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Connections on trivial vector bundles over projective schemes

Indranil Biswas [a], Phùng Hô Hai [b] and Joao Pedro dos Santos [*,c]

[a] School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

[b] Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi, Vietnam

[c] Institut de Mathématiques de Jussieu – Paris Rive Gauche, 4 place Jussieu, Case 247, 75252 Paris Cedex 5, France

E-mails: indranil@math.tifr.res.in (I. Biswas), phung@math.ac.vn (P. H. Hai), joao_pedro.dos_santos@yahoo.com (J. P. dos Santos)

Abstract. Over a smooth and proper complex scheme, the differential Galois group of an integrable connection may be obtained as the closure of the transcendental monodromy representation. In this paper, we employ a completely algebraic variation of this idea by restricting attention to connections on trivial vector bundles and replacing the fundamental group by a certain Lie algebra constructed from the regular forms. In more detail, we show that the differential Galois group is a certain "closure" of the aforementioned Lie algebra.

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1. Introduction

A fundamental result of modern differential Galois theory affirms that, for a proper ambient variety, the differential Galois group might be obtained as the Zariski closure of the monodromy group. Our objective here is to make a synthesis of results by other mathematicians and use this to throw light on a similar finding in the realm of connections on trivial vector bundles. In this case, the role of the fundamental group is played by a certain Lie algebra (see Definition 4) and the role of the Zariski closure by the group-envelope (see Definition 22).

Let us be more precise: consider a field K of characteristic zero, a smooth, geometrically connected and proper K-scheme X, and a K-point x_0 of X. In the special case $K = \mathbf{C}$, it is known, mainly due to GAGA, that the category of integrable connections on X is equivalent to the category of complex representations of the transcendental object $\pi_1(X(\mathbf{C}), x_0)$. In addition,

* Corresponding author.

for any such connection $(\mathscr{E},\nabla)$, the differential Galois group at the point x_0 (Definition 6) is the Zariski closure of the image $\mathrm{Im}(M_{\mathscr{E}})$, where $M_{\mathscr{E}}:\pi_1(X(\mathbf{C}),x_0)\longrightarrow \mathbf{GL}(\mathscr{E}|_{x_0})$ is the transcendental monodromy representation.

In this work, we wish to draw attention to the fact that the category of integrable connections $(\mathscr{E},\nabla)$ on trivial vector bundles (that is, $\mathscr{E}\simeq\mathscr{O}_X^{\oplus r}$) is equivalent to the category of representations of *a Lie algebra* $\mathfrak{L}_X$. Then, in the same spirit as the previous paragraph, the differential Galois group of $(\mathscr{E},\nabla)$ at the point x_0 will be the "closure of the image of $\mathfrak{L}_X$" in $\mathbf{GL}(\mathscr{E}|_{x_0})$ (see Definition 22). The advantage here is that, contrary to what happens with the computation of the monodromy representation in the case $K=\mathbf{C}$, the image of $\mathfrak{L}_X$ is immediately visible. See Theorem 25.

Once the above results have been put up, it is very simple to construct connections on curves with prescribed differential Galois groups. For this goal, we make use of the fact that semi-simple, respectively reductive, Lie algebras can be generated by solely two elements, respectively three (if the base field is sufficiently "large"). See Corollary 28 and Corollary 30.

Some notation and conventions

In all that follows, K is a field of characteristic zero. Vector spaces, associative algebras, Lie algebras, Hopf algebras, etc, are always to be considered over K.

(1) The category of finite dimensional vector spaces (over K) is denoted by **vect**.
(2) The category of Lie algebras is denoted by **LA**. The category of Hopf algebras [25, p. 71] is denoted by **Hpf**.
(3) All group schemes are to be affine; **GS** is the category of affine group schemes. Given $G\in\mathbf{GS}$, we let $\mathbf{Rep}\,G$ stand for the category of finite dimensional representations of G.
(4) If $\mathfrak{A}$ stands for an associative algebra, we let $\mathfrak{A}$-**mod** be the category of left $\mathfrak{A}$-modules which are of finite dimension over K. The same notation is invoked for Lie algebras.
(5) An *ideal* of an associative algebra is, unless otherwise specified, a two-sided ideal. The *tensor algebra* on a vector space V is denoted by $\mathbf{T}(V)$. The *free algebra* on a set $\{s_i\}_{i\in I}$ is denoted by $K\{s_i\}$.
(6) A *curve* C is a one dimensional, integral and smooth K-scheme.
(7) A *vector bundle* is a locally free sheaf of finite rank. A *trivial* vector bundle on X is a direct sum of a finite number of copies of $\mathscr{O}_X$.

2. Construction of a Hopf algebra

Let Φ and Ψ be two finite dimensional vector spaces, and let

$$\beta:\Phi\otimes\Phi\longrightarrow\Psi$$

be a K-linear map with transpose $\beta^*:\Psi^*\longrightarrow\Phi^*\otimes\Phi^*$. Let

$$\mathfrak{I}_\beta=\text{Ideal in }\mathbf{T}\left(\Phi^*\right)\text{ generated by }\mathrm{Im}\,\beta^*,$$

and define

$$\mathfrak{A}_\beta=\mathbf{T}\left(\Phi^*\right)/\mathfrak{I}_\beta. \tag{1}$$

It is useful at this point to note that $\mathfrak{I}_\beta$ is a homogeneous ideal so that $\mathfrak{A}_\beta$ has a natural grading. In more explicit terms, fix a basis $\{\varphi_i\}_{i=1}^r$ of Φ and a basis $\{\psi_i\}_{i=1}^s$ of Ψ. Write $\{\varphi_i^*\}_{i=1}^r$ and $\{\psi_i^*\}_{i=1}^s$ for the respective dual bases. If

$$\beta\left(\varphi_k\otimes\varphi_\ell\right)=\sum_{i=1}^s\beta_i^{(k\ell)}\cdot\psi_i,$$

then

$$\beta^*(\psi_i^*) = \sum_{1\le k,\ell\le r} \beta_i^{(k\ell)}\cdot\varphi_k^*\otimes\varphi_\ell^*.$$

Consequently, $\mathfrak{A}_\beta$ in (1) is the quotient of the free algebra $K\{t_1,\dots,t_r\}$ (identified with $\mathbf{T}(\Phi^*)$ in the obvious way) by the ideal generated by the s elements

$$\sum_{1\le k,\ell\le r} \beta_i^{(k\ell)} t_k t_\ell, \qquad i=1,\dots,s.$$

In particular, given $V\in\mathbf{vect}$ and elements $A_1,\dots,A_r\in\mathrm{End}(V)$, the association $t_i\longmapsto A_i$ defines a representation of $\mathfrak{A}_\beta$ if and only if

$$\sum_{1\le k,\ell\le r} \beta_i^{(k\ell)}\cdot A_k A_\ell = 0, \qquad \forall\ i=1,\dots,s.$$

It is worth pointing out that if β is alternating, then

$$\sum_{1\le k,\ell\le r} \beta_i^{(k\ell)} t_k t_\ell = \sum_{1\le k<\ell\le r} \beta_i^{(k\ell)}[t_k,t_\ell]. \tag{2}$$

This reformulation has useful consequences for the structure of $\mathfrak{A}_\beta$.

From now on, β is always assumed to be alternating.

Let $\mathbf{L}(\Phi^*)$ be the free Lie algebra on the vector space Φ^* so that $\mathbf{T}(\Phi^*)$ is the universal enveloping algebra of $\mathbf{L}(\Phi^*)$ [6, II.3.1, p. 32, Theorem 1]. Clearly, abbreviating φ_i^* to t_i,

$$\sum_{1\le k<\ell\le r} \beta_i^{(k\ell)}[t_k,t_\ell]\in\mathbf{L}\big(\Phi^*\big), \qquad \forall\ i=1,\dots,s.$$

Let

$$\mathfrak{K}_\beta = \quad \text{Lie ideal of } \mathbf{L}\big(\Phi^*\big) \text{ generated by the } s \text{ elements } \left\{\sum_{1\le k<\ell\le r}\beta_i^{(k\ell)}[t_k,t_\ell]\right\}_{i=1}^{s} \text{ in (2).}$$

Proposition 1. *The algebra $\mathfrak{A}_\beta$ in* (1) *is the universal enveloping algebra of the Lie algebra $\mathbf{L}(\Phi^*)/\mathfrak{K}_\beta$.*

Proof. This is a consequence of the following general observations. Let $\mathfrak{g}$ be a Lie algebra and $\iota:\mathfrak{g}\to U$ be the morphism into its universal enveloping algebra. Let $S\subset\mathfrak{g}$ be a subset and let $S_{\mathrm{Lie}}\subset\mathfrak{g}$, respectively $S_{\mathrm{alg}}\subset U$, be the Lie ideal generated by S, respectively the ideal generated by $\iota(S)$. A moment's thought proves that S_{alg} is the ideal of U generated by $\iota(S_{\mathrm{Lie}})$. We conclude that U/S_{alg} is the universal enveloping algebra of the Lie algebra $\mathfrak{g}/S_{\mathrm{Lie}}$ [6, I.2.3, Proposition 3]. Applying this to the Lie ideal $\mathfrak{K}_\beta$, the ideal $\mathfrak{I}_\beta$ and the Lie algebra $\mathbf{L}(\Phi^*)$, we arrive at a proof of the proposition. □

Definition 2. *The Lie algebra $\mathbf{L}(\Phi^*)/\mathfrak{K}_\beta$ shall be denoted by $\mathfrak{L}_\beta$.*

A simple remark should be recorded here.

