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On the boundedness of a family of oscillatory singular integrals

Hussain Al-Qassem *,a, Leslie Cheng b and Yibiao Pan c

a Mathematics Program, Department of Mathematics, Statistics and Physics, College of Arts and Sciences, Qatar University, 2713, Doha, Qatar
b Department of Mathematics, Bryn Mawr College, Bryn Mawr, PA 19010, U.S.A.
c Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, U.S.A.
E-mails: husseink@qu.edu.qa, lcheng@brynmawr.edu, yibiao@pitt.edu

Abstract. Let \( \Omega \in H^1(\mathbb{S}^{n-1}) \) with mean value zero, \( P \) and \( Q \) be polynomials in \( n \) variables with real coefficients and \( Q(0) = 0 \). We prove that
\[
\left| \operatorname{p.v.} \int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))} \frac{\Omega(x|x|)}{|x|^n} \, dx \right| \leq A \| \Omega \|_{H^1(\mathbb{S}^{n-1})}
\]
where \( A \) may depend on \( n \), \( \deg(P) \) and \( \deg(Q) \), but not otherwise on the coefficients of \( P \) and \( Q \).

The above result answers an open question posed in [13]. Additional boundedness results of similar nature are also obtained.

Keywords. oscillatory integrals, singular integrals, Calderón–Zygmund kernels, Hardy spaces.

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1. Introduction

The study of oscillatory singular integrals has a long-standing history ([1, 4, 7–9, 11, 12]). For the specific topic considered in this paper, we shall begin with a well-known result of Stein and Wainger in [12] and its extension by Stein in [10].

Let \( n \geq 2 \), \( K(x) \) be a Calderón–Zygmund kernel given by
\[
K(x) = \frac{\Omega(x|x|)}{|x|^n}
\]
where \( \Omega : \mathbb{S}^{n-1} \to \mathbb{C} \) is integrable over the unit sphere \( \mathbb{S}^{n-1} \) with respect to the induced Lebesgue measure \( \sigma \) and satisfies
\[
\int_{\mathbb{S}^{n-1}} \Omega(x) \, d\sigma(x) = 0.
\]
For $d \in \mathbb{N}$, let $\mathcal{P}_{n,d}$ denote the space of real-valued polynomials in $n$ variables whose degrees do not exceed $d$. It was proved in [10] that, if $\Omega \in L^\infty(\mathbb{S}^{n-1})$ and $P \in \mathcal{P}_{n,d}$, then

$$\left| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)} K(x) \, dx \right| \leq C_{n,d} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}$$

where $C_{n,d}$ is independent of the coefficients of $P$.

In the recent paper [13] the authors obtained an extension of the above result in which the phase functions belong to a certain class of rational functions while $\Omega$ is allowed to be in a block space $B^0_0(\mathbb{S}^{n-1})$. Their result can be described as follows.

**Theorem 1 ([13]).** Let $q > 1$ and $K(x)$ be a Calderón–Zygmund kernel given by (1)–(2). Let $P(x), Q(x) \in \mathcal{P}_{n,d}$ such that $Q(0) = 0$ and $\Omega \in B^0_0(\mathbb{S}^{n-1})$. Then

$$\left| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)/Q(x)} K(x) \, dx \right| \leq A$$

where $A$ may depend on $\|\Omega\|_{B^0_0(\mathbb{S}^{n-1})}$, $n$ and $d$ but not otherwise on the coefficients of $P$ and $Q$.

The definition of $B^0_0(\mathbb{S}^{n-1})$ for $v > -1$ and $q > 1$ can be found in [13]. It had been known that the bound (3) also holds for all $\Omega \in L^\infty(\mathbb{S}^{n-1})$, which was proved by Folch-Gabayet and Wright in [5].

Let $H^1(\mathbb{S}^{n-1})$ denote the Hardy space over the unit sphere. An important question, posed by the authors of [13], is whether the bound (3) continues to hold under the condition $\Omega \in H^1(\mathbb{S}^{n-1})$ (with the same phase functions $P(x) + 1/Q(x)$). This is a very natural question because both $B^0_0(\mathbb{S}^{n-1})$ and $L^\infty(\mathbb{S}^{n-1})$ are proper subspaces of $H^1(\mathbb{S}^{n-1})$.

