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Functional analysis / *Analyse fonctionnelle*

Separation ratios of maps between Banach spaces

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Abstract. Under the weak assumption on a Banach space E that $E \oplus E$ embeds isomorphically into E, we provide a characterisation of when a Banach space X coarsely embeds into E via a single numerical invariant.

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1. Introduction

The concept of coarse embeddability between metric spaces can be viewed as a large scale analogue of uniform embeddability and may most easily be understood in terms of the moduli associated with a map. However, as we are exclusively concerned with Banach spaces, these moduli can further be reduced to a couple of numerical invariants.

Definition 1. *For a (generally discontinuous and nonlinear) map $X \xrightarrow{\phi} E$ between two Banach spaces we define the* exact compression coefficient $\overline{\kappa}(\phi)$, *the* compression coefficient $\kappa(\phi)$ *and the* expansion coefficient $\omega(\phi)$ *by*

$$\overline{\kappa}(\phi) = \sup_{t<\infty} \inf_{\|x-y\|=t} \left\|\phi(x) - \phi(y)\right\|,$$

$$\kappa(\phi) = \sup_{t<\infty} \inf_{\|x-y\|\geqslant t} \left\|\phi(x) - \phi(y)\right\|,$$

and

$$\omega(\phi) = \inf_{t>0} \sup_{\|x-y\|\leqslant t} \left\|\phi(x) - \phi(y)\right\|.$$

To avoid certain trivialities, we shall tacitly assume that all Banach spaces have dimension at least 2 and hence, in particular, that the infima and suprema above are taken over non-empty sets. Let us first note the obvious fact that $\omega(\phi)=0$ if and only if ϕ is uniformly continuous. On the other hand, $\omega(\phi)<\infty$ if and only if ϕ is *Lipschitz for large distances*, that is,

$$\big\|\phi(x)-\phi(y)\big\| \leqslant K\|x-y\|+K$$

for some constant K and all $x,y\in X$. Similarly, assumptions on $\kappa(\phi)$ correspond to known conditions on the map ϕ. We summarise these as follows.

(1) $\omega(\phi)=0$, that is, ϕ is uniformly continuous,
(2) $\omega(\phi)<\infty$, that is, ϕ is Lipschitz for large distances,
(3) $\kappa(\phi)=\infty$, that is, ϕ is *expanding*,
(4) $\kappa(\phi)>0$, that is, ϕ is *uncollapsed*.

Note that the three coefficients above are all positive homogenous, in the sense that

$$\kappa(\lambda\phi)=\lambda\cdot\kappa(\phi)$$

for all $\lambda>0$ and similarly for $\overline{\kappa}(\phi)$ and $\omega(\phi)$. In particular, this means that the following quantity is invariant under rescaling ϕ.

Definition 2. *The* separation ratio *of a map $X\xrightarrow{\phi}E$ is the quantity*

$$\mathscr{R}(\phi)=\frac{\kappa(\phi)}{\omega(\phi)},$$

where we set $\frac{a}{\infty}=\frac{0}{a}=0$ for all $a\in[0,\infty]$ and $\frac{a}{0}=\frac{\infty}{a}=\frac{\infty}{0}=\infty$ for all $0<a<\infty$.

Whereas ϕ being a uniform embedding cannot be directly expressed via the coefficients above, we note that ϕ is a *coarse embedding* provided that $\omega(\phi)<\infty$ and $\kappa(\phi)=\infty$, that is, if ϕ is Lipschitz for large distances and is expanding. We thus see that

$$\mathscr{R}(\phi)=\infty$$

if and only if ϕ is either uniformly continuous and uncollapsed (e.g., a uniform embedding) or if ϕ is a coarse embedding.

Motivated in part by the still open problem of deciding whether a Banach space X coarsely embeds into a Banach space E if and only it uniformly embeds, the papers [3, 4, 8–11] contain various constructions for producing uniform and coarse embeddings or obstructions to the same. In particular, in [10] (see Theorem 1.16 therein) we showed that, provided that $E\oplus E$ isomorphically embeds into E, then a uniformly continuous and uncollapsed map $X\xrightarrow{\phi}E$ gives rise to a simultaneously uniform and coarse embedding of X into E. However, as shown by A. Naor [8], there are Lipschitz for large distance maps that are not even close to any uniformly continuous map. For the exclusive purpose of coarse embeddability, our main result, Theorem 3, removes the problematic assumption of uniform continuity of ϕ.

