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Boundedness of classical solutions to a chemotaxis consumption system with signal dependent motility and logistic source

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Abstract. We consider the chemotaxis system:

$$\begin{cases} u_t = \nabla\cdot\big(\gamma(v)\nabla u - u\xi(v)\nabla v\big) + \mu u(1-u), & x\in\Omega,\ t>0,\\ v_t = \Delta v - uv, & x\in\Omega,\ t>0,\end{cases}$$

under homogeneous Neumann boundary conditions in a bounded domain $\Omega\subset\mathbb{R}^n, n\geq 2$, with smooth boundary. Here, the functions $\gamma(v)$ and $\xi(v)$ are as:

$$\gamma(v) = (1+v)^{-k} \quad \text{and} \quad \xi(v) = -(1-\alpha)\gamma'(v),$$

where $k>0$ and $\alpha\in(0,1)$.

We prove that the classical solutions to the above system are uniformly-in-time bounded provided that $k(1-\alpha) < \frac{4}{n+5}$ and the initial value v_0 and μ satisfy the following conditions:

$$0 < \|v_0\|_{L^\infty(\Omega)} \leq \left[\frac{4\big[1-k\big(1-\alpha\big)\big]}{k(n+1)(1-\alpha)}\right]^{\frac{1}{k}} - 1,$$

and

$$\mu > \frac{kn(1-\alpha)\|v_0\|_{L^\infty(\Omega)}}{(n+1)(1+\|v_0\|_{L^\infty(\Omega)})}.$$

This result improves the recent result obtained for this problem by Li and Lu (J. Math. Anal. Appl.) (2023).

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1. Introduction

In this paper, we study the following initial boundary value problem:

$$\begin{cases} u_t = \nabla\cdot\big(\gamma(v)\nabla u - u\xi(v)\nabla v\big) + \mu u(1-u), & x\in\Omega,\ t>0,\\ v_t = \Delta v - uv, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0,\ v(x,0) = v_0, & x\in\Omega,\end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n, n \geq 2$, is a bounded domain with smooth boundary, ν denotes the unit outward normal vector to $\partial\Omega$ and u_0 and v_0 are initial functions. Here, $u = u(x,t)$ denotes the cell density and $v = v(x,t)$ is the nutrient consumed chemical concentrations.

In mathematical biology, systems like (1) describe the mechanism of chemotaxis. The chemotaxis is the movement of cells towards a higher concentration of a chemical signal substance produced by the cells. If the second equation of problem (1) is changed and written as follows:

$$\begin{cases} u_t = \nabla\cdot\big(\gamma(v)\nabla u - u\xi(v)\nabla v\big) + \mu u(1-u), & x\in\Omega,\ t>0, \\ \tau v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases} \tag{2}$$

where $\tau \in \{0,1\}$, then this system is the classical chemotaxis system which has been introduced by Keller and Segel [15]. For problem (2), in the absence of logistic source, when the positive function $\gamma(v)$ belongs to $C^3((0,\infty))$ and $\xi(v) = -\gamma'(v)$ as well as

$$\gamma_\infty := \limsup_{s\to\infty} \gamma(s) < \frac{1}{\tau},$$

then for $n \geq 1$, the existence a unique global non-negative classical solution is proved [30]. Also, the uniform-in-time boundedness of classical solutions is proved in any dimension when the function γ has strictly positive lower and upper bounds [30]. This result also is proved for $n \geq 2$, when the function γ decays at a certain slow rate at infinity [30].

In the special case $\gamma(v) = c_0 v^{-k}$ with $k > 0$ and $c_0 > 0$, for $n \geq 1$, the global existence and boundedness of the solution is proved for all $k > 0$ under a smallness assumption on c_0 [31]. When $n \geq 2$, by removing the smallness condition on c_0, and applying the condition $k \in (0, \frac{2}{n-2})$, the same result is proved in cases $\tau = 0$ [1] and $\tau = 1$ [8].

In the other special case $\gamma(v) = e^{-\chi v}$ with $\chi > 0$, for $n = 2$, it is proved that the classical solutions for this problem are global and bounded if $\int_\Omega u_0\,\mathrm{d}x < \frac{4\pi}{\chi}$, whereas for $\int_\Omega u_0\,\mathrm{d}x > \frac{4\pi}{\chi}$ blow up occurs either in finite or infinite time [14]. For $n = 2$ and $\tau = 0$, it is proved that the blow up occurs in infinite time [9]. Also, for $n = 2$, it is proved that the classical solution is globally bounded if the positive function $\gamma(v)$ decreases slower than an exponential speed at high signal concentrations. For $n \geq 3$, this result is proved when $\gamma(v)$ decreases at certain algebraically speed [7]. Also, in the presence of logistic source, when $n = 2$ and the positive function $\gamma(v)$ belongs to $C^3([0,\infty))$, $\gamma'(v) < 0$, $\lim_{v\to\infty}\gamma(v) = 0$ and $\lim_{v\to\infty}\frac{\gamma'(v)}{\gamma(v)}$ exists, the existence of bounded classical solutions are proved in [12]. For $n \geq 3$, if the last condition is replaced with $|\gamma'(v)| \leq m$, where m is some positive constant, then the global existence and boundedness of the solution is proved when $\mu > 0$ is large [13].

