



INSTITUT DE FRANCE  
Académie des sciences

# *Comptes Rendus*

---


## *Mathématique*

Bertrand Deroin and Carlos Matheus

**Non-linear bi-algebraic curves and surfaces in moduli spaces of Abelian differentials**

Volume 361 (2023), p. 1691-1698

<https://doi.org/10.5802/crmath.529>

 This article is licensed under the  
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.  
<http://creativecommons.org/licenses/by/4.0/>



*Les Comptes Rendus. Mathématique* sont membres du  
Centre Mersenne pour l'édition scientifique ouverte

[www.centre-mersenne.org](http://www.centre-mersenne.org)

e-ISSN : 1778-3569



---

Algebraic geometry / Géométrie algébrique

# Non-linear bi-algebraic curves and surfaces in moduli spaces of Abelian differentials

*Courbes et surfaces bi-algébriques non-linéaires dans les espaces de modules de différentielles abéliennes*

Bertrand Deroin<sup>a</sup> and Carlos Matheus<sup>\*,b</sup>

<sup>a</sup> CNRS & Université de Cergy-Pontoise (UMR CNRS 8088), 95302, Cergy-Pontoise, France

<sup>b</sup> CNRS & École Polytechnique (UMR CNRS 7640), 91128, Palaiseau, France

E-mails: [bertrand.deroin@cyu.fr](mailto:bertrand.deroin@cyu.fr), [carlos.matheus@math.cnrs.fr](mailto:carlos.matheus@math.cnrs.fr)

**Abstract.** The strata of the moduli spaces of Abelian differentials are non-homogenous spaces carrying natural bi-algebraic structures. Partly inspired by the case of homogenous spaces carrying bi-algebraic structures (such as torii, Abelian varieties and Shimura varieties), Klingler and Lerer recently showed that any bi-algebraic curve in a stratum of the moduli space of Abelian differentials is linear provided that the so-called condition ( $\star$ ) is fulfilled.

In this note, we construct a non-linear bi-algebraic curve, resp. surface, of Abelian differentials of genus 7, resp. 10.

**Résumé.** Les strates des espaces de modules de différentielles abéliennes sont des espaces non-homogènes possédant des structures bi-algébriques naturelles. Partiellement inspirés par le cas des espaces homogènes bi-algébriques (comme les tores, les variétés abéliennes et les variétés de Shimura), Klingler et Lerer ont récemment montré qu'une courbe bi-algébrique dans un strate d'un espace de modules de différentielles abéliennes est linéaire pourvu que la soi-disant condition ( $\star$ ) est satisfaite.

Dans cette note, on construit une courbe, resp. surface, bi-algébrique non-linéaire de différentielles abéliennes de genre 7, resp. 10.

*Manuscript received 6 March 2023, revised 31 May 2023, accepted 24 June 2023.*

## 1. Introduction

The study of transcendence properties and unlikely intersections in Diophantine geometry is a fascinating topic possessing a vast literature developing in many directions including those related to the interplay between Hodge theory and the geometry of homogenous spaces such as Abelian varieties and their moduli spaces (that is, Shimura varieties). In this context, an

---

\* Corresponding author.

impressive number of heuristic principles and rigorous results<sup>1</sup> was discovered by many authors, and, as a way to unify these statements and also suggest new ones, the point of view of *bi-algebraic structures* became increasingly popular: see, for instance, the survey [10] and the articles [2] and [3].

