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On Bousfield's conjectures for the unstable Adams spectral sequence for SO and U

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Abstract. The unstable Adams spectral sequence is a spectral sequence that starts from algebraic information about the mod 2 cohomology $H^*(X)$ of a space X as an unstable algebra over the Steenrod algebra $\mathscr{A}$, and converges, in good cases, to the 2-localized homotopy groups of X. Bousfield and Don Davis looked at the case when X was either of the infinite matrix groups SO or U. Bousfield and Davis created algebraic spectral sequences and conjectured that they agreed with the unstable Adams spectral sequences for SO and U. To this end the following algebraic decomposition must hold

$$\mathrm{Ext}^s_{\mathscr{U}}\left(\widetilde{H}^*\left(\mathbb{R}P^\infty, \Sigma^t\mathbb{Z}/2\right)\right) \cong \bigoplus_n \mathrm{Ext}^s_{\mathscr{U}}\left(M_n/M_{n-1}, \Sigma^t\mathbb{Z}/2\right)$$

where $M_1 \subset M_2 \subset \cdots$ is the well known dyadic filtration of the $\mathscr{A}$-module $\widetilde{H}^*\left(\mathbb{R}P^\infty, \mathbb{Z}/2\right) \cong \mathbb{F}_2[u]$ given by the dyadic expansion of the powers of u. This paper aims at showing that this decomposition holds for numerous values of s and t.

Keywords. Injective resolution, Projective resolution, Unstable Adams spectral sequence, Unstable modules.

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1. Introduction

In this article, we study a conjecture of Bousfield on the unstable Adams spectral sequence for SO and U. This is first stated in the original paper [2] of Bousfield and Davis and is studied by Kathryn Lesh in [5]. We begin by recalling the precise statement of the conjecture. We work at the prime 2 and for a topological space X, we write $\mathrm{H}^*(X)$ for $\mathrm{H}^*(X;\mathbb{F}_2)$. Let $\mathscr{A}$ be the Steenrod algebra. An $\mathbb{N}$−graded commutative $\mathscr{A}$−algebra K is called an unstable algebra if the Steenrod action is unstable, represents the Frobenius twist

$$Sq^k x = \begin{cases} 0 & \text{if } k > n, \\ x^2 & \text{if } k = n, \end{cases}$$

for all $x \in K^n$, and satisfies the Cartan formula

$$Sq^k(ab) = \sum_{i=0}^{k} \left(Sq^i a\right)\left(Sq^{k-i} b\right)$$

for all $a, b \in K$. In particular, as Sq^0 is the identity element then $x = x^2$ for all x homogeneous of degree 0. In other words, the degree 0 elements of an unstable algebra form a Boolean algebra. If we no longer require the product structure, keeping only the instability condition of the Steenrod action, we get the notion of an unstable module. Denote by $\mathcal{K}$ the category of unstable algebras, and by $\mathcal{U}$ the category of unstable modules. The Cartan formula on the tensor product $M \otimes N$ of two unstable modules define another unstable one, making $\mathcal{U}$ a symmetric monoidal category. For an unstable module M, we write $S^*(M)$ for the symmetric power of M. This is an algebra but does not satisfy the Cartan formula. Folowing [10], we define

$$U(M) = \frac{S^*(M)}{\left\langle Sq^{|x|}x - x^2 \right\rangle}.$$

to get a functor $U : \mathcal{U} \to \mathcal{K}$, which is left adjoint to the forgetful functor $\mathcal{K} \to \mathcal{U}$. The importance of the categories $\mathcal{U}$ and $\mathcal{K}$ is justified by the unstable Adams spectral sequence, introduced by Massey and Peterson in [6], generalized by Bousfield and Curtis in [1], and generalized further by Bousfield and Kan in [3]. In general, the unstable Adams spectral sequence for a topological space X has the form

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{K}}\left(\mathrm{H}^*(X), \mathrm{H}^*\left(S^t\right)\right) \Longrightarrow \pi_{s-t}\left(\widehat{X_2}\right).$$

However, if the cohomology of the space X is of the form $U(M)$ for some unstable module M, then the derived functor Ext in the nonabelian category $\mathcal{K}$ can be computed as Ext-groups in the category $\mathcal{U}$:

$$\mathrm{Ext}^s_{\mathcal{K}}\left(\mathrm{H}^*(X), \mathrm{H}^*\left(S^t\right)\right) \cong \mathrm{Ext}^s_{\mathcal{U}}\left(M, \Sigma^t \mathbb{F}_2\right),$$

where $\Sigma^t \mathbb{F}_2 \cong \widetilde{\mathrm{H}}^*\left(S^t\right)$. So, in these cases, we write the unstable Adams spectral sequence for X as:

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{U}}\left(M, \Sigma^t \mathbb{F}_2\right) \Longrightarrow \pi_{s-t}\left(\widehat{X_2}\right).$$

We will be discussing the unstable Adams spectral sequence for the stable special orthogonal group SO and the stable unitary group U. As $\mathrm{H}^*(SO) \cong U\left(\widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty)\right)$ and $\mathrm{H}^*(U) \cong U\left(\widetilde{\mathrm{H}}^*\left(\Sigma\left(\mathbb{C}P^\infty_+\right)\right)\right)$, the spectral sequences have the following form:

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}^s_{\mathcal{U}}\left(\widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty), \Sigma^t \mathbb{F}_2\right) \Longrightarrow \pi_{s-t}\left(\widehat{SO_2}\right), \\ E_2^{s,t} &= \mathrm{Ext}^s_{\mathcal{U}}\left(\widetilde{\mathrm{H}}^*\left(\Sigma\left(\mathbb{C}P^\infty_+\right)\right), \Sigma^t \mathbb{F}_2\right) \Longrightarrow \pi_{s-t}\left(\widehat{U_2}\right). \end{aligned}$$

Recall that $\mathrm{H}^*(\mathbb{R}P^\infty)$ is the polynomial algebra $\mathbb{F}_2[u]$ on one generator of degree 1 and

$$\widetilde{\mathrm{H}}^*\left(\Sigma\left(\mathbb{C}P^\infty_+\right)\right) \cong \Sigma \mathbb{F}_2[v]$$

where v is of degree 2 and the Steenrod action is determined by:

$$Sq^k u^n = \binom{n}{k} u^{n+k},$$

$$Sq^{2k} v^n = \binom{n}{k} v^{n+k}.$$

Let $\alpha(i)$ be the number of 1's in the dyadic expansion of i and filter $\widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty)$ and $\widetilde{\mathrm{H}}^*\left(\Sigma\left(\mathbb{C}P^\infty_+\right)\right)$by:

$$R_n = \left\{u^d \mid \alpha(d) \le n\right\},$$

$$C_n = \left\{\Sigma v^d \mid \alpha(d) \le n\right\}.$$

In [2], the authors examine the Postnikov tower of SO and get a spectral sequence:

$$E_2^{s,t} = \bigoplus_{n \ge 1} \mathrm{Ext}^s_{\mathcal{U}}\left(\frac{C_n}{C_{n-1}}, \Sigma^t \mathbb{F}_2\right) \Longrightarrow \pi_{s-t}\left(\widehat{U_2}\right)$$

A similar reasoning is followed for SO:

$$E_2^{s,t} = \bigoplus_{n\geq 1} \mathrm{Ext}^s_{\mathscr{U}}\left(\frac{R_n}{R_{n-1}}, \Sigma^t \mathbb{F}_2\right) \Longrightarrow \pi_{s-t}\left(\widehat{SO_2}\right)$$

Moreover, the authors show that these new spectral sequences can be constructed algebraically and they hope to identify these spectral sequences to the unstable Adams spectral sequences for U and SO respectively. The first step of the identification is the comparison of the E_2 terms.