Theorem 3. *Suppose X and E are Banach spaces so that $E\oplus E$ isomorphically embeds into E. Then X coarsely embeds into E if and only if*

$$\sup_{\phi}\mathscr{R}(\phi)=\infty,$$

where the supremum is taken over all maps $X\xrightarrow{\phi}E$.

The proof of Theorem 3 also allows us to address another issue, namely, the preservation of cotype under different forms of embeddability. For this, consider the following conditions on a map $X\xrightarrow{\phi}E$.

(5) $\overline{\kappa}(\phi)=\infty$, that is, ϕ is *almost expanding*,

(6) $\overline{\kappa}(\phi) > 0$, that is, ϕ is *almost uncollapsed*.

Also, the map ϕ is said to be *solvent* provided that there are constants $R_1, R_2, \ldots$ so that

$$R_n \leqslant \|x-y\| \leqslant R_n + n \quad \Rightarrow \quad \big\|\phi(x)-\phi(y)\big\| \geqslant n.$$

Provided that ϕ is Lipschitz for large distances, ϕ is solvent if and only if it is almost expanding (see [9, Lemma 8]). In analogy with Definition 2, we then define the *exact separation ratio* of ϕ to be

$$\overline{\mathscr{R}}(\phi) = \frac{\overline{\kappa}(\phi)}{\omega(\phi)}.$$

As $\kappa(\phi) \leqslant \overline{\kappa}(\phi)$, we then have $\mathscr{R}(\phi) \leqslant \overline{\mathscr{R}}(\phi)$. Also, $\overline{\mathscr{R}}(\phi) = \infty$ exactly when ϕ is either uniformly continuous and almost uncollapsed or is Lipschitz for large distances and solvent.

In connection with this, B. Braga [4] strengthened work by M. Mendel and A. Naor [7] to show that, if X maps into a Banach space E with non-trivial type by a map that is either uniformly continuous and almost uncollapsed or is Lipschitz for large distances and solvent, then

$$\mathrm{cotype}(X) \leqslant \mathrm{cotype}(E).$$

The following statement therefore covers both cases of Braga's result and seemingly provides the ultimate extension in this direction.

Theorem 4. *Suppose X and E are Banach spaces so that*

$$\sup_{\phi} \overline{\mathscr{R}}(\phi) = \infty$$

and that E has non-trivial type. Then

$$\mathrm{cotype}(X) \leqslant \mathrm{cotype}(E).$$

Problem 7.4 in Braga's paper [4] asks what can be deduced about a space X that admits a map $X \xrightarrow{\phi} E$ that is just Lipschitz for large distances and almost uncollapsed, i.e. so that $\overline{\mathscr{R}}(\phi) > 0$. That is, will restrictions on the geometry of E also lead to information about the geometry of X? In Example 10, we show that this is not always so. Indeed, if X is separable and E is infinite-dimensional, one can always find a map $X \xrightarrow{\phi} E$ that is both Lipschitz for large distances and uncollapsed, i.e., so that $\mathscr{R}(\phi) > 0$, and after renorming E one can even obtain $\mathscr{R}(\phi) \geqslant 1$. On the other hand, Theorem 4 provides a positive answer to Braga's question under the alternative assumption $\sup_\phi \overline{\mathscr{R}}(\phi) = \infty$.

2. Proofs

Before proving our main results, let us introduce four functional moduli that lie behind the definitions of the (exact) compression and expansion coefficients.

Definition 5 (Compression moduli). *For a (generally discontinuous and nonlinear) map $X \xrightarrow{\phi} E$ between two Banach spaces we define the* exact compression modulus $\overline{\kappa}_\phi \colon [0,\infty[\to [0,\infty[$

$$\overline{\kappa}_\phi(t) = \inf\big\{\|\phi(x)-\phi(y)\| \;\big|\; \|x-y\| = t\big\}$$

and the compression modulus *by* $\overline{\kappa}_\phi \colon [0,\infty[\to [0,\infty[$ *by*

$$\kappa_\phi(t) = \inf\big\{\|\phi(x)-\phi(y)\| \;\big|\; \|x-y\| \geqslant t\big\}.$$

Thus, $\overline{\kappa}_\phi$ is the pointwise largest map so that $\overline{\kappa}_\phi\big(\|x-x\|\big) \leqslant \big\|\phi(x)-\phi(y)\big\|$ for all $x, y \in X$, while $\kappa_\phi(t) = \inf_{r \geqslant t} \overline{\kappa}_\phi(r)$ is the pointwise largest monotone map satisfying the same inequality.