Now, we want to write some results related to problem (1). But first, we explain the origin of the definition of this problem. Tuval et al. in [25] introduced the following chemotaxis-Navier–Stokes system which describes the motion of oxygen-driven swimming bacteria in an in-compressible fluid

$$\begin{cases} u_t + \omega\cdot\nabla u = \nabla\cdot\big(\nabla u - u\xi(v)\nabla v\big), & x\in\Omega,\ t>0, \\ v_t + \omega\cdot\nabla v = \Delta v - ug(v), & x\in\Omega,\ t>0, \\ \omega_t + (\omega\cdot\nabla)\omega = \Delta\omega - \nabla P + u\nabla\phi, & x\in\Omega,\ t>0,\ t>0, \\ \nabla\cdot\omega = 0, & x\in\Omega,\ t>0,\ t>0. \end{cases}$$

Here, u denotes the bacteria density and v is the oxygen concentration. Also, ω and P are the velocity and pressure of the fluid, respectively. The function ξ measures the chemotactic sensitivity, g is the consumption rate of the oxygen by the bacteria, and ϕ is a given potential function. We see that problem (1) can be obtained from the preceding chemotaxis-Navier–Stokes system upon the choice $\omega \equiv 0, \gamma(v) \equiv 1$ and $g(v) = v$. For the related results with the chemotaxis-Navier–Stokes systems, we refer the interested readers to [5, 6, 10, 29] and references therein. For the problem (1), in the absence of logistic source, when $\gamma(v) \equiv 1$, $\xi(v) \equiv \chi$, where χ is some positive constant, in the two-dimensional case for the bounded convex domains with smooth boundary, it is proved that the classical solutions for this problem are global and bounded [23]. Also, for $n \geq 3$, the classical solutions for this problem are global and bounded provided that $\|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{6(n+1)\chi}$ [22]. This condition extends to $\|v_0\|_{L^\infty(\Omega)} \leq \frac{\pi}{\chi\sqrt{2(n+1)}}$ in [4]. Also, in the presence of logistic source under this condition, the existence of bounded classical solutions is proved in [3].

The authors in [20] studied the problem (1) when the positive function $\gamma(v)$ belongs to $C^3([0,\infty))$ and $\gamma'(v) < 0$ for all $v \geq 0$ as well as $\xi(v) = -\gamma'(v)$. For $n = 2$ and $\mu > 0$, they proved the global existence and boundedness of solution. Also, when $n \geq 3$ and μ is suitably large, they obtained the same result. Besides, they showed the solution converges exponentially to $(1,0)$ when t tends to infinity. In the case of $\mu = 0$, under the same conditions on $\gamma(v)$, the authors in [19] proved the existence a unique global bounded classical solution with some suitable small initial data. Wang in [26] studied the above problem when the logistic source is as $f(u) = \alpha u - \mu u^\kappa$. He proved that this problem admits a global bounded classical solution if one of the cases $\big(n \leq 2, \kappa > 1;\ n \geq 3, \kappa > 2$ or $n \geq 3, \kappa = 2$ and μ is large$\big)$ holds.

In [21], the authors studied the problem (1). They assumed that the positive function $\gamma(v)$ belongs to $C^2([0,\infty))$ such that $\gamma'(v) < 0$, $\gamma''(v) \geq 0$ and $\xi(v) = -(1-\alpha)\gamma'(v)$ with $\alpha \in (0,1)$. Under the following conditions:

$$\frac{(\gamma'(v))^2}{\gamma''(v)} \leq \frac{n}{2(n+1)^3}, \qquad 0 < \|v_0\|_{L^\infty(\Omega)} \leq \gamma^{-1}\left(\frac{1}{n+1}\right)$$

and

$$\mu > \max_{0<v\leq\|v_0\|_{L^\infty(\Omega)}} \frac{-\gamma'(v)\,\|v_0\|_{L^\infty(\Omega)}}{\gamma(v)},$$

they proved that the problem (1) has a unique global classical solution that is uniformly in time bounded. Besides, under some conditions, they proved that the solution converges to $(1,0)$ when t tends to infinity. For this problem, there are other results. To see these results, we refer the interested readers to [18, 24, 27] and references therein. In this paper, we focus on the functions $\gamma(v)$ and $\xi(v)$ as follows:

$$\gamma(v) = (1+v)^{-k} \quad \text{and} \quad \xi(v) = -(1-\alpha)\gamma'(v) \tag{3}$$

where $k > 0$ and $\alpha \in (0,1)$. For these functions, we will prove the following theorem:

Theorem 1. *Let $u_0 \geq 0$ and $v_0 \geq 0$ satisfy $(u_0, v_0) \in (W^{1,q}(\Omega))^2$ for some $q > n$ and the functions $\gamma(v)$ and $\xi(v)$ are defined as* (3). *Also, assume that*

$$k(1-\alpha) < \frac{4}{n+5} \tag{4}$$

and the initial value v_0 and μ satisfy the following conditions:

$$0 < \|v_0\|_{L^\infty(\Omega)} \leq \left[\frac{4\big[1-k\big(1-\alpha\big)\big]}{k(n+1)(1-\alpha)}\right]^{\frac{1}{k}} - 1 \tag{5}$$

and

$$\mu > \frac{kn(1-\alpha)\|v_0\|_{L^\infty(\Omega)}}{(n+1)(1+\|v_0\|_{L^\infty(\Omega)})}. \tag{6}$$

