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A formula for the sum of n weak* closed sets in L^∞

Mohsen Zivari-Rezapour [a], Yichen Liu [*, b] and Behrouz Emamizadeh [c, d]

[a] Department of Mathematics, Faculty of Mathematical Sciences and Computer, Shahid Chamran University of Ahvaz, Ahvaz, Iran

[b] Department of Applied Mathematics, School of Mathematics and Physics, Xi'an Jiaotong-Liverpool University, Suzhou, China

[c] Department of Mathematical Sciences, University of Nottingham Ningbo China, Ningbo, China

[d] Institute for Research in Fundamental Sciences (IPM), Tehran, Iran

E-mails: mzivari@scu.ac.ir, yichen.liu01@xjtlu.edu.cn, behrouz.emamizadeh@nottingham.edu.cn, behrouz.emamizadeh@ipm.ir

Abstract. In this note, we derive an equation which describes the closure of a particular set comprising $n-$valued functions. This result provides an answer to a long standing question for which the particular case $n=2$ had been known and used frequently in the optimal control problems.

Keywords. $n-$valued functions, formulation, weak* closure, rearrangements, eigenvalues.

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1. Introduction

In many optimal control problems, the admissible set is defined as follows

$$A := \left\{ f \in L^\infty(D) : f(f-1) = 0, \int_D f \,\mathrm{d}x = \alpha \right\} \tag{1}$$

where D is a bounded (measurable) set in $\mathbb{R}^N$, $\alpha \in (0,|D|)$, and $|D|$ denotes the Lebesgue measure of D, see for example [7]. Henceforth, we identity L^∞ as $\left(L^1\right)^\star$, the dual of L^1. Then, it is widely known that

$$w^\star - \text{closure of } A = \left\{ f \in L^\infty(D) : 0 \le f \le 1, \int_D f \,\mathrm{d}x = \alpha \right\}, \tag{2}$$

[*] Corresponding author.

see Proposition 2.4 in [3]. This latter set is drastically more convenient to work with compared to the former. For example, it is convex and weak* compact in $L^\infty(D)$, whereas A fails to retain either of these two properties.

In this note, we answer a long standing question; mainly, there is an analogy of (2) for n−valued functions. More precisely, we replace the set of $\{0,1\}$−valued functions (verifying the integral constraint $\int_D f\,\mathrm{d}x = \alpha$), i.e. A, by an appropriate set of n−valued functions. A natural generalization of (1) is the following set

$$\widetilde{A} = \left\{ f \in L^\infty(D) : (f-\gamma_1)(f-\gamma_2)\cdots(f-\gamma_n) = 0, \int_D f\,\mathrm{d}x = \alpha \right\}.$$

However, a function $h \in \widetilde{A}$ may not satisfy the condition $|\{x \in D : f(x) = \gamma_i\}| > 0,\ i = 1,2,\ldots,n$, which we require, as illustrated by the following example:

Example. Let $D = (0,1)$, $\alpha = 1$ and $\gamma_i = i-1$, $i = 1,2,3$. Clearly, the set $\widetilde{A}$ contains the following 2-valued and 3-valued controls:

$$f_1 = 2\chi_{(\frac{1}{2},1)}, \quad f_2 = \chi_{(0,\frac{1}{3})} + 2\chi_{(\frac{2}{3},1)}.$$

Note that this drawback is irrelevant to A due to imposing the condition $\alpha \in (0,|D|)$. To overcome this obstacle, we first make the following observation that

$$A = \left\{ \chi_E : E \subseteq D, |E| = \alpha \right\}$$

where χ_E denotes the characteristic function of E. Furthermore, we make a second observation that

$$\left\{ \chi_E : E \subseteq D, |E| = \alpha \right\} = \mathscr{R}(\chi_F)$$

where F can be any subset of D verifying $|F| = \alpha$. Here, $\mathscr{R}(\chi_F)$ denotes the classical rearrangement class generated by χ_F. In the following lines, we briefly recall the definition of two functions being rearrangements of each other.

For two (measurable) functions $f, g : D \to \mathbb{R}$, we say f and g are rearrangements of each other if

$$\lambda_f(\alpha) = \lambda_g(\alpha) \quad \text{for all } \alpha \in \mathbb{R},$$

where λ_f (similarly λ_g) denotes the distribution function of f, i.e.

$$\lambda_f(\alpha) = |\{x \in D : f(x) \geq \alpha\}|.$$

We use the notation $\mathscr{R}(f)$ to indicate the set of functions defined on D which are rearrangements of f; $\mathscr{R}(f)$ is called the rearrangement class generated by f. Note that if f and g are rearrangements of each other, they will generate the same class of rearrangements, i.e. $\mathscr{R}(f) = \mathscr{R}(g)$.

