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Transversely product singularities of foliations in projective spaces

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Abstract. We prove that a transversely product component of the singular set of a holomorphic foliation on $\mathbb{P}^n$ is necessarily a Kupka component.

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1. Introduction

Let U be an open set of a complex manifold M and let $k \in \mathbb{N}$. Let η be a holomorphic k-form on U and let $\operatorname{Sing}\eta := \{p \in U : \eta(p) = 0\}$ denote the singular set of η. We say that η is integrable if each point $p \in U \setminus \operatorname{Sing}\eta$ has a neighborhood V supporting holomorphic 1-forms $\xi_1, \dots, \xi_k$ with $\eta|_V = \xi_1 \wedge \dots \wedge \xi_k$, such that $\mathrm{d}\xi_j \wedge \eta = 0$ for each $j = 1, \dots, k$. In this case the distribution

$$\mathscr{D}_\eta : \ \mathscr{D}_\eta(p) = \{v \in T_p M : i_v \eta(p) = 0\}, \quad p \in U \setminus \operatorname{Sing}\eta$$

defines a holomorphic foliation of codimension k on $U \setminus \operatorname{Sing}\eta$. A singular holomorphic foliation $\mathscr{F}$ of codimension k on M can be defined by an open covering $(U_j)_{j \in J}$ of M and a collection of integrable k-forms $\eta_j \in \Omega^k(U_j)$ such that $\eta_i = g_{ij}\eta_j$ for some $g_{ij} \in \mathscr{O}^*(U_i \cap U_j)$ whenever $U_i \cap U_j \neq \emptyset$. The singular set $\operatorname{Sing}\mathscr{F}$ is the proper analytic subset of M given by the union of the sets $\operatorname{Sing}\eta_j$. From now on we only consider foliations $\mathscr{F}$ such that $\operatorname{Sing}\mathscr{F}$ has no component of codimension one.

Given a singular holomorphic foliation $\mathscr{F}$ of codimension k on M as above, the Kupka singular set of $\mathscr{F}$, denoted by $K(\mathscr{F})$, is the union of the sets

$$K(\eta_j) = \{p \in U_j : \eta_j(p) = 0, \mathrm{d}\eta_j(p) \neq 0\}.$$

This set does not depend on the collection (η_j) of k-forms used to define $\mathscr{F}$. It is well known (see [7, 9]) that, given $p \in K(\mathscr{F})$, the germ of $\mathscr{F}$ at p is holomorphically equivalent to the product of a one-dimensional foliation with an isolated singularity by a regular foliation of dimension $(\dim \mathscr{F} - 1)$. More precisely, if $\dim M = k + m + 1$, there exist a holomorphic vector field $X = X_1\partial_{x_1} + \dots + X_{k+1}\partial_{x_{k+1}}$ on $\mathbb{D}^{k+1}$ with a unique singularity at the origin, a neighborhood V of p in M and a biholomorphism $\psi : V \to \mathbb{D}^{k+1} \times \mathbb{D}^m$, $\psi(p) = 0$, which conjugates $\mathscr{F}$ with the foliation

$\mathscr{F}_X$ of $\mathbb{D}^{k+1} \times \mathbb{D}^m$ generated by the commuting vector fields $X, \partial_{y_1}, \ldots, \partial_{y_m}$, where $y = (y_1, \ldots, y_m)$ are the coordinates in $\mathbb{D}^m$. If $\mu = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{k+1}$, the foliation $\mathscr{F}_X$ is also defined by the k-form $\omega = i_X\mu$ and the Kupka condition $d\omega(0) \neq 0$ is equivalent to the inequality $\operatorname{div} X(0) \neq 0$.

Following [8], we say that $\mathscr{F}$ is a transversely product at $p \in \operatorname{Sing}\mathscr{F}$ if as above there exist a holomorphic vector field X and a biholomorphism $\psi : V \to \mathbb{D}^{k+1} \times \mathbb{D}^m$ conjugating $\mathscr{F}$ with $\mathscr{F}_X$, except that it is not assumed that $\operatorname{div} X(0) \neq 0$. We say that Γ is a local transversely product component of $\operatorname{Sing}\mathscr{F}$ if Γ is a compact irreducible component of $\operatorname{Sing}\mathscr{F}$ and $\mathscr{F}$ is a transversely product at each $p \in \Gamma$. In particular, if $\Gamma \subset K(\mathscr{F})$ we say that Γ is a Kupka component — for more information about Kupka singularities and Kupka components we refer the reader to [1–7]. If Γ is a transversely product component of $\operatorname{Sing}\mathscr{F}$, we can cover Γ by finitely many normal coordinates like ψ, with the same vector field X: that is, there exist a holomorphic vector field X on $\mathbb{D}^{k+1}$ with a unique singularity at the origin and a covering of Γ by open sets $(V_\alpha)_{\alpha \in A}$ such that each V_α supports a biholomorphism $\psi_\alpha : V_\alpha \to \mathbb{D}^{k+1} \times \mathbb{D}^m$ that maps $\Gamma \cap V_\alpha$ onto $\{0\} \times \mathbb{D}^m$ and conjugates $\mathscr{F}$ with the foliation $\mathscr{F}_X$. The sets (V_α) can be chosen arbitrarily close to Γ.

In [8], the author proves that a local transversely product component of a codimension one foliation on $\mathbb{P}^n$ is necessarily a Kupka component. The goal of the present paper is to generalize this theorem to foliations of any codimension.

Theorem 1. *Let $\mathscr{F}$ a holomorphic foliation of dimension ≥ 2 and codimension ≥ 1 on $\mathbb{P}^n$. Let Γ be a transversely product component of $\operatorname{Sing}\mathscr{F}$. Then Γ is a Kupka component.*

This theorem is a corollary of the following result.