Our first result answers the above question in the affirmative.

**Theorem 2.** Let $K(x)$ be a Calderón–Zygmund kernel given by (1)–(2). Let $P(x), Q(x) \in \mathcal{P}_{n,d}$ such that $Q(0) = 0$. Suppose that $\Omega \in H^1(\mathbb{S}^{n-1})$. Then

$$\left| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)/Q(x)} K(x) \, dx \right| \leq A\|\Omega\|_{H^1(\mathbb{S}^{n-1})}$$

where $A$ may depend on $n$ and $d$ but not otherwise on the coefficients of $P$ and $Q$.

As usual, Theorem 2 implies the uniform boundedness of oscillatory singular integral operators of the following type on $L^2(\mathbb{R}^m)$:

$$f \rightarrow \text{p.v.} \int_{\mathbb{R}^n} f(u_1 - P_1(y), \ldots, u_m - P_m(y)) e^{iQ(y)} |y|^{-n} \Omega(\frac{y}{|y|}) \, dy,$$

where $P_1, \ldots, P_m, Q$ are polynomials and $\Omega$ is a function in $H^1(\mathbb{S}^{n-1})$ with a zero mean-value. The proof of Theorem 2 will be given in Section 2.

The general question about whether the condition $Q(0) = 0$ can be removed is open. But for $\deg(Q) \leq 1$, this is known to be the case.

**Theorem 3 ([15, 13]).** Let $K(x)$ be a Calderón–Zygmund kernel given by (1)–(2). Let $P(x) \in \mathcal{P}_{n,d}$ and $Q(x) = a + v \cdot x$ where $a \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Suppose that $\Omega \in L^\infty(\mathbb{S}^{n-1})$ or $\Omega \in B^0_0(\mathbb{S}^{n-1})$ for some $q > 1$. Then

$$\left| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)/Q(x)} K(x) \, dx \right| \leq A$$

where $A$ may depend on $n$, $d$ and the respective norm of $\Omega$, but not otherwise on $a$, $v$ and the coefficients of $P$.

We have the following extension of Theorem 3:
The proof of Theorem 4 will be given in Section 3.

The following is an important estimate due to E. M. Stein:

**Theorem 5.** Let $\Omega \in L\log L(\mathbb{S}^{n-1})$ and $d \in \mathbb{N}$. For every homogeneous polynomial of degree $d$ on $\mathbb{R}^n$ $P(x) = \sum_{|\alpha| = d} a_\alpha x^\alpha$, let $m_P = \sum_{|\alpha| = d} |a_\alpha|$. Then there exists a constant $C_{n,d,\Omega} > 0$ which is independent of $\{a_\alpha\}$ such that

$$\int_{\mathbb{S}^{n-1}} |\Omega(x)| \left| \log \left( \frac{|P(x)|}{m_P} \right) \right| \, d\sigma(x) \leq C_{n,d,\Omega}$$

(7)

holds whenever $m_P \neq 0$.

What happens if $P(x)$ is a general polynomial instead of a homogeneous polynomial? For $P(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$, the direct analogue of (7), where $m_P$ is replaced by $\sum_{|\alpha| \leq d} |a_\alpha|$, is clearly false. This is due to the fact that, unlike $P \to m_P$ for homogeneous polynomials of a fixed degree,

$$\sum_{|\alpha| \leq d} a_\alpha x^\alpha \to \sum_{|\alpha| \leq d} |a_\alpha|$$

is not a norm on $\mathcal{P}_{n,d}|_{\mathbb{S}^{n-1}}$. To remedy this situation, we can simply replace the above with any norm on $\mathcal{P}_{n,d}|_{\mathbb{S}^{n-1}}$ (e.g. $\|\cdot\|_{\infty}$) to arrive at the following extension of Theorem 5:

**Theorem 6.** Let $\|\cdot\|$ be a norm on $\mathcal{P}_{n,d}|_{\mathbb{S}^{n-1}}$. Then there exists a positive constant $C$ which depends on $n, d$ and $\|\cdot\|$ only such that

$$\int_{\mathbb{S}^{n-1}} |\Omega(y)| \left| \log \left( \frac{|P(y)|}{\|P\|_{\mathbb{S}^{n-1}}} \right) \right| \, d\sigma(y) \leq C(1 + \||\Omega|\|_{L\log L(\mathbb{S}^{n-1})})$$

(8)

holds for all $\Omega \in L\log L(\mathbb{S}^{n-1})$ and all $P \in \mathcal{P}_{n,d}$ not vanishing identically over $\mathbb{S}^{n-1}$.

