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Some remarks on the companions conjecture for normal varieties

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Abstract. Drinfeld in 2010 proved the companions conjecture for smooth varieties over a finite field, generalizing L. Lafforgue's result for smooth curves. We study the obstruction to prove the conjecture for arbitrary normal varieties. To do this, we introduce a new property of morphisms. We verify this property in some cases, showing thereby the companions conjecture for some singular normal varieties.

Keywords. ℓ-adic representation, independence of ℓ, étale fundamental group.

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1. Introduction

1.1. *The companions conjecture*

Let $\mathbb{F}_q$ be a finite field of characteristic p and let X_0 be a connected normal variety over $\mathbb{F}_q$. For a prime ℓ different from p, the category of Weil lisse $\overline{\mathbb{Q}}_\ell$-sheaves over X_0 carries much information on the arithmetic and the geometry of X_0. An invariant that is associated to any Weil lisse $\overline{\mathbb{Q}}_\ell$-sheaf is the trace field $E \subseteq \overline{\mathbb{Q}}_\ell$ generated by the coefficients of the Frobenius polynomials at closed points. Deligne proved in [3] that if $\mathcal{V}_0$ is an irreducible Weil lisse $\overline{\mathbb{Q}}_\ell$-sheaf over X_0 with finite order determinant, then E is a finite extension of $\mathbb{Q}$. He showed this finiteness by reduction to the case of curves, where it was proven by L. Lafforgue in [6, Thm. VII.6] as a consequence of the Langlands correspondence. This property of the trace field was one of the conjectures proposed by Deligne in [2, Conj. 1.2.10]. In the same list, he also formulated the following conjecture.

Conjecture 1 (Companions conjecture). *After possibly replacing E with a finite field extension, for every finite place λ not dividing p, there exists a Weil lisse E_λ-sheaf E-compatible[1] with $\mathcal{V}_0$.*

When the dimension of X_0 is 1, the conjecture is again a consequence of the Langlands correspondence. For higher dimensional varieties, Drinfeld in [4] proved Conjecture 1 when X_0 is smooth. As noticed in [4, §6], his method cannot be applied directly to prove the full conjecture.

[1] Cf. [1, Def. 3.1.15].

1.2. *The obstruction*

Suppose for simplicity that the singular locus of X_0 consists of one closed point and that we can solve that singularity. In other words, suppose that there exists a smooth variety Y_0 and a proper morphism $h_0 : Y_0 \to X_0$ which sends a closed subscheme $i_0 : Z_0 \hookrightarrow Y_0$ to a closed point $x_0 \in |X_0|$ and such that h_0 is an isomorphism outside Z_0. Write $\mathbb{F}$ for an algebraic closure of $\mathbb{F}_q$ and suppose that $Z := Z_0 \otimes_{\mathbb{F}_q} \mathbb{F}$ is connected.

Lemma 2 ([5, Cor. IX.6.11]). *For every geometric point z of Z_0 there exists an exact sequence*

$$\pi_1^{\text{ét}}(Z,z) \xrightarrow{i_*} \pi_1^{\text{ét}}(Y_0,z) \xrightarrow{h_{0*}} \pi_1^{\text{ét}}(X_0,h(z)) \to 1$$

in the sense that the smallest normal closed subgroup containing the image of i_* is equal to the kernel of h_{0*}.

By the lemma, every étale Weil lisse sheaf $\mathcal{V}_0$ over Y_0 which is trivial over Z is the inverse image of an étale Weil lisse sheaf defined over X_0. Since we know the companions conjecture for Y_0, in order to deduce it for X_0 we have to verify the following property.

$\mathscr{P}(Z_0)$: For every pair $(\mathcal{V}_0, \mathcal{W}_0)$ of compatible absolutely irreducible Weil lisse sheaves with finite order determinant over Y_0, the sheaf $\mathcal{V}_0$ is trivial over Z if and only if the same is true for $\mathcal{W}_0$.

If $Z_0 \subseteq Y_0$ satisfies $\mathscr{P}(Z_0)$ we say that Z_0 is a *λ-uniform subvariety*. Thanks to Lemma 2 and the companions conjecture for smooth varieties, we have the following result.