Then the solution of the problem (1) *is global and bounded.*

We note that the authors in [21] in the case of $\gamma(v) = (1+v)^{-k}(k > 0)$ proved that the solution of the problem (1) is global and bounded provided that:

$$k < \frac{n}{2(n+1)^3 - n}, \qquad \mu > k\|v_0\|_{L^\infty(\Omega)} \qquad \text{and} \qquad \|v_0\|_{L^\infty(\Omega)} \leq \gamma^{-1}\left(\frac{1}{n+1}\right).$$

Because of $\gamma'(v) < 0$, the last condition is written as:

$$0 < \| v_0 \|_{L^\infty(\Omega)} \le (n+1)^{\frac{1}{k}} - 1.$$

In the following, we show that our result improves the obtained result in [21].

- For $\alpha \in (0,1)$, it is not difficult to see that:
$$\frac{n}{2(n+1)^3 - n} < \frac{4}{n+5} < \frac{4}{(n+5)(1-\alpha)}.$$
Thus, our result extends the range of k.
- We see that if $k < \frac{4}{((n+1)^2+4)(1-\alpha)}$, then:
$$n+1 < \frac{4\left[1 - k\left(1-\alpha\right)\right]}{k\,(n+1)\,(1-\alpha)}.$$
Because of
$$\frac{n}{2(n+1)^3 - n} < \frac{4}{(n+1)^2+4} < \frac{4}{((n+1)^2+4)(1-\alpha)},$$
therefore, our result extends the upper bound obtained for $\| v_0 \|_{L^\infty(\Omega)}$ corresponding to the range of k in [21].
- Also, we have
$$\frac{n\,(1-\alpha)}{(n+1)(1+\| v_0 \|_{L^\infty(\Omega)})} < 1.$$
Thus, if we take the values k and $\| v_0 \|_{L^\infty(\Omega)}$ in the range of obtained in [21], then the lower bound obtained in our result for μ is smaller than the lower bound obtained in [21].

2. Our results

Here, we state the standard well-posedness and classical solvability result.

Lemma 2. *Let $u_0 \ge 0$ and $v_0 \ge 0$ satisfy $(u_0, v_0) \in (W^{1,q}(\Omega))^2$ for some $q > n$. Then problem* (1) *has a unique local in time classical solution*

$$(u, v) \in \left(C\left([0, T_{max}); W^{1,q}(\Omega)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{max})\right)^2\right.$$

where T_{max} denotes the maximal existence time. In addition, if $T_{max} < +\infty$, then:

$$\limsup_{t \to T_{max}} \| u(\cdot, t) \|_{L^\infty(\Omega)} = +\infty.$$

Moreover, u and v satisfy the following inequalities:

$$u \ge 0 \quad and \quad 0 \le v \le \| v_0 \|_{L^\infty(\Omega)} \quad in \quad \Omega \times (0, T_{max}), \tag{7}$$

also,

$$\int_\Omega u(\cdot, t)\, \mathrm{d}x \le c, \tag{8}$$

where c is some positive constant.

For details of the proof, we refer the reader to [12, 21].

Based on main idea in [3, 4, 16, 17], we write the following key lemma.

Lemma 3. *Let (u, v) be the solution of problem* (1). *If there exists a smooth positive function $\varphi(v)$ such that for $p \ge 2$ the following inequality holds*

$$(B(v))^2 - 4A(v)C(v) \le 0, \tag{9}$$

where the functions A, B and C are defined as:

$$\begin{cases} A(v) = (p-1)\varphi(v)\gamma(v), \\ B(v) = (p-1)\varphi(v)\xi(v) - \varphi'(v)(\gamma(v)+1), \\ C(v) = \frac{1}{p}\varphi''(v) - \varphi'(v)\xi(v), \end{cases} \tag{10}$$

then:

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x \le -\int_\Omega \left[\mu\varphi(v) + \frac{1}{p}v\varphi'(v)\right]u^{p+1}\,\mathrm{d}x + \mu\int_\Omega u^p\varphi(v)\,\mathrm{d}x.$$

Proof. We assume that there exists a smooth positive function $\varphi(v)$ such that for $p \ge 2$, (9) holds. We take this function and use (1) and integration by parts to write:

$$\begin{aligned} \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x &= \int_\Omega u^{p-1}\varphi(v)\; u_t\,\mathrm{d}x + \frac{1}{p}\int_\Omega u^p\;\varphi'(v)v_t\,\mathrm{d}x \\ &= -(p-1)\int_\Omega u^{p-2}\varphi(v)\gamma(v)\,|\nabla u|^2\,\mathrm{d}x \\ &\quad + \int_\Omega u^{p-1}\Big[(p-1)\varphi(v)\xi(v) - \varphi'(v)(\gamma(v)+1)\Big]\big(\nabla u\cdot\nabla v\big)\,\mathrm{d}x \\ &\quad + \int_\Omega u^p\left[\varphi'(v)\xi(v) - \frac{1}{p}\varphi''(v)\right]|\nabla v|^2\,\mathrm{d}x \\ &\quad - \int_\Omega \left[\mu\varphi(v) + \frac{1}{p}v\varphi'(v)\right]u^{p+1}\,\mathrm{d}x + \mu\int_\Omega u^p\varphi(v)\,\mathrm{d}x. \end{aligned} \tag{11}$$