Lemma 3. *The above Lie algebra $\mathfrak{L}_\beta$ is a quotient of the free Lie algebra $\mathbf{L}(\Phi^*)$. In particular, $\mathfrak{L}_\beta$ is generated by the image of Φ^*.*

Recall that for a Lie algebra L, the universal enveloping algebra $\mathbf{U}L$ has a natural structure of Hopf algebra [25, 3.2.2, p. 58] and hence from Proposition 1 it follows that $\mathfrak{A}_\beta$ has the structure of a Hopf algebra. Similarly, $\mathbf{T}(\Phi^*)$ is also a Hopf algebra and the quotient map

$$\mathbf{T}\big(\Phi^*\big)\longrightarrow\mathfrak{A}_\beta \tag{3}$$

is an arrow of Hopf algebras.

In what follows, we give the category $\mathfrak{A}_\beta$-**mod** the tensor product explained in [20, 1.8.1, p. 14]. To wit, if V and W are $\mathfrak{A}_\beta$–modules, then $V\otimes_K W$ is an $\mathfrak{A}_\beta$–module by means of the composition ① below:

$$\begin{tikzcd}
\mathfrak{A}_\beta\otimes V\otimes W \arrow[r, "\text{comult.}\otimes\text{id}\otimes\text{id}"] \arrow[rrd, bend right, "①"'] & \mathfrak{A}_\beta\otimes\mathfrak{A}_\beta\otimes V\otimes W \arrow[r, equal] & \left(\mathfrak{A}_\beta\otimes V\right)\otimes\left(\mathfrak{A}_\beta\otimes W\right) \arrow[d, "\text{mult.}\otimes\text{mult.}"] \\
 & & V\otimes W.
\end{tikzcd}$$

It turns out that the canonical equivalence

$$\mathfrak{L}_\beta\text{-}\mathbf{mod}\xrightarrow{\sim}\mathfrak{A}_\beta\text{-}\mathbf{mod} \tag{4}$$

is actually a tensor equivalence.

The only case in which $\mathfrak{A}_\beta$ will interest us is that of:

Definition 4. *Let X be a smooth, connected and projective K-scheme. Let*

$$\beta: H^0\left(X,\Omega_X^1\right)\otimes H^0\left(X,\Omega_X^1\right)\longrightarrow H^0\left(X,\Omega_X^2\right)$$

be the wedge product of differential forms. We put

$$\mathfrak{A}_\beta=\mathfrak{A}_X\quad\text{and}\quad\mathfrak{L}_\beta=\mathfrak{L}_X.$$

3. Connections

We shall begin this section by establishing the notation and pointing out references. We fix a smooth and connected K-scheme X. Soon, we shall assume X to be projective.

Definition 5. *We let* **MC** *be the category of K–linear connections on coherent $\mathscr{O}_X$–modules and* **MIC** *the full subcategory of* **MC** *whose objects are integrable connections [15, 1.0]. We let* $\mathbf{MC}^{\text{tr}}$ *be the full subcategory of* **MC** *having as objects pairs $(\mathscr{E},\nabla)$ in which $\mathscr{E}$ is a trivial vector bundle. The category* $\mathbf{MIC}^{\text{tr}}$ *is defined analogously: it is the full subcategory of* **MIC** *having as objects pairs $(\mathscr{E},\nabla)$ in which $\mathscr{E}$ is a trivial vector bundle.*

A fundamental result of the theory of connections is that for each $(\mathscr{E},\nabla)$, the coherent sheaf $\mathscr{E}$ is actually locally free [15, Proposition 8.8]. Using this and the reconstruction theorem of Tannakian categories [9, Theorem 2.11], it is possible to show that, given $x_0\in X(K)$, the functor "taking the fibre at x_0" defines a K-linear tensor equivalence

$$\bullet|_{x_0}:\mathbf{MIC}\xrightarrow{\sim}\mathbf{Rep}\,\Pi(X,x_0), \tag{5}$$

where $\Pi(X,x_0)$ is a group scheme over K. This group scheme is sometimes called the "differential fundamental group scheme of X at x_0". It is in rare cases that $\Pi(X,x_0)$ will be an algebraic group (if, for example, $K=\mathbf{C}$, X is proper and $\pi_1(X^{\text{an}})$ is finite), and hence it is important to turn it into a splice of smaller pieces. This motivates the following definition.

Definition 6 (The differential Galois group). *Let $(\mathscr{E},\nabla)\in$* **MIC** *be given, and let $\rho_{\mathscr{E}}:\Pi(X,x_0)\longrightarrow$* $\mathbf{GL}(\mathscr{E}|_{x_0})$ *be the representation associated to $\mathscr{E}$ via the equivalence in* (5). *The image of $\rho_{\mathscr{E}}$ in* $\mathbf{GL}(\mathscr{E}|_{x_0})$ *is the differential Galois group of $(\mathscr{E},\nabla)$ at the point x_0.*

Remark 7. For $(\mathscr{E},\nabla)\in$ **MIC**, the category of representations of the differential Galois group of $(\mathscr{E},\nabla)$ at x_0 is naturally a full subcategory of **MIC**. For each vector bundle, let us agree to denote by $\check{\mathscr{F}}$ its dual $\mathscr{H}om(\mathscr{F},\mathscr{O}_X)$, and endow it with the canonical connection [15, 1.1]. It is then not difficult to see that

$$\langle(\mathscr{E},\nabla)\rangle_\otimes = \left\{\mathscr{M}'/\mathscr{M}'' \in \mathbf{MIC} : \begin{array}{c}\text{there exist } a_i, b_i \in \mathbf{N} \text{ such that}\\ \mathscr{M}'' \subset \mathscr{M}' \subset \bigoplus_i \mathscr{E}^{\otimes a_i} \otimes \check{\mathscr{E}}^{\otimes b_i}\end{array}\right\}.$$

See [27, 3.4 and 3.5].

From now on, X is in addition *projective*. Let us be more explicit about objects in $\mathbf{MC}^{\mathrm{tr}}$. Fix $E \in \mathbf{vect}$ and let

$$\begin{aligned} A \in \mathrm{Hom}_{K\text{-}\mathbf{alg}}(\mathbf{T}\left(H^0\left(X,\Omega^1_X\right)^*\right), \mathrm{End}(E)) &= \mathrm{Hom}(H^0\left(X,\Omega^1_X\right)^*, \mathrm{End}(E)) \\ &= \mathrm{End}(E) \otimes H^0\left(X,\Omega^1_X\right). \end{aligned}$$

(We note that in order to make the final identification above, we relied on the fact that $\dim H^0(\Omega^1_X) < \infty$.) Hence, A gives rise to an $\mathrm{End}(E)$–valued 1–form on X which, in turn, gives rise to a connection

$$d_A : \mathscr{O}_X \otimes E \longrightarrow (\mathscr{O}_X \otimes E) \underset{\mathscr{O}_X}{\otimes} \Omega^1_X \tag{6}$$

on the trivial vector bundle $\mathscr{O}_X \otimes E$. Explicitly, let $\{\theta_i\}_{i=1}^g$ be a basis of $H^0(X,\Omega^1_X)$ with dual basis $\{\varphi_i\}_{i=1}^g$ and let $A_i := A(\varphi_i) \in \mathrm{End}(E)$; we arrive at

$$d_A(1\otimes e) = \sum_{i=1}^g (1 \otimes A_i(e)) \otimes \theta_i$$

for all $e \in E$.

Definition 8. *The above pair consisting of* $(\mathscr{O}_X \otimes E, d_A)$ *shall be denoted by* $\mathscr{V}(E, A)$.

Let now $\{\sigma_i\}_{i=1}^h$ be a basis of $H^0(X,\Omega^2_X)$ and write

$$\theta_k \wedge \theta_\ell = \sum_{i=1}^h \beta_i^{(k\ell)} \cdot \sigma_i.$$

Since X is proper, Hodge theory tells us that all global 1-forms are closed [7, Theorem 5.5] and hence the curvature

$$R_{d_A} : \mathscr{O}_X \otimes E \longrightarrow (\mathscr{O}_X \otimes E) \otimes_{\mathscr{O}_X} \Omega^2_X$$

of the connection d_A in (6) satisfies

$$R_{d_A}(1\otimes e) = \sum_{i=1}^h \sum_{1\le k,\ell\le g} \left(1 \otimes \beta_i^{(k\ell)} A_k A_\ell(e)\right) \otimes \sigma_i.$$

Hence, $R_{d_A} = 0$ if and only if

$$\sum_{1\le k,\ell\le g} \beta_i^{(k\ell)} A_k A_\ell = 0$$

for each $i \in \{1,\dots,h\}$. Also, since β in Definition 4 is alternating, we conclude that $R_{d_A} = 0$ if and only if for each $i \in \{1,\dots,h\}$,

$$\sum_{1\le k,\ell\le g} \beta_i^{(k\ell)} A_k A_\ell = \sum_{1\le k<\ell\le g} \beta_i^{(k\ell)} [A_k, A_\ell] = 0.$$

These considerations form the main points of the proof of the following result, whose thorough verification is left to the interested reader. (It is worth noting that $K = H^0(X,\mathscr{O}_X)$ since X is proper, integral and has a K-point.)