Proposition 3. *If $Z_0 \subseteq Y_0$ is λ-uniform, X_0 satisfies the companions conjecture.*

1.3. *Main results*

The aim of this text is to shed some new lights on the companions conjecture for normal varieties. For this scope, we focus on λ-uniformity. We extend the notion of λ-uniform subvarieties in Section 1.2 to the one of *λ-uniform morphisms* of varieties (Definition 9) and we investigate the following conjecture.

Conjecture 4 (Conjecture 10). *Let Y_0 and Z_0 be varieties over $\mathbb{F}_q$. If Y_0 is normal, every morphism $f_0 : Z_0 \to Y_0$ is λ-uniform (cf. Definition 9).*

We shall verify Conjecture 4 in some particular cases.

Theorem 5 (Theorem 14, Theorem 16). *Let $f_0 : Z_0 \to Y_0$ be a morphism of geometrically connected varieties over $\mathbb{F}_q$ with Y_0 normal and let z be a geometric point of Z_0. The morphism f_0 is λ-uniform in the following cases.*

(i) *If Z_0 has a simply connected dual complex and normal irreducible components.*
(ii) *If the smallest closed normal subgroup of $\pi_1^{\text{ét}}(Y_0, f(z))$ containing the image of $\pi_1^{\text{ét}}(Z,z)$ is open inside $\pi_1^{\text{ét}}(Y, f(z))$.*
(iii) *If $\pi_1^{\text{ét}}(Y, f(z))$ contains an open solvable profinite subgroup.*

Combining the previous results we get the following consequence.

Corollary 6. *Let Y_0 be a smooth geometrically connected variety over $\mathbb{F}_q$. If X_0 is a normal variety that can be written as a contraction of a geometrically connected subvariety $Z_0 \subseteq Y_0$ satisfying one of the conditions of Theorem 5, then X_0 verifies Conjecture 1.*

An independent property we prove in this text is a property of invariance of λ-uniformity "under deformations" (Theorem 12). This might be useful for further developments in the direction of Conjecture 4. Moreover, we present in Section 3.3 some variants of Conjecture 4 and we present a concrete example, proposed by de Jong, where these conjectures are not known.

1.4. *Notation and conventions*

For us, a *variety* over a field k is a separated scheme of finite type over k. We write $X_0, Y_0, Z_0, \dots$ for varieties over $\mathbb{F}_q$ and $X, Y, Z, \dots$ for the base change to $\mathbb{F}$. Further, we put a subscript $_0$ to indicate objects and morphisms defined over $\mathbb{F}_q$ and the suppression of this subscript shall mean that we are extending the scalars to $\mathbb{F}$. If E is a number field we write $|E|_{\neq p}$ for the set of finite places of E which do not divide p. For every $\lambda \in |E|_{\neq p}$, we denote by E_λ the completion of E with respect to λ.

We use the notation for Weil lisse sheaves as in [1, §2.2]. We say that a Weil lisse E_λ-sheaf $\mathcal{V}_0$ is *split untwisted* if every irreducible subquotient of $\mathcal{V}_0$ is absolutely irreducible and has finite order determinant. We say instead that $\mathcal{V}_0$ is *untwisted* if it is split untwisted after possibly extending E_λ. Recall that if Y_0 is a normal variety over $\mathbb{F}_q$, every untwisted Weil lisse sheaf over Y_0 is pure of weight 0 and geometrically semi-simple by [2, Thm. 3.4.1] and [3, Thm. 1.6]. In addition, by [2, Prop. 1.3.14] and [1, Prop. 3.1.16], every untwisted Weil lisse sheaf is étale.

An E-*compatible system* over X_0, denoted by $\underline{\mathcal{V}_0}$, is a family $\{\mathcal{V}_{\lambda,0}\}_{\lambda\in|E|_{\neq p}}$ where each $\mathcal{V}_{\lambda,0}$ is an E-rational Weil lisse E_λ-sheaf and such that all sheaves are pairwise E-compatible. Each $\mathcal{V}_{\lambda,0}$ is called the λ-*component* of $\underline{\mathcal{V}_0}$. We say that a compatible system is *semi-simple, untwisted,* or *split untwisted* if each λ-component has the respective property.