In their recent work [9], Klingler and Lerer proposed<sup>2</sup> to extend the bi-algebraic point of view to the non-homogenous setting of moduli spaces of Abelian differentials. More concretely, the moduli space of Abelian differentials of genus  $g$  is stratified into complex quasi-projective algebraic orbifolds  $H(\kappa)$  parametrising non-trivial Abelian differentials whose zeroes have multiplicities prescribed by a list  $\kappa = (k_1, \dots, k_\sigma)$  such that  $k_1 + \dots + k_\sigma = 2g - 2$ . As it was proved by Veech and Masur, the relative periods of the elements of  $H(\kappa)$  can be used to define the so-called *period charts* inducing a linear integral structure on the analytification of  $H(\kappa)$ . In particular, after projectivising the stratum  $H(\kappa)$ ,  $\kappa = (k_1, \dots, k_\sigma)$ , we obtain a quasi-projective orbifold  $\mathcal{H}(\kappa)$  of dimension  $2g - 2 + \sigma$  whose analytification possesses a linear projective structure. In this setting, a closed, irreducible, algebraic subvariety  $W$  of  $\mathcal{H}(\kappa)$  is *bi-algebraic* if its analytification  $W^{\text{an}}$  is algebraic in period charts<sup>3</sup>, cf. [9, Def. 1.1], and Klingler and Lerer proved that all bi-algebraic curves in the strata  $\mathcal{H}(2)$  and  $\mathcal{H}(1, 1)$  of Abelian differentials of genus two are (projectively) linear (in period charts), and, in general, any bi-algebraic curve in  $\mathcal{H}(\kappa)$  is linear *provided* their condition  $(\star)$  is satisfied (cf. [9, Thm. 2.8 & 2.10]). Furthermore, they asked whether the bi-algebraicity of a subvariety of  $\mathcal{H}(\kappa)$  is already enough to automatically ensure its linearity without extra conditions (cf. [9, Conj. 2.7]). The main results of this paper say that some bi-algebraic subvarieties can be non-linear:

**Theorem 1.** *The projectivisation of the family  $\{(C_u, \omega_u)\}_{u \in \mathbb{C} \setminus \{0, \pm 1\}}$  of Abelian differentials defined by*

$$C_u := \overline{\{y^6 = x(x-1)(x+1)(x-u)\}} \text{ and } \omega_u := x^2 dx / y^5, \tag{1}$$

*is a bi-algebraic curve in  $\mathcal{H}(12)$  which is not linear.*

**Remark 2.** Note that  $C_u$  is a branched cover of  $\{z^2 = x(x-1)(x+1)(x-u)\}$ . In particular, the family of Abelian differentials  $\{(C_u, [dx/y^3])\}_{u \in \mathbb{C} \setminus \{0, \pm 1\}}$  corresponds to an arithmetic Teichmüller curve (in the sense of [8, §5]).

**Theorem 3.** *Given a generic algebraic curve  $C \subset \mathbb{C}^3$ , the projectivisation of the family  $\{(C_{a,b,c}, \omega_{a,b,c})\}_{\substack{(a,b,c) \in C, \\ a,b,c \in \mathbb{C} \setminus \{0,1\} \\ \text{distinct}}}$  of Abelian differentials defined by*

$$C_{a,b,c} := \overline{\{y^6 = x(x-1)(x-a)(x-b)(x-c)\}} \text{ and } \omega_{a,b,c} := dx / y^5, \tag{2}$$

*is a bi-algebraic curve in  $\mathcal{H}(18)$  which is not linear.*

In particular, observe that if  $S \subset \mathbb{C}^3$  is a generic algebraic surface, then the projectivisation of the family  $\{(C_{a,b,c}, \omega_{a,b,c})\}_{\substack{(a,b,c) \in S, \\ a,b,c \in \mathbb{C} \setminus \{0,1\} \\ \text{distinct}}}$  is a bi-algebraic surface in  $\mathcal{H}(18)$  which is not linear. We

do not have any example of a non linear bi-algebraic subvariety of dimension at least 3 in some stratum of abelian differentials.

The proofs of these statements occupy the rest of the paper. More precisely, after a brief discussion of the variations of Hodge structures associated to the families of curves  $C_u$  and  $C_{a,b,c}$  in Section 3, we establish Theorem 1, resp. 3, in Section 5, resp. 6.

<sup>1</sup>Such as Ax–Schanuel conjectures, Ax–Lindemann type theorems, André–Oort and Zilber–Pink conjectures.

<sup>2</sup>This is a natural goal because the moduli spaces of Abelian differentials seem to “behave” like homogenous spaces: for example, the celebrated breakthroughs by Eskin, Mirzakhani and Mohammadi [5], [6] show that this is the case from the point of view of Dynamical Systems.

<sup>3</sup>That is, the relative periods of Abelian differentials projectively lying in  $W^{\text{an}}$  satisfy exactly  $\text{codim}_{\mathcal{H}(\kappa)}(W)$  independent algebraic relations (over  $\mathbb{C}$ ).

