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On a problem of Nathanson related to minimal asymptotic bases of order h

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Abstract. For integer $h \geq 2$ and $A \subseteq \mathbb{N}$, we define hA to be the set of all integers which can be written as a sum of h, not necessarily distinct, elements of A. The set A is called an asymptotic basis of order h if $n \in hA$ for all sufficiently large integers n. An asymptotic basis A of order h is minimal if no proper subset of A is an asymptotic basis of order h. For $W \subseteq \mathbb{N}$, denote by $\mathscr{F}^*(W)$ the set of all finite, nonempty subsets of W. Let $A(W)$ be the set of all numbers of the form $\sum_{f \in F} 2^f$, where $F \in \mathscr{F}^*(W)$. In this paper, we give some characterizations of the partitions $\mathbb{N} = W_1 \cup \cdots \cup W_h$ with the property that $A = A(W_1) \cup \cdots \cup A(W_h)$ is a minimal asymptotic basis of order h. This generalizes a result of Chen and Chen, recent result of Ling and Tang, and also recent result of Sun.

Keywords. Asymptotic bases, minimal asymptotic bases, binary representation.

2020 Mathematics Subject Classification. 11B34.

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1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. For an integer $h \geq 2$ and $A \subseteq \mathbb{N}$, we define

$$hA = \{n : n = a_1 + \cdots + a_h, a_i \in A, i = 1, 2, \ldots, h\}.$$

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The set A is called an asymptotic basis of order h if $n \in hA$ for all sufficiently large integers n. An asymptotic basis A of order h is minimal if no proper subset of A is an asymptotic basis of order h. This means that for any $a \in A$, the set $E_a = hA \setminus h(A \setminus \{a\})$ is infinite.

Let W be a nonempty subset of $\mathbb{N}$. Denote by $\mathscr{F}^*(W)$ the set of all finite, nonempty subsets of W. Let $A(W)$ be the set of all numbers of the form $\sum_{f \in F} 2^f$, where $F \in \mathscr{F}^*(W)$.

In 1988, Nathanson [8] gave a construction of minimal asymptotic bases of order h.

Theorem A. *Let $h \geq 2$ and let $W_i = \{n \in \mathbb{N} : n \equiv i \pmod h\}$ for $i = 0,1,\ldots,h-1$. Let $A = A(W_0) \cup A(W_1) \cup \cdots \cup A(W_{h-1})$. Then A is a minimal asymptotic basis of order h.*

Let $h \geq 2$ and $\mathbb{N} = W_1 \cup \cdots \cup W_h$ be a partition; that is, $W_i \cap W_j = \emptyset$ if $i \neq j$. Nathanson also posed the following open problem:

Problem 1. *Characterise the partitions $\mathbb{N} = W_1 \cup \cdots \cup W_h$ with the property that $A = A(W_1) \cup \cdots \cup A(W_h)$ is a minimal asymptotic basis of order h.*

In 2011, Chen and Chen [1] solved Problem 1 for $h = 2$. They proved the next theorem.

Theorem B. *Let $\mathbb{N} = W_1 \cup W_2$ be a partition with $0 \in W_1$ such that W_1 and W_2 are infinite. Then $A = A(W_1) \cup A(W_2)$ is a minimal asymptotic basis of order 2 if and only if either W_1 contains no consecutive integers or W_2 contains consecutive integers or both.*

In 2020, Ling and Tang [7] focus on Problem 1 for $h = 3$; they proved the following result:

Theorem C. *For any $i \in \{0,1,2,3,4,5\}$, if $W_0 = \{n \in \mathbb{N} : n \equiv i, i+1 \pmod 6\}$, $W_1 = \{n \in \mathbb{N} : n \equiv i+2, i+4 \pmod 6\}$ and $W_2 = \{n \in \mathbb{N} : n \equiv i+3, i+5 \pmod 6\}$, then $A = A(W_0) \cup A(W_1) \cup A(W_2)$ is a minimal asymptotic basis of order three.*

In 2021, Sun [11] gave a generalization of Theorem A.

Theorem D. *Let h and t be two positive integers with $h \geq 2$. Let*

$$W_j = \bigcup_{i=0}^{\infty} [iht + jt, iht + jt + t - 1]$$

for $j = 0,1,\ldots,h-1$. Then $A = A(W_0) \cup A(W_1) \cup \cdots \cup A(W_{h-1})$ is a minimal asymptotic basis of order h.