Now, for $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, we introduce the set

$$\mathscr{A} = \left\{ \sum_{i=1}^n c_i \chi_{E_i} : E_i \subseteq D, \{E_i\} \text{ are mutually disjoint}, |E_i| = \alpha_i \right\},$$

where $\alpha_i > 0$ are prescribed satisfying $\sum_{i=1}^n \alpha_i = |D|$. Without loss of generality, we suppose $\{c_i\}$ is strictly increasing. Our main result is the following

Theorem. *Using the notations mentioned above, the following equation holds*

$$\overline{\mathscr{A}} = \sum_{i=1}^n \mathscr{K}_i \tag{3}$$

where $\overline{\mathscr{A}}$ denotes the weak* closure of $\mathscr{A}$,

$$\mathscr{K}_i = \left\{ f \in L^\infty(D) : 0 \leq f \leq c_i - c_{i-1}, \int_D f\,\mathrm{d}x = (c_i - c_{i-1}) \sum_{k=i}^n \alpha_k \right\} \quad \textit{for all } i = 2,\ldots,n,$$

and $\mathscr{K}_1 = \{c_1\}$. Here, c_1 denotes the constant function.

Remark 1. A direct consequence of the Theorem is the following useful observation. Suppose $\beta_i \in \mathbb{R}$ satisfies $0 < \beta_n < \beta_{n-1} < \cdots < \beta_1 < |D|$ and

$$\mathscr{K}_i = \left\{ f \in L^\infty(D) : 0 \le f \le 1, \int_D f \,\mathrm{d}x = \beta_i \right\} \quad \text{for all } i = 1, \ldots, n.$$

If we set $\beta_0 = |D|$, $\beta_{n+1} = 0$ and

$$\mathscr{A} = \left\{ \sum_{i=1}^{n+1} (i-1)\chi_{E_i} : E_i \subseteq D, \{E_i\} \text{ are mutually disjoint}, |E_i| = \beta_{i-1} - \beta_i \right\},$$

then we have

$$\overline{\mathscr{A}} = \sum_{i=1}^{n} \mathscr{K}_i.$$

2. Proof of the Theorem

In the proof of the Theorem, we use the following important lemma from [2] which we include a sketch of its proof for the convenience of readers.

Lemma. *Let $f_1, f_2, \ldots, f_n \in L^\infty(D)$ such that $\int_D f_i f_j \,\mathrm{d}x = \int_0^{|D|} f_i^\triangle f_j^\triangle \,\mathrm{d}s$ for all $i, j = 1, 2, \ldots, n$. Then,*

$$\sum_{i=1}^{n} \overline{\mathscr{R}(f_i)} = \overline{\mathscr{R}\left(\sum_{i=1}^{n} f_i \right)}.$$

Here, $f_i^\triangle$ denotes the decreasing rearrangement on $(0, |D|)$, i.e. $f_i^\triangle(s) := \max\{\alpha : \lambda_{f_i}(\alpha) \ge s\}$.

Proof. We only sketch the proof here; the readers can refer to Theorem 3 in [2] for details.

The inclusion $\overline{\mathscr{R}\left(\sum_{i=1}^n f_i\right)} \subseteq \sum_{i=1}^n \overline{\mathscr{R}(f_i)}$ follows from Theorem 18.10 in [5]. Now, we show the inclusion in the other direction. To this end, we set $f = \sum_{i=1}^n f_i$ and utilize the condition that $\int_D f_i f_j \,\mathrm{d}x = \int_0^{|D|} f_i^\triangle f_j^\triangle \,\mathrm{d}s$ to show the following

$$0 \le \int_0^{|D|} \left(f^\triangle - \sum_{i=1}^{n} f_i^\triangle \right)^2 \mathrm{d}s \le \int_D \left(f - \sum_{i=1}^{n} f_i \right)^2 \mathrm{d}x = 0.$$

This, in turn, implies

$$\left(\sum_{i=1}^{n} f_i \right)^\triangle = \sum_{i=1}^{n} f_i^\triangle. \tag{4}$$

Next, let $h = \sum_{i=1}^n h_i \in \sum_{i=1}^n \overline{\mathscr{R}(f_i)}$ with $h_i \in \overline{\mathscr{R}(f_i)}$ for $i = 1, 2, \ldots, n$. We can use the standard rearrangement techniques and (4) to derive

$$\int_0^{|D|} h^\triangle \,\mathrm{d}s = \int_0^{|D|} \left(\sum_{i=1}^{n} f_i \right)^\triangle \mathrm{d}s \quad \text{and} \quad \int_0^t h^\triangle \,\mathrm{d}s \le \int_0^t \left(\sum_{i=1}^{n} f_i \right)^\triangle \mathrm{d}s,$$

for all $t \in (0, |D|)$. This implies $h \in \overline{\mathscr{R}\left(\sum_{i=1}^n f_i\right)}$, by [1] or Proposition 5 in [2]. The proof of the Lemma is complete. □

Proof of the Theorem. The proof of the Theorem is in fact an immediate consequence of the Lemma.