### 2. Eigencohomology of cyclic covers

Let  $t = (t_1, \dots, t_n) \in \mathbb{C}^n$  be a collection of distinct complex numbers. The plane algebraic curve defined by the equation

$$y^d = (x - t_1) \dots (x - t_n)$$

compactifies as a smooth compact Riemann surface  $C_t$  by adding  $a = \gcd(d, n)$  points at infinity. The function  $X_t : C_t \rightarrow \mathbb{P}^1$  defined by  $X_t(x, y) = x$  is a ramified covering over the sphere having  $n$  critical points of order  $d$  and  $a$  critical point(s) of order  $d/a$ . So by Riemann–Hurwitz, the genus  $g$  of  $C_t$  is given by the formula:

$$g = \frac{(n - 1)(d - 1) - (a - 1)}{2}.$$

In particular, if  $(d, n) = (6, 4)$ ,  $g = 7$ , and if  $(d, n) = (6, 5)$ ,  $g = 10$ .

The covering  $X_t$  is an abelian covering whose Galois group is generated by the transformation  $\pi_t(x, y) = (x, \zeta y)$  where  $\zeta = \exp(2i\pi/d)$ . We denote by  $H_\zeta^*(C_t, \mathbb{C})$  the eigenspace  $\text{Ker}(\pi_t^* - \zeta) \subset H^*(C_t, \mathbb{C})$ . Unless specified, a holomorphic/meromorphic eigenform on  $C_t$  is a form  $\eta$  such that  $\pi_t^*\eta = \zeta\eta$ . The following result is well-known, see, e.g., [12, §3] for more details. We provide the proof for completeness.

**Lemma 4.** *In the regime  $n < d$ , we have*

$$H_\zeta^{1,0}(C_t, \mathbb{C}) = \left\{ U(x) \frac{dx}{y^{d-1}} \mid U \text{ polynomial of degree } \leq n - 2 \right\}.$$

Moreover, the eigenspace  $H_\zeta^1(C_t, \mathbb{C})$  is made of cohomology classes of holomorphic eigenforms, i.e.

$$H_\zeta^1(C_t, \mathbb{C}) \simeq H_\zeta^{1,0}(C_t, \mathbb{C}) \text{ and } H_\zeta^{0,1}(C_t, \mathbb{C}) = \overline{H_\zeta^{1,0}(C_t, \mathbb{C})} = 0.$$

**Proof.** A form in  $H_\zeta^{1,0}(C_t)$  can be expressed as  $\eta = U(x) \frac{dx}{y^{d-1}}$  where  $U$  is a meromorphic function on  $\mathbb{P}^1$ , which is holomorphic except possibly at the points  $t_i$ 's and the point at infinity. In the sequel assume that  $U$  does not vanish identically.

Around the point  $t_i$ , we can write  $U(x) \sim \text{cst}(x - t_i)^{k_i}$  for some integer  $k_i$  and a non zero constant. We can also take  $y$  as a holomorphic coordinates and we have  $x - t_i \sim \text{cst } y^d$ , so

$$\eta \sim_{(t_i, 0)} \text{cst } y^{dk_i} dy$$

We deduce for  $\eta$  to be holomorphic,  $k_i$  must be non negative for every  $i = 1, \dots, n$ , hence  $U$  is a polynomial of  $x$ .

It will be convenient to introduce the integers  $b, c$  such that  $n = ab$  and  $d = ac$ . Around a point  $\infty$  of  $C_t$  at infinity, we have a coordinate  $z$  so that  $x \sim \text{cst } z^{-c}$  and  $y \sim \text{cst } z^{-b}$ . We then have (denoting  $\text{deg}(U)$  the degree of  $U$ )

$$\eta \sim_\infty \text{cst } z^{-c \text{deg}(U) - c - 1 + b(d-1)}$$

which shows that  $\eta$  is holomorphic iff  $\text{deg}(U) \leq \frac{b(d-1)-c-1}{c} = n - 1 - \frac{(b+1)}{c}$ , or equivalently, since  $b < c$ , iff  $\text{deg}(U) \leq n - 2$ . In particular  $\dim H_\zeta^{1,0}(C_t, \mathbb{C}) = n - 1 = \dim H_\zeta^1(C_t, \mathbb{C})$  so we are done.  $\square$