In 2011, Chen and Chen [1] gave the following sufficient condition for Problem 1.

Theorem E. *Let $h \geq 2$ and r be the least integer with $r > \log h / \log 2$. Let $\mathbb{N} = W_1 \cup \cdots \cup W_h$ be a partition such that each set W_i is infinite and contains r consecutive integers for $i = 1,\ldots,h$. Then $A = A(W_1) \cup \cdots \cup A(W_h)$ is a minimal asymptotic basis of order h.*

For other related results about minimal asymptotic bases, see [2–5, 10, 12].

In this paper, we continue to focus on Problem 1. First, we give a generalization of Theorem C and Theorem D. For $W \subseteq \mathbb{N}$, set $W(x) = |\{n \in W : n \leq x\}|$.

Theorem 2. *Let $h \geq 2$ be an integer and $\mathbb{N} = W_1 \cup \cdots \cup W_h$ be a partition such that each set W_j $(1 \leq j \leq h)$ satisfies $|W_j(ht-1)| = t$ for infinitely many integers t. Then $A = A(W_1) \cup \cdots \cup A(W_h)$ is a minimal asymptotic basis of order h.*

Our second theorem is a generalization of Theorem E.

Theorem 3. *Let h be an integer, $h \geq 2$, and $\mathbb{N} = W_1 \cup \cdots \cup W_h$ be a partition such that each set W_i is infinite for $i = 1,\ldots,h$, $0 \in W_1$ and every W_i contains $\lceil \frac{\log(h+1)}{\log 2} \rceil$ consecutive integers for $i = 2,\ldots,h$. Then $A = A(W_1) \cup \cdots \cup A(W_h)$ is a minimal asymptotic basis of order h.*

Remark 4. It is easy to see that $\lceil \frac{\log(h+1)}{\log 2} \rceil$ is the least integer greater than $\log h / \log 2$.

There is a more limited variant of Problem 1, see [6, 9, 12]

2. Lemmas

Lemma 5 (see [8, Lemma 1]). *Let* $\mathbb{N} = W_1 \cup \cdots \cup W_h$, *where* $W_i \neq \emptyset$ *for* $i = 1, \ldots, h$. *Then* $A = A(W_1) \cup \cdots \cup A(W_h)$ *is an asymptotic basis of order* h.

Lemma 6. *Let* $w_1, \ldots, w_s$ *be* s *distinct nonnegative integers. If*

$$\sum_{i=1}^{s} 2^{w_i} \equiv \sum_{j=1}^{t} 2^{x_j} \pmod{2^{w_s+1}},$$

where $0 \le x_1, \ldots, x_t < w_s + 1$ *are integers not necessarily distinct, then there exist nonempty disjoint sets* $J_1, \ldots, J_s$ *of* $\{1, 2, \ldots, t\}$ *such that*

$$2^{w_i} = \sum_{j \in J_i} 2^{x_j}$$

for $i = 1, \ldots, s$.

Proof. By the proof of Lemma 2 from [8], there exist nonempty subsets $J_1, \ldots, J_s$ of $\{1, 2, \ldots, t\}$ such that

$$2^{w_i} = \sum_{j \in J_i} 2^{x_j}$$

for $i = 1, \ldots, s$.

The result is trivial for $s = 1$. Now we assume that $s \ge 2$. Since there exists a subset J_1 of $\{1, 2, \ldots, t\}$ such that

$$2^{w_1} = \sum_{j \in J_1} 2^{x_j},$$

it follows that

$$\sum_{2 \le i \le s} 2^{w_i} \equiv \sum_{j \in \{1, \ldots, t\} \setminus J_1} 2^{x_j} \pmod{2^{w_s+1}},$$

which implies that there exists a subset J_2 of $\{1, 2, \ldots, t\} \setminus J_1$ such that

$$2^{w_2} = \sum_{j \in J_2} 2^{x_j},$$

and so $J_1 \cap J_2 = \emptyset$. Continuing this process, it follows that there exist nonempty disjoint subsets $J_1, \ldots, J_s$ of $\{1, 2, \ldots, t\}$ such that

$$2^{w_i} = \sum_{j \in J_i} 2^{x_j},$$

for $i = 1, \ldots, s$.

