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Salem numbers of automorphisms of K3 surfaces with Picard number 4

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Abstract. We construct automorphisms of positive entropy of K3 surfaces of Picard number 4 with certain Salem numbers. We also prove that there is a fixed point free automorphism of positive entropy on a K3 surface of Picard number 4 with Salem degree 4.

Keywords. K3 surface, Automorphism, Salem number.

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1. Introduction

A complex number λ is an *algebraic integer* if $S(\lambda) = 0$ for some irreducible monic polynomial $S(x) \in \mathbb{Z}[x]$; it is a *unit* if λ^{-1} is also an algebraic integer. A *Salem number* is a unit $\lambda > 1$ whose conjugates other than $\lambda^{\pm 1}$ lie on the unit circle. The irreducible monic integer polynomial $S(x)$ of λ is a *Salem polynomial* and the degree of $S(x)$ is the degree of Salem number λ. In this paper, we allow quadratic Salem numbers; these were excluded in Salem's original definition [14, III.3]. A *Salem trace* is an algebraic integer $\tau > 2$ whose other conjugates lie in $(-2, 2)$; the minimal polynomial of such a number is called a *Salem trace polynomial*. Salem traces and Salem numbers correspond bijectively, via the relation $\tau = \lambda + \lambda^{-1}$.

All Salem polynomials satisfy the equation $x^n S(\frac{1}{x}) = S(x)$, where $n = \deg S(x)$. This simply means that its coefficients form a palindromic sequence, that is, they read the same backwards as forwards. Hence any Salem polynomial of degree 4 is as follows

$$S(x) = x^4 - ax^3 - bx^2 - ax + 1 \in \mathbb{Z}[x]. \tag{1}$$

Since $S(x) = x^2((x + \frac{1}{x})^2 - a(x + \frac{1}{x}) - b - 2)$, the associated Salem trace polynomial is $R(y) = y^2 - ay - b - 2 \in \mathbb{Z}[y]$.

A compact complex surface X is called a *K3 surface* if it is simply connected and has a nowhere vanishing holomorphic 2-form ω_X. We consider the second integral cohomology $H^2(X, \mathbb{Z})$ with the cup product as a lattice. It is known that $H^2(X, \mathbb{Z})$ is an even unimodular lattice of signature $(3, 19)$, which is unique up to isomorphism and is called the *K3 lattice*.

For an automorphism $f \in \mathrm{Aut}(X)$, it is known that either all eigenvalues of $f^*|H^2(X,\mathbb{Z})$ are roots of unity or there is a unique eigenvalue of modulus > 1, this eigenvalue is simple, and it is a Salem number λ ([6, Theorem 3.2]). This number λ is the dynamical degree of f. The degree of its minimal polynomial is bounded from above by the Picard number of X. An automorphism f of X is called *symplectic* (resp. *anti-symplectic*) if its action on the vector space $H^{2,0}(X) = \mathbb{C}\omega_X$ is trivial, i.e., $f^*\omega_X = \omega_X$ (resp. $f^*\omega_X = -\omega_X$).

In this paper, we show the following.

Theorem 1. *Let λ be a Salem number of degree 4, and let $S(x) = x^4 - ax^3 - bx^2 - ax + 1$ be its minimal polynomial.*

(1) *If a is even and b is odd, λ^6 is a Salem number of a symplectic automorphism of a projective K3 surface with Picard number* 4.
(2) *If both a and b are even, λ^8 is a Salem number of a symplectic automorphism of a projective K3 surface with Picard number* 4.
(3) *If both a and b are odd, λ^{10} is a Salem number of a symplectic automorphism of a projective K3 surface with Picard number* 4.
(4) *If a is odd and b is even, λ^{12} is a Salem number of a symplectic automorphism of a projective K3 surface with Picard number* 4.

Similarly, for a Salem polynomial $x^2 - ax + 1$ with Salem number λ, λ^2 is a Salem number of symplectic automorphism of a projective K3 surface with Picard number 2. ([5])

Remark 2. By the fundamental result of Gromov and Yomdin ([4, 15]), the topological entropy of an automorphism of a compact Kähler surface is the logarithm of a Salem number. For dynamics of automorphisms of compact complex surfaces, we refer to [3]. For minimal Salem numbers of automorphisms of K3 surfaces, we refer to [7, 8]. Moreover, for similar results on complex tori, Enriques surfaces and K3 surfaces, we refer to [1, 2, 12, 13, 16].