### 3. Finite monodromies

The condition  $(\star)$  of Klingler and Lerer (cf. [9, Def. 6.5]) indicates that the families of curves whose Jacobians possess non-trivial fixed parts are potential sources of non-linear bi-algebraic subvarieties: see [9, §6.4] for more explanations. In this direction, we note that such families of curves were previously studied by several authors in the context of Teichmüller dynamics (e.g., [1, 7, 11–13] among many others), and, in fact, the families of curves  $C_u$  and  $C_{a,b,c}$  are particular instances of the families of curves described in [12, Thm. 8.3]. For later reference, we recall below some of the remarkable properties of the families of Jacobians of  $C_u$  and  $C_{a,b,c}$ .

**Proposition 5.** *If  $(d, n) = (6, 4)$  or  $(d, n) = (6, 5)$ , the monodromy of the Gauss–Manin connection on the bundle  $H^1_\zeta$  is finite. In particular, the families of forms  $\{(C_u, [\omega_u])\}_{u \neq 0, \pm 1}$ , lying in  $\mathcal{H}(12)$ , and  $\{(C_{a,b,c}, [\omega_{a,b,c}])\}_{a,b,c \neq 0, 1}$ , lying in  $\mathcal{H}(18)$ , are bi-algebraic.*

*distinct*

Here  $[\omega]$  is the projective class of  $\omega$  in the projectivization of the Hodge bundle.

**Proof.** This statement is included in Theorem 8.3 of [12]. Let us provide the proof for completeness. Lemma 4 ensures that the Gauss–Manin invariant hermitian form  $\frac{i}{2} \int \omega \wedge \bar{\omega}$  on  $H^1_\zeta$  (resp.  $H^1_{\zeta^5} = \overline{H^1_\zeta}$ ) is positive definite, so its monodromy is unitary. Moreover, the bundle  $H^1(C_t, \mathbb{C})_\zeta \oplus H^1(C_t, \mathbb{C})_{\zeta^5}$  is defined over  $\mathbb{Q}$  (because  $\zeta, \zeta^5$  are the sole primitive sixth roots of unity). So the monodromy is integral (it is the restriction of the symplectic integral representation to a rational subspace) and preserves a unitary form; hence it is finite. In particular, as it is explained in [12, pp. 929] for instance, since the periods of the corresponding families of forms are holomorphic (multivalued) functions of  $t$  which are algebraic when the monodromy is finite, one has that the corresponding families of forms are bi-algebraic. □

#### 4. Picard–Fuchs theory: computation of the Gauss–Manin connection

The vector bundle  $H^1_\zeta$  over the configuration space  $B$  of  $n$  distinct complex numbers is isomorphic, thanks to Lemma 4, to the product  $B \times \mathbb{C}_{n-2}[x]$ , where  $\mathbb{C}_{n-2}[x]$  stands for the space of polynomials of degree  $\leq n - 2$ . More precisely, for  $U \in \mathbb{C}_{n-2}[x]$ , we denote

$$\omega_{t,U} = U(x) \frac{dx}{y^{d-1}} \tag{3}$$

the corresponding holomorphic eigenform on  $C_t$ . The map  $(t, U) \mapsto \omega_{t,U}$  gives the desired trivialization. In the following, we compute the Gauss–Manin connection on  $H^1_\zeta$  in this latter: it is defined as follows

$$\nabla.(\omega_{t,U}) = \omega_{t,\nabla.U}.$$

We denote

$$P(x) = (x - t_1) \dots (x - t_n). \tag{4}$$

**Proposition 6.** *In the trivialization  $H^1_\zeta \simeq B \times \mathbb{C}_{n-2}[x]$  given by Lemma 4, the Gauss–Manin connection is given by*

$$\nabla_{\partial_{t_k}} U = -\frac{U(t_k)}{P'(t_k)} \frac{P(x) - P(t_k) - P'(t_k)(x - t_k)}{(x - t_k)^2} + \frac{1}{d} \frac{U(t_k)}{P'(t_k)} \frac{P'(x) - P'(t_k)}{x - t_k} + \frac{d - 1}{d} \frac{U(x) - U(t_k)}{x - t_k}$$

for any  $U \in \mathbb{C}_{n-2}[x]$ .