Conjecture 1 (Bousfield's splitting conjecture). *There are isomorphisms:*

$$\bigoplus_{n\geq 1} \mathrm{Ext}^s_{\mathscr{U}}\left(\frac{R_n}{R_{n-1}}, \Sigma^t \mathbb{F}_2\right) \cong \mathrm{Ext}^s_{\mathscr{U}}\left(\widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right), \Sigma^t \mathbb{F}_2\right),$$

$$\bigoplus_{n\geq 1} \mathrm{Ext}^s_{\mathscr{U}}\left(\frac{C_n}{C_{n-1}}, \Sigma^t \mathbb{F}_2\right) \cong \mathrm{Ext}^s_{\mathscr{U}}\left(\widetilde{\mathrm{H}}^*\left(\Sigma\left(\mathbb{C}P^\infty_+\right)\right), \Sigma^t \mathbb{F}_2\right).$$

Mahowald suggests that the spectral sequences constructed in [2] can be demystified by taking the destabilization of the stable Adams resolutions of the connective so and u spectra. Following Mahowald's suggestions, in [5] Kathryn Lesh constructs other spectral sequences:

$$E_2^{s,t} = \bigoplus_{n\geq 1} \mathrm{Ext}^s_{\mathscr{U}}\left(\frac{R_n}{R_{n-1}}, \Sigma^t \mathbb{F}_2\right) \Longrightarrow \pi_{s-t}\left(\widehat{SO_2}\right),$$

$$E_2^{s,t} = \bigoplus_{n\geq 1} \mathrm{Ext}^s_{\mathscr{U}}\left(\frac{C_n}{C_{n-1}}, \Sigma^t \mathbb{F}_2\right) \Longrightarrow \pi_{s-t}\left(\widehat{U_2}\right)$$

Lesh also conjectures that these are other models for the unstable Adams spectral sequences of SO and U.

In this paper, we focus on the Bousfield splitting conjecture and get the following result.

Theorem 2 (Corollaries 17 and 41). *Conjecture 1 holds if either $s \leq 1$ or $s \geq [t/2]$.*

We will verify Conjecture 1 by computing both sides of the isomorphims. The key observation is that the Ext-groups appearing in this conjecture are $\mathbb{F}_2$−vector spaces of finite dimension. Therefore, in order to verify these isomorphisms, we only need to count the dimension of these vector spaces.

2. Unstable Modules

This section provides some basis facts of the category $\mathscr{U}$ of unstable modules used throughout the present paper. We refer to [9] for a thorough treatment of unstable modules.

2.1. *Projective unstable modules*

Denote by $\mathscr{V}_{\mathbb{F}_2}$ the category of $\mathbb{F}_2$−vector spaces. The functor $F_n : \mathscr{U} \to \mathscr{V}_{\mathbb{F}_2}, M \mapsto M^n$ is representable, and we denote by $F(n)$ the representing object. As the functor F_n is exact, $F(n)$ is a projective unstable module. For all unstable modules M, there is an epimorphism

$$\bigoplus_{\substack{x\in M^n,\\ n\geq 0}} F(n) \twoheadrightarrow M.$$

Therefore, the $F(n)$'s, $n \geq 0$ form a system of projective generators for the category $\mathscr{U}$ of unstable modules.

Definition 3. *For a set X of nonnegative integers, denote by $F(X)$ the direct sum of $F(n), n \in X$.*

Remark 4. As a morphism $F(a) \to F(b)$ is determined by a Steenrod operation of degree $a - b$, if X, Y are two sets of nonnegative integers of cardinal m and n, respectively, then we represent a morphism $\varphi : F(X) \to F(Y)$ by a matrix $M \in \mathrm{Mat}_{n,m}(\mathscr{A})$.

2.2. *The module $F(1)$ and the dyadic filtration*

It is straightforward from the definition that $F(n)$ has the following description:

$$F(n) \cong \frac{\Sigma^n \mathscr{A}}{\left\langle Sq^I \Sigma^n 1 \,\middle|\, I \text{ is admissible and } e(I) > n \right\rangle}.$$

Hereby, we denote the generator $\Sigma^n 1$ of $F(n)$ by ι_n.

From this description of $F(n)$, we can identify $F(1)$ as the submodule of $\widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty)$ generated by the variable u, that is $F(1) = \mathbb{F}_2\langle u, u^2, u^4, \dots\rangle$ and

$$Sq^n u^{2^k} = \begin{cases} u^{2^{k+1}} & \text{if } n = 2^k \\ u^{2^k} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We can use the module $F(1)$ to describe the composition series of the dyadic filtration on $\widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty)$ and $\widetilde{\mathrm{H}}^*\left(\Sigma\left(\mathbb{C}P^\infty_+\right)\right)$. Recall from the introduction that for all $n \geq 1$, we have

$$R_n := \left\{u^d \,\middle|\, \alpha(d) \leq n\right\}.$$

Note that $R_1 = F(1)$. Moreover, the morphism

$$R_m \otimes R_n \to \widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right) \otimes \widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right) \xrightarrow{\text{mult}} \widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right)$$

is surjective onto R_{m+n}, for all $m, n \geq 1$. Denote by Λ^n the n-th exterior power functor, then the following result is straightforward.

Lemma 5. *The epimorphism $F(1)^{\otimes n} \to R_n$ induces an isomorphism of unstable modules:*

$$\Lambda^n(F(1)) \xrightarrow{\cong} \frac{R_n}{R_{n-1}}$$

for all $n \geq 1$.

There is a clean way to describe the dyadic filtration of $\widetilde{\mathrm{H}}^*\left(\Sigma\left(\mathbb{C}P^\infty_+\right)\right)$ using that of $\widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty)$. To this end, we will need some endofunctors of the category $\mathscr{U}$: the suspension functor Σ and the doubling functor Φ.

The suspension functor $\Sigma : \mathscr{U} \to \mathscr{U}$ is easy to describe. What ΣM is to the unstable module M is what $\widetilde{\mathrm{H}}^*(\Sigma X)$ is to $\widetilde{\mathrm{H}}^*(X)$, that is

$$(\Sigma M)^n = \begin{cases} M^{n-1} & \text{if } n \geq 1, \\ \{0\} & \text{if } n = 0, \end{cases}$$

and the Steenrod action is defined by

$$Sq^n \Sigma x = \Sigma Sq^n x.$$

The functor Σ is exact and admits a left adjoint Ω, called the loop functor of unstable modules. The loop functor Ω is right exact, and we denote by Ω_s the $s-$th left derived functor of Ω.

To describe the doubling functor $\Phi : \mathscr{U} \to \mathscr{U}$, recall that the Steenrod operations represent the Frobenius twist of an unstable algebra. Therefore, we can use the Steenrod action to define the Frobenius twist of an unstable module as follows. For an unstable module M, let Sq_0 be the operator that associates an element x with $Sq^{|x|}x$. The operator Sq_0 defines a map $M \to M$. However, this map is not of degree 0. For this reason, we introduce the functor $\Phi : \mathscr{U} \to \mathscr{U}$, associating an unstable module M with ΦM, such that:

$$(\Phi M)^n = \begin{cases} M^k & \text{if } n = 2k, \\ \{0\} & \text{otherwise,} \end{cases}$$

and the Steenrod action is defined by

$$Sq^n\Phi x = \begin{cases} \Phi Sq^k x & \text{if } n = 2k, \\ \{0\} & \text{otherwise.} \end{cases}$$

Now, we can represent the operator Sq_0 as the natural transformation $\lambda : \Phi \to \mathrm{Id}$, sending each unstable module M to the morphism

$$\begin{aligned} \lambda_M : \Phi M &\longrightarrow M, \\ \Phi x &\longmapsto Sq_0 x. \end{aligned}$$

It turns out that the kernel, as well as the cokernel of λ_M can be determined using Σ and Ω.