Proposition 9. *The functor*

$$\mathscr{V} : \mathbf{T}\Big(H^0\left(\Omega^1_X\right)^*\Big)\text{-}\mathbf{mod} \longrightarrow \mathbf{MC}^{\mathrm{tr}}$$

is an equivalence of K-linear categories. Under this equivalence, $\mathscr{V}(E, A)$ lies in $\mathbf{MIC}^{\mathrm{tr}}$ *if and only if (E, A) is in fact a representation of $\mathfrak{A}_X$ (see Definition 4).*

Let us now discuss tensor products. Given representations

$$A : \mathbf{T}\Big(H^0\left(\Omega^1_X\right)^*\Big) \longrightarrow \mathrm{End}(E) \text{ and } B : \mathbf{T}\Big(H^0\left(\Omega^1_X\right)^*\Big) \longrightarrow \mathrm{End}(F),$$

we obtain a new representation $A \boxtimes B : \mathbf{T}(H^0(\Omega^1_X)^*) \longrightarrow \mathrm{End}(E \otimes F)$ by putting

$$A \boxtimes B(\varphi) = A(\varphi) \otimes \mathrm{id}_F + \mathrm{id}_E \otimes B(\varphi), \qquad \forall \ \varphi \in H^0\left(\Omega^1_X\right)^*.$$

This is of course only the tensor structure on the category $\mathbf{T}(H^0(\Omega^1_X)^*)$-**mod** defined by the Hopf algebra structure of $\mathbf{T}(H^0(\Omega^1_X)^*)$ [25, p. 58]. With this, it is not hard to see that the canonical isomorphism of $\mathscr{O}_X$-modules

$$\mathscr{O}_X \otimes (E \otimes F) \xrightarrow{\sim} (\mathscr{O}_X \otimes E) \otimes_{\mathscr{O}_X} (\mathscr{O}_X \otimes F)$$

is horizontal with respect to the tensor product connection on the right [15, Section 1.1] and the connection $d_{A\boxtimes B}$ on the left (it is the connection induced by the connections d_A and d_B). We then arrive at equivalences of tensor categories:

Theorem 10.

(i) *The functor*

$$\mathscr{V} : \mathbf{T}\Big(H^0\left(\Omega^1_X\right)^*\Big)\text{-}\mathbf{mod} \longrightarrow \mathbf{MC}^{\mathrm{tr}}$$

is an equivalence of K-linear tensor categories.

(ii) *The restriction*

$$\mathscr{V} : \mathfrak{A}_X\text{-}\mathbf{mod} \longrightarrow \mathbf{MIC}^{\mathrm{tr}}$$

is also an equivalence of K-linear tensor categories. In addition, the composition $(\bullet|_{x_0}) \circ \mathscr{V}$ is naturally isomorphic to the forgetful functor, where $\bullet|_{x_0}$ is constructed in (5).

(iii) *The composition of the equivalence in* (4) *with $\mathscr{V} : \mathfrak{A}_X$-**mod** $\longrightarrow$ $\mathbf{MIC}^{\mathrm{tr}}$ defines a K-linear tensor equivalence*

$$\mathfrak{L}_X\text{-}\mathbf{mod} \xrightarrow{\sim} \mathbf{MIC}^{\mathrm{tr}}$$

(see Definition 4 for $\mathfrak{L}_X$).

Making use again of the main theorem of (categorical) Tannakian theory, [9, p. 130, Theorem 2.11], we obtain an equivalence of K-linear tensor categories:

$$\bullet|_{x_0} : \mathbf{MIC}^{\mathrm{tr}} \xrightarrow{\sim} \mathbf{Rep}\,\Theta(X, x_0), \tag{7}$$

where $\Theta(X, x_0)$ is a group scheme. In addition, the inclusion map

$$\mathbf{MIC}^{\mathrm{tr}} \longrightarrow \mathbf{MIC}$$

defines a morphism

$$\mathbf{q}_X : \Pi(X, x_0) \longrightarrow \Theta(X, x_0),$$

where $\Pi(X, x_0)$ and $\Theta(X, x_0)$ are constructed in (5) and (7) respectively. Along the lines of Proposition 3.1 of [1], we have:

Proposition 11. *The above morphism $\mathbf{q}_X$ is in fact a quotient morphism.*

Proof. Let $\mathscr{E} \longrightarrow \mathscr{Q}$ be an epimorphism of **MIC** with $\mathscr{E} \in \mathbf{MIC}^{\mathrm{tr}}$; write e for the rank of $\mathscr{E}$ and q for that of $\mathscr{Q}$.

Let G stand for the Grassmannian variety of q–dimensional quotients of $K^{\oplus e}$, and let $\mathscr{O}_G^{\oplus e} \longrightarrow \mathscr{U}$ stand for the universal epimorphism [23, 5.1.6]. We then obtain a morphism $f : X \longrightarrow G$ such that $f^*\mathscr{U} = \mathscr{Q}$. For each projective curve $\gamma : C \longrightarrow X$, the vector bundle $\gamma^*\mathscr{Q} = (f\circ\gamma)^*\mathscr{U}$, which carries a connection, has degree zero [2, Remark 3.3]; in particular, $(f\circ\gamma)^*\det\mathscr{U}$ has also degree zero. As $\det\mathscr{U}$ is a very ample invertible sheaf on G [23, p. 114], from $\mathrm{degree}((f\circ\gamma)^*\det\mathscr{U}) = \mathrm{degree}((f\circ\gamma)^*\mathscr{U}) = 0$ we conclude that $(f\circ\gamma)^*\det\mathscr{U}$ is trivial, and hence the schematic image of $f\circ\gamma$ is a (closed) point of G [17, p. 331, Exercise 8.1.7 (a)]. Now, Ramanujam's Lemma (see Remark 12 below) can be applied to show that any two closed points x_1 and x_2 of X belong to the image of a morphism $\gamma : C \longrightarrow X$ from a projective curve. Therefore, the schematic image of X under f is a single point (necessarily closed) and hence f factors as $X \longrightarrow \operatorname{Spec} K' \longrightarrow G$, with K' a finite extension of K. Since $H^0(X, \mathscr{O}_X) = K$, it must be the case that $K' = K$, and hence f factors through the structural morphism $X \longrightarrow \operatorname{Spec} K$. Consequently, $f^*\mathscr{U} = \mathscr{Q}$ is a trivial vector bundle. The standard criterion for a morphism of group schemes to be a quotient morphism (see [9, p. 139, Proposition 2.21 (a)] for the criterion) can be applied to complete the proof. □

Remark 12 (Ramanujam's Lemma). Let Z be a geometrically integral projective K-scheme and z_1, z_2 two closed points on it. We contend that there exists a projective curve C together with a morphism $\gamma : C \longrightarrow X$ such that z_1 and z_2 belong to the image of γ. The proof is the same as in [21, p. 56], but the Bertini theorem necessary for our purpose comes from [14, Cor. 6.11].

If $\dim Z = 1$, it is sufficient to chose C to be the normalisation of Z. Let $\dim Z := d \geq 2$ and suppose that the result holds for all geometrically integral and projective schemes of dimension strictly smaller than d. We only need now to find a geometrically integral closed subscheme $Y \subset Z$ containing z_1 and z_2 and having dimension strictly smaller than d. Let $\pi : Z' \longrightarrow Z$ be the blow up of the closed subscheme $\{z_1, z_2\}$. Note that Z' is geometrically integral [17, 8.1.12 (c) and (d), p. 322]. In addition, the fibres of π above z_1 and z_2 are Cartier subschemes of Z' and hence of dimension at least 1 [17, 2.5.26, p. 74]. Let $Z' \longrightarrow \mathbf{P}^N$ be a closed immersion, and let $H \subset \mathbf{P}^N$ be a hyperplane such that

- $Z' \cap H$ is geometrically integral [14, 6.11 (2)-(3)] of dimension $\dim Z - 1$, and
- $Z' \cap \pi^{-1}(z_i) \neq \varnothing$, loc. cit, (1)-(b).

Then, the schematic image of $\pi : Z' \cap H \longrightarrow Z$ is the scheme Y that we are seeking.

Remark 13. In [17, Ch. 8, p. 331, Exercise 1.5], the reader shall find a useful, but slightly weaker version of Ramanujam's Lemma.

Remark 14. The idea to consider certain connections as representations of a Lie algebra can be found at least on [8, 12.2–5].

4. The Tannakian envelope of a Lie algebra

All that follows in this section is, in essence, due to Hochschild [12]; since he expressed himself without using group schemes and his ideas are spread out in several papers, we shall briefly condense his theory in what follows. The reader should also consult [22], where some results reviewed here also appear.

Our objective in this section is to give a construction of the affine envelope of a Lie algebra. One can, of course, employ the categorical Tannakian theory [9, p. 130, Theorem 2.11] to the category $\mathfrak{L}$-**mod** to obtain such a construction, but we prefer to draw the reader's attention to something which is less widespread than [9] and more concrete.