**Proof.** Let  $N_i$  be simply connected neighborhoods of the  $t_i$ 's, chosen sufficiently small to be disjoint. Denoting by  $N = \cup N_i$ , and  $Y = \mathbb{C} \setminus N$ , the Riemann surfaces  $X_{t'}^{-1}(Y)$  for  $t' = (t'_1, \dots, t'_n) \in \prod_i N_i$ , are naturally identified to  $X_t^{-1}(Y)$  via a holomorphic family of biholomorphisms  $\Phi_{t',t} : X_t^{-1}(Y) \rightarrow X_{t'}^{-1}(Y)$  that satisfy  $X_{t'} \circ \Phi_{t',t} = X_t$  and  $\Phi_{t,t} = \text{id}$ . Using this family it is possible to derivate a holomorphic family of holomorphic 1-forms  $\omega_t$  on  $X_t^{-1}(Y)$  with respect to the  $t_k$ -variables, by the formula

$$\partial_{t_k} \omega_t := \left( \partial_{t'_k} \Phi_{t',t}^* \omega_{t'} \right)_{|t'=t};$$

by shrinking  $N$ , this defines a meromorphic 1-form on  $C_t$  which is holomorphic apart from the points  $(t_i, 0) \in C_t$  but that might develop a pole at each  $(t_i, 0) \in C_t$ . These poles have no residue at the points  $(t_i, 0)$ , so the period of  $\partial_{t_k} \omega_t$  is well-defined as an element of  $H^1(C_t, \mathbb{C})$ : for more discussion about this construction (leading to the Kodaira–Spencer map and the Gauss–Manin connection), see Clemens' book [4, §2.10] and Voisin's book [14, Ch. 9].

Applied to the family of eigenforms  $\{(C_t, \omega_{t,U})\}_{t \in \mathbb{C}^n \setminus \Delta}$ , where  $U \in \mathbb{C}_{n-2}[x]$ , the derivative  $\partial_{t_k} \omega_t$  is a meromorphic eigenform, which is cohomologous to a holomorphic eigenform  $\omega_{t,V_t}$  with  $V_t \in \mathbb{C}_{n-2}[x]$  by Lemma 4. By definition

$$\nabla_{t_k} U := V_t. \tag{5}$$

This equation is characterized by the existence of a (unique) meromorphic eigenfunction  $f : C_t \rightarrow \mathbb{P}^1$  which is such that

$$\partial_{t_k} \omega_{t,U} - \omega_{t,V_t} = df.$$

Here eigenfunction means  $f \circ \pi_t = \zeta f$ . Let us now make the computations and find the function  $f$ .

Derivating the multi-valued function  $y = \prod_j (x - t_j)^{1/d}$  with respect to the variables  $t_k$  yields  $\partial_{t_k} y = \frac{-y}{d(x-t_k)}$  from which we get

$$\partial_{t_k} \left( U(x) \frac{dx}{y^{d-1}} \right) = \frac{d-1}{d} \frac{U(x)}{x-t_k} \frac{dx}{y^{d-1}}. \tag{6}$$

This meromorphic eigenform has a pole at the point  $(t_k, 0)$ , which in the coordinates  $y$  of  $C_t$  at  $(t_k, 0)$  has the following local behaviour (assuming that  $U(t_k) \neq 0$ )

$$\partial_{t_k} \left( U(x) \frac{dx}{y^{d-1}} \right) \sim_{(t_k,0)} (d-1) \frac{U(t_k)}{y^d} dy. \tag{7}$$

The meromorphic function  $f : C_t \rightarrow \mathbb{P}^1$  defined by

$$f(x, y) := \frac{y}{x-t_k} \tag{8}$$

has a unique pole at the point  $(t_k, 0)$  and it has the following local behaviour

$$f \sim_{(t_k,0)} \frac{P'(t_k)}{y^{d-1}} \tag{9}$$

(it has no pole at infinity since  $d \geq n$ ). In particular, the eigenform

$$\partial_{t_k} \left( U(x) \frac{dx}{y^{d-1}} \right) + \frac{U(t_k)}{P'(t_k)} df$$