Proposition 6 ([8, Proposition 2.4],[9, Proposition 1.7.3]). *Let M be an unstable module. Then $\Omega_s M = \{0\}$ for all $s > 1$. Moreover, $\Omega_1 M$ and ΩM fit in the following natural exact sequence:*

$$0 \to \Sigma\Omega_1 M \to \Phi M \xrightarrow{\lambda_M} M \xrightarrow{\sigma_M} \Sigma\Omega M \to 0,$$

where σ_M is the unit of the adjunction $(\Omega \dashv \Sigma)$.

With the help of Σ and Φ, we can write:

$$\widetilde{\mathrm{H}}^*\left(\Sigma\left(\mathbb{C}P^\infty_+\right)\right) = \Sigma\Phi\mathrm{H}^*\left(\mathbb{R}P^\infty\right)$$

and

$$C_n = \left\{\Sigma v^d \,\middle|\, \alpha(d) \le n\right\} = \Sigma\Phi\left(R_n \oplus \mathbb{F}_2\right).$$

It follows from Lemma 5 that we have:

Lemma 7. *There are isomorphisms of unstable modules:*

$$\Sigma\Phi\Lambda^n\left(F(1)\right) \xrightarrow{\cong} \frac{C_n}{C_{n-1}}$$

for all $n \ge 1$.

2.3. *The loop functor of unstable modules*

Now, we compute the loop functor Ω on $\widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right)$ and $\Lambda^n\left(F(1)\right)$ for all $n \ge 1$.

Proposition 8. *We have an isomorphism of unstable modules:*

$$\Omega\mathrm{H}^*\left(\mathbb{R}P^\infty\right) \cong \Phi\mathrm{H}^*\left(\mathbb{R}P^\infty\right).$$

Proof. Recall that $\mathrm{H}^*\left(\mathbb{R}P^\infty\right)$ is the polynomial algebra $\mathbb{F}_2[u]$ on one generator u of degree 1. Define an $\mathbb{F}_2$−linear transformation $\alpha : \mathbb{F}_2[u] \to \Sigma\Phi\mathbb{F}_2[u]$ by:

$$\alpha\left(u^n\right) = \begin{cases} \Sigma\Phi u^k & \text{if } n = 2k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Because of the formulae

$$Sq^k u^{2n} = \begin{cases} \binom{n}{m} u^{2m+2n} & \text{if } k = 2m, \\ 0 & \text{otherwise,} \end{cases}$$

$$Sq^k\Sigma\Phi u^n = \begin{cases} \Sigma\Phi\binom{n}{m} u^{m+n} & \text{if } k = 2m, \\ 0 & \text{otherwise,} \end{cases}$$

then α is also a morphism over the Steenrod algebra. It is clear that α is surjective and the sequence

$$0 \to \Phi\mathbb{F}_2[u] \xrightarrow{\lambda_{\mathbb{F}_2[u]}} \mathbb{F}_2[u] \xrightarrow{\alpha} \Sigma\Phi\mathbb{F}_2[u] \to 0$$

is exact. Therefore, the conclusion follows from Proposition 6. □

Corollary 9. *We have an isomorphism of unstable modules:*

$$\Omega\widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right) \cong \Phi\mathrm{H}^*\left(\mathbb{R}P^\infty\right).$$

Proof. It follows from the fact that $\mathrm{H}^*(\mathbb{R}P^\infty) \cong \widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty) \oplus \mathbb{F}_2$ and $\Omega\mathbb{F}_2 \cong \{0\}$. □

Similarly, we have:

Proposition 10. *We have an isomorphism of unstable modules $\Omega R_n \cong \Phi R_{n-1}$ for all $n \geq 1$.*

Corollary 11. *We have an isomorphism of unstable modules $\Omega\Lambda^n(F(1)) \cong \Phi\Lambda^{n-1}(F(1))$ for all $n \geq 1$.*

2.4. *Brown–Gitler modules*

For an $\mathbb{F}_2$−vector space M, denote by $M^\sharp$ the linear dual $\mathrm{Hom}_{\mathbb{F}_2}(M,\mathbb{F}_2)$. The functor $J^n : \mathscr{U} \to \mathscr{V}_{\mathbb{F}_2}, M \mapsto M^{n,\sharp}$ from the category $\mathscr{U}$ of unstable modules to the category of $\mathbb{F}_2$−vector spaces is representable for all natural numbers n (see [9, Chapter 2]). Denote by $J(n)$ the representing object. As the functor J^n is exact, $J(n)$ in an injective unstable module for all integers $n \geq 0$. It is cogenerated in an unstable way by an element ι_n of degree n, that is,

$$J(n)^k = \begin{cases} \{0\} & \text{if } k > n, \\ \mathbb{F}_2\langle\iota_n\rangle & \text{if } k = n, \\ \mathbb{F}_2\left\langle\theta_I \,\middle|\, Sq^I \text{ is admissible of degree } n-k, e\left(Sq^I\right) \leq k\right\rangle & \text{if } k < n, \end{cases}$$

where $Sq^I\theta_I = \iota_n$. As

$$\mathrm{Hom}_{\mathscr{U}}(J(n), J(m)) \cong J(n)^\sharp,$$

a morphism from $J(n)$ to $J(m)$ is determined by a Steenrod operation θ. Follows [9], we denote such morphism by $\bullet\theta$. The composition law is

$$(\bullet\omega)\circ(\bullet\sigma) = \bullet(\sigma\omega).$$

For all unstable modules M, the morphism

$$M \hookrightarrow \prod_{\substack{x\in M^{n,\sharp},\\ n\geq 0}} J(n)$$

is injective. It follows that the $J(n)$'s form a system of injective cogenerators for the category $\mathscr{U}$.

3. Injective resolutions and the conjecture

In [7], the author develops a new technique to construct the minimal injective resolution of $\Sigma^t\mathbb{F}_2$. In this section, we will use these constructions to verify Conjecture 1 for $s \geq [t/2]$.

Recall that the squaring of the Bockstein operation Sq^1 is trivial. Therefore, the Bockstein operation Sq^1 defines the following complex:

Definition 12. *The sequence*

$$\cdots \xrightarrow{\bullet Sq^1} J(4k) \xrightarrow{\binom{\bullet Sq^1}{\bullet Sq^{2k}}} \begin{matrix} J(4k-1) \\ \oplus \\ J(2k) \end{matrix} \xrightarrow{\left(\bullet Sq^1, \bullet Sq^{2k}\right)} J(4k-2) \xrightarrow{\bullet Sq^1} J(4k-3) \xrightarrow{\bullet Sq^1} \cdots$$

is called the Bockstein complex of Brown–Gitler modules.

Here the same matrix notation as for direct sums of free modules is used for morphisms between direct sum of Brown–Gitler modules. We write the Bockstein complex cohomologically as $\left(\mathscr{B}^{-k},\beta^{-k}:\mathscr{B}^{-k}\to\mathscr{B}^{1-k}\right)_{k\geq 1}$, where

$$\mathscr{B}^{-k}=\begin{cases}J(4m-1)\oplus J(2m) & \text{if } k=4m-1,\\ J(k) & \text{otherwise.}\end{cases}$$

Lemma 13 ([7, Proposition 7.8]). *The Bockstein complex is exact.*

It turns out that the Bockstein complex plays an important role in the minimal injective resolution of $\Sigma^t\mathbb{F}_2$.