Let $\mathfrak{L}$ be a Lie algebra with universal enveloping algebra $\mathbf{U}\mathfrak{L}$. Note that $\mathbf{U}\mathfrak{L}$ is not only an algebra, but also a cocommutative Hopf algebra; see [25, p. 58, Section 3.2.2] and [20, p. 72, Example 1.5.4]. Consequently, the Hopf dual

$$(\mathbf{U}\mathfrak{L})^\circ = \left\{\varphi : \mathbf{U}\mathfrak{L} \to K : \begin{array}{c}\varphi \text{ vanishes on a subspace}\\ \text{of finite codimension}\end{array}\right\}$$

is a commutative Hopf algebra (see [25, Section 6.2, pp. 122-3] or [20, Theorem 9.1.3]). This means that

$$\mathbf{G}(\mathfrak{L}) := \operatorname{Spec}(\mathbf{U}\mathfrak{L})^\circ$$

is a group scheme, which we call the *affine envelope of* $\mathfrak{L}$. Let us show that this construction gives a left adjoint to the functor

$$\mathrm{Lie} : \mathbf{GS} \longrightarrow \mathbf{LA}.$$

We start by noting that $\mathbf{G}$ is indeed a functor; given a morphism of Lie algebras $\mathfrak{G} \longrightarrow \mathfrak{H}$, the associated morphism $\mathbf{U}\mathfrak{G} \longrightarrow \mathbf{U}\mathfrak{H}$ gives rise to a map of coalgebras $(\mathbf{U}\mathfrak{H})^\circ \longrightarrow (\mathbf{U}\mathfrak{G})^\circ$; see [25, p. 114, Remark 1]. The fact that the algebra structures are also preserved is a consequence of the fact that $\mathbf{U}\mathfrak{G} \longrightarrow \mathbf{U}\mathfrak{H}$ is also an arrow of coalgebras.

Let G be a group scheme, and let $\rho : \mathfrak{L} \longrightarrow \mathrm{Lie}\, G$ be a morphism of Lie algebras; we write ρ for the map induced between universal enveloping algebras as well. Interpreting elements in $\mathrm{Lie}\, G$ as elements of $\mathrm{End}_K(\mathscr{O}(G))$, we obtain a morphism of K-algebras $\mathbf{U}(\mathrm{Lie}\, G) \longrightarrow \mathrm{End}_K(\mathscr{O}(G))$. To continue, we recall that a module over an algebra is *locally finite* if it is the union of finite dimensional submodules. In addition, such a notion is easily adapted to include modules over Lie algebras or comodules over coalgebras. Because of [27, 3.3 Theorem], the $\mathscr{O}(G)$-comodule $\mathscr{O}(G)$ is locally finite and hence is also locally finite as a $\mathrm{Lie}\, G$–module, $\mathfrak{L}$–module or $\mathbf{U}\mathfrak{L}$–module. Let

$$\varphi_\rho : \mathscr{O}(G) \longrightarrow (\mathbf{U}\mathfrak{L})^*$$

be defined by

$$\varphi_\rho(a) : u \longmapsto \varepsilon\big(\rho(u)(a)\big), \tag{8}$$

where

$$\varepsilon : \mathscr{O}(G) \longrightarrow K \tag{9}$$

is the counit and $u \in \mathbf{U}\mathfrak{L}$.

Lemma 15.

(1) *For each $a \in \mathscr{O}(G)$, the element $\varphi_\rho(a)$ in* (8) *lies in the Hopf dual* $(\mathbf{U}\mathfrak{L})^\circ$.

(2) *The map φ_ρ is a morphism of Hopf algebras.*

Proof. (1) Because $\mathscr{O}(G)$ is a locally finite $\mathbf{U}\mathfrak{L}$–module, as explained above, the element a belongs to a finite dimensional subspace V which is stable under $\mathbf{U}\mathfrak{L}$. Let $I \subset \mathbf{U}\mathfrak{L}$ be the kernel of the induced map of K-algebras $\mathbf{U}\mathfrak{L} \xrightarrow{\rho} \mathbf{U}(\mathrm{Lie}\, G) \longrightarrow \mathrm{End}(V)$; it follows that $I \subset \mathrm{Ker}\,\varphi_\rho(a)$ and $\varphi_\rho(a) \in (\mathbf{U}\mathfrak{L})^\circ$.

(2) This verification is somewhat lengthy, but straightforward once the right path has been found. We shall only indicate the most important ideas. Let us write φ instead of φ_ρ and consider elements of $\mathbf{U}\mathfrak{L}$ as G-invariant linear operators [27, Section 12.1] on $\mathscr{O}(G)$. In what follows, we shall use freely the symbol Δ to denote comultiplication on different coalgebras.

Compatibility with multiplication. We must show that

$$[\varphi(a) \otimes \varphi(b)]\,(\Delta(u)) = \varphi(ab)(u) \tag{10}$$

for all $a, b \in \mathscr{O}(G)$ and $u \in \mathbf{U}\mathfrak{L}$. Obviously, eq. (10) holds for $u \in K \subset \mathbf{U}\mathfrak{L}$. In case $u \in \mathfrak{L}$, the validity of eq. (10) is an easy consequence of the fact that $u : \mathscr{O}(G) \longrightarrow \mathscr{O}(G)$ is a derivation and $\Delta u = u \otimes 1 + 1 \otimes u$. We then prove that if eq. (10) holds for u then, for any given $\delta \in \mathfrak{L}$, eq. (10) holds for $u\delta$. Since $\mathbf{U}\mathfrak{L}$ is generated by $\mathfrak{L}$, we are done.

Compatibility with comultiplication For $\zeta \in (\mathbf{U}\mathfrak{L})^\circ$, we know that $\Delta_{(\mathbf{U}\mathfrak{L})^\circ}(\zeta)$ is defined by

$$u \otimes v \longmapsto \zeta(uv)$$

for $u, v \in \mathbf{U}\mathfrak{L}$. We need to prove that

$$\varepsilon(uv(a)) = \big(\varphi \otimes \varphi\big) \circ \Delta a$$

for every triple $u, v \in \mathbf{U}\mathfrak{L}$ and $a \in \mathscr{O}(G)$, where ε is the homomorphism in (9). This follows from the invariance formulas $\Delta u = (\mathrm{id} \otimes u)\Delta$.

Compatible with unit and co-unit. This is much simpler and we omit its verification.

Compatibility with antipode. Since φ respects multiplication and comultiplication, unit and co-unit, it is a morphism of bialgebras. Now, [25, Lemma 4.0.4] guarantees that φ is compatible with the antipode. □

Proposition 16. *The above construction establishes a bijection*

$$\begin{aligned}\varphi : \mathrm{Hom}_{\mathbf{LA}}(\mathfrak{L}, \mathrm{Lie}\, G) &\longrightarrow \mathrm{Hom}_{\mathbf{Hpf}}(\mathscr{O}(G), (\mathbf{U}\mathfrak{L})^\circ)\\ &= \mathrm{Hom}_{\mathbf{GS}}(\mathbf{G}(\mathfrak{L}), G),\end{aligned}$$

rendering $\mathbf{G} : \mathbf{LA} \longrightarrow \mathbf{GS}$ *a left adjoint to* $\mathrm{Lie} : \mathbf{GS} \longrightarrow \mathbf{LA}$.

Proof. We construct the inverse of φ and leave the reader with all verifications. Let $f : \mathscr{O}(G) \longrightarrow (\mathbf{U}\mathfrak{L})^\circ$ be a morphism of Hopf algebras. Let $x \in \mathfrak{L}$ be given, and define

$$\psi_f(x) : \mathscr{O}(G) \longrightarrow K, \quad a \longmapsto f(a)(x). \tag{11}$$

It is a simple matter to show that $\psi_f(x)$ is an ε-derivation (see [27, 12.2] for the definition), which is then interpreted as an element of $\mathrm{Lie}\, G$ in a standard fashion. In addition, $\psi_f : \mathfrak{L} \longrightarrow \mathrm{Lie}\, G$ gives a morphism of Lie algebras (the reader might use the bracket as explained in [27, Section 12.1, p. 93]). Then $f \longmapsto \psi_f$ and $\rho \longmapsto \varphi_\rho$ are mutually inverses; the verification of this fact consists of a chain of simple manipulations and we contend ourselves in giving some elements of the equations to be verified. That $\psi_{\varphi_\rho} = \rho$ is in fact immediate. On the other hand, the verification of

$$\varphi_{\psi_f}(a)(u) = f(a)(u), \qquad \forall\ a \in \mathscr{O}(G), \forall\ u \in \mathbf{U}\mathfrak{L}$$

requires the ensuing observations. (We shall employ Sweedler's notation for the Hopf algebra $\mathscr{O}(G)$ [25, Section 1.2, 10ff].)

(1) For $\delta \in \mathfrak{L}$, the derivation $\mathscr{O}(G) \longrightarrow \mathscr{O}(G)$ associated to $\psi_f(\delta)$ is determined by $a \longmapsto \sum_{(a)} a_{(1)} \cdot [f(a_{(2)})(\delta)]$.
(2) The axioms for the coproduct and co-unit show that $\sum_{(a)} \varepsilon(a_{(1)}) a_{(2)} = a$.
(3) Suppose that for $u \in \mathbf{U}\mathfrak{L}$ and $\delta \in \mathbf{U}\mathfrak{L}$ we know that, for all $a \in \mathscr{O}(G)$,

$$\varphi_{\psi_f}(a)(u) = f(a)(u) \quad \text{and} \quad \varphi_{\psi_f}(a)(\delta) = f(a)(\delta).$$

Then $\varphi_{\psi_f}(a)(u\delta) = f(a)(u\delta)$ because of the equations

$$f(a)(xy) = \sum_{(a)} f\big(a_{(1)}\big)(x) \cdot f\big(a_{(2)}\big)(y), \qquad \forall\ x, y \in \mathbf{U}\mathfrak{L},$$

which are a consequence of the fact that f is a map of Hopf algebras.