Definition 6 (Expansion moduli). *For a map $X \xrightarrow{\phi} E$ between Banach spaces, the* exact expansion modulus $\overline{\omega}_\phi\colon [0,\infty[\to[0,\infty]$ *is defined by*

$$\overline{\omega}_\phi(t)=\sup\big\{\|\phi(x)-\phi(y)\|\bigm| \|x-y\|=t\big\},$$

and the expansion modulus $\omega_\phi\colon [0,\infty[\to[0,\infty]$ *by*

$$\omega_\phi(t)=\sup\big\{\|\phi(x)-\phi(y)\|\bigm| \|x-y\|\leqslant t\big\}.$$

The following are evident.

$$\kappa_\phi(t)\leqslant\overline{\kappa}_\phi(t)\leqslant\overline{\omega}_\phi(t)\leqslant\omega_\phi(t).$$

We recall that, to avoid trivialities, all Banach spaces are assumed to have dimension at least 2. Thus, suppose $X \xrightarrow{\phi} E$ is a map and that $t>0$ and $x,y\in X$. Let $n\geqslant 1$ be minimal so that $\|x-y\|\leqslant nt$, whereby $(n-1)t\leqslant\|x-y\|$ and pick $z_0=x,z_1,z_2,\dots,z_n=y$ so that $\|z_{i-1}-z_i\|=t$ for $i=1,\dots,n$. Then

$$\big\|\phi(x)-\phi(y)\big\|\ \leqslant\ \sum_{i=1}^{n}\big\|\phi(z_{i-1})-\phi(z_i)\big\|\ \leqslant\ n\cdot\overline{\omega}_\phi(t)\leqslant\frac{\overline{\omega}_\phi(t)}{t}\|x-y\|+\overline{\omega}_\phi(t).$$

In turn, this shows that

$$\omega_\phi(s)\leqslant\frac{\overline{\omega}_\phi(t)}{t}s+\overline{\omega}_\phi(t)$$

for all $s,t>0$ and so $\limsup_{s\to 0_+}\omega_\phi(s)\leqslant\inf_{t>0}\overline{\omega}_\phi(t)$. Because ω_ϕ is non-decreasing, the limit $\lim_{s\to 0_+}\omega_\phi(s)=\inf_{s>0}\omega_\phi(s)$ exists, whereby

$$\inf_{t>0}\overline{\omega}_\phi(t)\leqslant\liminf_{t\to 0_+}\overline{\omega}_\phi(t)\leqslant\limsup_{t\to 0_+}\overline{\omega}_\phi(t)\leqslant\lim_{t\to 0_+}\omega_\phi(t)\leqslant\inf_{t>0}\overline{\omega}_\phi(t).$$

All in all, we find that

$$\omega(\phi)=\inf_{t>0}\omega_\phi(t)=\lim_{t\to 0_+}\omega_\phi(t)=\lim_{t\to 0_+}\overline{\omega}_\phi(t)=\inf_{t>0}\overline{\omega}_\phi(t).$$

In particular, we would obtain nothing new by introducing an *exact expansion coefficient* by $\overline{\omega}(\phi)=\inf_{t>0}\overline{\omega}_\phi(t)$, since this is just the expansion coefficient itself. Furthermore, if $\omega(\phi)<\infty$, then ϕ is Lipschitz for large distances, that is,

$$\big\|\phi(x)-\phi(y)\big\|\leqslant K\|x-y\|+K$$

for some constant K and all $x,y\in X$.

Next, the definition of the separation ratio may initially be difficult to parse, so let us briefly restate it more explicitly.

Lemma 7. *For a map $X \xrightarrow{\phi} E$ and a constant $K>0$, we have*

$$\mathscr{R}(\phi)>K$$

if and only if there are constants $\Delta,\delta,\Lambda,\lambda>0$ so that

$$\|x-y\|\geqslant\Delta\ \Rightarrow\ \big\|\phi(x)-\phi(y)\big\|\geqslant\delta,$$
$$\|x-y\|\leqslant\Lambda\ \Rightarrow\ \big\|\phi(x)-\phi(y)\big\|\leqslant\lambda$$

and $\frac{\delta}{\lambda}>K$.

Proof. Note that, if $\mathscr{R}(\phi)>K$, we may find $\Delta,\Lambda>0$ so that $\frac{\kappa_\phi(\Delta)}{\omega_\phi(\Lambda)}>K$. Letting $\delta=\kappa_\phi(\Delta)$ and $\lambda=\omega_\phi(\Lambda)$, the two implications follow.