Theorem 14 ([7, Proposition 7.8]). *For all integers $t\geq 1$, the injective dimension of $\Sigma^t\mathbb{F}_2$ is $t-1$. Moreover, if $\left(I^k,\partial^k\right)_{t-1\geq k\geq 0}$ is the minimal injective resolution of $\Sigma^t\mathbb{F}_2$, then each term I^k is a finite direct sum of Brown–Gitler modules and for $s\geq[t/2]$, we have:*

$$I^s\cong\mathscr{B}^{s-t}.$$

Corollary 15. *For all integers $t\geq 1$ and $t-1\geq s\geq[t/2]$, we have:*

$$\operatorname{Ext}^s_{\mathscr{U}}\left(\widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right),\Sigma^t\mathbb{F}_2\right)=\begin{cases}\mathbb{F}_2 & \textit{if } t-s\equiv 3(4),\\ \{0\} & \textit{otherwise,}\end{cases}$$

$$\operatorname{Ext}^s_{\mathscr{U}}\left(\frac{R_n}{R_{n-1}},\Sigma^t\mathbb{F}_2\right)=\begin{cases}\mathbb{F}_2 & \textit{if } t-s\equiv 3(4) \textit{ and } \alpha(t-s)=n,\\ \{0\} & \textit{otherwise.}\end{cases}$$

Proof. Denote $\widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right)$ by $\widetilde{H}$. It follows from Theorem 14 that we have

$$\begin{aligned}\operatorname{Hom}_{\mathscr{U}}\left(\widetilde{H},I^s\right)&\cong\operatorname{Hom}_{\mathscr{U}}\left(\widetilde{H},\mathscr{B}^{s-t}\right)\\ &\cong\begin{cases}\widetilde{H}^{t-s,\sharp}\oplus\widetilde{H}^{\left(\frac{t-s+1}{2}\right),\sharp} & \text{if } t-s\equiv 3(4),\\ \widetilde{H}^{t-s,\sharp} & \text{otherwise,}\end{cases}\\ &\cong\begin{cases}\mathbb{F}_2^{\oplus 2} & \text{if } t-s\equiv 3(4),\\ \mathbb{F}_2 & \text{otherwise,}\end{cases}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Hom}_{\mathscr{U}}\left(\widetilde{H},\partial^s\right)&\cong\operatorname{Hom}_{\mathscr{U}}\left(\widetilde{H},\beta^{s-t}\right)\\ &\cong\begin{cases}\binom{\mathrm{Id}}{\mathrm{Id}}:\mathbb{F}_2\to\mathbb{F}_2^{\oplus 2} & \text{if } t-s\equiv 0(4),\\ 0:\mathbb{F}_2\to\mathbb{F}_2 & \text{if } t-s\equiv 1(4),\\ \mathrm{Id}:\mathbb{F}_2\to\mathbb{F}_2 & \text{if } t-s\equiv 2(4),\\ (0,0):\mathbb{F}_2^{\oplus 2}\to\mathbb{F}_2 & \text{if } t-s\equiv 3(4).\end{cases}\end{aligned}$$

It follows that we have

$$\operatorname{Ext}^s_{\mathscr{U}}\left(\widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right),\Sigma^t\mathbb{F}_2\right)=\begin{cases}\mathbb{F}_2 & \text{if } t-s\equiv 3(4),\\ \{0\} & \text{otherwise.}\end{cases}$$

Now, as $R_n/R_{n-1}\cong\Lambda^n(F(1))$, we have:

$$\begin{aligned}\operatorname{Hom}_{\mathscr{U}}\left(\frac{R_n}{R_{n-1}},I^s\right)&\cong\operatorname{Hom}_{\mathscr{U}}\left(\Lambda^n(F(1)),\mathscr{B}^{s-t}\right)\\ &\cong\begin{cases}\Lambda^n(F(1))^{t-s,\sharp}\oplus\Lambda^n(F(1))^{\left(\frac{t-s+1}{2}\right),\sharp} & \text{if } t-s\equiv 3(4),\\ \Lambda^n(F(1))^{t-s,\sharp} & \text{otherwise,}\end{cases}\\ &\cong\begin{cases}\mathbb{F}_2 & \text{if } \alpha(t-s)=n,\\ \mathbb{F}_2 & \text{if } \alpha(t-s+1)=n, \text{ and } t-s\equiv 3\,(4),\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_{\mathcal{U}}\left(\widetilde{H},\partial^{s}\right) &\cong \mathrm{Hom}_{\mathcal{U}}\left(\widetilde{H},\beta^{s-t}\right)\\ &\cong \begin{cases}\mathrm{Id}:\mathbb{F}_2\to\mathbb{F}_2 & \text{if } t-s\equiv 0(4),\quad \alpha(t-s)=n,\\ \mathrm{Id}:\mathbb{F}_2\to\mathbb{F}_2 & \text{if } t-s\equiv 2(4),\quad \alpha(t-s)=n,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

□

Similarly, we have:

Corollary 16. *For all integers $t \geq 1$ and $t-1 \geq s \geq [t/2]$, we have:*

$$\mathrm{Ext}^{s}_{\mathcal{U}}\left(\widetilde{\mathrm{H}}^{*}\left(\Sigma\left(\mathbb{C}P^{\infty}_{+}\right)\right),\Sigma^{t}\mathbb{F}_2\right) = \begin{cases}\mathbb{F}_2 & \text{if } t-s\equiv 1(2),\\ \{0\} & \text{otherwise,}\end{cases}$$

$$\mathrm{Ext}^{s}_{\mathcal{U}}\left(\frac{C_n}{C_{n-1}},\Sigma^{t}\mathbb{F}_2\right) = \begin{cases}\mathbb{F}_2 & \text{if } t-s\equiv 1(2) \text{ and } \alpha(t-s-1)=n,\\ \{0\} & \text{otherwise.}\end{cases}$$

We can now make the following conclusion.

Corollary 17. *For all integers $t \geq 1$ and $t-1 \geq s \geq [t/2]$, we have:*

$$\bigoplus_{n\geq 1}\mathrm{Ext}^{s}_{\mathcal{U}}\left(\frac{R_n}{R_{n-1}},\Sigma^{t}\mathbb{F}_2\right) \cong \mathrm{Ext}^{s}_{\mathcal{U}}\left(\widetilde{\mathrm{H}}^{*}\left(\mathbb{R}P^{\infty}\right),\Sigma^{t}\mathbb{F}_2\right),$$

$$\bigoplus_{n\geq 1}\mathrm{Ext}^{s}_{\mathcal{U}}\left(\frac{C_n}{C_{n-1}},\Sigma^{t}\mathbb{F}_2\right) \cong \mathrm{Ext}^{s}_{\mathcal{U}}\left(\widetilde{\mathrm{H}}^{*}\left(\Sigma\left(\mathbb{C}P^{\infty}_{+}\right)\right),\Sigma^{t}\mathbb{F}_2\right).$$

4. Projective resolutions and the conjecture

In this section, we study the minimal projective resolution of the objects of interest and provide some other computations supporting Conjecture 1, namely the computations of Hom and Ext^1 groups.

4.1. *Minimal projective resolutions*

We recall the definition of the minimal projective resolution of an unstable module and some basic properties.

Definition 18. *Let M be an unstable module. A projective resolution $(P_k,\partial_k)_{k\geq 0}$ of M is called* minimal *if P_0 is the projective cover of M, P_1 is the projective cover of* $\mathrm{Ker}(P_0 \twoheadrightarrow M)$*, and P_k is that of* $\mathrm{Ker}(\partial_{k-1})$ *for all integers $k \geq 2$.*

Remark 19. The projective cover of a given module is unique up to isomorphisms so it follows from standard argument that two minimal projective resolutions of the same unstable module are isomorphic.

Recall that an unstable module M is of finite type if M^n is an $\mathbb{F}_2$−vector space of finite dimension for all $n \geq 0$.

