Rudy Rosas

Transversely product singularities of foliations in projective spaces

Volume 361 (2023), p. 1785-1787

Published online: 21 December 2023

https://doi.org/10.5802/crmath.528

This article is licensed under the Creative Commons Attribution 4.0 International License.
http://creativecommons.org/licenses/by/4.0/
Transversely product singularities of foliations in projective spaces

Rudy Rosas

Abstract. We prove that a transversely product component of the singular set of a holomorphic foliation on $\mathbb{P}^n$ is necessarily a Kupka component.

Funding. The author was supported by Vicerrectorado de Investigación de la Pontificia Universidad Católica del Perú.

1. Introduction

Let $U$ be an open set of a complex manifold $M$ and let $k \in \mathbb{N}$. Let $\eta$ be a holomorphic $k$-form on $U$ and let $\text{Sing}\eta := \{p \in U : \eta(p) = 0\}$ denote the singular set of $\eta$. We say that $\eta$ is integrable if each point $p \in U \setminus \text{Sing}\eta$ has a neighborhood $V$ supporting holomorphic 1-forms $\xi_1, \ldots, \xi_k$ with $\eta|_V = \xi_1 \wedge \cdots \wedge \xi_k$, such that $d\xi_j \wedge \eta = 0$ for each $j = 1, \ldots, k$. In this case the distribution $D_\eta := \{v \in T_p M : i_v \eta(p) = 0\}, \quad p \in U \setminus \text{Sing}\eta$ defines a holomorphic foliation of codimension $k$ on $U \setminus \text{Sing}\eta$. A singular holomorphic foliation $F$ of codimension $k$ on $M$ can be defined by an open covering $(U_j)_{j \in J}$ of $M$ and a collection of integrable $k$-forms $\eta_j \in \Omega^k(U_j)$ such that $\eta_i = g_{ij} \eta_j$ for some $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ whenever $U_i \cap U_j \neq \emptyset$. The singular set $\text{Sing}F$ is the proper analytic subset of $M$ given by the union of the sets $\text{Sing}\eta_j$. From now on we only consider foliations $F$ such that $\text{Sing}F$ has no component of codimension one.

Given a singular holomorphic foliation $F$ of codimension $k$ on $M$ as above, the Kupka singular set of $F$, denoted by $K(F)$, is the union of the sets

$$K(\eta_j) = \{p \in U_j : \eta_j(p) = 0, d\eta_j(p) \neq 0\}.$$ 

This set does not depend on the collection $(\eta_j)$ of $k$-forms used to define $F$. It is well known (see [7,9]) that, given $p \in K(F)$, the germ of $F$ at $p$ is holomorphically equivalent to the product of a one-dimensional foliation with an isolated singularity by a regular foliation of dimension $(\dim F - 1)$. More precisely, if $\dim M = k + m + 1$, there exist a holomorphic vector field $X = X_1 \partial_{x_1} + \cdots + X_{k+1} \partial_{x_{k+1}}$ on $\mathbb{D}^{k+1}$ with a unique singularity at the origin, a neighborhood $V$ of $p$ in $M$ and a biholomorphism $\psi : V \to \mathbb{D}^{k+1} \times \mathbb{D}^m$, $\psi(p) = 0$, which conjugates $F$ with the foliation...
$\mathcal{F}_X$ of $\mathbb{D}^{k+1} \times \mathbb{D}^m$ generated by the commuting vector fields $X, \partial_{y_1}, \ldots, \partial_{y_m}$, where $y = (y_1, \ldots, y_m)$ are the coordinates in $\mathbb{D}^m$. If $\mu = dx_1 \wedge \cdots \wedge dx_{k+1}$, the foliation $\mathcal{F}_X$ is also defined by the $k$-form $\omega = i_X \mu$ and the Kupka condition $d\omega(0) \neq 0$ is equivalent to the inequality $\text{div} X(0) \neq 0$.