Conversely, if the two implications hold for some $\Delta,\delta,\Lambda,\lambda>0$ so that $\frac{\delta}{\lambda}>K$, then

$$\mathscr{R}(\phi)=\frac{\sup_{t<\infty}\kappa_\phi(t)}{\inf_{t>0}\omega_\phi(t)}\geqslant\frac{\kappa_\phi(\Delta)}{\omega_\phi(\Lambda)}\geqslant\frac{\delta}{\lambda}>K,$$

which verifies the lemma. □

Proof of Theorem 3. As noted, if $X \xrightarrow{\phi} E$ is a coarse embedding between arbitrary Banach spaces, then $\mathscr{R}(\phi) = \infty$, which proves one direction of implication. Also, under the stated assumption on E, by Theorem 1.16 in [10], we have that X coarsely embeds into E if and only if $\mathscr{R}(\phi) = \infty$ for some map $X \xrightarrow{\phi} E$. So suppose instead only that $\sup_\phi \mathscr{R}(\phi) = \infty$. We then construct a coarse embedding $X \xrightarrow{\psi} E$ as follows.

Because $E \oplus E$ embeds isomorphically into E, we may inductively construct two sequences E_n, Z_n of closed linear subspaces of E all isomorphic to E so that

$$E_{n+1} \oplus Z_{n+1} \subseteq Z_n.$$

Concretely, we simply begin with an isomorphic copy $E \oplus E$ inside of E and let E_1 and Z_1 be respectively the first and second summand. Again, pick an isomorphic copy of $E \oplus E$ inside of Z_1 with first and second summand denoted respectively E_2 and Z_2, etc. It thus follows that

$$E \supseteq E_1 \oplus Z_1 \supseteq E_1 \oplus E_2 \oplus Z_2 \supseteq E_1 \oplus E_2 \oplus E_3 \oplus Z_3 \supseteq \ldots$$

is a decreasing sequence of closed linear subspaces of E. Let

$$V_n = E_1 \oplus E_2 \oplus \cdots \oplus E_n \oplus Z_n$$

and set $V = \bigcap_{n=1}^\infty V_n$. We note that V is a closed linear subspace of E in which each E_n is a closed subspace complemented by a bounded projection $V \xrightarrow{P_n} E_n$ so that $E_m \subseteq \ker P_n$ whenever $n \neq m$. On the other hand, we have no uniform bound on the norms $\|P_n\|$.

Fix now a sequence of isomorphisms $E \xrightarrow{T_n} E_n$ and find maps $X \xrightarrow{\theta_n} E$ with $\mathscr{R}(\theta_n) > n2^n\|P_n\|\|T_n\|\|T_n^{-1}\|$. Observe that, for all $t > 0$,

$$\kappa_{T_n\circ\theta_n}(t) \geqslant \frac{\kappa_{\theta_n}(t)}{\|T_n^{-1}\|},$$

whereas

$$\omega_{T_n\circ\theta_n}(t) \leqslant \|T_n\| \cdot \omega_{\theta_n}(t),$$

which shows that

$$\mathscr{R}(T_n \circ \theta_n) \geqslant \frac{\mathscr{R}(\theta_n)}{\|T_n\|\|T_n^{-1}\|} \geqslant n2^n\|P_n\|.$$

Setting $\phi_n = T_n \circ \theta_n$, we find that $\lim_n \frac{\mathscr{R}(\phi_n)}{2^n\|P_n\|} = \infty$. The conclusion of the theorem therefore follows directly from Lemma 8 below. $\square$

Lemma 8. *Suppose X and E are Banach spaces and $E \xrightarrow{P_n} E$ is a sequence of bounded linear projections onto subspaces $E_n \subseteq E$ so that $E_m \subseteq \ker P_n$ for all $m \neq n$. Assume also that there is a sequence of maps*

$$X \xrightarrow{\phi_n} E_n$$

so that

$$\lim_n \frac{\mathscr{R}(\phi_n)}{2^n\|P_n\|} = \infty.$$

Then X coarsely embeds into E.