Lemma 20. *Let M be an unstable module of finite type. Suppose that the sequence*

$$P_1 \xrightarrow{\partial} P_0 \to M \to 0$$

is exact where P_0 and P_1 are projective unstable modules. Then, P_0 is the projective cover of M if and only if for all integers $n \geq 0$, there are no morphisms $\phi: F(n) \to P_1$ and $\psi: P_0 \to F(n)$ such that $\psi\circ\partial\circ\phi$ is an isomorphism.

Proof. Suppose that P_0 is the projective cover of M, then P_0 is a direct sum of free unstable modules $F(m)$. Assume that there are morphisms $\phi : F(n) \to P_1$ and $\psi : P_0 \to F(n)$ such that $\psi \circ \partial \circ \phi$ is an isomorphism. Then, it follows that ψ is surjective. So ψ has a section, called $\overline{\psi}$. Therefore, there is an exact sequence

$$\operatorname{Coker}(\phi) \to \operatorname{Coker}(\overline{\psi}) \to M \to 0.$$

On the one hand, $\operatorname{Coker}(\overline{\psi})$ is a projective unstable module. On the other hand, as M is of finite type, then the number of summands $F(n)$ of P_0 is finite. Therefore, the number of summands $F(n)$ of $\operatorname{Coker}(\overline{\psi})$ is strictly less than that of P_0. It follows that there is no surjection from $\operatorname{Coker}(\overline{\psi})$ to M, whence a contradiction.

Now, if P_0 is not the projective cover of M, then there is an epimorphism from P_0 to the projective cover P of M with nontrivial kernel. Let $F(n)$ be a direct summand of P_0 lying in the kernel of the projection $P_0 \to P$, then the composition

$$P_1 \xrightarrow{\partial} P \to F(n)$$

is surjective. It follows that P_1 contains a direct summand $F(n)$ such that the composition

$$F(n) \hookrightarrow P_1 \xrightarrow{\partial} P_0 \twoheadrightarrow F(n)$$

is an isomorphism. The conclusion follows. □

Therefore, for unstable modules of finite type, we get the following simple characterization of minimal projective resolutions.

Proposition 21. *Let M be an unstable module of finite type. A projective resolution $(P_k, \partial_k)_{k\geq 0}$ of M is minimal if and only if for all integers $n, k \geq 0$, there are no morphisms $\phi : F(n) \to P_{k+1}$ and $\psi : P_k \to F(n)$ such that $\psi \circ \partial_{k+1} \circ \phi$ is an isomorphism.*

Corollary 22. *Let M be an unstable module of finite type and let $(P_k, \partial_k)_{k\geq 0}$ be the minimal projective resolution of M. Then, there are isomorphisms*

$$\operatorname{Ext}^s_{\mathcal{U}}\left(M, \Sigma^n \mathbb{F}_2\right) \cong \operatorname{Hom}_{\mathcal{U}}\left(P_s, \Sigma^n \mathbb{F}_2\right)$$

of $\mathbb{F}_2-$vector spaces for all integers $s \geq 0$.

Proof. Note that

$$\operatorname{Hom}_{\mathcal{U}}\left(F(k), \Sigma^n \mathbb{F}_2\right) \cong \begin{cases} \mathbb{F}_2 & \text{if } k = n, \\ \{0\} & \text{otherwise.} \end{cases}$$

Following Proposition 21, if $\left(P^k, \partial_k\right)_{k\geq 0}$ is the minimal projective resolution of M, then

$$\operatorname{Hom}_{\mathcal{U}}\left(\partial^\bullet, \Sigma^n \mathbb{F}_2\right) = 0.$$

The conclusion follows. □

Corollary 23. *Let M be an unstable module of finite type such that $\Omega_1 M = \{0\}$. If $(P_k, \partial_k)_{k\geq 0}$ is the minimal projective resolution of M, then $(\Omega P_k, \Omega\partial_k)_{k\geq 0}$ is the minimal projective resolution of ΩM.*

Proof. From [8, Lemma 2.8], $(\Omega P_k, \Omega\partial_k)_{k\geq 0}$ is a projective resolution of ΩM because $\Omega_1 M = 0$. Due to Proposition 21, this resolution is minimal. □

4.2. *Pseudo-hyperresolutions*

In this paragraph, we use pseudo-hyperresolutions to construct projective resolutions of the objects of interest: $\widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty)$, $\Sigma\Phi\mathrm{H}^*(\mathbb{R}P^\infty)$, $\Lambda^n(F(1))$ and $\Sigma\Phi\Lambda^n(F(1))$ for all $n \geq 1$. Our aim is modest: we only compute the first and second terms of the minimal projective resolutions of these objects for they give access to the computations of Hom and Ext^1 groups. Because of the following lemma, if we can verify Conjecture 1 for $\mathbb{R}P^\infty$, then the result for $\mathbb{C}P^\infty$ also holds. For this reason, we focus only on $\mathbb{R}P^\infty$.

Lemma 24. *If the isomorphism*

$$\bigoplus_{n\geq 1}\mathrm{Ext}^s_{\mathscr{U}}\left(\frac{R_n}{R_{n-1}},\Sigma^t\mathbb{F}_2\right)\cong \mathrm{Ext}^s_{\mathscr{U}}\left(\widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right),\Sigma^t\mathbb{F}_2\right),$$

holds, then so does Conjecture 1.

Proof. It follows from [8, Lemma 3.4] that we have:

$$\begin{aligned}
\mathrm{Ext}^s_{\mathscr{U}}\left(\Sigma\Omega\widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right),\Sigma^t\mathbb{F}_2\right) &\cong \mathrm{Ext}^s_{\mathscr{U}}\left(\Omega\widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right),\Sigma^{t-1}\mathbb{F}_2\right)\oplus \mathrm{Ext}^{s-1}_{\mathscr{U}}\left(\Omega\Phi\widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right),\Sigma^{t-1}\mathbb{F}_2\right)\\
&\cong \mathrm{Ext}^s_{\mathscr{U}}\left(\widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right),\Sigma^{t}\mathbb{F}_2\right)\oplus \mathrm{Ext}^{s-1}_{\mathscr{U}}\left(\Phi\widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right),\Sigma^{t}\mathbb{F}_2\right),\\
\mathrm{Ext}^s_{\mathscr{U}}\left(\Sigma\Omega\frac{R_n}{R_{n-1}},\Sigma^t\mathbb{F}_2\right) &\cong \mathrm{Ext}^s_{\mathscr{U}}\left(\Omega\frac{R_n}{R_{n-1}},\Sigma^{t-1}\mathbb{F}_2\right)\oplus \mathrm{Ext}^{s-1}_{\mathscr{U}}\left(\Omega\Phi\frac{R_n}{R_{n-1}},\Sigma^{t-1}\mathbb{F}_2\right)\\
&\cong \mathrm{Ext}^s_{\mathscr{U}}\left(\frac{R_n}{R_{n-1}},\Sigma^{t}\mathbb{F}_2\right)\oplus \mathrm{Ext}^{s-1}_{\mathscr{U}}\left(\Phi\frac{R_n}{R_{n-1}},\Sigma^{t}\mathbb{F}_2\right).
\end{aligned}$$

Note that by Corollary 9 and Proposition 10 together with the right exactness of Ω we have

$$\Omega\widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right)\cong \Phi\mathrm{H}^*\left(\mathbb{R}P^\infty\right)$$
$$\Omega\frac{R_n}{R_{n-1}}\cong\frac{\Phi R_n}{\Phi R_{n-1}}.$$

Now, it follows from Lemma 7 that we have

$$\frac{\Phi R_n}{\Phi R_{n-1}}\cong\frac{C_n}{C_{n-1}}$$

Then, thanks to Corollary 23, we have an isomorphism

$$\bigoplus_{n\geq 1}\mathrm{Ext}^s_{\mathscr{U}}\left(\frac{C_n}{C_{n-1}},\Sigma^t\mathbb{F}_2\right)\cong \mathrm{Ext}^s_{\mathscr{U}}\left(\Phi\mathrm{H}^*\left(\mathbb{R}P^\infty\right),\Sigma^t\mathbb{F}_2\right).$$

The conclusion follows. □

Thanks to this lemma, it suffices to study the minimal projective resolution of $\widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty)$ and $\Lambda^n(F(1))$ for all $n \geq 1$. Note that $\Lambda^1(F(1)) \cong F(1)$ is projective, so we only need to study the cases $n \geq 2$.