Following [8], we say that $\mathcal{F}$ is a transversely product at $p \in \text{Sing} \mathcal{F}$ if as above there exist a holomorphic vector field $X$ and a biholomorphism $\psi : V \to \mathbb{D}^{k+1} \times \mathbb{D}^m$ conjugating $\mathcal{F}$ with $\mathcal{F}_X$, except that it is not assumed that $\text{div} X(0) \neq 0$. We say that $\mathcal{F}$ is a locally transversely product component of $\text{Sing} \mathcal{F}$ if $\Gamma$ is a compact irreducible component of $\text{Sing} \mathcal{F}$ and $\mathcal{F}$ is a transversely product at each $p \in \Gamma$. In particular, if $\Gamma \subset K(\mathcal{F})$ we say that $\Gamma$ is a Kupka component — for more information about Kupka singularities and Kupka components we refer the reader to [1–7]. If $\Gamma$ is a transversely product component of $\text{Sing} \mathcal{F}$, we can cover $\Gamma$ by finitely many normal coordinates like $\psi$, with the same vector field $X$; that is, there exist a holomorphic vector field $X$ on $\mathbb{D}^{k+1}$ with a unique singularity at the origin and a covering of $\Gamma$ by open sets $(V_a)_{a \in A}$ such that each $V_a$ supports a biholomorphism $\psi_a : V_a \to \mathbb{D}^{k+1} \times \mathbb{D}^m$ that maps $\Gamma \cap V_a$ onto $[0] \times \mathbb{D}^m$ and conjugates $\mathcal{F}$ with the foliation $\mathcal{F}_X$. The sets $(V_a)$ can be chosen arbitrarily close to $\Gamma$.

In [8], the author proves that a local transversely product component of a codimension one foliation on $\mathbb{P}^n$ is necessarily a Kupka component. The goal of the present paper is to generalize this theorem to foliations of any codimension.

**Theorem 1.** Let $\mathcal{F}$ a holomorphic foliation of dimension $\geq 2$ and codimension $\geq 1$ on $\mathbb{P}^n$. Let $\Gamma$ be a transversely product component of $\text{Sing} \mathcal{F}$. Then $\Gamma$ is a Kupka component.

This theorem is a corollary of the following result.

**Theorem 2.** Let $\mathcal{F}$ a holomorphic foliation of dimension $\geq 2$ and codimension $k \geq 1$ on a complex manifold $M$. Suppose that $\mathcal{F}$ is defined by an open covering $(U_j)_{j \in J}$ of $M$ and a collection of $k$-forms $\eta_j \in \Omega^k(U_j)$. Let $L$ be the line bundle defined by the cocycle $(g_{ij})$ such that $\eta_i = g_{ij} \eta_j$, $g_{ij} \in \Theta^1(U_i \cap U_j)$. Let $\Gamma$ be a transversely product component of $\text{Sing} \mathcal{F}$ that is not a Kupka component. Then $c_1(L|_\Gamma) = 0$.

2. Proof of the results

**Proof of Theorem 2.** Let $\dim M = k + m + 1$. As explained in the introduction, there exist a holomorphic vector field $X$ on $\mathbb{D}^{k+1}$ with a unique singularity at the origin and a covering of $\Gamma$ by open sets $(V_a)_{a \in A}$ such that each $V_a$ is contained in $V$ and supports a biholomorphism $\psi_a : V_a \to \mathbb{D}^{k+1} \times \mathbb{D}^m$ that maps $\Gamma \cap V_a$ onto $[0] \times \mathbb{D}^m$ and conjugates $\mathcal{F}$ with the foliation $\mathcal{F}_X$ generated by the commuting vector fields $X, \partial_{y_1}, \ldots, \partial_{y_m}$. Notice that $\text{div}(X)(0) = 0$, because $\Gamma$ is not a Kupka component. Since $\mathcal{F}_X$ is defined by the $k$-form $\omega = i_X \mu$, where $\mu = dx_1 \wedge \cdots \wedge dx_{k+1}$, we have that $\mathcal{F}|_{V_a}$ is defined by the $k$-form $\psi_a^*(\omega)$. If $V_a \cap V_\beta \neq \emptyset$, there exists $\theta_{a\beta} \in \Theta^1(V_a \cap V_\beta)$ such that

$$
\psi_a^*(\omega) = \theta_{a\beta} \psi_\beta^*(\omega).
$$

(1)