Proof. By composing with a translation, we may suppose that $\phi_n(0) = 0$ for each n. Because $\lim_n \frac{\mathscr{R}(\phi_n)}{2^n\|P_n\|} = \infty$, we may also find constants $\Delta_n, \delta_n, \Lambda_n, \lambda_n > 0$ so that

$$\|x - y\| \geqslant \Delta_n \Rightarrow \left\|\phi_n(x) - \phi_n(y)\right\| \geqslant \delta_n$$

and

$$\|x - y\| \leqslant \Lambda_n \Rightarrow \left\|\phi_n(x) - \phi_n(y)\right\| \leqslant \lambda_n,$$

while

$$\lim_n \frac{\delta_n}{\lambda_n 2^n\|P_n\|} = \infty.$$

For every n, we let

$$\psi_n(x) = \frac{1}{\lambda_n 2^n} \cdot \phi_n\Big(\tfrac{\Lambda_n}{n} \cdot x\Big).$$

Then

$$\begin{aligned} \|x-y\| \leqslant n \;&\Rightarrow\; \big\|\tfrac{\Lambda_n}{n}\cdot x - \tfrac{\Lambda_n}{n}\cdot y\big\| \leqslant \Lambda_n \\ &\Rightarrow\; \Big\|\phi_n\big(\tfrac{\Lambda_n}{n}\cdot x\big) - \phi_n\big(\tfrac{\Lambda_n}{n}\cdot y\big)\Big\| \leqslant \lambda_n \\ &\Rightarrow\; \|\psi_n(x)-\psi_n(y)\| \leqslant 2^{-n}. \end{aligned} \tag{1}$$

Similarly,

$$\begin{aligned} \|x-y\| \geqslant \frac{n\Delta_n}{\Lambda_n} \;&\Rightarrow\; \big\|\tfrac{\Lambda_n}{n}\cdot x - \tfrac{\Lambda_n}{n}\cdot y\big\| \geqslant \Delta_n \\ &\Rightarrow\; \Big\|\phi_n\big(\tfrac{\Lambda_n}{n}\cdot x\big) - \phi_n\big(\tfrac{\Lambda_n}{n}\cdot y\big)\Big\| \geqslant \delta_n \\ &\Rightarrow\; \big\|\psi_n(x)-\psi_n(y)\big\| \geqslant \frac{\delta_n}{\lambda_n 2^n}. \end{aligned} \tag{2}$$

In particular, if $\|x-y\| \leqslant m$, then $\|x-y\| \leqslant n$ for all $n \geqslant m$, whereby

$$\sum_{n=1}^{\infty} \big\|\psi_n(x)-\psi_n(y)\big\| \leqslant \sum_{n=1}^{m-1} \big\|\psi_n(x)-\psi_n(y)\big\| + \sum_{n=m}^{\infty} 2^{-n} < \infty.$$

Also, $\psi_n(0) = 0$ for all n, which shows that, for all $x \in X$,

$$\sum_{n=1}^{\infty} \big\|\psi_n(x)\big\| < \infty$$

and so the series $\sum_{n=1}^{\infty} \psi_n(x)$ is absolutely convergent in E. We may therefore define a map $X \xrightarrow{\psi} E$ by letting

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n(x).$$

We now verify that ψ is a coarse embedding of X into E. First, let $m \geqslant 1$ be any given natural number and suppose that $x, y \in X$ satisfy $\|x-y\| \leqslant m$. Then we may find $z_0 = x, z_1, z_2, \dots, z_m = y$ so that $\|z_{i-1} - z_i\| \leqslant 1$ for all i and so, in particular, $\|\psi_n(z_{i-1}) - \psi_n(z_i)\| \leqslant 2^{-n}$ for all n. It thus follows that

$$\begin{aligned} \big\|\psi(x)-\psi(y)\big\| &= \Big\|\sum_{n=1}^{\infty}\psi_n(x) - \sum_{n=1}^{\infty}\psi_n(y)\Big\| \\ &\leqslant \sum_{n=1}^{m-1}\big\|\psi_n(x)-\psi_n(y)\big\| + \sum_{n=m}^{\infty}\big\|\psi_n(x)-\psi_n(y)\big\| \\ &\overset{(1)}{\leqslant} \sum_{n=1}^{m-1}\big\|\psi_n(z_0)-\psi_n(z_m)\big\| + \sum_{n=m}^{\infty} 2^{-n} \\ &\leqslant \sum_{n=1}^{m-1}\Big\|\sum_{i=1}^{m}\big(\psi_n(z_{i-1})-\psi_n(z_i)\big)\Big\| + 2^{-m+1} \\ &\leqslant \sum_{n=1}^{m-1}\sum_{i=1}^{m}\big\|\psi_n(z_{i-1})-\psi_n(z_i)\big\| + 2^{-m+1} \\ &\leqslant \sum_{n=1}^{m-1}\sum_{i=1}^{m} 2^{-n} + 2^{-m+1} \\ &= \sum_{n=1}^{m-1} m2^{-n} + 2^{-m+1} \\ &< m + 2^{-m+1}. \end{aligned}$$