This completes the proof. □

In the proof of Proposition 16 we defined a bijection

$$\psi : \mathrm{Hom}_{\mathbf{GS}}(\mathbf{G}(\mathfrak{L}), G) \longrightarrow \mathrm{Hom}_{\mathbf{LA}}(\mathfrak{L}, \mathrm{Lie}\, G)$$

by means of eq. (11). In case $G = \mathbf{GL}(V)$ and in the light of the identification $\mathrm{Lie}\, \mathbf{GL}(V) = \mathfrak{gl}(V)$, ψ has a rather useful description. Let $f : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(V)$ be a representation and let $c_f : V \to V \otimes (\mathbf{U}\mathfrak{L})^\circ$ be the associated comodule morphism. It then follows that

$$(\mathrm{id}_V \otimes \text{evaluate at } x) \circ c_f = \psi_f(x). \tag{12}$$

Corollary 17. *Let V be a finite dimensional vector space and $f : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(V)$ a representation. Write $\psi_f : \mathfrak{L} \longrightarrow \mathfrak{gl}(V)$ for the morphism of Lie algebras mentioned above. Then, this gives rise to a K-linear equivalence of tensor categories*

$$\mathbf{Rep}\,\mathbf{G}(\mathfrak{L}) \longrightarrow \mathfrak{L}\text{-}\mathbf{mod}.$$

Proof. To define a functor $\mathbf{Rep}\,\mathbf{G}(\mathfrak{L}) \longrightarrow \mathfrak{L}$-**mod** it is still necessary to define the maps between sets of morphisms.

Let $f : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(V)$ and $g : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(W)$ be representations, and let $T \in \mathrm{Hom}_K(V, W)$. We shall show that $T \in \mathrm{Hom}_{\mathbf{G}(\mathfrak{L})}(V, W)$ if and only if $T \in \mathrm{Hom}_{\mathfrak{L}}(V, W)$.

Consider $\widehat{T} = \left(\begin{smallmatrix} I & 0 \\ T & I \end{smallmatrix}\right) \in \mathrm{GL}(V \oplus W)$ and denote by

$$C_T : \mathbf{GL}(V \oplus W) \longrightarrow \mathbf{GL}(V \oplus W)$$

the conjugation by $\widehat{T}$. Then T is G–equivariant if and only if $C_T \circ \left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right) = \left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right)$. Similarly let us write $c_T : \mathfrak{gl}(V \oplus W) \longrightarrow \mathfrak{gl}(V \oplus W)$ to denote conjugation by $\widehat{T}$. Then, for given representations $\rho : \mathfrak{L} \longrightarrow \mathfrak{gl}(V)$ and $\sigma : \mathfrak{L} \longrightarrow \mathfrak{gl}(W)$, the arrow T is a morphism of $\mathfrak{L}$-modules if and only if $c_T\left(\begin{smallmatrix} \rho & 0 \\ 0 & \sigma \end{smallmatrix}\right) = \left(\begin{smallmatrix} \rho & 0 \\ 0 & \sigma \end{smallmatrix}\right)$. Employing equation (12), we verify readily that

$$\begin{tikzcd}
\mathrm{Hom}(\mathbf{G}(\mathfrak{L}), \mathbf{GL}(V \oplus W)) \arrow[r, "C_T \circ (-)"] \arrow[d, "\psi"'] & \mathrm{Hom}(\mathbf{G}(\mathfrak{L}), \mathbf{GL}(V \oplus W)) \arrow[d, "\psi"] \\
\mathrm{Hom}(\mathfrak{L}, \mathfrak{gl}(V \oplus W)) \arrow[r, "c_T \circ (-)"'] & \mathrm{Hom}(\mathfrak{L}, \mathfrak{gl}(V \oplus W))
\end{tikzcd}$$

commutes. We then see that ψ becomes a functor, which is K-linear, exact and fully-faithful.

Let us now deal with the tensor product. Given representations $f : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(V)$ and $g : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(W)$, let us write

$$t : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(V \otimes W)$$

for the tensor product representation. We then obtain on $V \otimes W$ the structure of an $\mathfrak{L}$-module via ψ_t and it is to be shown that this is precisely the $\mathfrak{L}$-module structure coming from the tensor product of $\mathfrak{L}$-modules. In other words, we need to show that for any $x \in \mathfrak{L}$, the equation $\psi_f(x) \otimes \mathrm{id}_W + \mathrm{id}_V \otimes \psi_g(x) = \psi_t(x)$ holds. We make use of eq. (12) again. Let $v \in V$ and $w \in W$ be such that $c_f(v) = \sum_i v_i \otimes f_i$ and $c_g(w) = \sum_j w_j \otimes g_j$. Then $c_t(v \otimes w) = \sum_{i,j} v_i \otimes w_j \otimes f_i g_j$ and hence

$$\psi_t(x)(v \otimes w) = \sum_{i,j} v_i \otimes w_j \cdot \big(f_i(x)\varepsilon(g_j) + \varepsilon(f_i) g_j(x)\big),$$

where $\varepsilon : (\mathbf{U}\mathfrak{L})^\circ \longrightarrow K$ is the co-unit defined by evaluating at $1 \in \mathbf{U}\mathfrak{L}$, and we have used that $\Delta x = x \otimes 1 + 1 \otimes x$ (which is true, by definition of the coproduct, for all elements of $\mathfrak{L}$ inside $\mathbf{U}\mathfrak{L}$). Now, $\sum_i v_i \varepsilon(f_i) = v$ and $\sum_j w_j \varepsilon(g_j) = w$. Hence, $\psi_t(x)(v \otimes w) = \sum_i f_i(x) v_i \otimes w + \sum_j v \otimes g_j(x) w_j$, as we wanted. □

Corollary 18. *Let G be an algebraic group scheme, and let $\mathbf{G}(\mathfrak{L}) \longrightarrow G$ be a quotient morphism. Then G is connected. Said differently, $\mathbf{G}(\mathfrak{L})$ is pro-connected.*

Proof. For the finite étale group scheme $\pi_0(G)$ [27, Section 6.7], the set

$$\mathrm{Hom}_{\mathbf{LA}}(\mathfrak{L}, \mathrm{Lie}(\pi_0 G)) = \mathrm{Hom}_{\mathbf{LA}}(\mathfrak{L}, 0)$$

is a singleton and hence $\mathrm{Hom}_{\mathbf{GS}}(\mathbf{G}(\mathfrak{L}), \pi_0(G))$ is a singleton because of Proposition 16. It then follows that $\pi_0(G)$ is trivial and G is connected [27, Theorem of 6.6]. □

In what follows, we denote by

$$\chi : \mathfrak{L} \longrightarrow \mathrm{Lie}\,\mathbf{G}(\mathfrak{L}) \tag{13}$$

the morphism of Lie algebras corresponding to the identity of $\mathrm{Hom}_{\mathbf{GS}}(\mathbf{G}(\mathfrak{L}), \mathbf{G}(\mathfrak{L}))$ under the bijection in Proposition 16. This is, of course, the unit of the adjunction [18, IV.1]. Let us profit to note that, as explained in [18, IV.1, eq. (5)], for each $f \in \mathrm{Hom}_{\mathbf{GS}}(\mathbf{G}(\mathfrak{L}), G)$, the equation

$$\psi_f = \big(\mathrm{Lie}\, f\big) \circ \chi. \tag{14}$$

is valid.

One fundamental property of χ needs to be expressed in terms of "algebraic density" [13, p. 175].

Definition 19. *Let G be a group scheme. A morphism $\rho : \mathfrak{L} \longrightarrow \mathrm{Lie}\, G$ is algebraically dense if the only closed subgroup scheme $H \subset G$ such that $\rho(\mathfrak{L}) \subset \mathrm{Lie}\, H$ is G itself. If ρ happens to be an injection, we shall simply say that $\mathfrak{L}$ is algebraically dense.*

Proposition 20. *The morphism $\chi : \mathfrak{L} \longrightarrow \mathrm{Lie}\,\mathbf{G}(\mathfrak{L})$ in* (13) *is algebraically dense.*

To prove Proposition 20, we require the following fact. (The proofs of the first claims are in [27, Corollary in 3.3, p. 24] and [27, Theorem of 14.1, p. 109], while the proof of the final claim can be found in [10, II.5, Proposition 5.3, p. 250].)