For convenience in calculations, we write (11) as follows:

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x = \int_\Omega J(u,v)\,\mathrm{d}x - \int_\Omega \left[\mu\varphi(v) + \frac{1}{p}v\varphi'(v)\right]u^{p+1}\,\mathrm{d}x + \mu\int_\Omega u^p\varphi(v)\,\mathrm{d}x \tag{12}$$

with

$$\begin{aligned} J(u,v) &= -(p-1)u^{p-2}\varphi(v)\gamma(v)\,|\nabla u|^2 \\ &\quad + u^{p-1}\Big[(p-1)\varphi(v)\xi(v) - \varphi'(v)(\gamma(v)+1)\Big]\big(\nabla u\cdot\nabla v\big) \\ &\quad + u^p\left[\varphi'(v)\xi(v) - \frac{1}{p}\varphi''(v)\right]|\nabla v|^2 \\ &= -u^{p-2}A(v)|\nabla u|^2 + u^{p-1}B(v)\big(\nabla u\cdot\nabla v\big) - u^pC(v)|\nabla v|^2, \end{aligned} \tag{13}$$

where A, B and C are defined as (10). Now, by considering (13), we can write

$$\begin{aligned} J(u,v) &= -\left(\sqrt{u^{p-2}A(v)}\nabla u - \frac{u^{p-1}B(v)}{2\sqrt{u^{p-2}A(v)}}\nabla v\right)\cdot\left(\sqrt{u^{p-2}A(v)}\nabla u - \frac{u^{p-1}B(v)}{2\sqrt{u^{p-2}A(v)}}\nabla v\right) \\ &\quad + u^p\left[\frac{(B(v))^2}{4A(v)} - C(v)\right]|\nabla v|^2 \\ &\le u^p\left[\frac{(B(v))^2 - 4A(v)C(v)}{4A(v)}\right]|\nabla v|^2. \end{aligned}$$

In view of the condition (9), we see that $J \le 0$. Thus, the equality (12) becomes

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x \le -\int_\Omega \left[\mu\varphi(v) + \frac{1}{p}v\varphi'(v)\right]u^{p+1}\,\mathrm{d}x + \mu\int_\Omega u^p\varphi(v)\,\mathrm{d}x.$$

This completes our proof. □

In the following lemma, we present a function φ and show that for this function, the relation (9) holds.

Lemma 4. *Let $u_0 \geq 0$ and $v_0 \geq 0$ satisfy $(u_0, v_0) \in (W^{1,q}(\Omega))^2$ for some $q > n$ and the functions $\gamma(v)$ and $\xi(v)$ are defined as* (3)*. Also, assume that* (4)*,* (5) *and* (6) *hold. Then there exists some positive constant c such that the first component of problem* (1) *for all $t \in (0, T_{\max})$ satisfies*

$$\|u(\cdot,t)\|_{L^{n+1}(\Omega)} \leq c. \tag{14}$$

Proof. We want to apply Lemma 3. Hence, at first, we take $p = n+1$ and define the function φ as:

$$\varphi(v) = (1+v)^{-k\lambda} \quad \text{with} \quad \lambda = n(1-\alpha).$$

For this function, we have:

$$\varphi'(v) = -k\lambda(1+v)^{-k\lambda-1}$$

and

$$\varphi''(v) = k\lambda(k\lambda+1)(1+v)^{-k\lambda-2}.$$

In the following, we show that for this function φ, the relation (9) holds. We know from (3) that $\gamma(v) = (1+v)^{-k}$ and $\xi(v) = k(1-\alpha)(1+v)^{-k-1}$. By considering these, we compute:

$$\begin{aligned}
&(B(v))^2 - 4A(v)C(v)\\
&= n^2(\varphi(v))^2(\xi(v))^2 + (\varphi'(v))^2(\gamma(v)+1)^2\\
&\quad - 2n\varphi(v)\varphi'(v)\xi(v)(1-\gamma(v)) - \frac{4n}{n+1}\varphi(v)\varphi''(v)\gamma(v)\\
&= k^2 n^2(1-\alpha)^2(1+v)^{-2(k+1)-2kn(1-\alpha)}\\
&\quad + k^2 n^2(1-\alpha)^2(1+v)^{-2(kn(1-\alpha)+1)}\left[(1+v)^{-2k} + 2(1+v)^{-k} + 1\right]\\
&\quad + 2k^2 n^2(1-\alpha)^2(1+v)^{-2kn(1-\alpha)-k-2} - 2k^2 n^2(1-\alpha)^2(1+v)^{-2kn(1-\alpha)-2k-2}\\
&\quad - \frac{4kn^2(1-\alpha)}{n+1}(kn(1-\alpha)+1)(1+v)^{-2kn(1-\alpha)-k-2}\\
&= kn^2(1-\alpha)(1+v)^{-2(kn(1-\alpha)+1)}\Big\{k(1-\alpha)(1+v)^{-2k} + k(1-\alpha)\left[(1+v)^{-2k} + 2(1+v)^{-k} + 1\right]\\
&\quad + 2k(1-\alpha)(1+v)^{-k} - 2k(1-\alpha)(1+v)^{-2k} - \frac{4}{n+1}\big(kn(1-\alpha)+1\big)(1+v)^{-k}\Big\}\\
&= kn^2(1-\alpha)(1+v)^{-2(kn(1-\alpha)+1)}\left\{4\left[k(1-\alpha) - \frac{kn(1-\alpha)+1}{n+1}\right](1+v)^{-k} + k(1-\alpha)\right\}\\
&= \frac{kn^2(1-\alpha)(1+v)^{-2(kn(1-\alpha)+1)}}{n+1}\left\{k(n+1)(1-\alpha) - 4[1-k(1-\alpha)](1+v)^{-k}\right\}\\
&\leq \frac{kn^2(1-\alpha)(1+v)^{-2(kn(1-\alpha)+1)}}{n+1}\left\{k(n+1)(1-\alpha) - 4[1-k(1-\alpha)](1+\|v_0\|_\infty)^{-k}\right\}.
\end{aligned}$$