For $f \in \mathrm{Aut}(X)$, let

$$X^f := \{x \in X \mid f(x) = x\}. \tag{2}$$

If $X^f = \emptyset$, then f is said to be *fixed point free* or *free* for short. In [10], Oguiso first showed that there is a projective K3 surface with a free automorphism of positive entropy when the Picard number is 2. Conversely, he also showed that a smooth compact Kähler surface with a free automorphism of positive entropy is birational to a projective K3 surface of Picard number ≥ 2. In [5], the authors showed that for Picard number 2, the example in [10] is the unique free automorphism.

For Picard number > 2, one can construct free automorphisms of positive entropy by extending this isometry to a Picard lattice of higher Picard number. All the known examples of free automorphisms are constructed in this way. In this paper, however, we give examples of free automorphisms of positive entropy of a K3 surface with Picard number 4 with Salem degree 4 as well as 2.

Theorem 3. *There are free automorphisms of positive entropy on K3 surfaces with Picard number* 4 *with Salem degree* 2 *or* 4.

The paper is organized as follows. We recall some results on lattices and K3 surfaces in Section 2. We review *$P(x)$-lattices* in [6] in Section 3 and prove theorems in Section 4.

2. Preliminaries

2.1. *Lattices*

A *lattice* is a free $\mathbb{Z}$-module L of finite rank equipped with a symmetric bilinear form $\langle\cdot,\cdot\rangle : L \times L \to \mathbb{Z}$. If $x^2 := \langle x, x\rangle \in 2\mathbb{Z}$ for any $x \in L$, a lattice L is said to be even. For any given lattice L, the *twist*

$L(m)$ is obtained by changing the intersection form $\langle\cdot,\cdot\rangle$ of L by the integer m, i.e., $L(m) = L$ as $\mathbb{Z}$-modules but

$$\langle\cdot,\cdot\rangle_{L(m)} := m\cdot\langle\cdot,\cdot\rangle_L.$$

We fix a $\mathbb{Z}$-basis of L and identify the lattice L with its intersection matrix Q_L under this basis. The *discriminant* $\operatorname{disc}(L)$ of L is defined as $\det(Q_L)$, which is independent of the choice of basis. A lattice L is called *non-degenerate* if $\operatorname{disc}(L) \neq 0$ and *unimodular* if $\operatorname{disc}(L) = \pm 1$. For a non-degenerate lattice L, the signature of L is defined as (s_+, s_-), where s_+ (resp. s_-) denotes the number of the positive (resp. negative) eigenvalues of Q_L. An isometry of L is an automorphism of the $\mathbb{Z}$-module L preserving the bilinear form. The *orthogonal group* $O(L)$ of L consists of the isometries of L and we have the following identification:

$$O(L) = \left\{g \in GL_n(\mathbb{Z}) \,\middle|\, g^T\cdot Q_L\cdot g = Q_L\right\}, \quad n = \operatorname{rank} L. \tag{3}$$

For a non-degenerate lattice L, the *discriminant group* $A(L)$ of L is defined by

$$A(L) := L^*/L, \quad L^* := \left\{x \in L\otimes\mathbb{Q} \,\middle|\, \langle x, y\rangle \in \mathbb{Z} \ \ \forall\, y \in L\right\}. \tag{4}$$

Let K be a sublattice of a lattice L, that is, K is a $\mathbb{Z}$-submodule of L equipped with the restriction of the bilinear form of L to K. If L/K is torsion-free as a $\mathbb{Z}$-module, K is said to be *primitive*.

For a non-degenerate lattice L of signature $(1,k)$ with $k \geq 1$, we have the decomposition

$$\left\{x \in L\otimes\mathbb{R} \,\middle|\, x^2 > 0\right\} = C_L \sqcup (-C_L) \tag{5}$$

into two disjoint cones. Here C_L and $-C_L$ are connected components and C_L is called the *positive cone*. We define

$$O^+(L) := \left\{g \in O(L) \,\middle|\, g(C_L) = C_L\right\}, \quad SO^+(L) := O^+(L)\cap SO(L), \tag{6}$$

where $SO(L)$ is the subgroup of $O(L)$ consisting of isometries of determinant 1. The group $O^+(L)$ is a subgroup of $O(L)$ of index 2.