Lemma 21. *Let G be a group scheme. Then there exists a projective system of algebraic group schemes $\{G_i, u_{ij} : G_j \longrightarrow G_i\}$ where each u_{ij} is faithfully flat and an isomorphism $G \simeq \varprojlim_i G_i$. In addition, all arrows $\mathrm{Lie}\, u_{ij} : \mathrm{Lie}\, G_j \longrightarrow \mathrm{Lie}\, G_i$ are surjective.* □

Proof of Proposition 20. Let $u : H \longrightarrow \mathbf{G}(\mathfrak{L})$ be a closed immersion and let $\rho : \mathfrak{L} \longrightarrow \mathrm{Lie}\, H$ be a morphism of Lie algebras such that

$$(\mathrm{Lie}\, u) \circ \rho = \chi.$$

Let $f : \mathbf{G}(\mathfrak{L}) \longrightarrow H$ be a morphism of group schemes such that $\rho = \psi_f$. From eq. (14), we have

$$\rho = \big(\mathrm{Lie}\, f\big) \circ \chi.$$

Hence, $\chi = (\mathrm{Lie}(u \circ f)) \circ \chi$, which proves that $u \circ f = \mathrm{id}_{\mathbf{G}(\mathfrak{L})}$ (see eq. (14)). In particular, $\mathrm{Lie}\, u$ is surjective.

Let us now write

$$\mathbf{G}(\mathfrak{L}) = \varprojlim_i G_i$$

as in Lemma 21. Define H_i as being the image of H in G_i; a moment's thought shows that

$$H = \varprojlim_i H_i,$$

and that the the transition arrows of the projective system $\{H_i\}$ are also faithfully flat. This being so, the morphisms between Lie algebras in the projective system $\{H_i\}$ are all surjective [10, II.5, Proposition 5.3, p. 250]. Consequently, the obvious morphisms $\mathrm{Lie}\,\mathbf{G}(\mathfrak{L}) \longrightarrow \mathrm{Lie}\, G_i$ and $\mathrm{Lie}\, H \longrightarrow \mathrm{Lie}\, H_i$ are always surjective. Hence, the natural morphisms $\mathrm{Lie}\, H_i \longrightarrow \mathrm{Lie}\, G_i$ are always surjective.

Using [10, Proposition II.6.2.1, p. 259] and the fact that each G_i is connected, we conclude that $H_i = G_i$, and $H = \mathbf{G}(\mathfrak{L})$. This proves Proposition 20. □

Let G be an algebraic group scheme with Lie algebra $\mathfrak{g}$. Recall that a *Lie subalgebra of* $\mathfrak{g}$ *is algebraic* if it is the Lie subalgebra of a closed subgroup scheme of G [10, Definition II.6.2.4]. As argued in [10, II.6.2, p. 262], given an arbitrary Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, there exists a smallest algebraic Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$: it is the (algebraic) *envelope* of $\mathfrak{h}$ inside $\mathfrak{g}$. Allied with [10, II.6.2.1a, p. 259], it then follows that there exists a *smallest closed and connected subgroup scheme* of G whose Lie algebra contains $\mathfrak{h}$. This group carries no name in [10], so we shall allow ourselves to put forward:

Definition 22. *Let G be an algebraic group scheme and $\mathfrak{h} \subset \operatorname{Lie} G$ a Lie subalgebra. The group-envelope of $\mathfrak{h}$ is the smallest closed subgroup scheme of G whose Lie algebra contains $\mathfrak{h}$. We also define the group-envelope of a subspace $V \subset \operatorname{Lie} G$ as being the group-envelope of the Lie algebra generated by V in $\operatorname{Lie} G$.*

Theorem 23 ([12, Theorem 1, §3]). *Let $f : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(E)$ be the representation associated to the $\mathfrak{L}$–module $\rho : \mathfrak{L} \longrightarrow \mathfrak{gl}(E)$, that is, $\psi_f = \rho$. Then the image $I = \operatorname{image}(f)$ of $\mathbf{G}(\mathfrak{L})$ in $\mathbf{GL}(E)$ is the group-envelope of $\rho(\mathfrak{L}) \subset \mathfrak{gl}(E)$.*

Proof. Consider a factorization $\rho : \mathfrak{L} \longrightarrow \operatorname{Lie} H$, where $H \subset \mathbf{GL}(E)$ is closed. Because $\rho = (\operatorname{Lie} f) \circ \chi$ (see eq. (14)), it follows that $\chi(\mathfrak{L}) \subset (\operatorname{Lie} f)^{-1}(\operatorname{Lie} H)$. We now observe that the natural inclusion

$$\operatorname{Lie}\left(f^{-1}(H)\right) \subset \left(\operatorname{Lie} f\right)^{-1}(\operatorname{Lie} H)$$

is an *equality*, see Lemma 24 below. (We are unable to find a reference for this simple fact.) Hence, $f^{-1}(H) = \mathbf{G}(\mathfrak{L})$ because $\chi : \mathfrak{L} \longrightarrow \operatorname{Lie} \mathbf{G}(\mathfrak{L})$ is algebraically dense. This implies that $I \subset H$. Because $\rho(\mathfrak{L}) = \operatorname{Lie}(f) \circ \chi(\mathfrak{L})$, we deduce that $\operatorname{Lie} I \supset \rho(\mathfrak{L})$, so that I is the group-envelope. □

Lemma 24. *Let $f : G' \longrightarrow G$ be a morphism of group schemes. Let $H \subset G$ be a closed subgroup. Denote by H' its inverse image in G'. Then, $\operatorname{Lie} H' = \operatorname{Lie}(f)^{-1}(\operatorname{Lie} H)$.*

Proof. It is only needed to show the inclusion $\operatorname{Lie} H' \supset (\operatorname{Lie} f)^{-1}(\operatorname{Lie} H)$. We consider elements of Lie algebras as ε-derivations, cf. the proof of [27, Theorem 12.2]. Let $\partial : \mathscr{O}(G') \longrightarrow K$ be an ε-derivation whose image in $\operatorname{Lie} G$ is induced by an ε-derivation $\mathscr{O}(H) \longrightarrow K$. Now, if $I \subset \mathscr{O}(G)$ is the ideal of H, we conclude that

$$\mathscr{O}(G) \longrightarrow \mathscr{O}(G') \xrightarrow{\partial} K$$

annihilates I, and hence that ∂ annihilates $I\mathscr{O}(G)$. But this is the ideal of $f^{-1}(H)$ and hence $\partial : \mathscr{O}(G') \longrightarrow K$ comes from an ε-derivation $\mathscr{O}(H') \longrightarrow K$. □

5. The differential Galois group

In this section, X is assumed to be a projective, connected and smooth K-scheme and x_0 a K-point of X. We recall that $\Theta(X, x_0)$ is the group scheme constructed in eq. (7).

Using the tensor equivalences

$$\mathbf{Rep}\,\mathbf{G}(\mathfrak{L}_X) \xrightarrow{\sim} \mathfrak{L}_X\text{-}\mathbf{mod} \xrightarrow{\sim} \mathfrak{A}_X\text{-}\mathbf{mod} \xrightarrow{\mathscr{V}} \mathbf{MIC}^{\mathrm{tr}} \xrightarrow{\bullet|_{x_0}} \mathbf{Rep}\,\Theta(X, x_0)$$

obtained by eq. (4), Theorem 10 and Corollary 17, we derive an isomorphism

$$\gamma : \Theta(X, x_0) \xrightarrow{\sim} \mathbf{G}(\mathfrak{L}_X)$$

such that the corresponding functor $\gamma^\# : \mathbf{Rep}\,\mathbf{G}(\mathfrak{L}_X) \longrightarrow \mathbf{Rep}\,\Theta(X, x_0)$ is naturally isomorphic to the above composition.

Let $(E, A) \in \mathfrak{A}_X$-**mod** be given. With an abuse of notation, we shall let A denote the linear map $H^0(\Omega^1_X)^* \longrightarrow \operatorname{End}(E)$, the morphism of associative algebras $\mathfrak{A}_X \longrightarrow \operatorname{End}(E)$ or the morphism of Lie algebras $\mathfrak{L}_X \longrightarrow \operatorname{End}(E)$.

Theorem 25. *The differential Galois group of $\mathscr{V}(E, A) = (\mathscr{O}_X \otimes E, d_A)$ is the group-envelope of $A(H^0(X, \Omega^1_X)^*)$.*

Said otherwise, given a trivial vector bundle $\mathscr{E}$ or rank r with global basis $\{e_i\}_{i=1}^r$, an integrable connection

$$\nabla : \mathscr{E} \longrightarrow \mathscr{E} \otimes \Omega^1_X$$

and a basis $\{\theta_j\}_{j=1}^{g}$ *of* $H^0(X,\Omega^1_X)$, *define matrices* $A_k = (a^{(k)}_{ij})_{1\le i,j\le r} \in \mathrm{M}_r(K)$ *by*

$$\nabla e_j = \sum_{k=1}^{g}\sum_{i=1}^{r} a^{(k)}_{ij}\cdot e_i\otimes\theta_k.$$

Then, the differential Galois group of $\mathscr{E}$ *at the point* x_0 *is isomorphic to the group-envelope in* $\mathbf{GL}_r$ *of the Lie algebra generated by* $\{A_k\}_{k=1}^{g}$.

Proof. We note that the Lie subalgebra of $\mathrm{End}(E)$ generated by $A(H^0(\Omega^1_X)^*)$ is the image of $A(\mathfrak{L}_X)$; indeed, as a Lie algebra, $\mathfrak{L}_X$ is generated by $H^0(\Omega^1_X)^*$ (see Lemma 3). Now we apply Theorem 23 to conclude that the image of $\mathbf{G}(\mathfrak{L}_X)$ in $\mathbf{GL}(E)$ is the group-envelope of the Lie algebra generated by $A(H^0(\Omega^1_X)^*)$. Because of Proposition 11, the image of $\Theta(X,x_0)\simeq\mathbf{G}(\mathfrak{L}_X)$ is the image of $\Pi(X,x_0)$, which is the differential Galois group. $\square$

Remark 26. In "birational" differential Galois theory, one can find a result reminiscent of Theorem 25; see [24, p. 25, Remarks 1.33].