This completes the proof of Lemma 6. □

3. Proof of Theorem 2

By Lemma 5, it follows that A is an asymptotic basis of order h. For any $a \in A$, there exists $j \in \{1, 2, \ldots, h\}$ such that $a \in A(W_j)$. Without loss of generality, we may assume that $a \in A(W_1)$. Then there exists a set $K \subseteq W_1$ such that $a = \sum_{i \in K} 2^i$. Since there exist infinitely many t such that

$$|W_j(ht - 1)| = t$$

for $j = 1, 2, \ldots, h$, it follows that there exists an integer t_n such that

$$t_n h \le \min K < t_{n+1} h.$$

Let

$$n_T = a + \sum_{j=2}^{h} \sum_{v \in W_j \cap [0, t_{n+1}h - 1]} 2^v + \sum_{j=2}^{h} \sum_{v \in W_j \cap [t_{n+1}h, T]} 2^v,$$

where T is an integer such that $2^T > a$. Next we shall prove that $n_T \in E_a$. Note that $K(t_{n+1}h-1) \neq \emptyset$ and

$$n_T = \sum_{j\in K(t_{n+1}h-1)} 2^j + \sum_{j=2}^{h} \sum_{v\in W_j\cap[0,t_{n+1}h-1]} 2^v + 2^{t_{n+1}h}m$$

for some $m \geq 0$. Let $n_T = b_1 + b_2 + \cdots + b_h$ be any representation of n_T as a sum of h elements of A and let

$$b_i = \sum_{i\in S_i} 2^i$$

for $i = 1,2\ldots,h$. Let

$$c_i = \sum_{j\in S_i(t_{n+1}h-1)} 2^j$$

for $i = 1,2,\ldots,h$. Then

$$c_i \equiv b_i \pmod{2^{t_{n+1}h}}$$

and

$$|S_i(t_{n+1}h-1)| \leq t_{n+1}$$

for $i = 1,2,\ldots,h$, which implies that

$$\sum_{k\in K(t_{n+1}h-1)} 2^k + \sum_{j=2}^{h} \sum_{v\in W_j\cap[0,t_{n+1}h-1]} 2^v \equiv \sum_{1\leq i\leq h} c_i \equiv \sum_{1\leq i\leq h} \sum_{j\in S_i(t_{n+1}h-1)} 2^j \pmod{2^{t_{n+1}h}}.$$

Let

$$\sum_{k\in K(t_{n+1}h-1)} 2^k + \sum_{j=2}^{h} \sum_{v\in W_j\cap[0,t_{n+1}h-1]} 2^v \equiv \sum_{j=1}^{s} 2^{x_j} \pmod{2^{t_{n+1}h}},$$

where s is an integer such that $s \leq t_{n+1}h$. By Lemma 6, there exist nonempty disjoint subsets $J_0, J_1, \ldots, J_{t_{n+1}(h-1)}$ of $\{1,2,\ldots,s\}$ such that

$$2^{\min K} + \sum_{j=2}^{h} \sum_{v\in W_j\cap[0,t_{n+1}h-1]} 2^v = \sum_{j\in J_0} 2^{x_j} + \sum_{j\in J_1} 2^{x_j} + \sum_{j\in J_2} 2^{x_j} + \cdots + \sum_{j\in J_{t_{n+1}(h-1)}} 2^{x_j}.$$

Therefore,

$$1 + t_{n+1}(h-1) \leq 1 + |J_1| + \cdots + |J_{t_{n+1}(h-1)}| \leq s \leq t_{n+1}h,$$

and so

$$t_{n+1}(h-1) \leq |J_1| + \cdots + |J_{t_{n+1}(h-1)}| \leq t_{n+1}h - 1. \tag{1}$$

Since $|W_j(t_{n+1}h-1)| = t_{n+1}$ for any $j \geq 2$, it follows from (1) and Lemma 6 that for any $j \geq 2$, there exist $w \in W_j$, $J \subseteq \{1,2,\ldots,s\}$ and $|J| = 1$ such that

$$2^w = \sum_{j\in J} 2^{x_j},$$

which implies that $x_j = w \in W_j$, and so

$$\{b_1,\ldots,b_h\} \not\subseteq \bigcup_{j=1,j\neq m}^{h} A(W_j)$$

for any $m \geq 2$. Renumbering the indexes, we always assume that

$$b_i \in A(W_i), \quad i \geq 2.$$

Then

$$b_1 = a + \left(\sum_{j=2}^{h} \sum_{v\in W_j\cap[0,t_{n+1}h-1]} 2^v + \sum_{j=2}^{h} \sum_{v\in W_j\cap[t_{n+1}h,T]} 2^v - \sum_{2\leq i\leq h} b_i\right).$$

Since the binary representation of b_1 is unique, it follows that $b_1 = a$, that is $n_T \in E_a$. Noting that T is infinite, we have that A is minimal.