Since any two norms on a finite dimensional space are equivalent, one recovers (7) when applying (8) to homogeneous polynomials.

More broadly, results such as Theorem 6 can be framed in terms of functions of finite type and compactness, as is done in the theorem below.

**Theorem 7.** Let $M$ be a compact smooth submanifold of $\mathbb{R}^n$, $\sigma = \sigma_M$ be the induced Lebesgue measure on $M$ and $U$ be an open subset of $\mathbb{R}^m$. Let $f \in C^\infty(M \times U)$ such that, for every $u \in U$, $f(\cdot, u)$ does not vanish to infinite order at any point in $M$. Then, for every compact subset $W$ of $U$, there exists a positive constant $C = C(M, n, m, f, W)$ such that

$$\sup_{u \in W} \int_M |\Omega(y)| \left| \log |f(y, u)| \right| \, d\sigma(y) \leq C(1 + \||\Omega|\|_{L\log L(M)})$$

(9)

holds for all $\Omega \in L\log L(M)$.

The proof of Theorem 7 will be based on Malgrange preparation theorem. It will be given in Section 4 where one will also see how Theorem 6 follows as a simple consequence. As an application of Theorem 7, we have the following:
Then we write \( x \). After a rotation if necessary, we may also assume that 

\[
\Omega = \{ \langle x, y \rangle | y = x + \delta \} = \{ \langle x, y \rangle | y - \delta x \leq 1 \},
\]

for some \( \delta \). Under the assumption that \( \delta \) and \( \Omega \) enjoy the following additional properties:

(a) \( \text{supp}(\Omega) \subseteq \mathbb{R}^{n-1} \cap B(0, \delta) \) for some \( \delta > 0 \) where \( B(0, \delta) = \{ y \in \mathbb{R}^n : |y - \delta x| < \delta \} \); and 

(b) \( \| \Omega \|_{\infty} \leq \delta^{-n+1} \).

If \( \delta \geq 1/4 \), (12) follows from (b) and Theorem 1. Thus, we may assume that \( 0 < \delta < 1/4 \). By using a rotation if necessary, we may also assume that \( \Omega = (0, \ldots, 0, 1) \). For any \( x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n \), we write \( x = (\bar{x}, x_n) \) where \( \bar{x} = (x_1, \ldots, x_{n-1}) \). We also extend the definition of \( \Omega(\cdot) \) from \( \mathbb{R}^{n-1} \) to \( \mathbb{R}^n \setminus \{0\} \) by using \( \Omega(x) = \Omega(x/|x|) \). We define \( \Omega_\delta : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{C} \) by

\[
\Omega_\delta(x) = (\delta^{-n+1}|x|^n) \frac{\Omega(\delta \bar{x}, x_n)}{|(\delta \bar{x}, x_n)|^n}.
\]

Then \( \Omega_\delta(\cdot) \) is homogeneous of degree 0. It is well-known that, by the theory of Calderón–Zygmund operators, (2) implies that

\[
\int_{\mathbb{R}^{n-1}} \Omega_\delta(y) d\sigma(y) = 0.
\]

Next we will show that \( \| \Omega_\delta \|_{\infty} \lesssim 1 \). To see this, we assume that \( \Omega_\delta(x) \neq 0 \) for some \( x \in \mathbb{R}^n \setminus \{0\} \). Then

\[
\eta := \frac{|(\delta \bar{x}, x_n)|}{|\delta \bar{x}, x_n|} - \xi_0 < \delta.
\]