2. λ-uniform morphisms

2.1. *General properties*

Definition 7. *Let Z_0 be a connected variety. A compatible system $\underline{\mathcal{V}_0}$ over Z_0 is λ-*uniform *if one of the following disjoint conditions is verified.*

(i) *For every $\lambda \nmid p$, the lisse sheaf $\mathcal{V}_{\lambda,0}$ is geometrically trivial.*
(ii) *For every $\lambda \nmid p$, the lisse sheaf $\mathcal{V}_{\lambda,0}$ is geometrically non-trivial.*

We say that $\underline{\mathcal{V}_0}$ is strongly λ-uniform *if the dimension of $H^0(Z, \mathcal{V}_\lambda)$ does not depend on λ. Strongly λ-uniform compatible systems are clearly λ-uniform. If Z_0 is not connected we say that a compatible system is* λ-uniform *(resp.* strongly λ-uniform*) if it is λ-uniform (resp. strongly λ-uniform) over every connected component.*

Proposition 8. *Let Z_0 be a normal variety over $\mathbb{F}_q$. Every untwisted E-compatible system $\underline{\mathcal{V}_0}$ over Z_0 is strongly λ-uniform.*

Proof. After extending the base field, we may assume that Z_0 is geometrically connected. Since Z_0 is a normal variety, each λ-component of $\underline{\mathcal{V}_0}$ is pure of weight 0 by [3, Thm. 1.6]. Also, if U_0 is the smooth locus of Z_0, by [5, Prop. V.8.2], the étale fundamental group of U maps surjectively onto the étale fundamental group of Z. Therefore, we have a canonical isomorphism

$$H^0(Z, \mathcal{V}_\lambda) = H^0(U, \mathcal{V}_\lambda|_U)$$

for every λ. By [6, Cor. VI.3], we know that the dimension of $H^0(U, \mathcal{V}_\lambda|_U)$ can be recovered from the L-function of $\mathcal{V}_{\lambda,0}|_{U_0}$, thus we obtain the desired result. □

If we do not assume Z_0 normal, Proposition 8 becomes false in general (Example 15). The issue is a different behaviour of weights for non-normal varieties. In what follows, we want to understand whether a weaker variant of Proposition 8 is still true for singular varieties.

Definition 9. *Let $f_0 : Z_0 \to Y_0$ be a morphism of varieties over $\mathbb{F}_q$. We say that f_0 is a* λ-uniform morphism *if for every untwisted compatible system $\underline{\mathcal{V}_0}$ over Y_0, the pullback $f_0^* \underline{\mathcal{V}_0}$ is λ-uniform. If f_0 is a closed immersion we say that Z_0 is a* λ-uniform subvariety *of Y_0.*

Conjecture 10. *Let Y_0 and Z_0 be varieties over $\mathbb{F}_q$. If Y_0 is normal, every morphism $f_0 : Z_0 \to Y_0$ is λ-uniform.*

2.2. *Homotopic invariance*

Let us look more closely at λ-uniform morphisms by analysing the relation with the induced morphism on fundamental groups.

Let $f_0 : Z_0 \to Y_0$ be a morphism of geometrically connected varieties over $\mathbb{F}_q$ with Y_0 normal. If we choose a geometric point z of Z_0 we have a morphism

$$\pi_1^{\text{ét}}(Z,z) \xrightarrow{f_*} \pi_1^{\text{ét}}(Y_0, f(z)).$$

For every étale compatible system $\underline{\mathcal{V}_0}$ over Y_0 we denote by $\{\rho_{\lambda,0}\}_{\lambda \in |E|_{\neq p}}$ the associated family of ℓ-adic representations of $\pi_1^{\text{ét}}(Y_0, f(z))$. Let $\overline{\text{Im}(f_*)}$ be the smallest normal closed subgroup of $\pi_1^{\text{ét}}(Y_0, f(z))$ containing the image of f_*. The following lemma is a direct consequence of the definition of a λ-uniform morphism.

