathematischen Wissenschaften, vol. 303, Springer, 1993.
- [7] C. F. Dunkl, "Integral kernels with reflection group invariance", *Can. J. Math.* **43** (1991), no. 6, p. 1213-1227.
- [8] E. K. Ifantis, P. D. Siafarikas, "A differential equation for the zeros of Bessel functions", *Appl. Anal.* **20** (1985), p. 269-281.
- [9] M. F. E. de Jeu, "The Dunkl transform", *Invent. Math.* **113** (1993), no. 1, p. 147-162.
- [10] A. Laforgia, "Sugli zeri delle funzioni di Bessel", *Calcolo* **17** (1980), p. 211-220.
- [11] Z. Li, J. Liao, "Harmonic analysis associated with the one-dimensional Dunkl transform", *Constr. Approx.* **37** (2013), no. 2, p. 233-281.
- [12] S. S. Platonov, "Bessel harmonic analysis and approximation of functions on the half-line", *Izv. Ross. Akad. Nauk, Ser. Mat.* **71** (2007), no. 5, p. 149-196, in Russian; translated in *Izv. Math.*
- [13] M. Rosenblum, "Generalized Hermite polynomials and the Bose-like oscillator calculus", in *Nonselfadjoint operators and related topics*, Operator Theory: Advances and Applications, vol. 73, Birkhäuser, 1994, p. 369-396.
- [14] M. Rösler, "Bessel-type signed hypergroups on  $\mathbb{R}$ ", in *Probability measures on groups and related structures. XI (Oberwolfach, 1994)*, World Scientific, 1994, p. 292-304.
- [15] ———, "Dunkl operators: Theory and applications", in *Orthogonal polynomials and special functions (Leuven, 2002)*, Lecture Notes in Mathematics, vol. 1817, Springer, 2002, p. 93-135.

- [16] ———, “A positive radial product formula for the Dunkl kernel”, *Trans. Am. Math. Soc.* **355** (2003), no. 6, p. 2413-2438.
- [17] S. Thangavelu, Y. Xu, “Riesz transform and Riesz potentials for Dunkl transform”, *J. Comput. Appl. Math.* **199** (2007), no. 1, p. 181-195.
- [18] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press; Macmillan, 1948.