Under the condition (5), we see that

$$(B(v))^2 - 4A(v)C(v) \leq 0.$$

Thus, the relation (9) holds. We now can apply Lemma 3 and write:

$$\begin{aligned}
&\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{n+1}(1+v)^{-kn(1-\alpha)}\,\mathrm{d}x + \mu\int_\Omega u^{n+1}(1+v)^{-kn(1-\alpha)}\,\mathrm{d}x\\
&\quad \leq -\int_\Omega\left[\mu(1+v) - \frac{kn(1-\alpha)}{n+1}v\right](1+v)^{-kn(1-\alpha)-1}u^{n+2}\,\mathrm{d}x + 2\mu\int_\Omega u^{n+1}(1+v)^{-kn(1-\alpha)}\,\mathrm{d}x.
\end{aligned} \tag{15}$$

The Young inequality allows us to write:

$$2\mu\int_\Omega u^{n+1}(1+v)^{-kn(1-\alpha)}\,\mathrm{d}x \le \epsilon\int_\Omega u^{n+2}(1+v)^{-kn(1-\alpha)}\,\mathrm{d}x + c(\epsilon)\int_\Omega (1+v)^{-kn(1-\alpha)}\,\mathrm{d}x$$
$$\le \epsilon\int_\Omega u^{n+2}(1+v)^{-kn(1-\alpha)}\,\mathrm{d}x + c(\epsilon)\,|\Omega|, \qquad (16)$$

where ϵ is chosen as follows:

$$0<\epsilon<\mu-\frac{kn(1-\alpha)\|v_0\|_{L^\infty(\Omega)}}{(n+1)(1+\|v_0\|_{L^\infty(\Omega)})}$$

and:

$$c(\epsilon)=\frac{1}{n+2}\left[\frac{n+1}{\epsilon(n+2)}\right]^{n+1}(2\mu)^{n+2}.$$

We now combine the inequality (16) with (15) and use from $0\le v\le\|v_0\|_{L^\infty(\Omega)}$ and (6) to obtain:

$$\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{n+1}(1+v)^{-kn(1-\alpha)}\,\mathrm{d}x+\mu\int_\Omega u^{n+1}(1+v)^{-kn(1-\alpha)}\,\mathrm{d}x$$
$$\le\int_\Omega\left[\epsilon-\mu+\frac{kn(1-\alpha)v}{(n+1)(1+v)}\right](1+v)^{-kn(1-\alpha)}u^{n+2}\,\mathrm{d}x+c(\epsilon)|\Omega|$$
$$\le\int_\Omega\left[\epsilon-\mu+\frac{kn(1-\alpha)\|v_0\|_{L^\infty(\Omega)}}{(n+1)(1+\|v_0\|_{L^\infty(\Omega)})}\right](1+v)^{-kn(1-\alpha)}u^{n+2}\,\mathrm{d}x+c(\epsilon)\,|\Omega|.$$

We put:

$$y(t)=\int_\Omega u^{n+1}(1+v)^{-kn(1-\alpha)}\,\mathrm{d}x.$$

We see that the value of ϵ allows us to write:

$$y'(t)+\mu(n+1)\,y(t)\le c(\epsilon)\,(n+1)|\Omega|.$$

This yields:

$$y(t)\le\max\left\{y(0),\frac{c(\epsilon)\,|\Omega|}{\mu}\right\}. \qquad (17)$$

Making use of $0\le v\le\|v_0\|_{L^\infty(\Omega)}$ and (17), we have:

$$\int_\Omega u^{n+1}\,\mathrm{d}x\le\left(1+\|v_0\|_{L^\infty(\Omega)}\right)^{kn(1-\alpha)}\max\left\{y(0),\frac{c(\epsilon)\,|\Omega|}{\mu}\right\}.$$

Thus, we obtain the desired result. □

The proof of the following lemma is the same as [22, Lemma 3.2]. But, we write it to complement our content.

Lemma 5. *Let $u_0\ge0$ and $v_0\ge0$ satisfy $(u_0,v_0)\in(W^{1,q}(\Omega))^2$ for some $q>n$. Also, assume that* (4)*,* (5) *and* (6) *hold. Then there exists some positive constant C such that*

$$\|\nabla v\|_{L^\infty(\Omega)}\le C \qquad (18)$$

for all $t\in(0,T_{max})$.