Lemma 11. *A morphism f_0 is λ-uniform if and only if for every untwisted compatible system $\underline{\mathcal{V}_0}$ over Y_0, if $\overline{\text{Im}(f_*)} \subseteq \text{Ker}(\rho_{\lambda,0})$ for one λ then the same is true for every other $\lambda \in |E|$. In particular, the property of a morphism of being λ-uniform depends only on the inclusion $\overline{\text{Im}(f_*)} \subseteq \pi_1^{\text{ét}}(Y_0, y)$ as topological groups together with the assignment of the conjugacy classes of the Frobenii at closed points of $\pi_1^{\text{ét}}(Y_0, y)$ and their degrees.*

As a consequence of the previous lemma, we prove an "homotopic invariance" of λ-uniformity. Let T_0 and S_0 be geometrically connected varieties over $\mathbb{F}_q$ and $h_0 : T_0 \to S_0$ a proper and flat morphism with connected and reduced geometric fibres. Let s_0 and s'_0 be closed points of S_0 and write $\iota_0 : Z_0 \hookrightarrow T_0$ and $\iota'_0 : Z'_0 \hookrightarrow T_0$ for the closed immersions of the fibres of h_0 above s_0 and s'_0 respectively.

Theorem 12. *For every morphism $\widetilde{f}_0 : T_0 \to Y_0$, the restriction $f_0 := \widetilde{f}_0|_{Z_0}$ is λ-uniform if and only if $f'_0 := \widetilde{f}_0|_{Z'_0}$ is λ-uniform.*

Proof. Let z and z' be geometric points of Z_0 and Z'_0 respectively. By [10, Tag 0C0J], we have exact sequences

$$\pi_1^{\text{ét}}(Z,z) \xrightarrow{\iota_*} \pi_1^{\text{ét}}(T,z) \xrightarrow{h_*} \pi_1^{\text{ét}}(S,h(z)) \to 1$$
$$\pi_1^{\text{ét}}(Z',z') \xrightarrow{\iota'_*} \pi_1^{\text{ét}}(T,z') \xrightarrow{h_*} \pi_1^{\text{ét}}(S,h(z')) \to 1.$$

The choice of an étale path γ joining z with z' induces isomorphisms $\gamma : \pi_1^{\text{ét}}(T,z) \xrightarrow{\sim} \pi_1^{\text{ét}}(T',z')$ and $h_*(\gamma) : \pi_1^{\text{ét}}(S,h(z)) \xrightarrow{\sim} \pi_1^{\text{ét}}(S',h(z'))$. Thanks to the two exact sequences this implies that γ restricts to an isomorphism $\overline{\text{Im}(\iota_*)} \xrightarrow{\sim} \overline{\text{Im}(\iota'_*)}$. In turn, this implies that the induced isomorphism $f_*(\gamma)$: $\pi_1^{\text{ét}}(Y_0, f(z)) \xrightarrow{\sim} \pi_1^{\text{ét}}(Y_0, f'(z'))$ restricts to an isomorphism $\overline{\text{Im}(f_*)} \xrightarrow{\sim} \overline{\text{Im}(f'_*)}$. By construction, $f_*(\gamma)$ respects the conjugacy classes of Frobenii at closed points and their degrees. We conclude applying Lemma 11. □

3. Some examples

In this section, we verify Conjecture 10 in some cases. Note that by virtue of Proposition 8 we already know the conjecture when Z_0 is normal.

3.1. *Simply connected dual complex*

Let Z_0 be a geometrically connected variety over $\mathbb{F}_q$. Write $Z^{(i)}$ where $1 \le i \le n$ for the irreducible components of Z. Suppose that for every i, the irreducible component $Z^{(i)}$ is normal. Let z be a geometric point of Z_0 and for every $1 \le i \le n$, let $z^{(i)}$ be a generic geometric point of $Z^{(i)}$. We denote by Γ the dual complex of Z and by P the point of Γ associated to the irreducible component where z lies. Write Π for the free product $\pi_1^{\text{ét}}(Z^{(1)}, z^{(1)}) * \cdots * \pi_1^{\text{ét}}(Z^{(n)}, z^{(n)})$ and $\pi_1(\Gamma, P)^\wedge$ for the profinite completion of the topological fundamental group of Γ at P.

Theorem 13 (Stix). *The choice of étale paths $\{\gamma^{(i)}\}_{1\le i\le n}$ joining z to $z^{(i)}$ for every i determines an exact sequence*

$$\Pi \xrightarrow{\alpha} \pi_1^{\text{ét}}(Z, z) \xrightarrow{\beta} \pi_1(\Gamma, P)^\wedge \to 1,$$

in the sense that the smallest normal closed subgroup containing the image of α is equal to the kernel of β.