is holomorphic on  $C_t$ , we thus have

$$\partial_{t_k} \left( U(x) \frac{dx}{y^{d-1}} \right) + \frac{U(t_k)}{P'(t_k)} df = \omega_{t,V_t}, \tag{10}$$

with  $\nabla_{t_k} U = V_t$  by (5), so it remains to compute (10). We have

$$\begin{aligned} df = f \frac{df}{f} &= \frac{y}{x-t_k} \left( \frac{dy}{y} - \frac{dx}{x-t_k} \right) = \frac{y}{x-t_k} \left( \frac{1}{d} \sum_{j \neq k} \frac{dx}{x-t_j} + \left( \frac{1}{d} - 1 \right) \frac{dx}{x-t_k} \right) \\ &= \frac{P(x)}{x-t_k} \left( \frac{1}{d} \sum_{j \neq k} \frac{1}{x-t_j} - \frac{d-1}{d} \frac{1}{x-t_k} \right) \frac{dx}{y^{d-1}}. \end{aligned}$$

So using the identity

$$\frac{P(x)}{x-t_k} \sum_{j \neq k} \frac{1}{x-t_j} = \left( \frac{P(x)}{x-t_k} \right)' = \frac{P'(x)}{x-t_k} - \frac{P(x)}{(x-t_k)^2}$$

we deduce

$$V_t(x) = \frac{d-1}{d} \left( \frac{U(x)}{x-t_k} - \frac{U(t_k)}{P'(t_k)} \frac{P(x)}{(x-t_k)^2} \right) + \frac{1}{d} \frac{U(t_k)}{P'(t_k)} \left( \frac{P'(x)}{x-t_k} - \frac{P(x)}{(x-t_k)^2} \right).$$

Rearranging terms and using  $P(t_k) = 0$  we find the expression of the Gauss–Manin connection given by Proposition 6. □

**5. Proof of Theorem 1**

In view of Proposition 5, our task is reduced to check that the lines spanned by  $\omega_u$ ,  $u \neq 0, 1$ , sit “generically” on  $H^1_\zeta(C_u, \mathbb{C})$ . A direct application of Proposition 6 shows that the Gauss–Manin on  $H^1_\zeta$  in the trivialization given by the sections  $p_k = x^k \frac{dx}{y^5}$  for  $k = 0, 1, 2$  is

$$\nabla = d + \frac{dt}{6t(1-u^2)} \begin{pmatrix} 5u^2 - 4 & u & u^2 \\ 5u & 5u^2 & 5u \\ 2 & 2u & 2u^2 \end{pmatrix} \tag{11}$$

By Proposition 5, it suffices to prove that the family of periods  $\{p_{2,u}\}_{u \neq 0, \pm 1}$  does not lie in a flat strict subbundle of  $H^1_\zeta$ , which will be achieved by proving that at a generic parameter  $u$  the periods  $p_{2,u}, \nabla_u p_{2,u}, \nabla_u^2 p_{2,u}$  are linearly independent in  $H^1_\zeta(C_u, \mathbb{C})$ . We use the basis  $p_{k,u}$  of  $H^1_\zeta(C_u, \mathbb{C})$  to perform the computations, together with the explicit expression of the Gauss–Manin connection (11). We have in this basis

$$p_{2,u} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \nabla_u p_{2,u} = \frac{1}{6u(1-u^2)} \begin{pmatrix} u^2 \\ 5u \\ 2u^2 \end{pmatrix}$$

and

$$\nabla_u (6(1-u^2)\nabla_u p_{2,u}) = \frac{1}{6(1-u^2)} \begin{pmatrix} 7+u^2 \\ 40u \\ 24-8u^2 \end{pmatrix}.$$

Those three vectors are generically linearly independent, and by Leibniz rule, so are the vectors  $p_{2,u}, \nabla_u p_{2,u}, \nabla_u^2 p_{2,u}$ .

**Remark 7.** The family  $(\{y^{10} = x(x-1)(x-v)\}, [dx/y^9])_{v \neq 0,1}$  is a bi-algebraic curve in  $\mathcal{H}(16)$  because  $dx/y^9$  lies in a fixed part of dimension four (cf. [12, Thm. 8.3]). Nonetheless, a direct calculation using Proposition 6 reveals that this bi-algebraic curve is linear.

