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Fundamental groups of proper varieties are finitely presented

Le groupe fondamental d'une variété propre est de présentation finie

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Abstract. It was proven in [1], that the étale fundamental group of a connected smooth projective variety over an algebraically closed field k is topologically finitely presented. In this note, we extend this result to all connected proper schemes over k.

Résumé. Il a été prouvé dans [1], que le groupe fondamental étale d'une variété projective lisse connexe sur un corps algébriquement clos k est topologiquement de présentation finie. Dans cette note, nous étendons ce résultat à tous les schémas propres connexes sur k.

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1. Introduction

For a connected algebraic variety X over an algebraically closed field k of characteristic 0, the étale fundamental group $\pi_1^{\text{ét}}(X,\overline{x})$ of X is a topologically finitely presented profinite group. This is proven by first reducing to the case of $k = \mathbb{C}$ and then applying the Riemann Existence Theorem [3, Exp. XII, Thm. 5.1] together with the fact that the topological fundamental group $\pi_1^{\text{top}}(X(\mathbb{C}),x)$ is of finite presentation as a discrete group (see e.g. [7] or [4]).

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In characteristic $p > 0$, the picture is much more subtle due to the existence of Artin–Schreier covers of affine schemes, which makes $\pi_1^{\text{ét}}(\text{Spec}(A), \bar{x})$ typically not even topologically finitely generated. Remark 5.7 of [3, Exp. IX] raised doubts whether $\pi_1^{\text{ét}}(X, \bar{x})$ is topologically finitely presented for proper varieties, even for proper smooth curves. In recent work, however, Shusterman (in the case of curves [9]), and Esnault, Shusterman and the second named author [1] have shown that for smooth projective varieties $\pi_1^{\text{ét}}(X, \bar{x})$ is still topologically finitely presented. Both results are based on a criterion for finite presentation of profinite groups due to Lubotzky [8].

From now on, we will omit the base points.

Theorem 1 (part of Thm. 1.1 of [1]). *Let X be a connected smooth projective variety over an algebraically closed field k. Then the étale fundamental group $\pi_1^{\text{ét}}(X)$ is topologically finitely presented.*

Our goal is to generalize Theorem 1 to all connected schemes that are proper over $\text{Spec}(k)$ (which is new only if k has characteristic $p > 0$). Such a generalization responds affirmatively to a question raised by Esnault.

Theorem 2. *Let X be a connected scheme that is proper over* $\text{Spec}(k)$ *for an algebraically closed field k. Then $\pi_1^{\text{ét}}(X)$ is topologically finitely presented.*

To prove the theorem, we use descent along an alteration map to X and the van Kampen presentation of $\pi_1^{\text{ét}}(X)$ arising in this way. More precisely, we use this trick twice.

2. The proof

For a scheme T, let $\mathsf{F\acute{E}t}_T$ denote the category of finite étale covers of T. This gives rise to a category fibred over schemes. Recall that a morphism $g : T' \to T$ of schemes is said to be of effective descent for $\mathsf{F\acute{E}t}$, if g^* induces an equivalence of categories between $\mathsf{F\acute{E}t}_T$ and the category of descent data in $\mathsf{F\acute{E}t}$ along g.

Proposition 3 ([3, Exp. IX, Thm. 4.12]). *Let $f : X' \to X$ be a proper surjective morphism of finite presentation. Then f is of effective descent for* $\mathsf{F\acute{E}t}$.

Morphisms of effective descent $f : X' \to X$ for $\mathsf{F\acute{E}t}$ give rise to a van Kampen-like presentation of $\pi_1^{\text{ét}}(X)$ as the profinite completion of a quotient of the free topological product of the étale fundamental groups of the connected components of X' and the usual topological fundamental group of a suitably defined "dual graph". This goes back to [3, Exp. IX, §5] and has been worked out in detail in [10, Cor. 5.3].

The existence of such a presentation allows one to "descend" finite generation/presentation of the fundamental groups involved, as made precise in the following proposition. Every statement below is about *topological* finite generation/presentation.