Proof. The result follows from [11, Cor. 3.3]. □

Theorem 14. *If Γ is simply connected, every untwisted compatible system over Z_0 is λ-uniform.*

Proof. Let $\underline{\mathcal{V}_0}$ be an untwisted compatible system over Z_0. If $\mathcal{V}_{\lambda,0}$ is geometrically trivial for one λ, then it remains geometrically trivial when restricted to every irreducible component of Z_0. By Proposition 8, for every other $\lambda' \in |E|_{\neq p}$, the restriction of $\mathcal{V}_{\lambda',0}$ to every irreducible component of Z_0 is geometrically trivial as well. By Theorem 13, since Γ is simply connected, the geometric étale fundamental group of Z_0 is the smallest normal subgroup containing the images of the étale fundamental groups of the irreducible components $Z^{(i)}$. This shows that for every λ' the Weil lisse sheaf $\mathcal{V}_{\lambda',0}$ is geometrically trivial over Z_0 and this yields the desired result. □

Example 15. If Z_0 is an irreducible split nodal cubic curve over $\mathbb{F}_q$ with nodal point z_0, then $\pi_1^{\text{ét}}(Z, z)$ is isomorphic to $\widehat{\mathbb{Z}}$. In addition, the action of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ on $\pi_1^{\text{ét}}(Z, z)$ is trivial. Therefore, $\pi_1^{\text{ét}}(Z_0, z)$ is isomorphic to $\widehat{\mathbb{Z}} \times \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$, where the embedding $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q) \subseteq \pi_1^{\text{ét}}(Z_0, z)$ is induced by the closed immersion of $z_0 \hookrightarrow Z_0$. The Frobenius elements of $\pi_1^{\text{ét}}(Z_0, z)$ correspond via this isomorphism to elements of $\widehat{\mathbb{Z}} \times \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ of the form $(0, F^d)$, where F is the geometric Frobenius of $\mathbb{F}_q$ and d is some positive integer. This implies that every pair of étale lisse sheaves over Z_0 which are trivial at z_0 are $\mathbb{Q}$-compatible with all the eigenvalues at closed points equal to 1.

On the other hand, for every prime number $\ell \neq p$ and every continuous group automorphism α of $\overline{\mathbb{Z}}_\ell^{\oplus r}$, where $\overline{\mathbb{Z}}_\ell$ is the ring of integers of $\overline{\mathbb{Q}}_\ell$, there exists an étale lisse $\overline{\mathbb{Q}}_\ell$-sheaf of this type such that the induced ℓ-adic representation sends $(1, \mathrm{id})$ to α. In particular, we may take $r = 1$ and as α the multiplication by a root of unit. This construction produces lots of examples of untwisted compatible systems over Z_0 which are not λ-uniform.

3.2. *Finite monodromy*

As in Proposition 8, we may use the theory of weights in order to prove that certain morphisms are λ-uniform. This strategy needs strong finiteness conditions.

Theorem 16. *Let $f_0 : Z_0 \to Y_0$ be a morphism of geometrically connected varieties over $\mathbb{F}_q$ with Y_0 normal. Let z be a geometric point of Z_0. The morphism f_0 is λ-uniform in the following cases.*

(i) *If the smallest closed normal subgroup of $\pi_1^{\text{ét}}(Y_0, f(z))$ containing the image of $\pi_1^{\text{ét}}(Z, z)$ is an open subgroup of $\pi_1^{\text{ét}}(Y, f(z))$.*

(ii) *If $\pi_1^{\text{ét}}(Y, f(z))$ contains an open solvable profinite subgroup.*

Proof. Let $\underline{\mathcal{V}_0}$ be an untwisted compatible system over Y_0. For every $\lambda \in |E|_{\neq p}$, write G_λ for the geometric monodromy group of $\mathcal{V}_{\lambda,0}$.