In other words, for all m and $x, y \in X$, we have

$$\|x-y\| \leqslant m \Rightarrow \left\|\psi(x)-\psi(y)\right\| < m+2^{-m+1}.$$

Conversely, if m is any given number, find n large enough so that $\frac{\delta_n}{\lambda_n 2^n \|P_n\|} \geqslant m$. Then, if $\|x-y\| \geqslant \frac{n\Delta_n}{\Lambda_n}$, we have

$$\begin{aligned}
\|\psi(x)-\psi(y)\| &\geqslant \frac{1}{\|P_n\|}\left\|P_n\psi(x)-P_n\psi(y)\right\| \\
&= \frac{1}{\|P_n\|}\left\|\psi_n(x)-\psi_n(y)\right\| \\
&\overset{(2)}{\geqslant} \frac{\delta_n}{\lambda_n 2^n \|P_n\|} \\
&\geqslant m.
\end{aligned}$$

Taken together, these two conditions show that ψ is a coarse embedding. □

The gluing presented in Lemma 8 may be contrasted with the so-called barycentric gluing technique discussed in detail in [1]. In our gluing above, the purpose is to improve the metric qualities of maps $X \to E$, whereas the barycentric gluing allows one to paste together a sequence of maps $nB_X \to E$ defined on larger and larger balls of X, but where, on the other hand, the metric qualities of the maps are not improved. It is not clear whether the two techniques may be combined.

Proof of Theorem 4. Suppose X and E are Banach spaces so that

$$\sup_{\phi} \overline{\mathscr{R}}(\phi) = \infty,$$

and E have non-trivial type, i.e., $\mathrm{type}(E) > 1$. We then note that also $\mathrm{type}\big(\ell_2(E)\big) = \mathrm{type}(E) > 1$ and $\mathrm{cotype}\big(\ell_2(E)\big) = \mathrm{cotype}(E)$. Thus, if we can show that X maps into $\ell_2(E)$ by a map that is Lipschitz for large distances and solvent, then, by the previously mentioned result of Braga ([4, Theorem 1.3]), we will have that

$$\mathrm{cotype}(X) \leqslant \mathrm{cotype}\big(\ell_2(E)\big) = \mathrm{cotype}(E).$$

So fix a sequence of maps $X \xrightarrow{\phi_n} E$ so that $\overline{\mathscr{R}}(\phi_n) > n2^n$ for all $n \geqslant 1$. This means that there are $\Delta_n, \delta_n, \Lambda_n, \lambda_n > 0$ so that

$$\|x-y\| = \Delta_n \Rightarrow \left\|\phi_n(x)-\phi_n(y)\right\| \geqslant \delta_n$$

and

$$\|x-y\| \leqslant \Lambda_n \Rightarrow \left\|\phi_n(x)-\phi_n(y)\right\| \leqslant \lambda_n$$

and $\frac{\delta_n}{\lambda_n} > n2^n$. We then define ψ_n by $\psi_n(x) = \frac{1}{2^n\lambda_n}\phi_n\big(\frac{\Lambda_n}{n}x\big)$ and note that, as in (1) and (2),

$$\|x-y\| \leqslant n \Rightarrow \left\|\psi_n(x)-\psi_n(y)\right\| \leqslant 2^{-n},$$

whereas

$$\|x-y\| = \tfrac{n\Delta_n}{\Lambda_n} \Rightarrow \left\|\psi_n(x)-\psi_n(y)\right\| \geqslant n.$$

We finally define $X \xrightarrow{\psi} \ell_2(E)$ by $\psi(x) = \big(\psi_1(x), \psi_2(x), \dots\big)$ and note that ψ is well-defined by the above and satisfies $\omega(\psi) \leqslant \omega_\psi(1) \leqslant 1$ and $\overline{\kappa}(\psi) \geqslant \overline{\kappa}_\psi\big(\frac{n\Delta_n}{\Lambda_n}\big) \geqslant n$ for all n. So ψ is Lipschitz for large distances and almost expanding, whence, by [9, Lemma 8], ψ is Lipschitz for large distances and solvent. □

Another way to prove Theorem 4 is first to establish an analogue to Theorem 3 for the quantity $\sup_\phi \overline{\mathscr{R}}(\phi)$ in place of $\sup_\phi \mathscr{R}(\phi)$. This is done by observing that the proof of Theorem 3 above can be changed to prove the following statement.