Proof. By considering Lemma 2, we see that it is sufficient to prove for any $\tau\in(0,T_{\max})$,

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}\le C\quad\text{for all } t\in(\tau,T_{max}). \qquad (19)$$

We use the representation formula for the second equation (1) to have:

$$v(\cdot,t)=\mathrm{e}^{t(\Delta-1)}v_0+\int_0^t\mathrm{e}^{(t-s)(\Delta-1)}(1-u(\cdot,s))\,v(\cdot,s)\,\mathrm{d}s,\quad t\in(0,T_{max}).$$

We now take $p = n+1$ and use from $0 \le v \le \|v_0\|_{L^\infty(\Omega)}$ and (14) to write:

$$\begin{aligned}\|(1-u(\cdot,s))v(\cdot,s)\|_{L^p(\Omega)} &\le \|v(\cdot,s)\|_{L^\infty(\Omega)}\left(\int_\Omega |1-u(\cdot,s))|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\le \|v(\cdot,s)\|_{L^\infty(\Omega)}\left(\int_\Omega \big(1+|u(\cdot,s))|\big)^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\le \|v(\cdot,s)\|_{L^\infty(\Omega)}\left(2^{p-1}\int_\Omega \big(1+|u(\cdot,s))|^p\big)\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\le 2^{\frac{p-1}{p}}\|v(\cdot,s)\|_{L^\infty(\Omega)}\left(|\Omega|^{\frac{1}{p}}+|\|u(\cdot,s))\|_{L^p(\Omega)}\right)\\ &\le c, \end{aligned} \tag{20}$$

where we have used the inequality $(a+b)^m \le 2^{m-1}(a^m+b^m)$ with $a,b \ge 0$ and $m>1$, also $(a+b)^{m'} \le (a^{m'}+b^{m'})$ with $0<m'<1$. In order to prove (19), we take $\tau \in (0,\min\{1,T_{\max}\})$ and $\theta \in \left(\frac{2n+1}{2(n+1)},1\right)$ and use the estimates (3.16) and (3.17) in [22], also (20) to obtain:

$$\begin{aligned}\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} &\le c\,\|(-\Delta+1)^\theta v(\cdot,t)\|_{L^p(\Omega)}\\ &\le c\,t^{-\theta}\mathrm{e}^{-\delta t}\|v_0\|_{L^p(\Omega)} + c\int_0^t (t-s)^{-\theta}\mathrm{e}^{-\delta(t-s)}\|(1-u(\cdot,s))v(\cdot,s)\|_{L^p(\Omega)}\,\mathrm{d}s\\ &\le c\,t^{-\theta} + c\int_0^t (t-s)^{-\theta}\mathrm{e}^{-\delta(t-s)}\mathrm{d}s\\ &\le c\,t^{-\theta} + c\int_0^{+\infty}\sigma^{-\theta}\mathrm{e}^{-\delta\sigma}\mathrm{d}\sigma\\ &\le c(\tau^{-\theta}+1), \quad t\in(\tau,T_{max}),\end{aligned}$$

where the constant c can vary from line to line. This completes our proof. □

Upon the well-known Moser Alikakos iteration procedure [2], we prove the following lemma similar to [22, Lemma 3.2].

Lemma 6. *Let $u_0 \ge 0$ and $v_0 \ge 0$ satisfy $(u_0,v_0) \in (W^{1,q}(\Omega))^2$ for some $q>n$. Also, assume that* (4), (5) *and* (6) *hold. Then there exists some positive constant c such that the first component of problem* (1) *for all $t \in (0,T_{\max})$ satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le c.$$

Proof. We take $p \ge 2$ and use from (1) and integration by parts to obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\,\mathrm{d}x &= p\int_\Omega u^{p-1}\Big[\nabla\cdot(\gamma(v)\nabla u - u\xi(v)\nabla v)+\mu u(1-u)\Big]\mathrm{d}x\\ &= -p(p-1)\int_\Omega \gamma(v)\,u^{p-2}|\nabla u|^2\,\mathrm{d}x + p(p-1)\int_\Omega u^{p-1}\xi(v)\nabla u\cdot\nabla v\,\mathrm{d}x + \mu p\int_\Omega u^p(1-u)\,\mathrm{d}x.\end{aligned} \tag{21}$$

Because of $0 \le v \le \|v_0\|_{L^\infty(\Omega)}$, we have:

$$\begin{aligned}\gamma(v) &= (1+v)^{-k} \ge (1+\|v_0\|_{L^\infty(\Omega)})^{-k} := c_1,\\ \xi(v) &= k(1-\alpha)(1+v)^{-k-1} \le k(1-\alpha) := c_2.\end{aligned}$$

Making use of these, (18) and Young's inequality, we can write (21) as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\,\mathrm{d}x &\le -c_1\,p(p-1)\int_\Omega u^{p-2}|\nabla u|^2\,\mathrm{d}x + C\,c_2\,p(p-1)\int_\Omega u^{p-1}|\nabla u|\,\mathrm{d}x + \mu p\int_\Omega u^p\,\mathrm{d}x\\ &= -\frac{4\,c_1(p-1)}{p}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2\mathrm{d}x + 2\,C\,c_2(p-1)\int_\Omega u^{\frac{p}{2}}\cdot\left|\nabla u^{\frac{p}{2}}\right|\mathrm{d}x + \mu p\int_\Omega u^p\,\mathrm{d}x\\ &\le -\frac{2\,c_1(p-1)}{p}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2\mathrm{d}x + p\left(\frac{(p-1)\,C^2c_2^2}{2\,c_1}+\mu\right)\int_\Omega u^p\,\mathrm{d}x.\end{aligned} \tag{22}$$