By (b) and \( x_n = |(\delta \bar{x}, x_n)|(1 - \eta^2/2) \),

\[
|\Omega_\delta(x)| \leq \left( \frac{|x|}{|(\delta \bar{x}, x_n)|} \right)^n = (\delta |(\delta \bar{x}, x_n)|)^{-n} \left( |(\delta \bar{x}, x_n)|^2 + (\delta^2 - 1)x_n^2 \right)^{n/2} = \delta^{-n} \left( \frac{1 + (\delta^2 - 1)(1 - \eta^2/2)^2}{\eta^2(1 - \eta^2/4)} \right)^{n/2} \lesssim 1.
\]
Let $P_δ(x) = P(δx, x_n)$ and $Q_δ(x) = Q(δx, x_n)$. Then $P_δ(·), Q_δ(·) \in \mathcal{D}_{n,d}$, $\deg(P_δ) = \deg(P)$, $\deg(Q_δ) = \deg(Q)$ and $Q_δ(0) = Q(0) = 0$. It follows from Theorem 1 that
\[
\left| \text{p.v.} \int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))} \frac{\Omega(x)}{|x|^n} \, dx \right| = \left| \text{p.v.} \int_{\mathbb{R}^n} e^{i(P_δ(x)+1/Q_δ(x))} \frac{\Omega(δx, x_n)}{|(δx, x_n)|^n} δ^{n-1} \, dx \right|
\]
\[
= \left| \text{p.v.} \int_{\mathbb{R}^n} e^{i(P_δ(x)+1/Q_δ(x))} \frac{\Omega_δ(x)}{|x|^n} \, dx \right| ≤ A
\]
where $A$ depends on $n$ and $d$ only. This proves Theorem 2.

3. Proof of Theorem 4

First let us recall the following version of van der Corput’s lemma.

**Lemma 9.**

(i) Let $φ$ be a real-valued $C^k$ function on $[a, b]$ satisfying $|φ^{(k)}(x)| ≥ 1$ for every $x \in [a, b]$. Suppose that $k ≥ 2$, or that $k = 1$ and $φ'$ is monotone on $[a, b]$. Then there exists a positive constant $c_k$ such that
\[
\left| \int_a^b e^{iλφ(x)} \, dx \right| ≤ c_k |λ|^{-1/k}
\]
for all $λ \in \mathbb{R}$. The constant $c_k$ is independent of $λ, a, b$ and $φ$.

(ii) Let $φ$ and $c_k$ be the same as in (i). If $ψ ∈ C^1([a, b])$, then
\[
\left| \int_a^b e^{iλφ(x)} ψ(x) \, dx \right| ≤ c_k |λ|^{-1/k} \left( \|ψ\|_{L^∞([a, b])} + \|ψ'\|_{L^1([a, b])} \right)
\]
holds for all $λ \in \mathbb{R}$.

We will now give the proof of Theorem 4. Since the case $a = 0$ is already covered by Theorem 5, we shall assume that $a ≠ 0$. Initially we will assume that $Ω ∈ L^∞(\mathbb{S}^{n-1})$.

For $ω ∈ \mathbb{S}^{n-1}$, let
\[
θ = θ(ω) = \left| \frac{a}{h(ω)} \right|^{1/l}.
\]
Then
\[
\text{p.v.} \int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))} K(x) \, dx = \int_{\mathbb{S}^{n-1}} Ω(ω) I(ω) \, dσ(ω)
\]
where
\[
I(ω) = \int_0^∞ e^{i(P(ωr)+1/(a+h(ω)r^l))} \frac{dr}{r} = I_1(ω) + I_2(ω) + I_3(ω)
\]
where
\[
I_1(ω) = \int_0^{αθ} e^{i(P(ωr)+1/(a+h(ω)r^l))} \frac{dr}{r}, \quad I_2(ω) = \int_{αθ}^{βθ} e^{i(P(ωr)+1/(a+h(ω)r^l))} \frac{dr}{r}, \quad I_3(ω) = \int_{βθ}^∞ e^{i(P(ωr)+1/(a+h(ω)r^l))} \frac{dr}{r}
\]
and
\[
I_j(ω) = \int_{\mathbb{S}^{n-1}} Ω(ω) I_j(ω) \, dσ(ω) = O(1)
\]
for $j = 1$ and $j = 3$. 

The estimate (14) for \( j = 3 \) follows from a slight modification of the proof of Theorem 1 in [5]. Details will be omitted. Below we shall show how to obtain (14) for \( j = 1 \) with an appropriate selection of \( \alpha \).