Lemma 4. *Let L be a non-degenerate even lattice of rank n. For $g \in O(L)$ and $\varepsilon \in \{\pm 1\}$, g acts on $A(L)$ as $\varepsilon\cdot\mathrm{id}$ if and only if $(g - \varepsilon\cdot I_n)\cdot Q_L^{-1}$ is an integer matrix.*

Proof. This follows from the fact that L^* is generated by the columns of Q_L^{-1}. □

2.2. *K3 surfaces*

A compact complex surface X is a K3 surface if it is simply connected and has a nowhere vanishing holomorphic 2-from ω_X. We consider the second integral cohomology $H^2(X,\mathbb{Z})$ with the cup product as a lattice. It is known that $H^2(X,\mathbb{Z})$ is an even unimodular lattice of signature $(3,19)$, which is unique up to isomorphism and is called the *K3 lattice*. We fix such a lattice and denote it by Λ_{K3}. The Picard lattice S_X and transcendental lattice T_X of X are defined as follows

$$S_X := \left\{x \in H^2(X,\mathbb{Z}) \,\middle|\, \langle x, \omega_X\rangle = 0\right\}, \tag{7}$$

$$T_X := \left\{x \in H^2(X,\mathbb{Z}) \,\middle|\, \langle x, y\rangle = 0 \ (\forall\, y \in S_X)\right\}. \tag{8}$$

Here ω_X is considered as an element in $H^2(X,\mathbb{C})$ and the bilinear form on $H^2(X,\mathbb{Z})$ is extended to that on $H^2(X,\mathbb{C})$ linearly. The Picard group of X is naturally isomorphic to S_X. It is known that X is projective if and only if S_X has signature $(1,\rho-1)$, where $\rho = \operatorname{rank} S_X$ is the Picard number of X.

Let X be a projective K3 surface. Since $H^2(X,\mathbb{Z})$ is unimodular and S_X is non-degenerate, we have the following natural identification:

$$A(S_X) = A(T_X) = H^2(X,\mathbb{Z})/(S_X\oplus T_X). \tag{9}$$

By the global Torelli theorem for K3 surfaces [11, Proposition in Section 7], the following map is injective:

$$\mathrm{Aut}(X) \ni \varphi \mapsto (g,h) := (\varphi^*|_{S_X}, \varphi^*|_{T_X}) \in O(S_X) \times O(T_X). \tag{10}$$

Moreover, $(g,h) \in O(S_X) \times O(T_X)$ is the image of some $\varphi \in \mathrm{Aut}(X)$ by the map (10) if and only if

(1) the linear extension of g (resp. h) preserves the ample cone C_X of X (resp. $\mathbb{C}\omega_X$), and
(2) the actions of g and h on $A(S_X) = A(T_X)$ coincide.

A line bundle L on X is *very ample* if there is a closed embedding of X into $\mathbb{P}^n$ such that $L = \mathcal{O}_{\mathbb{P}^n}(1)|_X$. A line bundle L is *ample* if $L^{\otimes m}$ is very ample for some $m > 0$. The *ample cone* $\mathrm{Amp}(X)$ is the set of all finite sums $\Sigma a_i L_i$ with $L_i \in S_X$ ample and $a_i \in \mathbb{R}_{>0}$. The ample cone of a projective K3 surface is as follows:

$$\mathrm{Amp}(X) = \left\{ L \in C_{S_X} \,\middle|\, \langle L, L\rangle > 0, \langle L, C\rangle > 0 \text{ for all } C \cong \mathbb{P}^1 \subset X \right\}, \tag{11}$$

where C_{S_X} is the positive cone of S_X as in (5). In particular, if X has no (-2) curves, then $\mathrm{Amp}(X) = C_{S_X}$.

We have the following existence theorem of K3 surfaces.

Theorem 5 ([9, Theorem 1.14.4]). *For any even lattice L of rank $r \le 10$ and signature $(1, r-1)$, there exists a (projective) K3 surface X such that $S_X \cong L$.*

3. $P(x)$-lattices

We review the definition of $P(x)$-lattices considered in [6, Section 8]. For a separable polynomial $P(x)$, a $P(x)$*-lattice* is a pair (L, f) consisting of a non-degenerate lattice L and an element $f \in O(L)$ such that $P(x) = \det(xI - f)$.

Let (L, f) be a $P(x)$-lattice. For any nonzero $a \in \mathbb{Z}[f + f^{-1}] \subset \mathrm{End}(L)$, the new inner product

$$\langle g_1, g_2\rangle_a = \langle a g_1, g_2\rangle \tag{12}$$

defines the *twisted* $P(x)$-lattice $(L(a), f)$.

This operation preserves even lattices. It is easy to see that if L is unimodular, then the glue group of its twist satisfies

$$G(L(a)) \equiv L/aL$$

as a $\mathbb{Z}[f]$-module.