We now add $p\int_\Omega u^p\,\mathrm{d}x$ on both sides of (22) to have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\mathrm{d}x + p\int_\Omega u^p\,\mathrm{d}x \le -\frac{2\,c_1\,(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2\mathrm{d}x + c_3\int_\Omega u^p\,\mathrm{d}x \tag{23}$$

with

$$c_3 = p\Big(\frac{(p-1)\,C^2 c_2^2}{2\,c_1} + \mu + 1\Big).$$

To estimate the last term on the right hand side of (23), we use the following known Gagliardo–Nirenberg inequality (see [11, 28], for instance):

$$\|\psi\|_{L^q(\Omega)} \le C_{GN}\Big(\|\nabla\psi\|_{L^2(\Omega)}^{\vartheta}\|\psi\|_{L^r(\Omega)}^{1-\vartheta} + \|\psi\|_{L^r(\Omega)}\Big),$$

where

$$q(n-2) < 2n, \quad r\in(0,q) \quad \text{and} \quad \vartheta = \frac{\frac{n}{r}-\frac{n}{q}}{1-\frac{n}{2}+\frac{n}{r}} \in (0,1),$$

and C_{GN} is the constant in the Gagliardo–Nirenberg inequality. Now, we apply the Gagliardo–Nirenberg inequality with $\psi = u^{\frac{p}{2}}, q = 2, r = 1$ and $\vartheta = \frac{n}{n+2}$, and then use the Young inequality with exponents $r = \frac{n+2}{n}$ and $s = \frac{n+2}{2}$ to obtain:

$$\begin{aligned}
c_3\int_\Omega u^p\,\mathrm{d}x = c_3\,\|u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 &\le c_3\,(C_{GN})^2\Big(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{n}{n+2}}\|u^{\frac{p}{2}}\|_{L^1(\Omega)}^{\frac{2}{n+2}} + \|u^{\frac{p}{2}}\|_{L^1(\Omega)}\Big)^2\\
&\le 2\,c_3\,(C_{GN})^2\Big(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2n}{n+2}}\|u^{\frac{p}{2}}\|_{L^1(\Omega)}^{\frac{4}{n+2}} + \|u^{\frac{p}{2}}\|_{L^1(\Omega)}^2\Big)\\
&\le \frac{2\,c_1\,(p-1)}{p}\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + \big(c_4 + 2\,c_3\,(C_{GN})^2\big)\|u^{\frac{p}{2}}\|_{L^1(\Omega)}^2\\
&= \frac{2\,c_1(p-1)}{p}\int_\Omega|\nabla u^{\frac{p}{2}}|^2\,\mathrm{d}x + c_5\Big(\int_\Omega u^{\frac{p}{2}}\,\mathrm{d}x\Big)^2
\end{aligned}$$

with

$$c_4 = \frac{1}{s}\Big(\frac{2\,c_1\,r\,(p-1)}{p}\Big)^{-\frac{s}{r}}\big(2\,c_3\,(C_{GN})^2\big)^s \quad \text{and} \quad c_5 = c_4 + 2\,c_3(C_{GN})^2.$$

Combining the last inequality with (23) yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\,\mathrm{d}x + p\int_\Omega u^p\,\mathrm{d}x \le c_5\Big(\int_\Omega u^{\frac{p}{2}}\,\mathrm{d}x\Big)^2.$$

For $0\le t\le T_{\max}$, we can write:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{p\,t}\int_\Omega u^p\,\mathrm{d}x\Big) \le c_5\,e^{p\,t}\Big(\int_\Omega u^{\frac{p}{2}}\,\mathrm{d}x\Big)^2.$$

Now, we integrate and use $e^{-p\,t}\le 1$ to get:

$$\begin{aligned}
\int_\Omega u^p\,\mathrm{d}x &\le \int_\Omega u_0^p\,\mathrm{d}x + \frac{c_5}{p}\sup_{0\le t\le T_{\max}}\Big(\int_\Omega u^{\frac{p}{2}}\,\mathrm{d}x\Big)^2\\
&\le |\Omega|\,\|u_0\|_{L^\infty(\Omega)}^p + \frac{c_5}{p}\sup_{0\le t\le T_{\max}}\Big(\int_\Omega u^{\frac{p}{2}}\,\mathrm{d}x\Big)^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Big(\int_\Omega u^p\,\mathrm{d}x\Big)^{\frac{1}{p}} &\le \Big[|\Omega|\,\|u_0\|_{L^\infty(\Omega)}^p + \frac{c_5}{p}\sup_{0\le t\le T_{\max}}\Big(\int_\Omega u^{\frac{p}{2}}\,\mathrm{d}x\Big)^2\Big]^{\frac{1}{p}}\\
&\le |\Omega|^{\frac{1}{p}}\,\|u_0\|_{L^\infty(\Omega)} + \big(\frac{c_5}{p}\big)^{\frac{1}{p}}\sup_{0\le t\le T_{\max}}\Big(\int_\Omega u^{\frac{p}{2}}\,\mathrm{d}x\Big)^{\frac{2}{p}}.
\end{aligned} \tag{24}$$