Let
\[
\phi_\omega(r) = P(r\omega) + \frac{1}{a + h(\omega)r^l}.
\]

In order to apply van der Corput's lemma, we shall need to obtain appropriate lower bounds for at least one of the derivatives of \( \phi_\omega(\cdot) \) near 0. When \( l = 1 \), this can be done with any derivative of \( \phi_\omega(\cdot) \) whose order exceeds the degree of \( P(\cdot) \). When \( l > 1 \), one needs to be more selective as demonstrated below.

Let \( g(t) = (1 + t^l)^{-1} \). Then, for \( k = 0, 1, 2, \ldots \),
\[
\left| \frac{d^k g(0)}{dt^k} \right| = \begin{cases} 
  s! & \text{if } l \mid s \\
  0 & \text{if } l \nmid s.
\end{cases}
\]

Let \( k_0 \in \mathbb{N} \) such that \( k_0 l > \max(\deg(P), 1) \). By \( |g^{(k_0 l)}(0)| = (k_0 l)! \geq 1 \), there exists an \( \alpha \in (0, 1) \) such that \( |g^{(k_0 l)}(0)| \geq 1/2 \) for \( |t| \leq \alpha \). By
\[
\phi_\omega(r) = P(r\omega) + a^{-1}g(r/\theta),
\]
we have
\[
|\phi_\omega^{(k_0 l)}(r)| = |a|^{k_0 l - 1} |g^{(k_0 l)}(r/\theta)|
\geq (2|a|^{k_0 l})^{-1}
\]
for \( r \in (0, a\theta) \).

Let \( b = \min(||a||, 1) \). If \( |a| \geq 1 \), then
\[
\int_{[b^{1/(k_0 l)}a\theta, a\theta]} e^{i\phi_\omega(r)} \frac{dr}{r} = 0.
\]

If \( |a| < 1 \), then \( b = |a| \) and by Lemma 9,
\[
\left| \int_{[b^{1/(k_0 l)}a\theta, a\theta]} e^{i\phi_\omega(r)} \frac{dr}{r} \right| \lesssim \frac{1}{(2|a|^{k_0 l})^{1/(k_0 l)}} \cdot \frac{1}{|a|^{1/(k_0 l)}a\theta} \lesssim 1.
\]

Thus, we always have
\[
\left| \int_{[b^{1/(k_0 l)}a\theta, a\theta]} e^{i\phi_\omega(r)} \frac{dr}{r} \right| \lesssim 1. \tag{15}
\]

Therefore, it suffices to show that
\[
\int_{S^{n-1}} \Omega(\omega) \left( \int_0^{a\theta b^{1/(k_0 l)}} e^{i\phi_\omega(r)} \frac{dr}{r} \right) d\sigma(\omega) = O(1). \tag{16}
\]

Let
\[
q(x) = \left( \frac{1}{a} \right)^{k_0 - 1} \sum_{j=0}^{k_0 - 1} \left( -\frac{h(x)}{a} \right)^j.
\]

For any \( \omega \in S^{n-1} \) and \( 0 < r \leq a\theta b^{1/(k_0 l)} \), by \( 0 \leq b \leq 1 \),
\[
\left| \frac{h(\omega)}{a} r^l \right| \leq b^{1/k_0} a' \left( \frac{h(\omega)}{a} r^l \right) \leq a' < 1,
\]
which implies that
\[
|\phi_\omega(r) - (P(r\omega) + q(r\omega))| \lesssim |a|^{-1} \left| \frac{h(\omega)}{a} r^l \right|^{|k_0|} \lesssim |a|^{-1} \left| \frac{h(\omega)}{a} r^l \right|^{|k_0|}.
\]
Thus,
\[
\left| \int_0^{a_b b_{1/(b_1)}} \left( e^{i\phi_\omega(r)} - e^{i(P(r) + q(r))} \right) \frac{dr}{r} \right| \lesssim |a|^{-1} \vartheta^{-k_0} \int_0^{a_b b_{1/(b_1)}} r^{k_0 - 1} dr \lesssim a^{k_0} |a|^{-1} b \lesssim 1,
\]
which immediately gives
\[
\int_{\mathbb{R}^{n-1}} \Omega(\omega) \int_0^{a_b b_{1/(b_1)}} \left( e^{i\phi_\omega(r)} - e^{i(P(r) + q(r))} \right) \frac{dr}{r} d\sigma(\omega) = O(1). \tag{17}
\]