Remark 27. One could hope for a straightforward way to compute differential Galois groups, as Theorem 25, in the case where X fails to be projective and $(\mathscr{E},\nabla)$ is taken to be a *regular-singular connection*. But this is certainly false: take $X=\operatorname{Spec}K[x,x^{-1}]$ and define $(\mathscr{O}_Xe,\nabla)$ by $\nabla e = ke\otimes\frac{dx}{x}$ for any given $k\in\mathbf{Z}\smallsetminus\{0\}$. In this case, the differential Galois group is trivial (since $\nabla(x^{-k}e)=0$), while the Lie algebra generated by $k\in K$ is not.

Fixing generators of the Lie algebra of a subgroup scheme of some general linear group allows us to construct connections with a prescribed differential Galois group. To state our results, we need the notion of semi-simple and reductive group schemes over a general field. Let $\overline{K}$ be an algebraic closure of K. An algebraic group scheme G (over K) is *semi-simple*, respectively *reductive*, if and only if $G\otimes\overline{K}$ is a semi-simple, respectively reductive, group scheme over $\overline{K}$ [19, 6.44, 6.46]. Since in the case of reductive group schemes our arguments require a bit more group theory, we treat the semi-simple and reductive cases separately.

Corollary 28. *Let* X *be a projective curve (smooth and integral, by assumption) over* K *of genus* g *and carrying a point* $x_0\in X(K)$. *Let* G *be a connected algebraic group scheme with Lie algebra* $\mathfrak{g}$.

(1) *Let*

$$\mu=\min\left\{\dim V:\ \begin{array}{l}V\text{ is a subspace of }\mathfrak{g}\text{ which}\\ \text{generates }\mathfrak{g}\text{ as a Lie algebra}\end{array}\right\}.$$

Then, there exists a trivial vector bundle with a connection having differential Galois group G *if* $\mu\le g$.

(2) *Suppose that* $g\ge 2$ *and that* G *is semi-simple. Then there exists a trivial vector bundle with a connection having differential Galois group* G.

(2bis) *In the setting of the previous item, if* G *is, in addition, split, then the connection can be written down explicitly. (For the definition of "split", see [19, Definition 19.22, p. 402].)*

Proof.

(1) Let $V\subset\mathfrak{g}$ be a subspace of $\mathfrak{g}$ of dimension μ which generates $\mathfrak{g}$ as a Lie algebra. Let $G\longrightarrow\mathbf{GL}(E)$ be a closed immersion and regard V as a subspace of $\mathrm{End}(E)$. We then pick any K-linear map $A: H^0(\Omega^1_X)^*\longrightarrow\mathrm{End}(E)$ such that $\mathrm{Im}(A)=V$. This map then becomes a morphism of K-algebras $A:\mathbf{T}H^0(\Omega^1_X)^*\longrightarrow\mathrm{End}(E)$. Note that G is the group-envelope of V and by Theorem 25, the differential Galois group of $\mathscr{V}(E,A)$ is isomorphic to G.

(2) We begin by recalling that a Lie algebra over K is semi-simple if and only its base change to $\overline{K}$ is likewise [6, I.6.10]. Since $G\otimes\overline{K}$ is semi-simple if and only if $\mathrm{Lie}(G\otimes\overline{K})\simeq(\mathrm{Lie}\,G)\otimes\overline{K}$ is semi-simple (see either [10, Corollary II.6.2.2] or [26, Proposition 27.2.2]), we can assure that $\mathfrak{g}:=\mathrm{Lie}(G)$ is semi-simple. According to Kuranishi's theorem (see [16, Theorem 1] or [6, VIII.2,

p. 221, Exercise 8]), there exists a two dimensional vector space $V \subset \mathfrak{g}$ generating $\mathfrak{g}$ as a Lie algebra so the previous item can be applied.
(2bis) Let T be a maximal torus of G which is split. Let us write $\mathfrak{t}$ for Lie T. Then, $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the eigenspace associated to a non-trivial character $\alpha : T \longrightarrow \mathbf{G}_{m,K}$ [19, 21.a]. For convenience, we shall also denote by α the differential $\mathfrak{t} \longrightarrow K$ obtained from α. From the direct sum decomposition above, we see that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\eta_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ and let

$$\xi \in \mathfrak{t} \setminus \bigcup_{\alpha \in R} \mathrm{Ker}(\alpha) \cup \bigcup_{\substack{\alpha,\beta \in R \\ \alpha \neq \beta}} \mathrm{Ker}(\alpha - \beta).$$

Then

$$\xi \quad \text{and} \quad \eta := \sum_\alpha \eta_\alpha$$

are generators of $\mathfrak{g}$ [6, VIII.2, p. 221, Exercise 8]. We can now proceed as in (1): Let $G \subset \mathbf{GL}(E)$ and let $\xi, \eta \in \mathfrak{g}$ be interpreted as endomorphisms of E. Then, if $\varphi, \psi \in H^0(\Omega^1_X)$ are linearly independent, define a connection on $\mathscr{O} \otimes E$ by

$$\nabla(1 \otimes e) = (1 \otimes \xi(e)) \otimes \varphi + (1 \otimes \eta(e)) \otimes \psi.$$

Its differential Galois group is isomorphic to G. □

Corollary 28 (2) can be modified to throw light on the case of reductive groups. Some preparatory material is necessary.

Let G be a reductive group scheme with Lie algebra $\mathfrak{g}$. Then $G \otimes \overline{K}$ is a reductive $\overline{K}$-group scheme and hence $\mathfrak{g} \otimes \overline{K} = \mathrm{Lie}(G \otimes \overline{K})$ is a reductive $\overline{K}$-Lie algebra [26, Proposition 27.2.2]. By [6, I.6.10] $\mathfrak{g}$ is reductive. Hence, $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$, where $\mathfrak{s}$ is semi-simple and $\mathfrak{z}$ is the center of $\mathfrak{g}$ [6, I.6.4, Proposition 5]. As is well-known, if Z stands for the *neutral component of the center* of G, then $\mathrm{Lie}\, Z = \mathfrak{z}$ [10, II.6, Proposition 2.1, p. 259]. Moreover, Z is a geometrically connected group scheme of multiplicative type (see Proposition 1.34 and Corollary 17.62 in [19]), and hence a *torus*.

Lemma 29. *Suppose that $[K : \mathbf{Q}] = \infty$. Then, there exists a subset $\{\xi, \eta, \zeta\} \subset \mathfrak{g}$ with the following property. The Lie algebra generated by $\{\xi, \eta, \zeta\}$ is algebraically dense in $\mathfrak{g}$. Said differently, any closed subgroup scheme $H \subset G$ such that $\{\xi, \eta, \zeta\} \subset \mathrm{Lie}\, H$ must actually coincide with G.*

Proof. Let $\xi, \eta \in \mathfrak{s}$ generate $\mathfrak{s}$ as a Lie algebra. We now show that there exists $\zeta \in \mathfrak{z}$ such that $K\zeta$ is algebraically dense in $\mathfrak{z}$. Since Z is a torus, the $\overline{K}$-group scheme $Z \otimes \overline{K}$ is isomorphic to $\mathbf{G}^r_{m,\overline{K}}$. From [26, 24.6.3] and the hypothesis that $[K : \mathbf{Q}] = \infty$, there exists $\zeta \in \mathfrak{z}$ such that $\overline{K} \cdot (\zeta \otimes 1)$ is algebraically dense in $\mathrm{Lie}(Z \otimes \overline{K})$: it is enough to pick $\zeta = (\zeta_1, \ldots, \zeta_r) \in K^r$ with $\{\zeta_i\}$ linearly independent over $\mathbf{Q}$. It is then a simple matter to see that $K\zeta \subset \mathfrak{z}$ is algebraically dense.

To end, let $H \subset G$ be a closed subgroup scheme whose Lie algebra $\mathfrak{h}$ contains $\{\xi, \eta, \zeta\}$. Then $\zeta \in \mathfrak{h} \cap \mathfrak{z} = \mathrm{Lie}(H \cap Z)$ [19, 10.14] and hence $\mathfrak{z} \subset \mathfrak{h}$, which shows, using the equality $\mathfrak{g} = \mathfrak{s} + \mathfrak{z}$, that $\mathfrak{h} = \mathfrak{g}$ and we conclude by [10, II.6, Proposition 2.1]. □

The same method used to establish Corollary 28 now gives the following.

Corollary 30. *Let X, x_0 and g be as in Corollary 28. Let G be a reductive group scheme. If $[K : \mathbf{Q}] = \infty$ and $g \geq 3$, then G is the differential Galois group of a connection on a trivial vector bundle. In addition, if G is split, the construction can be made explicit.*

Let us illustrate how explicit the constructions can be made with an example. All depends on the construction of generating elements in a Lie algebra.

Example 31. Let $K = \mathbf{Q}$ and X be an arbitrary smooth and projective curve carrying a K-rational point and having genus at least 2. We give ourselves non-proportional global differential forms φ and ψ. We wish to construct an explicit expression of a connection on a free vector bundle on X such that the associated differential Galois group is an "exceptional" group scheme. To explain what exceptional means requires some preliminary material from [3–6] and [19].