**6. Proof of Theorem 3**

In view of Proposition 5, our task is reduced to check that the lines spanned by  $\omega_{a,b,c}$ ,  $a, b, c \neq 0, 1$ , sit “generically” on the fibers of the vector bundle  $H^1_\zeta$ . As it turns out, this fact is a particular case of Theorem 6.1 in [12], where a result of Veech and Masur (about the period charts) is cleverly explored to derive that the (period) map taking the configuration of points  $\{0, 1, a, b, c\}$  to  $[dx/y^5]$  is a holomorphic local diffeomorphism.<sup>4</sup>

In the sequel, we give an alternative proof of the non-linearity of the bi-algebraic surface  $\{(C_{a,b,c}, \omega_{a,b,c})\}_{a,b,c \in \mathbb{C} \setminus \{0,1\}}$  via Proposition 6: in particular, we do not rely on the results of Veech and Masur, but rather perform direct calculations (which might be of independent interest).

Proposition 6 shows that

$$\nabla_{\partial_u} \frac{dx}{y^5} = \frac{1}{P'(u)} \left( \frac{1}{6} \frac{P'(x) - P'(u)}{x-u} - \frac{P(x) - P'(u)(x-u)}{(x-u)^2} \right) \frac{dx}{y^5}$$

(where  $u \in \{a, b, c\}$  and  $P(x) = x(x-1)(x-a)(x-b)(x-c)$ ).

At this point, it remains to check that

$$\frac{dx}{y^5}, \nabla_{\partial_a} \frac{dx}{y^5}, \nabla_{\partial_b} \frac{dx}{y^5}, \nabla_{\partial_c} \frac{dx}{y^5}$$

<sup>4</sup>Furthermore, McMullen also gets in [12, Thm. 9.1] that the periods of  $dx/y^5$  are algebraic functions of  $(a, b, c)$ .

are linearly independent. For this sake, let us write them in terms of the basis  $\{x^k dx/y^5 : k = 0, 1, 2, 3\}$ . If we denote  $\{u, v, w\} = \{a, b, c\}$ , then

$$\frac{P'(x) - P'(u)}{x - u} = (u^3 - u^2 + uv - u^2v + uw - u^2w - 2vw + uvw) + (u^2 - u + 3v - uv + 3w - uw + 3vw)x + (u - 4 - 4v - 4w)x^2 + 5x^3,$$

$$\frac{P(x) - P'(u)(x - u)}{(x - u)^2} = (u^3 - u^2 + uv - u^2v + uw - u^2w - vw + uvw) + (u^2 - u + v - uv + w - uw + vw)x + (u - 1 - v - w)x^2 + x^3,$$

so that

$$\begin{aligned} \frac{1}{6} \frac{P'(x) - P'(u)}{x - u} - \frac{P(x) - P'(u)(x - u)}{(x - u)^2} &= \frac{1}{6} (5u^2 - 5u^3 - 5uv + 5u^2v - 5uw + 5u^2w + 4vw - 5uvw) \\ &\quad + \frac{1}{6} (5u - 5u^2 - 3v + 5uv - 3w + 5uw - 3vw)x \\ &\quad + \frac{1}{6} (2 - 5u + 2v + 2w)x^2 - \frac{x^3}{6}. \end{aligned}$$

In particular, the matrix  $M$  whose column vectors are  $\frac{dx}{y^5}$ ,  $P'(a)\nabla_{\partial_a} \frac{dx}{y^5}$ ,  $P'(b)\nabla_{\partial_b} \frac{dx}{y^5}$ ,  $P'(c)\nabla_{\partial_c} \frac{dx}{y^5}$  written in the basis  $dx/y^5$ ,  $xdx/y^5$ ,  $x^2dx/y^5$ ,  $x^3dx/y^5$  is

$$M = \begin{pmatrix} 1 & * & * & * \\ 0 & \frac{1}{6}(5a-5a^2-3b+5ab-3c+5ac-3bc) & \frac{1}{6}(5b-5b^2-3a+5ab-3c-3ac+5bc) & \frac{1}{6}(5c-5c^2-3a+5ac-3b-3ab+5bc) \\ 0 & \frac{1}{6}(2-5a+2b+2c) & \frac{1}{6}(2+2a-5b+2c) & \frac{1}{6}(2+2a+2b-5c) \\ 0 & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \end{pmatrix}$$

Since the determinant of  $M$  is

$$\det(M) = -\frac{91}{216} (a - b)(a - c)(b - c),$$

the proof of Theorem 3 is now complete.