By an inequality on page 334 of [10],
\[
\left| \text{p.v.} \int_{|x|<\alpha} e^{i(P(x) + q(x))} K(x) dx \right| \leq A,
\]
i.e.
\[
\int_{\mathbb{R}^{n-1}} \Omega(\omega) \int_0^{a_b b_{1/(b_1)}} e^{i(P(r) + q(r))} \frac{dr}{r} d\sigma(\omega) = O(1). \tag{18}
\]

Trivially, we have
\[
\left| \int_{\mathbb{R}^{n-1}} \Omega(\omega) \int_0^{a_b b_{1/(b_1)}} e^{i(P(r) + q(r))} \frac{dr}{r} \right| \lesssim |\ln \left( \frac{|h(\omega)|}{m_h} \right)|.
\]
It follows from Theorem 5 that
\[
\int_{\mathbb{R}^{n-1}} \Omega(\omega) \int_0^{a_b b_{1/(b_1)}} e^{i(P(r) + q(r))} \frac{dr}{r} d\sigma(\omega) = O(1). \tag{19}
\]

By (17)–(19), we obtain (16). This proves (6) for $\Omega \in L^\infty(\mathbb{S}^{n-1})$. By applying the argument used in the proof of Theorem 2, one then obtains (6) for $\Omega \in H^1(\mathbb{S}^{n-1})$. Details are omitted.

4. Nonvanishing of infinite order

Let $M$ be a smooth $k$-dimensional submanifold of $\mathbb{R}^n$, $f : M \to \mathbb{R}$ be a $C^\infty$ function and $p \in M$. We say that $f$ does not vanish to infinite order at $p$ if there is a chart $(U_p, \varphi)$ around $p$ such that $\varphi(p) = 0$ and $D^\alpha (f \circ \varphi^{-1})(0) \neq 0$ for some $\alpha \in (\mathbb{N} \cup \{0\})^k$. For $r > 0$, let $B_k(r)$ denote the open ball in $\mathbb{R}^k$ which is centered at the origin and has radius $r$. We begin with the following:

**Lemma 10.** Let $k, m \in \mathbb{N}$, $x \in \mathbb{R}^k$, $y \in \mathbb{R}^m$ and $R > 0$. Let $g(x, y) \in C^\infty(B_k(R) \times B_m(R))$ such that
\[
\frac{\partial^\alpha g(0, 0)}{\partial x^\alpha} \neq 0 \tag{20}
\]
holds for some $\alpha \in (\mathbb{N} \cup \{0\})^k$. Then there exists an $r \in (0, R/3)$ such that, for every $\delta \in \{0, (\max(|\alpha|, 1))^{-1}\}$ and every $C^\infty$ function $h : \mathbb{R}^k \to \mathbb{R}$ which is supported in $B_k(r)$,
\[
\sup_{y \in B_m(r)} \int_{B_k(r)} |g(x, y)|^{-\delta} |h(x)| dx < \infty. \tag{21}
\]

**Proof.** If (20) holds with $|\alpha| = 0$, i.e. $g(0, 0) \neq 0$, then (21) follows trivially by continuity. Now suppose that $g(0, 0) = 0$ and let
\[
l = \min \left\{ |\alpha| : \alpha \in (\mathbb{N} \cup \{0\})^k \text{ and } \frac{\partial^\alpha g(0, 0)}{\partial x^\alpha} \neq 0 \right\}. \tag{22}
\]

By an argument on p. 317 of [10], we may assume that
\[
\frac{\partial^l g(0, 0)}{\partial x^l} \neq 0.
\]
By (22) we also have
\[
\frac{\partial^l g(0, 0)}{\partial x^l} = 0.
\]
for \( j = 0, 1, \ldots, l - 1 \). Let \( \bar{x} = (x_1, \ldots, x_{k-1}) \). By Malgrange preparation theorem ([6]), there exist \( r \in (0, R/3), \eta_0 > 0, a_0(\bar{x}, y), \ldots, a_{l-1}(\bar{x}, y) \in C^\infty(B_{k-1}(r) \times B_m(r)) \) and \( c(x, y) \in C^\infty(B_k(r) \times B_m(r)) \) such that, for all \((x, y) \in B_k(r) \times B_m(r), \)