This completes the proof of Theorem 2.

4. Proof of Theorem 3

By Lemma 5, it follows that A is an asymptotic basis of order h. For $a \in A$, we assume that $a \in A(W_i)$. Let $E_a = hA \setminus h(A \setminus \{a\})$. Now we prove that E_a is infinite.

Let

$$n_T = a + \sum_{w \in [0,T] \setminus W_i} 2^w,$$

where T is an integer with $T > a$ such that each $[0,T] \cap W_j$ contains $\lceil \frac{\log(h+1)}{\log 2} \rceil$ consecutive integers for $j = 2, \dots, h$. To prove that $n_T \in E_a$, it suffices to prove that if $n_T = a_1 + a_2 + \cdots + a_h$ with $a_i \in A$ for $1 \le i \le h$, then there exists at least one $a_k = a$.

We distinguish two cases according to whether $i = 1$ or $2 \le i \le h$.

Case 1. $i = 1$**.** Suppose that there exists an integer $j \ge 2$ such that

$$\{a_1, a_2, \dots, a_h\} \subseteq \bigcup_{1 \le l \le h, l \ne j} A(W_j).$$

Let $\left\{b+1, b+2, \dots, b + \lceil \frac{\log(h+1)}{\log 2} \rceil \right\} \subseteq [0,T] \cap W_j$. Then by Lemma 6 there exists

$$\{a'_1, \dots, a'_h\} \subseteq \bigcup_{1 \le l \le h, l \ne j} A(W_j) \cup \{0\}$$

such that

$$2^{b+1} + \cdots + 2^{b + \lceil \frac{\log(h+1)}{\log 2} \rceil} = a'_1 + \cdots + a'_h.$$

Since $a'_i \notin A(W_j)$ for $i = 1, \dots, h$, we have

$$a'_i \le 2^0 + 2^1 + \cdots + 2^b = 2^{b+1} - 1$$

for $i = 1, \dots, h$. It follows that

$$2^{b+1} + \cdots + 2^{b + \lceil \frac{\log(h+1)}{\log 2} \rceil} \le h(2^{b+1} - 1) < h 2^{b+1},$$

that is $2^{\lceil \frac{\log(h+1)}{\log 2} \rceil} - 1 < h$, a contradiction. Hence, for any integer $j \ge 2$, we have

$$\{a_1, a_2, \dots, a_h\} \nsubseteq \bigcup_{1 \le l \le h, l \ne j} A(W_l).$$

Renumbering the indexes, we may assume that $a_i \in A(W_i)$ for $i = 2, 3, \dots, h$. It follows that

$$a_1 = a + \sum_{2 \le j \le h} \left(\sum_{w \in [0,T] \cap W_j} 2^w - a_j \right).$$

Since $a \in A(W_1)$, $W_1, \dots, W_h$ are disjoint, and the binary representation of a_1 is unique, we have $a_1 = a$. Therefore $n_T \in E_a$, and E_a is infinite.

Case 2. $i \ge 2$**.** Since n_T is odd, therefore we may assume that $a_1 \in A(W_1)$. Let $2 \le j \le h$, $j \ne i$. Similar to the Case 1, we get that

$$\{a_1, \dots, a_h\} \nsubseteq \bigcup_{1 \le k \le h, k \ne j} A(W_k).$$

Renumbering the indexes, we may assume that $a_j \in A(W_j)$ for $j = 2, 3, \dots, i-1, i+1, \dots, h$. It follows that

$$a_i = a + \sum_{1 \le j \le h, j \ne i} \left(\sum_{w \in [0,T] \cap W_j} 2^w - a_j \right).$$

Since $a \in A(W_i)$, $W_1, \dots, W_h$ are disjoint, and the binary representation of a_i is unique, we have $a_i = a$. Therefore $n_T \in E_a$, and E_a is infinite.

This completes the proof.

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