Observe that if Sq^I is an admissible Steenrod operation, then $\bullet Sq^I : J(m) \to J(n)$ is nontrivial if and only if the excess of Sq^I is less than or equal to n. It follows that the equality $\bullet\alpha = \bullet\beta : J(m) \to J(n)$ implies that $\alpha - \beta$ is a sum of admissible operations of excess greater than n. That is what we use to verify the following technical computation of Steenrod operations.

Lemma 25. *Let Sq^aSq^b be admissible of excess n. Then, every admissible term in $Sq^aSq^b - Sq^{a+b-n}Sq^n$ is of excess strictly greater than n.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc}
\Sigma J(2a+2b+2n-1) & \xrightarrow{\Sigma \bullet Sq^{2a}Sq^{2b}} & \Sigma J(2n-1) \\
\downarrow & & \downarrow \\
J(2a+2b+2n) & \xrightarrow{\bullet Sq^{2a}Sq^{2b}} & J(2n) \\
\bullet Sq^{a+b+n} \downarrow & & \downarrow \bullet Sq^{n} \\
J(a+b+n) & \xrightarrow{\bullet Sq^{a}Sq^{b}} & J(n)
\end{array}$$

Note that $a-b=n$ then $a=b+n$ and $a+b+n=2b+2n$. Therefore, we have

$$Sq^{a+b+n}Sq^{a+b-n} = Sq^{2a}Sq^{2b}.$$

Hence, we have

$$\begin{aligned}
\left(\bullet Sq^{n}\right)\circ\left(\bullet Sq^{a+b-n}\right)\left(\bullet Sq^{a+b+n}\right) &= \left(\bullet Sq^{n}\right)\circ\left(\bullet Sq^{2a}Sq^{2b}\right) \\
&= \left(\bullet Sq^{a}Sq^{b}\right)\circ\left(\bullet Sq^{a+b+n}\right).
\end{aligned}$$

As $\bullet Sq^{a+b+n}$ is surjective, we have:

$$\left(\bullet Sq^{a}Sq^{b}\right) = \left(\bullet Sq^{n}\right)\circ\left(\bullet Sq^{a+b-n}\right) = \bullet\left(Sq^{a+b-n}Sq^{n}\right)$$

whence the conclusion. □

Lemma 26. *If a,b are two odd numbers such that $a<2b$, then*

$$Sq^{a}Sq^{b} = \sum_{i=0}^{([a/2]-1)/2}\binom{b-2i-2}{a-4i-2}Sq^{a+b-2i-1}Sq^{2i+1}.$$

Proof. Let $m > a+b$ be an integer and let $n = 2m+b$ and $2k = n+a$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
J(2k) & \xrightarrow{\bullet Sq^{a}} & J(n) & \xrightarrow{\bullet Sq^{b}} & J(2m) \\
\bullet Sq^{k} \downarrow & & & & \downarrow \bullet Sq^{m} \\
J(k) & \xrightarrow[\tilde{\Phi}(\bullet Sq^{a})]{} & 0 & \xrightarrow[\tilde{\Phi}(\bullet Sq^{b})]{} & J(m)
\end{array}$$

Note that, we have:

(i) $\widetilde{\Phi}\left(\bullet Sq^{a}\right) = \widetilde{\Phi}\left(\bullet Sq^{b}\right) = 0$;
(ii) $\widetilde{\Phi}\left(\bullet Sq^{b}\right)\circ\widetilde{\Phi}\left(\bullet Sq^{a}\right) = \widetilde{\Phi}\left(\bullet Sq^{a}Sq^{b}\right)$.

The conclusion follows. □

Lemma 27. *Let M be a connected unstable module, that is $M^0 = \{0\}$. If $\lambda_M : \Phi M \to M, x \mapsto Sq_0 x$ is surjective then $M = \{0\}$.*

Proof. As ΦM is concentrated in even degree and $\lambda_M : \Phi M \to M$ is surjective, then M is also concentrated in even degree. A simple induction on $k \geq 1$ shows that M is concentrated in degree divisible by 2^k. Hence, $M^n = \{0\}$ for all $n \geq 1$. Moreover, as M is connected, we have $M^0 = \{0\}$. Then, we have $M = \{0\}$. □

The following result is a mean to construct a projective resolution for M from that of ΩM.

Corollary 28. *Let M be a connected unstable module such that $\Omega_1 M = \{0\}$. Let $\mathscr{P}$ be a complex*

$$P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_n} P_0 \to M \tag{1}$$

in which each P_i, $0 \le i \le n$, is projective. If $\Omega\mathscr{P}$ is exact, then so is $\mathscr{P}$.

Proof. Note that the functors Φ and Σ are exact. Then, the long exact sequence associated with the following short exact sequence of complexes

$$0 \to \Phi\mathscr{P} \to \mathscr{P} \to \Sigma\Omega\mathscr{P} \to 0$$

yields the triviality of the homology groups $\mathrm{H}^s(\mathscr{P})$ for all $n-1 \ge s \ge 1$. Moreover, we get the following commutative diagram

$$\begin{array}{ccc}
\Phi\mathrm{H}^0(\mathscr{P}) & \xrightarrow{\Phi f} & \Phi M \\
\downarrow & & \downarrow \\
\mathrm{H}^0(\mathscr{P}) & \xrightarrow{f} & M \\
\downarrow & & \downarrow \\
\Sigma\mathrm{H}^0(\Omega\mathscr{P}) & \xrightarrow{\Sigma\Omega f} & \Sigma\Omega M
\end{array}$$

where both columns are short exact sequences. Now, it follows from the Snake lemma that both $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are connected. Moreover, by assumption $\Omega\mathscr{P}$ is exact so $\Omega(f)$ is an isomorphism. Therefore, we get the surjectivity of $\Phi\mathrm{Ker}(f) \to \mathrm{Ker}(f)$ and $\Phi\mathrm{Coker}(f) \to \mathrm{Coker}(f)$. It follows from Lemma 27 that f is an isomorphism whence the conclusion. □

Remark 29. Note that we have $\Omega\Phi F(n) \cong \Sigma\Phi F(n-1)$. Moreover, $\Omega_1\Phi F(n) = \{0\}$. Therefore, we can use Lemma 28 to construct projective resolutions of $\Phi F(n)$ by induction on n. We start with the case $n = 1$.