(i). If $f_0^*(\mathcal{V}_{\lambda,0})$ is geometrically trivial over Z_0 for one λ, then ρ_λ is trivial when restricted to the image of $\pi_1^{\text{ét}}(Z,z)$ in $\pi_1^{\text{ét}}(Y,f(z))$. Thanks to the hypothesis, we deduce that ρ_λ factors through a finite quotient of $\pi_1^{\text{ét}}(Y,f(z))$. This implies that G_λ is a finite algebraic group. By [7, Prop. 2.2] and [1, Thm. 4.1.1], the subgroups

$$\text{Ker}(\rho_{\lambda'}) \cap \pi_1^{\text{ét}}(Y,f(z)) \subseteq \pi_1^{\text{ét}}(Y,f(z))$$

are all equal when λ' varies in $|E|_{\neq p}$. Since the image of $\pi_1^{\text{ét}}(Z,z)$ in $\pi_1^{\text{ét}}(Y,f(z))$ is contained in $\text{Ker}(\rho_\lambda)$, it is also contained in $\text{Ker}(\rho_{\lambda'})$ for every $\lambda' \in |E|_{\neq p}$. Therefore, the lisse sheaf $f_0^*(\mathcal{V}_{\lambda',0})$ is geometrically trivial for every $\lambda' \nmid p$, as we wanted.

(ii). Since $\pi_1^{\text{ét}}(Y,f(z))$ contains an open solvable profinite subgroup, we know that for every λ the neutral component $G_\lambda^\circ \subseteq G_\lambda$ is solvable. Combining this with the fact that each $\mathcal{V}_{\lambda,0}$ is geometrically semi-simple, we deduce that G_λ° is a torus. By [2, Thm. 1.3.8], this implies that G_λ° is trivial so that G_λ is finite and we can proceed as in the previous case. □

Corollary 17. *A dominant morphism $f_0 : Z_0 \to Y_0$ is λ-uniform. In particular, if Y_0 is a smooth curve, every morphism with target Y_0 is λ-uniform.*

Proof. Let $\eta \hookrightarrow Y$ be the generic point of Y and write Z_η the preimage of η via f. Since f is dominant, the scheme Z_η is a non-empty variety over the function field of Y. Fix a closed point η' of Z_η and choose a geometric point $\overline{\eta}$ over η'. We have the following commutative diagram

$$\begin{array}{ccc} \pi_1^{\text{ét}}(\eta',\overline{\eta}) & \hookrightarrow & \pi_1^{\text{ét}}(\eta,f(\overline{\eta})) \\ \downarrow & & \twoheaddownarrow \\ \pi_1^{\text{ét}}(Z,\overline{\eta}) & \longrightarrow & \pi_1^{\text{ét}}(Y,f(\overline{\eta})). \end{array}$$

Note that $\pi_1^{\text{ét}}(\eta',\overline{\eta})$ maps to a finite index subgroup of $\pi_1^{\text{ét}}(\eta,f(\overline{\eta}))$. Therefore, the image of $\pi_1^{\text{ét}}(Z,\overline{\eta})$ in $\pi_1^{\text{ét}}(Y,f(\overline{\eta}))$ has finite index as well. Thanks to this, we may apply Theorem 16 (i) to conclude. □

3.3. *Final comments*

For a Weil lisse sheaf, the property of being geometrically trivial can be thought as the combination of two properties: being geometrically unipotent[2] and geometrically semi-simple. For this reason, it seems pretty natural to split Conjecture 10 in two parts. Let $f_0 : Z_0 \to Y_0$ be a morphism of varieties over $\mathbb{F}_q$ with Y_0 normal.

Conjecture 18 (Unipotency). *For every pair $(\mathcal{V}_0, \mathcal{W}_0)$ of compatible Weil lisse sheaf over Y_0, if $f_0^*\mathcal{V}_0$ is geometrically unipotent the same is true for $f_0^*\mathcal{W}_0$.*

Conjecture 19 (Semi-simplicity). *For every geometrically semi-simple Weil lisse sheaf $\mathcal{V}_0$ over Y_0, the inverse image $f_0^*\mathcal{V}_0$ is geometrically semi-simple.*

These two conjectures have different natures. The first one is a numerical property of the traces of the elements of the geometric étale fundamental group. The second one can not be read looking at the traces and it would be a generalisation of [1, Cor. 3.6.7]. Let us focus on this second conjecture. Suppose that Z_0 is a semi-stable curve with all the irreducible components isomorphic to $\mathbb{P}^1_{\mathbb{F}_q}$. In light of [8, Thm. 1.1], one might be tempted to hope that for every Weil lisse sheaf over Y_0, the inverse image to Z_0 is geometrically finite, which would imply Conjecture 19 for this choice of Z_0. Unluckily, this does not seem to be true as pointed out by de Jong.