Theorem 9. *Suppose X and E are Banach spaces so that $E \oplus E$ isomorphically embeds into E. Assume also that*

$$\sup_{\phi} \overline{\mathscr{R}}(\phi) = \infty,$$

then there is a map $X \xrightarrow{\phi} E$ that is Lipschitz for large distances and solvent.

In order to obtain Theorem 4, one then notes that $\ell_2(E) \oplus \ell_2(E) \cong \ell_2(E)$ and so, if $\sup_\phi \overline{\mathscr{R}}(\phi) = \infty$, where the supremum is taken over all maps $X \xrightarrow{\phi} E$, we have a map $X \xrightarrow{\psi} \ell_2(E)$ that is both Lipschitz for large distances and solvent.

3. Examples

For every pair of Banach spaces X and E, we define the *coarse embeddability ratio* of X in E to be the numerical invariant

$$\mathscr{CR}(X,E) = \sup\{\mathscr{R}(\phi) \mid \phi \colon X \to E \text{ is a map}\}.$$

This is simply the quantity appearing in Theorem 3, which therefore states that, under very mild assumptions on E, we have $\mathscr{CR}(X,E) = \infty$ if and only if X coarsely embeds into E. As the next example shows, the main interest lies in the case when $\mathscr{CR}(X,E) > 1$, whereas $\mathscr{CR}(X,E) = 1$ is easily obtained.

Example 10. If X is separable and E is a Banach space that admits an infinite *equilateral* set, that is, an infinite subset $A \subseteq E$ so that, for some $\delta > 0$,

$$\|x - y\| = \delta$$

for all distinct $x, y \in A$, then we have $\mathscr{CR}(X,E) \geqslant 1$. To see this, let $(Y_x)_{x\in A}$ be a partition of X indexed by the set A into subsets $Y_x \subseteq X$ of diameter at most 1 and let $X \xrightarrow{\phi} E$ be defined by

$$\phi(y) = x \Leftrightarrow x \in A \;\&\; y \in Y_x.$$

Observe that, if $\|y - y'\| > 1$, then y and y' must belong to different pieces of the partition and so $\|\phi(y) - \phi(y')\| = \delta$. On the other hand, $\|\phi(y) - \phi(y')\| \leqslant \delta$ for all $y, y' \in X$, so we see that $\kappa_\phi(t) = \delta$ for all $t > 1$, whereas $\omega_\phi(t) \leqslant \delta$ for all $t > 0$. So $\mathscr{R}(\phi) \geqslant 1$.

In particular, this reasoning applies when E is one of the classical Banach spaces ℓ_p, c_0, L_p or even the Tsirelson space T. Indeed, in these spaces, the standard unit basis $(e_n)_{n=1}^\infty$ is an infinite equilateral set (or, in the case of Tsirelson's space, $(e_n)_{n=2}^\infty$ is equilateral). Here we remark that T^* is the reflexive space originally constructed and described by B. S. Tsirelson [12], while T is its ℓ_1-asymptotic dual whose explicit construction was given by T. Figiel and W. B. Johnson [6].

Let us also observe that, if E is infinite-dimensional, then E admits an equivalent renorming with respect to which it has an infinite equilateral set. Indeed, since E is infinite-dimensional, it contains a normalised basic sequence $(e_n)_{n=1}^\infty$. We define a new equivalent norm $|||\cdot|||$ on the closed linear space $[e_n]_{n=1}^\infty$ by letting

$$\left|\left|\left|\sum_{n=1}^\infty a_n e_n\right|\right|\right| = \sup\left\{\left\|\sum_{n\in I} a_n e_n\right\| + \left\|\sum_{n\in J} a_n e_n\right\| \;\middle|\; I, J \text{ are intervals and } i < j \text{ for all } i \in I \text{ and } j \in J\right\}.$$

As $\|e_n\| = 1$ for all n, we find that $|||e_i - e_j||| = 2$ for all $i < j$ and so $(e_n)_{n=1}^\infty$ is an equilateral set of the norm $|||\cdot|||$. It now suffices to notice that $|||\cdot|||$ extends to an equivalent norm on all of E.