**Remark 8.** The determinant of  $\frac{dx}{y^5}$ ,  $\nabla_{\partial_a} \frac{dx}{y^5}$ ,  $\nabla_{\partial_b} \frac{dx}{y^5}$ ,  $\nabla_{\partial_c} \frac{dx}{y^5}$  in the basis  $dx/y^5$ ,  $xdx/y^5$ ,  $x^2dx/y^5$ ,  $x^3dx/y^5$ , which is somehow more natural, is given by the expression

$$-\frac{91}{216} \frac{(a - b)(a - c)(b - c)}{P'(a)P'(b)P'(c)} = \frac{91}{216} \frac{1}{abc(a - 1)(b - 1)(c - 1)(a - b)(a - c)(b - c)}$$

### Acknowledgements

This work began at the workshop *Finiteness results for special subvarieties: Hodge theory, minimality, dynamics* organised by S. Filip, D. Fisher and B. Klingler held in Como (Italy) from October 16 to 22, 2022. We are indebted to several participants of this workshop (including S. Cantat, S. Filip, B. Klingler, L. Lerer and M. Möller) for many interesting discussions.

### References

- [1] A. Avila, C. Matheus, J.-C. Yoccoz, “The Kontsevich-Zorich cocycle over Veech-McMullen family of symmetric translation surfaces”, *J. Mod. Dyn.* **14** (2019), p. 21-54.
- [2] B. Bakker, B. Klingler, J. Tsimerman, “Tame topology of arithmetic quotients and algebraicity of Hodge loci”, *J. Am. Math. Soc.* **33** (2020), no. 4, p. 917-939.
- [3] G. Baldi, B. Klingler, E. Ullmo, “On the distribution of the Hodge locus”, 2021, <https://arxiv.org/abs/2107.08838>.
- [4] C. H. Clemens, *A scrapbook of complex curve theory*, University Series in Mathematics, Plenum Press, 1980, ix+186 pages.
- [5] A. Eskin, M. Mirzakhani, “Invariant and stationary measures for the  $SL(2, \mathbb{R})$  action on moduli space”, *Publ. Math., Inst. Hautes Étud. Sci.* **127** (2018), p. 95-324.



- [6] A. Eskin, M. Mirzakhani, A. Mohammadi, “Isolation, equidistribution, and orbit closures for the  $SL(2, \mathbb{R})$  action on moduli space”, *Ann. Math.* **182** (2015), no. 2, p. 673-721.
- [7] G. Forni, C. Matheus, A. Zorich, “Square-tiled cyclic covers”, *J. Mod. Dyn.* **5** (2011), no. 2, p. 285-318.
- [8] E. Gutkin, C. Judge, “Affine mappings of translation surfaces: geometry and arithmetic”, *Duke Math. J.* **103** (2000), no. 2, p. 191-213.
- [9] B. Klingler, L. A. Lerer, “Abelian differentials and their periods: the bi-algebraic point of view”, 2022, <https://arxiv.org/abs/2202.06031>.
- [10] B. Klingler, E. Ullmo, A. Yafaev, “Bi-algebraic geometry and the André-Oort conjecture”, in *Algebraic geometry: Salt Lake City 2015*, Proceedings of Symposia in Pure Mathematics, vol. 97.2, American Mathematical Society, 2018, p. 319-359.
- [11] C. Matheus, J.-C. Yoccoz, “The action of the affine diffeomorphisms on the relative homology group of certain exceptionally symmetric origamis”, *J. Mod. Dyn.* **4** (2010), no. 3, p. 453-486.
- [12] C. T. McMullen, “Braid groups and Hodge theory”, *Math. Ann.* **355** (2013), no. 3, p. 893-946.
- [13] M. Möller, “Shimura and Teichmüller curves”, *J. Mod. Dyn.* **5** (2011), no. 1, p. 1-32.
- [14] C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*, Cours Spécialisés (Paris), vol. 10, Société Mathématique de France, 2002, viii+595 pages.