Lemma 30. *The following sequence is exact*

$$\cdots \to F(n) \xrightarrow{Sq^1} F(n-1) \xrightarrow{Sq^1} \cdots \xrightarrow{Sq^1} F(3) \xrightarrow{Sq^1} F(2) \to \Phi F(1) \to 0$$

Proof. Recall that the module $F(n)$ has an additive basis

$$\left\{Sq^I \iota_n \,\middle|\, I \text{ is admissible and } ex(I) \le n\right\}$$

and the morphism $Sq^1 : F(n) \to F(n-1)$ sends $Sq^I\iota_n$ to $Sq^I Sq^1 \iota_{n-1}$. If $I = (I_1, 1)$ then $Sq^I Sq^1 = 0$. Otherwise, if $I = (i_1, i_2, \ldots, i_k)$ is admissible of excess less than or equal to n and $i_k \ge 2$ then $Sq^I Sq^1$ is admissible of excess less than or equal to $n-1$. Therefore, the kernel of $Sq^1 : F(n) \to F(n-1)$ has an additive basis

$$\left\{Sq^I \iota_n \,\middle|\, I = (i_1, i_2, \ldots, i_k, 1) \text{ is admissible and } ex(I) \le n\right\}$$

This is also an additive basis of the image of the morphism $Sq^1 : F(n+1) \to F(n)$. Therefore, the following complex is acyclic

$$\cdots \to F(n) \xrightarrow{Sq^1} F(n-1) \xrightarrow{Sq^1} \cdots \xrightarrow{Sq^1} F(2) \xrightarrow{Sq^1} F(1)$$

Because the image of $Sq^1 : F(2) \to F(1)$ is $\Phi F(1)$, then we get the desired conclusion. □

The sequence

$$\cdots \to F(n) \xrightarrow{Sq^1} F(n-1) \xrightarrow{Sq^1} \cdots \xrightarrow{Sq^1} F(3) \xrightarrow{Sq^1} F(2) \to \Phi F(1) \to 0$$

is the minimal projective resolution of $\Phi F(1)$. We now use pseudo-hyperresolutions to give access to projective resolutions of $\Phi F(n)$ for $n \ge 2$. For this, we introduce the following notations.

Definition 31. *Let $P(1) = F(4)$, $Q(1) = F(3)$. For $n \geq 2$, define:*

$$\begin{aligned}
&A_n = \{4n+1, 4n+5, \dots, 8n-7\}, && \\
&B_n = \{4n, 4n+2, \dots, 8n-4\}, && D_n = \{4n-1\}, \\
&C_n = \{k+2 \mid k \in X_{n-1}\}, && E_n = \left\{k+2 \mid k \in Q_{n-1}\right\}, \\
&P_n = A_n \sqcup B_n \sqcup C_n, && Q_n = D_n \sqcup E_n,
\end{aligned}$$

and for $T \in \left\{A, B, C, D, E, P, Q\right\}$, let $T(n) = F(T_n)$. We also write:

$$\begin{aligned}
a_n &:= \left(Sq^2, Sq^6, \dots, Sq^{4n-6}\right) &&: A(n) \to D(n), \\
d_n &:= \operatorname{diag}\left(Sq^{2n}, Sq^{2n+2}, \dots, Sq^{4n-4}\right) &&: A(n) \to E(n), \\
\tau_n &:= \left(Sq^1, Sq^3, \dots, Sq^{2n-1}\right) &&: Q(n) \to F(2n).
\end{aligned}$$

Lemma 32. *For all $n \geq 1$, there exists an exact sequence*

$$P(n) \xrightarrow{\sigma_n} Q(n) \xrightarrow{\tau_n} F(2n) \to \Phi F(n) \to 0 \tag{2}$$

such that $\sigma_1 = Sq^1$ and for $n \geq 2$, we have

$$\sigma_n = \begin{pmatrix} \tau_{2n-1} & *_1 \\ a_n & d_n \\ *_2 & \sigma_{n-1} \end{pmatrix} : B(n) \oplus A(n) \oplus C(n) \to D(n) \oplus E(n).$$

Proof. We proceed by induction on n. The case $n = 1$ holds by Remark 29. The induction argument goes as follows. Assume that the result holds for $n \leq m$. We now consider the case $n = m+1$. The sequence

$$0 \to \Phi F\left(q\right) \xrightarrow{\Phi x \mapsto Sq_0 x} F\left(q\right) \xrightarrow{\iota_q \mapsto \Sigma \iota_{q-1}} \Sigma F\left(q-1\right) \to 0$$

is exact by Proposition 6. Therefore, the following sequence

$$F\left(2q\right) \xrightarrow{\iota_{2q} \mapsto Sq^q \iota_q} F\left(q\right) \to \Sigma F\left(q-1\right) \to 0$$

provides the first and the second term in the minimal projective resolution of $\Sigma F\left(q-1\right)$ for all integers $q \geq 1$. If X is a set of integers, and $s \in \mathbb{Z}$, then we write $s + X = \{s + k | k \in X\}$. Consider the following diagram:

$$\begin{array}{ccccc}
 & & & & F(1+B_{m+1}) \\
 & & & & \downarrow{\scriptstyle \tau_{2m+1}} \\
 & & F(-1+A_{m+1}) & \xrightarrow{a_{m+1}} & F(4m+2) \\
 & & {\scriptstyle d_{m+1}}\downarrow & & \downarrow{\scriptstyle Sq^{2m+1}} \\
F(1+P_m) & \xrightarrow[\sigma_m]{} & F(1+Q_m) & \xrightarrow[\tau_m]{} & F(2m+1) \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma P(m) & \xrightarrow[\Sigma\sigma_m]{} & \Sigma Q(m) & \xrightarrow[\Sigma\tau_m]{} & \Sigma F(2m)
\end{array}$$

Denote

$$P := F(1+B_{m+1}) \oplus F(-1+A_{m+1}) \oplus F(1+P_m).$$

It follows from [7, Corollary 4.8] that there exists a morphism

$$\partial : F(1+P_m) \to F(4n+2)$$

such that the sequence

$$P \xrightarrow{\begin{pmatrix} \tau_{2m+1} & 0 \\ a_{m+1} & d_{m+1} \\ \partial & \sigma_m \end{pmatrix}} F(4n+2) \oplus F(1+Q_m) \xrightarrow{\left(Sq^{2m+1} \ \tau_m\right)} F(2m+1) \to 0 \tag{3}$$

is exact. Remark that $\Omega P(m+1) \cong P$ and $\Omega Q_{m+1} \cong F(4n+2) \oplus F(1+Q_m)$. Now, note that

$$\tau_{2m+1} = \left(Sq^1, Sq^3, \dots, Sq^{4m+1}\right).$$

It follows from Lemma 26 that there exists $\delta : F(1+B_{m+1}) \to F(1+Q_m)$ such that

$$Sq^{2m+1} \circ \tau_{2m+1} = \tau_m \circ \delta.$$

Therefore, the sequence

$$P(m+1) \xrightarrow{\begin{pmatrix} \tau_{2m+1} & \delta \\ a_{m+1} & d_{m+1} \\ \partial & \sigma_m \end{pmatrix}} Q(m+1) \xrightarrow{\tau_{m+1}} F(2m+2) \to 0$$

is an exact sequence after Corollary 3. □

Remark 33. A simple induction on n shows that all coefficients in the matrix form of σ_n are Steenrod squares.