By setting $c_0 = 0$, the key to the proof will be the decomposition in (5). We define

$$f_i = (c_i - c_{i-1})\chi_{\bigcup_{k=i}^n E_k},$$

$i = 1, 2, \ldots, n$. We then set

$$f = \sum_{i=1}^{n} f_i. \tag{5}$$

Clearly, $\mathscr{A} = \mathscr{R}(f)$. On the other hand, for almost every $s \in (0,|D|)$, we have

$$f_i^{\triangle}(s) = (c_i - c_{i-1})\chi_{(0,\sum_{k=i}^{n}\alpha_k)}(s),$$

$i = 1,2,\ldots,n$. For $i, j \in \{1,2,\ldots,n\}$ with $i \le j$, then we deduce, by direct computations, that

$$\int_D f_i(x) f_j(x)\,\mathrm{d}x = (c_i - c_{i-1})(c_j - c_{j-1}) \sum_{k=j}^{n} \alpha_k = \int_0^{|D|} f_i^{\triangle}(s) f_j^{\triangle}(s)\,\mathrm{d}s. \tag{6}$$

From (5), (6) and the Lemma, we infer that

$$\overline{\mathscr{A}} = \overline{\mathscr{R}(f)} = \overline{\mathscr{R}\left(\sum_{i=1}^{n} f_i\right)} = \sum_{i=1}^{n} \overline{\mathscr{R}(f_i)}.$$

Moreover, by Proposition 2.4 in [3], we have

$$\mathscr{K}_i := \overline{\mathscr{R}(f_i)} = \left\{ g \in L^\infty(D) : 0 \le g \le c_i - c_{i-1}, \int_D g\,\mathrm{d}x = (c_i - c_{i-1}) \sum_{k=i}^{n} \alpha_k \right\},$$

$i = 1,2,\ldots,n$. Notice that $\mathscr{R}(f_1) = \{f_1\} = \{c_1\}$, hence we infer $\mathscr{K}_1 = \overline{\mathscr{R}(f_1)} = \{c_1\}$. Therefore, the decomposition (3) follows. □

Remark 2. It is important to mention that everything stated in this paper holds true if the Lebesgue measure is replaced by any other (finite) non-atomic measures.

This theorem has nice applications, for example, in spectral theory which we only outline here. Consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda \rho(x) u, & \text{in } D \\ u = 0, & \text{on } \partial D, \end{cases} \tag{7}$$

where this boundary value problem models the displacement, denoted u, of a clamped membrane with density $\rho(x)$ on the bounded domain D from the rest position. It is well known that if ρ is a non-trivial function in $L_+^\infty(D)$, then the eigenvalue problem (7) has a sequence of eigenvalues such that

$$0 < \lambda_1 < \lambda_2 \le \ldots \to \infty,$$

see for example [6]. $\lambda_1(\rho)$, to emphasize its dependence on ρ, is called the principal eigenvalue and it can be formulated in terms of the Rayleigh quotient

$$\lambda_1(\rho) := \inf \left\{ \frac{\int_D |\nabla v|^2\,\mathrm{d}x}{\int_D \rho v^2\,\mathrm{d}x} : v \in H_0^1(D), \int_D \rho v^2\,\mathrm{d}x > 0 \right\}.$$

One is interested in the following maximization and minimization problems

$$\sup_{\rho \in \mathscr{K}_1 + \mathscr{K}_2 + \cdots + \mathscr{K}_n} \lambda_1(\rho) \quad \text{and} \quad \inf_{\rho \in \mathscr{K}_1 + \mathscr{K}_2 + \cdots + \mathscr{K}_n} \lambda_1(\rho). \tag{8}$$

By using the Theorem, we can transform the above problems to the following rearrangement optimization problems

$$\sup_{\rho \in \overline{\mathscr{A}}} \lambda_1(\rho) \quad \text{and} \quad \inf_{\rho \in \overline{\mathscr{A}}} \lambda_1(\rho)$$

which can be solved by the results in [4, 8]. Moreover, one can prove that the optimization problems in (8) have optimal solutions which, in turn, give rise to free boundary problems related to the eigenvalue problem (7).

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