Therefore the cocycle $(\theta_{a\beta})$ define the line bundle $L$ restricted to some neighborhood of $\Gamma$. Thus, in order to prove that $c_1(L|_\Gamma) = 0$ it is enough to show that each $\theta_{a\beta}|_{\Gamma}$ is locally constant. Fix some $a, \beta \in A$ such that $V_a \cap V_\beta \neq \emptyset$. If we set $\psi = \psi_a \circ \psi_\beta^{-1}$ and $\theta = \theta_{a\beta} \circ \psi_\beta^{-1}$, from (1) we have that $\psi^*(\omega) = \theta(s)$, which means that $\psi$ preserves the foliation $\mathcal{F}_X$. It suffices to prove that the derivatives $\theta_{y_1}(p), \ldots, \theta_{y_m}(p)$ vanish if $p \in [0] \times \mathbb{D}^m$. Since $\partial_{y_1}$ is tangent to $\mathcal{F}_X$, then the vector field $\psi^*(\partial_{y_1})$ is tangent to $\mathcal{F}_X$ and so we can express

$$
\psi^*(\partial_{y_1})(\omega) = \mathcal{L}_{\omega} \omega = \lambda X + \lambda_1 \partial_{y_1} + \cdots + \lambda_m \partial_{y_m},
$$

where $\lambda, \lambda_1, \ldots, \lambda_m$ are holomorphic. Then

$$
\mathcal{L}_{\psi^*(\partial_{y_1})} \omega = \mathcal{L}_{\lambda X} \omega + \lambda \mathcal{L}_X \omega + \omega = \lambda \mathcal{L}_X \omega = \lambda \text{div}(X) \omega,
$$

Therefore, $\mathcal{F}$ is transversely a product at $p \in \text{Sing} \mathcal{F}$.
where the last equality follows from the identity $\omega = i_X \mu$. Thus, since

$$\psi^* \left( \mathcal{L}_{\mathcal{L}_\psi \partial \gamma_1} \omega \right) = \mathcal{L}_{\partial \gamma_1} \psi^* \omega = \mathcal{L}_{\partial \gamma_1} (\theta \omega) = \theta \gamma_1 \omega,$$

we obtain that

$$\theta \gamma_1 \omega = \psi^* (\lambda \text{div}(X) \omega) = \lambda(\psi) \text{div}(X)(\psi) \theta \omega$$

and therefore $\theta \gamma_1 (p) = 0$ if $p \in \{0\} \times \mathbb{D}^m$, because $\text{div}(X)$ vanishes along $\{0\} \times \mathbb{D}^m$. In the same way we prove that $\theta \gamma_1 (p) = \cdots = \theta \gamma_m (p) = 0$ if $p \in \{0\} \times \mathbb{D}^m$, which finishes the proof. \hfill \Box

**Proof of Theorem 1.** Suppose that $\Gamma$ is not a Kupka component. Let $L$ be the line bundle associated to $\mathcal{F}$ as in the statement of Theorem 2. We notice that $c_1(L) \neq 0$, otherwise $\mathcal{F}$ will be defined by a global $k$-form on $\mathbb{P}^n$, which is impossible. Then, if we take an algebraic curve $\mathcal{C} \subset \Gamma$, we have $c_1(L) \cdot \mathcal{C} 
eq 0$. Therefore, if $\Omega$ is a 2-form on $\mathbb{P}^n$ in the class $c_1(L)$ and $V$ is a tubular neighborhood of $\Gamma$,

$$c_1(L|_\Gamma) \cdot \mathcal{C} = \int_{\mathcal{C}} \Omega|_{\Gamma} = \int_{\mathcal{C}} \Omega = c_1(L) \cdot \mathcal{C} \neq 0,$$

which contradicts Theorem 2. \hfill \Box

**References**