Example 10 illustrates that the embeddability ratio $\mathscr{CR}(X,E)$ is sensitive to the specific norm on E, but not to the choice of norm on X. On the other hand, the condition $\mathscr{CR}(X,E) = \infty$ only depends on the isomorphism class of E. Note also that, if X, Y and Z are Banach spaces so that $\mathscr{CR}(X,Y)=\infty$, then

$$\mathscr{CR}(X,Z)\geqslant\mathscr{CR}(Y,Z).$$

An important non-embeddability result was recently established by F. Baudier, G. Lancien and T. Schlumprecht [2], who showed that the separable Hilbert space ℓ_2 does not coarsely embed into Tsirelson's space T^*. It is known that T^* is minimal, that is, T^* embeds isomorphically into all of its infinite-dimensional subspaces (see [5, Chapter VI]). Also, T^* has an unconditional basis and can therefore be written as a direct sum of two infinite-dimensional subspaces. It therefore follows that $T^*\oplus T^*$ embeds isomorphically into T^* and thus $E=T^*$ satisfies the assumption of Theorem 3. It follows that the coarse embeddability ratio $\mathscr{CR}(\ell_2,T^*)$ is finite and we now proceed to give an upper bound.

Proposition 11. *If T^* denotes Tsirelson's space, then*

$$\mathscr{CR}(\ell_2,T^*)\leqslant 4.$$

Proof. We rely on the analysis of [2], which also contains additional details about the construction below. For the proof, assume towards a contradiction that $\ell_2\xrightarrow{\phi}E$ satisfies $\mathscr{R}(\phi)>4$. Then by pre and post-composing ϕ with dilations we can suppose that, for some constants $\Delta>0$ and $\delta>4$, we have

$$\|x-y\|\geqslant\Delta\ \Rightarrow\ \big\|\phi(x)-\phi(y)\big\|\geqslant\delta$$

and

$$\|x-y\|\leqslant\sqrt{2}\ \Rightarrow\ \big\|\phi(x)-\phi(y)\big\|\leqslant 1.$$

Let $(e_n)_{n=1}^\infty$ be the standard unit vector basis for ℓ_2 and set $\epsilon=\frac{\delta-4}{2}$. Let also k be large enough so that $\sqrt{2k}\geqslant\Delta$ and let $[\mathbb{N}]^k$ be the collection of all k-element subsets of $\mathbb{N}$ equipped with the *Johnson metric*,

$$d_J(A,B)=\frac{|A\triangle B|}{2}.$$

Observe that d_J is simply the shortest-path metric on the graph whose vertices is $[\mathbb{N}]^k$ and where two vertices A and B are connected by an edge provided that $|A\triangle B|=2$. Let then $f\colon[\mathbb{N}]^k\to T^*$ be defined by

$$f(A)=\phi\Big(\sum_{n\in A}e_n\Big).$$

Observe that, if $d_J(A,B)=1$, then

$$\Bigg\|\sum_{n\in A}e_n-\sum_{n\in B}e_n\Bigg\|=\sqrt{|A\triangle B|}=\sqrt{2}$$

and so $\|f(A)-f(B)\|\leqslant 1$. Thus, f is Lipschitz with constant 1.

By Proposition 4.1 of [2] there is an infinite subset $\mathbb{M}\subseteq\mathbb{N}$ and some $y\in T^*$ so that, for any $A\in[\mathbb{N}]^k$ with $A\subseteq\mathbb{M}$, there are vectors $y_1^A,\dots,y_k^A\in T^*$ with $\|y_i^A\|\leqslant 1$ so that $y,y_1^A,\dots,y_k^A$ form a finite block basis of the standard unit vector basis for T^*, $k\leqslant\min\operatorname{supp}(y_1^A)$ and

$$\big\|f(A)-(y+y_1^A+\cdots+y_k^A)\big\|<\epsilon.$$

In particular, for all $A,B\in[\mathbb{N}]^k$, $A,B\subseteq\mathbb{M}$, we have that

$$\begin{aligned}\|f(A)-f(B)\|&<\big\|y_1^A+\cdots+y_k^A\big\|+\big\|y_1^B+\cdots+y_k^B\big\|+2\epsilon\\&\leqslant 2+2+2\epsilon\\&\leqslant\delta,\end{aligned}$$

where the second bound follows from (2.13) in [2]. On the other hand, for any two disjoint $A, B \in [\mathbb{N}]^k$, we have

$$\left\| \sum_{n \in A} e_n - \sum_{n \in B} e_n \right\| = \sqrt{2k} \geqslant \Delta,$$

which implies that $\| f(A) - f(B) \| \geqslant \delta$ and thus contradicts the preceding upper bound. □

The following still unsolved problem provides the main theoretical motivation for our investigations here.

Problem 12. *Suppose X and E are Banach spaces. Is it true that X coarsely embeds into E if and only if it uniformly embeds?*

Problem 13. *Suppose X and E are Banach spaces so that $\mathscr{CR}(X, E) > 1$. Does it follow that $\mathscr{CR}(X, E) = \infty$?*

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