Lemma 34. *Let $n \geq 1$ be an odd integer, then we can write Sq^n as a sum of products $Sq^a Sq^{2b+1}$, where $a > 0$, if and only if $n+1$ is not a power of 2.*

Proof. Remark that if $n+1$ is not a power of 2 then n is of the form $2^\ell k + 2^{\ell-1} - 1$ for some integers $\ell \geq 2$ and $k \geq 1$. We will now show that:

$$Sq^{2^\ell k + 2^{\ell-1} - 1} = Sq^{2^\ell k} Sq^{2^{\ell-1}-1} + \sum_{s=1}^{\ell-1} Sq^{2^\ell - 2^s} Sq^{2^\ell (k-1) + 2^{\ell-1} + 2^s - 1} \tag{4}$$

In fact, for $1 \leq s \leq \ell - 1$, we have

$$Sq^{2^\ell - 2^s} Sq^{2^\ell (k-1) + 2^{\ell-1} + 2^s - 1} = \sum_{t=\max(2^{\ell-1} - 2^{s+1} + 2, 0)}^{2^{\ell-1} - 2^{s-1}} \binom{2^\ell (k-1) + 2^{\ell-1} + 2^s - 2 - t}{2^\ell - 2^s - 2t} Sq^{2^\ell + 2^{\ell-1} - 1 - t} Sq^t.$$

On the other hand, we have

$$\binom{2^{\ell-1} + 2^s - 2 - t}{2^\ell - 2^s - 2t} = \binom{2^{\ell-1} + 2^{s-1} - 2 - t}{2^\ell - 2^{s-1} - 2t}$$

for all $2^{\ell-1} - 2^s + 2 \leq t \leq 2^{\ell-1} - 2^{s-1}$, and

$$\binom{2^\ell k - 2}{2^{\ell-1}} = 1.$$

Then, Equation 4 follows. The remaining conclusion follows from the fact that

$$\binom{2b}{1 + 2^k - 2b} = 0$$

for all $1 \leq b$. □

Proposition 35 (Compare with [4, Proposition 6.2.1]). *The following sequence provides the first two terms of the minimal projective resolution of $\Phi F(n)$ for $n \geq 1$:*

$$\bigoplus_{1 \leq d < \log_2(n)} F\left(2n + 2^d - 1\right) \xrightarrow{\omega} F(2n) \xrightarrow{p} \Phi F(n) \to 0.$$

where $\omega\left(\iota_{2n+2^d-1}\right) = Sq^{2^d-1}\iota_{2n}$, and $p(\iota_{2n}) = \Phi\iota_n$.

Proof. It follows from Lemma 32 that $\left\{Sq^{2k-1}\iota_n \mid 1 \leq k \leq n\right\}$ is a set of generators of $\mathrm{Ker}(p)$. By Lemma 34, $\left\{Sq^{2^d-1}\iota_n \mid 1 \leq d \leq \log_2(n)\right\}$ form a minimal set of generators of $\mathrm{Ker}(p)$, whence the conclusion. □

The first two terms of the minimal projective resolution of $\Phi F(n)$ can also be found in [4, Proposition 6.2.1]. However, in [4], instead of ω, the morphism from the second to the first term of the resolution sends ι_{2n+2^d-1} to $Q_{d-1}\iota_{2n}$. In this paper, we choose Sq^{2^d-1} instead of the Milnor operation Q_{d-1} because it gives us the following result, which can also be found in [11, Theorem 5].

Corollary 36. *For every odd number n, the operation Sq^n is equal to a sum of products of the form $Sq^{n-2^d+1}Sq^{2^d-1}$.*

Similarly, we get the following result.

Proposition 37. *The following sequence provides the first two terms of the minimal projective resolution of $\Lambda^n(F(1))$ for $n \geq 1$:*

$$\bigoplus_{0\leq d\leq n-2} F\left(2^n+2^d-1\right) \xrightarrow{\partial} F\left(2^n-1\right) \to \Lambda^n(F(1)) \to 0.$$

where $\partial\left(\iota_{2^n+2^d-1}\right) = Sq^{2^d}\iota_{2^n-1}$.

In what follows, denote by $(\mathfrak{P}_k, \mathfrak{d}_k)_{k\geq 0}$ the minimal projective resolution of $\widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty)$.

Lemma 38. *The first term $\mathfrak{P}_0$ of the minimal projective resolution of $\widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty)$ is isomorphic to the direct sum of $F(2^n-1)$ with $n \geq 1$.*

Proof. It is clear that the minimal set of $\mathscr{A}$−generators of $\widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty)$ is $\left\{u^{2^n-1} \mid n \geq 1\right\}$. The conclusion follows. □

Proposition 39. *There exists a projective unstable module Q such that*

$$\mathfrak{P}_1 \cong Q \oplus \left(\bigoplus_{2\leq n}\left(\bigoplus_{0\leq d\leq n-2} F\left(2^n+2^d-1\right)\right)\right).$$

Proof. Denote by φ the projection

$$\begin{array}{ccc} \mathfrak{P}_0 \cong \bigoplus_{1\leq n} F\left(2^n-1\right) & \longrightarrow & \widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty) \\ \iota_{2^n-1} & \longmapsto & u^{2^n-1} \end{array}$$

and by $\omega_{n,d}$ the element:

$$\omega_{n,d} := Sq^{2^d}\iota_{2^n-1} + Sq^{2^{n-1}}\cdots Sq^{2^{d+1}}Sq^{2^d}\iota_{2^{d+1}-1}$$

for all $2 \leq n$ and $0 \leq d \leq n-2$. It is clear that $\varphi\left(\omega_{n,d}\right) = 0$. As Sq^{2^d} is indecomposable, $\omega_{n,d}$ is $\mathscr{A}$−indecomposable. It follows that $\omega_{n,d}$ must be hit by the generator of a free module in a minimal resolution, whence the conclusion. □

Corollary 40. *The following sequence provides the first two terms of the minimal projective resolution of $\widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty)$:*

$$\bigoplus_{2\leq n}\left(\bigoplus_{0\leq d\leq n-2} F\left(2^n+2^d-1\right)\right) \xrightarrow{\partial} \bigoplus_{1\leq n} F\left(2^n-1\right) \to \widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right) \to 0.$$

where

$$\partial\left(\iota_{2^n+2^d-1}\right) = Sq^{2^d}\iota_{2^n-1} + Sq^{2^{n-1}}\cdots Sq^{2^{d+1}}Sq^{2^d}\iota_{2^{d+1}-1}.$$

Proof. We will show that $Q \cong \{0\}$. Let I be a set of integers such that

$$Q \cong \bigoplus_{k\in I} F(k).$$

From the connectivity of the kernel of the map

$$\bigoplus_{1\leq n} F\left(2^n-1\right) \to \widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right),$$

the set I cannot contain $0,1,2,3$. By constructing the pseudo-hyperresolution for $\Phi \mathrm{H}^*(\mathbb{R}P^\infty)$, it is easy to see that

$$\{n-1 \mid n \in I\} = \{2n \mid n \in I\}.$$

Therefore, if I is not empty, then it must contain 1, whence a contradiction. It follows that $I = \emptyset$, whence $Q \cong \{0\}$. □

Corollary 41. *Conjecture 1 holds for $s \leq 1$.*

Proof. Note that we need to verify the following isomorphisms for $s \leq 1$:

$$\bigoplus_{n\geq 1} \mathrm{Ext}^s_{\mathscr{U}}\left(\frac{R_n}{R_{n-1}}, \Sigma^t \mathbb{F}_2\right) \cong \mathrm{Ext}^s_{\mathscr{U}}\left(\widetilde{\mathrm{H}}^*\left(\mathbb{R}P^\infty\right), \Sigma^t \mathbb{F}_2\right),$$

As $\Sigma^t\mathbb{F}_2$ is a simple object, the Ext-group $\mathrm{Ext}^s_{\mathscr{U}}\left(M, \Sigma^t\mathbb{F}_2\right)$ is isomorphic to the Hom-group $\mathrm{Hom}_{\mathscr{U}}\left(P_s, \Sigma^t\mathbb{F}_2\right)$ where P_s is the $s-$th term of the minimal projective resolution of the unstable module M. On the one hand, Proposition 37 provides the first two terms of the minimal projective resolution of $\Lambda^n(F(1)) \cong R_n/R_{n-1}$ for $n \geq 1$. On the other hand, Corollary 40 provides the first two terms of the minimal projective resolution of $\widetilde{\mathrm{H}}^*(\mathbb{R}P^\infty)$. Then, the conclusion follows from counting the dimension of both sides of the isomorphisms. □

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