[2] A geometrically unipotent Weil lisse sheaf is a Weil lisse sheaf $\mathcal{V}_0$ such that $\mathcal{V}$ is a successive extension of trivial lisse sheaves.

Example 20 (de Jong). For an integer $n \geq 3$ such that $(n, p) = 1$, we write $Y^{(n)}$ for the moduli scheme of principally polarized abelian surfaces over $\mathbb{F}$ with a symplectic level-n-structure. Let $Z^{(n)} \subseteq Y^{(n)}$ be the supersingular locus of $Y^{(n)}$. Since we are working with abelian surfaces, this coincides with the Moret–Bailly locus of $Y^{(n)}$. Therefore, by [9, Prop. 7.3], the variety $Z^{(n)}$ is connected for every choice of n. Let N be a positive multiple of n prime to p. The preimage of the natural finite étale Galois cover $Y^{(N)} \to Y^{(n)}$ is $Z^{(N)}$. As $Z^{(N)}$ is connected, the restriction $Z^{(N)} \to Z^{(n)}$ is a finite étale Galois cover with the same Galois group as $\mathrm{Gal}(Y^{(N)}/Y^{(n)})$. If z is a geometric point of $Z^{(n)}$, we have the following commutative diagram

$$\begin{array}{ccc} \pi_1^{\text{ét}}(Z^{(n)}, z) & \longrightarrow & \pi_1^{\text{ét}}(Y^{(n)}, z) \\ \downarrow & & \downarrow \\ \mathrm{Gal}(Z^{(N)}/Z^{(n)}) & \xrightarrow{\sim} & \mathrm{Gal}(Y^{(N)}/Y^{(n)}). \end{array}$$

When N goes to infinity, the cardinality of $\mathrm{Gal}(Y^{(N)}/Y^{(n)})$ goes to infinity as well. This implies that the image of $\pi_1^{\text{ét}}(Z^{(n)}, z) \to \pi_1^{\text{ét}}(Y^{(n)}, z)$ is infinite. The varieties $Z^{(n)}$ and $Y^{(n)}$ descend to geometrically connected varieties $Z_0^{(n)}$ and $Y_0^{(n)}$ over some finite field $\mathbb{F}_q$.

Example 20 might be a nice concrete example to analyse for further developments on Conjecture 10.

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References

[1] M. D'Addezio, "The monodromy groups of lisse sheaves and overconvergent F-isocrystals", *Sel. Math., New Ser.* **26** (2020), no. 3, article no. 45 (41 pages).

[2] P. Deligne, "La Conjecture de Weil. II", *Publ. Math., Inst. Hautes Étud. Sci.* **52** (1980), p. 137-252.

[3] ———, "Finitude de l'extension de $\mathbb{Q}$ engendrée par des traces de Frobenius, en caractéristique finie", *Mosc. Math. J.* **12** (2012), no. 3, p. 497-514.

[4] V. Drinfeld, "On a conjecture of Deligne", *Mosc. Math. J.* **12** (2012), p. 515-542.

[5] A. Grothendieck, M. Raynaud, *Revêtements étales et groupe fondamental (SGA 1)*, Lecture Notes in Mathematics, vol. 224, Springer, 1971.

[6] L. Lafforgue, "Chtoucas de Drinfeld et correspondance de Langlands", *Invent. Math.* **147** (2002), no. 1, p. 1-241.

[7] M. Larsen, R. Pink, "Abelian varieties, ℓ-adic representations, and ℓ-independence", *Math. Ann.* **302** (1995), no. 3, p. 561-580.

[8] B. Lasell, M. Ramachandran, "Observations on harmonic maps and singular varieties", *Ann. Sci. Éc. Norm. Supér.* **29** (1996), no. 2, p. 135-148.

[9] F. Oort, "A Stratification of a Moduli Space of Abelian Varieties", in *Moduli of Abelian varieties*, Progress in Mathematics, vol. 195, Birkhäuser, 2001, p. 345-416.

[10] The Stacks Project Authors, "Stacks Project", http://stacks.math.columbia.edu.

[11] J. Stix, "A general Seifert-Van Kampen theorem for algebraic fundamental groups", *Publ. Res. Inst. Math. Sci.* **42** (2006), no. 3, p. 763-786.