Proposition 4 ([3, Exp. IX, Cor. 5.2 + Cor. 5.3]). *Let $f : X' \to X$ be a morphism of effective descent for* $\mathsf{F\acute{E}t}$. *We denote $X' \times_X X'$ by X'' and $X' \times_X X' \times_X X'$ by X'''.*

- (a) *Assume that X', X'' have finite π_0's and that $\pi_1^{\text{ét}}$'s of the connected components of X' are finitely generated. Then $\pi_1^{\text{ét}}(X)$ is finitely generated.*
- (b) *Assume that X', X'', X''' have finite π_0's, that $\pi_1^{\text{ét}}$'s of the connected components of X' are of finite presentation and that $\pi_1^{\text{ét}}$'s of the connected components of X'' are finitely generated. Then $\pi_1^{\text{ét}}(X)$ is of finite presentation.*

Let us now recall a result of de Jong specialized to our setting.

Proposition 5 (see [6, Thm. 4.1]). *Let X be a scheme that is proper over* $\text{Spec}(k)$ *for an algebraically closed field k. Then there exists a proper surjective morphism (of finite presentation) $f : X' \to X$ from a smooth projective variety X' over* $\text{Spec}(k)$.

Proof. Let $\nu : X^\nu \to X$ be the normalization of X. The map ν is finite, and thus the scheme X^ν is still proper over $\mathrm{Spec}(k)$.

Let then $f_1 : X' \to X^\nu$ be the alteration map of [6, Thm. 4.1] applied to each connected component of X^ν. The map f_1 is proper, dominant, and thus surjective. Moreover, *loc. cit.* guarantees that X' is regular and projective (and not merely proper!) over $\mathrm{Spec}(k)$. Now, as k is algebraically closed, X' is smooth over $\mathrm{Spec}(k)$. The composition $f = \nu \circ f_1 : X' \to X$ has all the requested properties. □

We are now ready to finish the proof of the main result.

Proof of Theorem 2. Take $f : X' \to X$ as in Proposition 5. By Theorem 1 and Proposition 3 the map f satisfies the assumptions for Proposition 4 (a). This shows that $\pi_1^{\text{ét}}(X)$ is finitely generated, for any scheme X that is proper over $\mathrm{Spec}(k)$. In fact, finite generation was already proven in [3, Exp. X, Thm. 2.9], and we included the argument here for the convenience of the reader.

We are going to apply finite generation to the connected components of $X'' = X' \times_X X'$, which are connected proper schemes over $\mathrm{Spec}(k)$. Indeed, using Theorem 1 (this time crucially!) and Proposition 3 again, the map f now satisfies the assumptions of Proposition 4 (b). This shows that $\pi_1^{\text{ét}}(X)$ is finitely presented. □

3. More general base fields

Similarly to [1, §5], our main result extends to more arithmetic settings. We thank Peter Haine for essentially suggesting the following corollary.

Corollary 6. *Let X be a connected scheme that is proper over* $\mathrm{Spec}(k)$ *for a field k. Then $\pi_1^{\text{ét}}(X)$ is finitely presented if and only if the absolute Galois group* Gal_k *is finitely presented.*

Proof. We may assume X is reduced and thus $k' = \mathrm{H}^0(X, \mathscr{O}_X)$ is a finite field extension of k. Let $\bar{k}$ be an algebraic closure of k containing k'. Then $X \to \mathrm{Spec}(k')$ being the Stein factorization of $X \to \mathrm{Spec}(k)$ implies that $\overline{X} = X \times_{k'} \bar{k}$ is connected. By Theorem 2, the group $\pi_1(\overline{X})$ is finitely presented. The fundamental exact sequence [3, Exp. IX, Thm. 6.1]

$$1 \to \pi_1(\overline{X}) \to \pi_1(X) \to \mathrm{Gal}_{k'} \to 1$$

shows that $\pi_1(X)$ is finitely presented if and only if $\mathrm{Gal}_{k'}$ is finitely presented. The latter is equivalent to Gal_k being finitely presented, see for example [2, Prop. 2.3]. □

Remark 7. Examples of fields k with finitely presented Gal_k include: fields algebraic over a finite field, local p-adic fields, $\mathbb{R}$ and more generally real closed fields, $K((t))$ for a field of characteristic 0 with Gal_K of finite presentation, and by [5, Thm. 5.1] for a hilbertian field k, probabilistically almost always (for the Haar measure on Gal_k) the fixed field k^Σ in the separable closure $\bar{k}$ of a finite subset $\Sigma \subset \mathrm{Gal}_k$.

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