Theorem 2. *Let $\mathscr{F}$ a holomorphic foliation of dimension ≥ 2 and codimension $k \geq 1$ on a complex manifold M. Suppose that $\mathscr{F}$ is defined by an open covering $(U_j)_{j\in J}$ of M and a collection of k-forms $\eta_j \in \Omega^k(U_j)$. Let L be the line bundle defined by the cocycle (g_{ij}) such that $\eta_i = g_{ij}\eta_j$, $g_{ij} \in \mathscr{O}^*(U_i \cap U_j)$. Let Γ be a transversely product component of $\operatorname{Sing}\mathscr{F}$ that is not a Kupka component. Then $c_1(L|_\Gamma) = 0$.*

2. Proof of the results

Proof of Theorem 2. Let $\dim M = k + m + 1$. As explained in the introduction, there exist a holomorphic vector field X on $\mathbb{D}^{k+1}$ with a unique singularity at the origin and a covering of Γ by open sets $(V_\alpha)_{\alpha\in A}$ such that each V_α is contained in V and supports a biholomorphism $\psi_\alpha : V_\alpha \to \mathbb{D}^{k+1} \times \mathbb{D}^m$ that maps $\Gamma \cap V_\alpha$ onto $\{0\} \times \mathbb{D}^m$ and conjugates $\mathscr{F}$ with the foliation $\mathscr{F}_X$ generated by the commuting vector fields $X, \partial_{y_1}, \ldots, \partial_{y_m}$. Notice that $\operatorname{div}(X)(0) = 0$, because Γ is not a Kupka component. Since $\mathscr{F}_X$ is defined by the k-form $\omega = i_X\mu$, where $\mu = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{k+1}$, we have that $\mathscr{F}|_{V_\alpha}$ is defined by the k-form $\psi_\alpha^*(\omega)$. If $V_\alpha \cap V_\beta \neq \emptyset$, there exists $\theta_{\alpha\beta} \in \mathscr{O}^*(V_\alpha \cap V_\beta)$ such that

$$\psi_\alpha^*(\omega) = \theta_{\alpha\beta}\psi_\beta^*(\omega). \tag{1}$$

Therefore the cocycle $(\theta_{\alpha\beta})$ define the line bundle L restricted to some neighborhood of Γ. Thus, in order to prove that $c_1(L|_\Gamma) = 0$ it is enough to show that each $\theta_{\alpha\beta}|_\Gamma$ is locally constant. Fix some $\alpha, \beta \in A$ such that $V_\alpha \cap V_\beta \neq \emptyset$. If we set $\psi = \psi_\alpha \circ \psi_\beta^{-1}$ and $\theta = \theta_{\alpha\beta} \circ \psi_\beta^{-1}$, from (1) we have that $\psi^*(\omega) = \theta\omega$, which means that ψ preserves the foliation $\mathscr{F}_X$. It suffices to prove that the derivatives $\theta_{y_1}(p), \ldots, \theta_{y_m}(p)$ vanish if $p \in \{0\} \times \mathbb{D}^m$. Since ∂_{y_1} is tangent to $\mathscr{F}_X$, then the vector field $\psi_*(\partial_{y_1})$ is tangent to $\mathscr{F}_X$ and so we can express

$$\psi_*(\partial_{y_1}) = \lambda X + \lambda_1\partial_{y_1} + \cdots + \lambda_m\partial_{y_m},$$

where $\lambda, \lambda_1, \ldots, \lambda_m$ are holomorphic. Then

$$\mathscr{L}_{\psi_*(\partial_{y_1})}\omega = \mathscr{L}_{\lambda X}\omega = \lambda\mathscr{L}_X\omega + \mathrm{d}\lambda \wedge i_X\omega = \lambda\mathscr{L}_X\omega = \lambda \operatorname{div}(X)\omega,$$

where the last equality follows from the identity $\omega = i_X\mu$. Thus, since

$$\psi^*\left(\mathscr{L}_{\psi_*(\partial_{y_1})}\omega\right) = \mathscr{L}_{\partial_{y_1}}\psi^*\omega = \mathscr{L}_{\partial_{y_1}}(\theta\omega) = \theta_{y_1}\omega,$$

we obtain that

$$\theta_{y_1}\omega = \psi^*\left(\lambda\,\mathrm{div}(X)\omega\right) = \lambda(\psi)\,\mathrm{div}(X)(\psi)\theta\omega$$

and therefore $\theta_{y_1}(p) = 0$ if $p \in \{0\}\times\mathbb{D}^m$, because $\mathrm{div}(X)$ vanishes along $\{0\}\times\mathbb{D}^m$. In the same way we prove that $\theta_{y_2}(p) = \cdots = \theta_{y_m}(p) = 0$ if $p \in \{0\}\times\mathbb{D}^m$, which finishes the proof. □

Proof of Theorem 1. Suppose that Γ is not a Kupka component. Let L be the line bundle associated to $\mathscr{F}$ as in the statement of Theorem 2. We notice that $c_1(L) \neq 0$, otherwise $\mathscr{F}$ will be defined by a global k-form on $\mathbb{P}^n$, which is impossible. Then, if we take an algebraic curve $\mathscr{C} \subset \Gamma$, we have $c_1(L)\cdot\mathscr{C} \neq 0$. Therefore, if Ω is a 2-form on $\mathbb{P}^n$ in the class $c_1(L)$ and V is a tubular neighborhood of Γ,

$$c_1(L|_\Gamma)\cdot\mathscr{C} = \int_{\mathscr{C}}\Omega|_\Gamma = \int_{\mathscr{C}}\Omega = c_1(L)\cdot\mathscr{C} \neq 0,$$

which contradicts Theorem 2. □

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