\[
g(x, y) = c(x, y)(x_k^l + a_{l-1}(\bar{x}, y)x_k^{l-1} + \cdots + a_0(\bar{x}, y))
\]

and \( |c(x, y)| \geq \eta_0 \). Thus, for any \( \delta \in (0, 1/l) \) and any \( C^\infty \) function \( h(x) \) supported on \( B_k(r) \),

\[
\sup_{y \in B_m(r)} \int_{B_k(r)} |g(x, y)|^{-\delta} |h(x)| \, dx \lesssim \sup_{y \in B_m(r)} \int_{B_k(r)} \int_{|x| < r} (x_k^l + a_{l-1}(\bar{x}, y)x_k^{l-1} + \cdots + a_0(\bar{x}, y))^{-\delta} \, dx \, dy < \infty. \tag{23}
\]

**Proof of Theorem 7.** Let \( M \) be a compact smooth submanifold of \( \mathbb{R}^n \) and \( U \) be an open subset of \( \mathbb{R}^m \). Let \( f \in C^\infty(M \times U) \) such that, for every \( u \in U, f(\cdot, u) \) does not vanish to infinite order at any point in \( M \). Suppose that \( W \) is a compact subset of \( U \). By Lemma 10 and the compactness of \( M \) and \( W \), there exist \( \delta = \delta_{f,W} > 0 \) and \( C = C(M, n, m, f, W) \) such that

\[
\frac{\sup_{u \in W} \int_{M} |f(y, u)|^{-\delta} \, d\sigma(y)}{\sup_{y \in M} |\log(|f(y, u)|)| \, d\sigma(y)} \lesssim 1.
\]

On the other hand, we have trivially that

\[
\int_{y \in M : |\Omega(y)| \leq |f(y, u)|^{-\delta/2}} |\Omega(y)| \, d\sigma(y) \lesssim \|\Omega\|_{L^\infty(M)}.
\]

Thus (9) holds and the proof of Theorem 7 is now complete. \( \square \)

**Proof of Theorem 6.** Let \( m = \dim(P_{n,d}|_{\mathbb{S}^{n-1}}) \) and \( p_1(x), \ldots, p_m(x) \in P_{n,d}|_{\mathbb{S}^{n-1}} \) such that \( \{p_j|_{\mathbb{S}^{n-1}} : 1 \leq j \leq m\} \) forms a basis for \( P_{n,d}|_{\mathbb{S}^{n-1}} \). Define \( f : \mathbb{S}^{n-1} \times (\mathbb{R}^m \setminus \{0\}) \to \mathbb{R} \) by

\[
f(x, u) = \sum_{j=1}^{m} u_j p_j(x)
\]

for \( x \in \mathbb{S}^{n-1} \) and \( u = (u_1, \ldots, u_m) \in \mathbb{R}^m \setminus \{0\} \). For each \( u \in \mathbb{R}^m \setminus \{0\}, f(\cdot, u) \) is not identically zero on \( \mathbb{S}^{n-1} \) which, by real-analyticity, implies that it does not vanish to infinite order at any point in \( \mathbb{S}^{n-1} \). By Theorem 7,

\[
\sup_{u \in \mathbb{S}^{m-1}} \int_{\mathbb{S}^{n-1}} |\Omega(x)| \, d\sigma(x) \lesssim C(1 + \|\Omega\|_{L^\infty(M, \mathbb{S}^{n-1})}), \tag{24}
\]

which implies that (8) holds when the norm is given by

\[
\sum_{j=1}^{m} u_j p_j \bigg|_{\mathbb{S}^{n-1}} \to \left( \sum_{j=1}^{m} u_j^2 \right)^{1/2}.
\]

Since any two norms on \( P_{n,d}|_{\mathbb{S}^{n-1}} \) are equivalent, Theorem 6 is proved. \( \square \)

**Proof of Theorem 8.** By assumption and Theorem 7,

\[
\sup_{u \in W} \int_{\mathbb{S}^{n-1}} |\Omega(\omega)| \, d\sigma(\omega) \leq C(1 + \|\Omega\|_{L^\infty(M, \mathbb{S}^{n-1})}). \tag{25}
\]

One can then adopt the arguments in the proof of Theorem 1 in [5], at times using (25) instead of (7), to finish the proof. Details are omitted. \( \square \)
References