We note that

$$\begin{aligned}
c_5 &= c_4 + 2\,c_3 (C_{GN})^2 \\
&= \frac{1}{s}\left(\frac{2\,c_1\,r\,(p-1)}{p}\right)^{-\frac{s}{r}} \left(2\,c_3\,(C_{GN})^2\right)^s + 2\,c_3 (C_{GN})^2 \\
&= \frac{1}{s}\left(2\,c_1\,r\right)^{-\frac{s}{r}} \left(2\,(C_{GN})^2\right)^s \left(\frac{p}{p-1}\right)^{\frac{n}{2}} (c_3)^s + 2\,c_3\,(C_{GN})^2 \\
&\le m\left[\left(\frac{p}{p-1}\right)^{\frac{n}{2}} (c_3)^s + c_3\right] \\
&\le m\left[\left(\frac{p}{p-1}\right)^{\frac{n}{2}} + 1\right] (c_3)^s
\end{aligned}$$

with

$$m = \max\left\{\frac{1}{s}\left(2\,c_1\,r\right)^{-\frac{s}{r}} \left(2\,(C_{GN})^2\right)^s, 2\,(C_{GN})^2\right\}.$$

Here, we have used from $c_3 > 1$ and $s > 1$. By inserting c_3 and using $p \ge 2$, we obtain:

$$\begin{aligned}
\frac{c_5}{p} &\le m\left[\left(\frac{p}{p-1}\right)^{\frac{n}{2}} + 1\right] \left(\frac{(p-1)\,C^2 c_2^2}{2\,c_1} + \mu + 1\right)^{\frac{n}{2}+1} p^{\frac{n}{2}} \\
&\le 2\,m\left(\frac{C^2 c_2^2}{2\,c_1} + \mu + 1\right)^{\frac{n}{2}+1} \left(\frac{p}{p-1}\right)^{\frac{n}{2}} (p-1)^{\frac{n}{2}+1} p^{\frac{n}{2}} \\
&= c_6\,(p-1)\,p^n \\
&\le c_6\,p^{n+1}
\end{aligned} \tag{25}$$

with

$$c_6 = 2\,m\left(\frac{C^2 c_2^2}{2\,c_1} + \mu + 1\right)^{\frac{n}{2}+1}.$$

Making use of (25) and $p^{\frac{n+1}{p}} > 1$, we can write (24) as follows:

$$\begin{aligned}
\left(\int_\Omega u^p \,\mathrm{d}x\right)^{\frac{1}{p}} &\le |\Omega|^{\frac{1}{p}} \,\|u_0\|_{L^\infty(\Omega)} + \left(c_6\,p^{n+1}\right)^{\frac{1}{p}} \sup_{0\le t\le T_{\max}} \left(\int_\Omega u^{\frac{p}{2}} \,\mathrm{d}x\right)^{\frac{2}{p}} \\
&\le c_7^{\frac{1}{p}}\, p^{\frac{n+1}{p}} \left(\|u_0\|_{L^\infty(\Omega)} + \sup_{0\le t\le T_{\max}} \left(\int_\Omega u^{\frac{p}{2}} \,\mathrm{d}x\right)^{\frac{2}{p}}\right)
\end{aligned} \tag{26}$$

with $c_7 = |\Omega| + c_6$. We now define:

$$M(p) = \max\left\{\|u_0\|_{L^\infty(\Omega)}, \sup_{0\le t\le T_{\max}} \left(\int_\Omega u^p \mathrm{d}x\right)^{\frac{1}{p}}\right\}.$$

This allows us to write (26) as:

$$M(p) \le 2\,c_7^{\frac{1}{p}}\, p^{\frac{n+1}{p}}\, M\left(\frac{p}{2}\right).$$

We now take $p = 2^i$ $(i \in \mathbb{N})$ to obtain:

$$\begin{aligned}
M(2^i) &\le 2\,c_7^{2^{-i}}\, 2^{\frac{(n+1)i}{2^i}}\, M(2^{i-1}) \\
&\le 2\,c_7^{2^{-i}+2^{-i+1}}\, 2^{(n+1)\left(\frac{i}{2^i}+\frac{i-1}{2^{i-1}}\right)} M(2^{i-2}) \\
&\le \cdots \\
&\le 2\,c_7^{2^{-i}+2^{-i+1}+\cdot+2^{-1}}\, 2^{(n+1)\left(\frac{i}{2^i}+\frac{i-1}{2^{i-1}}+\cdot+\frac{1}{2}\right)} M(1).
\end{aligned} \tag{27}$$

We now compute the following elementary series

$$S := \sum_{i=1}^{\infty} \frac{i}{2^i} = \sum_{i=0}^{\infty} \frac{i+1}{2^{i+1}} = \sum_{i=0}^{\infty} \left(\frac{i}{2^{i+1}} + \frac{1}{2^{i+1}} \right) = \frac{1}{2} \sum_{i=1}^{\infty} \frac{i}{2^i} + \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = \frac{1}{2} S + 1.$$

Thus, $S = 2$. Making use of this, $\lim_{i\to\infty} \|u(\cdot,t)\|_{L^{2^i}(\Omega)} = \|u(\cdot,t)\|_{L^\infty(\Omega)}$ and (8), by letting $i \to \infty$ in (27), we obtain the desired result. □

Proof of Theorem 1. By considering the extensibility criterion provided by Lemma 2, the proof is a consequence of (9) and Lemma 6. □

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