Recall from the "Isogeny Theorem" that there exists a semi-simple and split group scheme G whose root system is G_2 and, in addition, such group is unique up to isomorphism. This is clearly proved as [19, Theorem 23.25, pp 492-3], albeit in terms of root data. The (well-known) link between root data and root systems is explained by the concept of semi-simple root data [19, Definition C.34, p. 615] and "diagrams" [19, Definition C.27, p. 613]. See also [19, Theorem 23.58, p. 501] for a concise statement.

We set out to construct a connection on X whose differential Galois group is isomorphic to G. To employ our method, we need to realise $\mathrm{Lie}\, G$ explicitly as an algebra of matrices and then spot generators; for the first task we follow [11] because of the explicitness of its constructions over $\mathbf{Q}$. Let V be an *eight*-dimensional vector space with basis $\{b_i\}_{i=0}^{7}$. For $0 \le i, j \le 7$, define $e_{ij} \in \mathrm{End}(V)$ by

$$b_k \longmapsto \begin{cases} 0, & \text{if } k \neq j, \\ b_i, & \text{if } k = j. \end{cases}$$

We then define "upper-triangular maps"

$$\begin{aligned} &x_0 = e_{01} + e_{23} - e_{24} + e_{35} - e_{45} - e_{67}, \qquad x_1 = e_{12} - e_{56}, \\ &x_2 = -e_{02} + e_{13} - e_{14} + e_{36} - e_{46} + e_{57}, \\ &x_3 = e_{03} - e_{04} - e_{15} + e_{26} + e_{37} - e_{47}, \qquad x_4 = -e_{05} + e_{27}, \quad x_5 = -e_{06} + e_{17}, \end{aligned}$$

"diagonal maps"

$$x_6 = e_{00} + e_{11} - e_{66} - e_{77}, \quad x_7 = e_{00} + e_{22} - e_{55} - e_{77},$$

and "lower-triangular maps"

$$\begin{aligned} &x_8 = e_{60} - e_{71}, && x_9 = e_{50} - e_{72}, \\ &x_{10} = -e_{30} + e_{40} + e_{51} - e_{62} - e_{73} + e_{74}, && \\ &x_{11} = e_{20} - e_{31} + e_{41} - e_{63} + e_{64} - e_{75}, && x_{12} = -e_{21} + e_{65}, \\ &x_{13} = -e_{10} - e_{32} + e_{42} - e_{53} + e_{54} + e_{76}. && \end{aligned}$$

(See p. 631 and section 3.3 in [11], but beware that he uses a capital X to denote endomorphisms.) By computer-algebra, one can verify that the span of $\{x_i\}_{i=0}^{13}$ is a Lie algebra $\mathfrak{G}$ of dimension 14 and that $\mathfrak{T} := Kx_6 + Kx_7$ is an abelian subalgebra. If $\{x_6^*, x_7^*\}$ is the dual basis of $\{x_6, x_7\}$, then define

$$\begin{aligned} &\alpha_0 = x_7^*, \qquad \alpha_1 = x_6^* - x_7^*, \quad \alpha_2 = x_6^*, \\ &\alpha_3 = x_6^* + x_7^*, \quad \alpha_4 = x_6^* + 2x_7^*, \quad \alpha_5 = 2x_6^* + x_7^*. \end{aligned}$$

By computer-algebra, we verify that for each $i \in \{0, \dots, 5\}$, we have

$$\mathrm{ad}_t(x_i) = \alpha_i(t) x_i, \qquad \forall\ t \in \mathfrak{T}$$

and

$$\mathrm{ad}_t(x_{13-i}) = -\alpha_i(t) x_{13-i}, \qquad \forall\ t \in \mathfrak{T}.$$

Consequently,

$$\mathfrak{G} = \mathfrak{T} \oplus \bigoplus_{i=0}^{5} Kx_i \oplus Kx_{13-i}$$

is the root decomposition of $(\mathfrak{G}, \mathfrak{T})$ and defines a root system of type G_2 on the vector space $\mathfrak{T}^* = Kx_6^* + Kx_7^*$, see [11, Figure 2, p. 634].

Following the method described in Corollary 28 (2bis), we know that the connection on $V \otimes \mathscr{O}_X$ determined by the End(V)-valued form

$$\underbrace{\begin{pmatrix} 0 & 1 & -1 & 1 & -1 & -1 & -1 & 0 \\ -1 & 0 & 1 & 1 & -1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & -1 & 1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & 1 & -1 & 1 & 0 \end{pmatrix}}_{\sum_{i=0}^{5} x_i + x_{13-i}} \otimes \varphi + \underbrace{\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{pmatrix}}_{x_6 + 3x_7} \otimes \psi$$

has differential Galois group isomorphic to G. Indeed, as $\mathfrak{G}$ is semi-simple, it must be the Lie algebra of a closed and connected subgroup scheme of $\mathbf{GL}(V)$ [10, Proposition II.6.2.6, p. 262], which must be isomorphic to G.

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Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

[1] I. Biswas, J. P. dos Santos, S. Dumitrescu, S. Heller, "On certain Tannakian categories of integrable connections over Kähler manifolds", *Can. J. Math.* **74** (2022), no. 4, p. 1034-1061.

[2] I. Biswas, S. Subramanian, "Vector bundles on curves admitting a connection", *Q. J. Math.* **57** (2006), no. 2, p. 143-150.

[3] N. Bourbaki, *Éléments de Mathématiques. Groupes et algèbres de Lie. Chapitres 2 et 3*, reprint of the 1972 original ed., Springer, 2006.

[4] ———, *Éléments de Mathématiques. Groupes et algèbres de Lie. Chapitres 7 et 8*, reprint of the 1975 original ed., Springer, 2006.

[5] ———, *Élément de mathématique. Groupes et algèbres de Lie. Chapitres 4 à 6*, reprint of the 1968 original ed., Springer, 2007.

[6] ———, *Éléments de mathématique. Groupes et algèbres de Lie. Chapitre 1*, reprint of the 1972 original ed., Springer, 2007.

[7] P. Deligne, "Théorème de Lefschetz et critères de dégénérescence de suites spectrales", *Publ. Math., Inst. Hautes Étud. Sci.* **35** (1968), p. 107-126.

[8] ———, "Le groupe fondamental de la droite projective moins trois points", in *Galois Groups over* **Q**, *Berkeley, CA*, Mathematical Sciences Research Institute Publications, vol. 16, Springer, 1987, p. 79-297.

[9] P. Deligne, J. S. Milne, "Tannakian categories", in *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, vol. 900, Springer, 1982, p. 101-228.

[10] M. Demazure, P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique. Généralités. Groupes commutatifs*, Masson; North-Holland, 1970, avec un appendice 'Corps de classes local' par Michiel Hazewinkel.

[11] W. H. Hesselink, "The nullcone of the Lie algebra of G_2", *Indag. Math., New Ser.* **30** (2019), no. 4, p. 623-648.

[12] G. P. Hochschild, "Algebraic Lie algebras and representative functions", *Ill. J. Math.* **3** (1959), p. 499-529.

[13] ———, "Lie algebra cohomology and affine algebraic groups", *Ill. J. Math.* **18** (1974), p. 170-176.

[14] J.-P. Jouanolou, *Théorèmes de Bertini et applications*, Progress in Mathematics, vol. 42, Birkhäuser, 1983.

[15] N. M. Katz, "Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin", *Publ. Math., Inst. Hautes Étud. Sci.* **39** (1970), p. 175-232.

[16] M. Kuranishi, "Two elements generations on semi-simple Lie groups", *Kōdai Math. Semin. Rep.* **1949** (1949), no. 5-6, p. 89-90.

[17] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford Science Publications, 2002.

[18] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics, vol. 5, Springer, 1971.

[19] J. Milne, *Algebraic Groups. The Theory of Group Schemes of Finite Type Over a Field*, Cambridge Studies in Advanced Mathematics, vol. 170, Cambridge University Press, 2017.

[20] S. Montgomery, *Hopf algebras and their actions on rings*, Regional Conference Series in Mathematics, vol. 82, American Mathematical Society, 1993, Expanded version of ten lectures given at the CBMS Conference on Hopf algebras and their actions on rings, which took place at DePaul University in Chicago, USA, August 10-14, 1992.

[21] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research. Studies in Mathematics, vol. 5, Tata Institute of Fundamental Research; Oxford University Press, 1970.

[22] N. Nahlus, "Lie algebras of pro-affine algebraic groups", *Can. J. Math.* **54** (2002), no. 3, p. 595-607.

[23] N. Nitsure, "Construction of Hilbert and Quot schemes", in *Fundamental algebraic geometry*, Mathematical Surveys and Monographs, vol. 123, American Mathematical Society, 2005, p. 105-137.

[24] M. van der Put, M. F. Singer, *Galois theory of linear differential equations*, Grundlehren der Mathematischen Wissenschaften, vol. 328, Springer, 2003.

[25] M. Sweedler, *Hopf algebras*, Benjamin, 1969.

[26] P. Tauvel, R. W. T. Yu, *Lie algebras and algebraic groups*, Springer Monographs in Mathematics, Springer, 2005.

[27] W. C. Waterhouse, *Introduction to affine group schemes*, Graduate Texts in Mathematics, vol. 66, Springer, 1979.