We define the *principal* $P(x)$-lattice (L, f) by

$$L = \mathbb{Z}[x]/P(x) \subset K$$

with the inner product

$$\langle g_1, g_2\rangle = tr_{K/\mathbb{Q}}\left(\frac{g_1 g_2^\sigma}{R'(y)}\right) = \sum_{i=1}^{d} \frac{g_1(x_i) g_2(x_i^{-1})}{R'(x_i + x_i^{-1})}, \tag{13}$$

where $\{x_i\}_1^d$ are the roots of $P(x)$. The action of f is given by multiplication by x. The lattice L is even with

$$\det(L) = |P(-1)P(1)|. \tag{14}$$

Now let λ be a solution of $S(x)$ in (1) and $\mathbb{Z}[\lambda] = \mathbb{Z}[x]/(S(x))$ and $\mathbb{Q}(\lambda)$ its field of fraction. Then $[\mathbb{Q}(\lambda) : \mathbb{Q}(\tau)] = 2$ and $\sigma : \lambda \mapsto \lambda^{-1}$ is a natural involution of $\mathbb{Q}(\lambda)$ over $\mathbb{Q}(\tau)$. Moreover $R(y) = y^2 - ay - b - 2$ and $R'(y) = 2y - a$. Hence we have

$$\langle 1, 1\rangle = \langle \lambda, \lambda\rangle = \langle \lambda^2, \lambda^2\rangle = \langle \lambda^3, \lambda^3\rangle = 0 \tag{15}$$

$$\langle 1, \lambda\rangle = 1, \langle 1, \lambda^2\rangle = a, \langle 1, \lambda^3\rangle = a^2 + b - 1 \tag{16}$$

and its intersection matrix is

$$Q_L := \begin{bmatrix} 0 & 1 & a & a^2+b-1 \\ 1 & 0 & 1 & a \\ a & 1 & 0 & 1 \\ a^2+b-1 & a & 1 & 0 \end{bmatrix}. \tag{17}$$

Remark 6. Note that the signature of L is $(1,3)$ by Corollary 8.4 in [6]. Generally, the signature of $P(x)$-lattice is $(d,d)+(p,-p)+(-q,q)$, where p is the number of roots of $R(x)$ in $[-2,2]$ satisfying $R'(\tau)>0$, and q is the number satisfying $R'(\tau)<0$. Hence if $S(x)$ has degree $2d$ and d is even, then $R(y)$ has an odd number of zeros in the range $[-2,2]$, and therefore the lattice L has signature $(d-1,d+1)$.

4. Proof

4.1. *Proof of Theorem 1*

Let $S_4(x) = x^4 - ax^3 - bx^2 - ax + 1$ be a Salem polynomial of degree 4. Then for $y = x + x^{-1}$, $R_2(y)$ is the corresponding Salem trace polynomial of degree 2, i.e., $R_2(y) = y^2 - ay - b - 2$. Using $P(x)$-lattice in Section 3, we have a lattice L_4 whose intersection matrix is

$$Q_{L_4} = \begin{pmatrix} 0 & 1 & a & a^2+b-1 \\ 1 & 0 & 1 & a \\ a & 1 & 0 & 1 \\ a^2+b-1 & a & 1 & 0 \end{pmatrix}. \tag{18}$$

Define g as the multiplication by x on this lattice. Then we have the following

Lemma 7. *For the lattice in* (18) *and g as above,* $(g^6-\mathrm{id})(Q_{L_4(2)})^{-1}$ *is integral if a is even and b is odd and* $(g^8-\mathrm{id})(Q_{L_4(2)})^{-1}$ *is integral if both a and b are even. Moreover,* $(g^{10}-\mathrm{id})(Q_{L_4(2)})^{-1}$ *is integral if both a and b are odd and* $(g^{12}-\mathrm{id})(Q_{L_4(2)})^{-1}$ *is integral if a is odd and b are even.*

Proof. Note that g is given by the following matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \end{pmatrix}.$$

Now Lemma follows from the matrix multiplications. □

Since L_4 is an even lattice of signature $(1,3)$ as in Remark 6, there exists a projective K3 surface X with Picard lattice L_4 (Theorem 5 or [7, Theorem 6.1]).

Moreover, $L_4(2)$ has no (-2) curves, hence the ample cone is the same with positive cone. Note that for $L_4(2)$, the ample divisor $(1,1,0,0)$ maps to the ample divisor $(0,1,1,0)$ via g. Since g^6, g^8, g^{10}, and g^{12} preserve the positive cone, by the global Torelli theorem, each of these induces a symplectic automorphism on X. Hence the theorem is proved.

Remark 8. For S_6 and L_6, we have the similar result as in the lemma, but the signature L_6 is $(2,4)$ which is not hyperbolic.

4.2. *Examples of free automorphisms with positive entropy*

In this subsection, we give several free automorphisms of positive entropy of K3 surface with Picard number 4.

4.2.1. *Case 1*

We construct one along the following steps.

Step 1. Let $P(x) = (x^2 - x - 1)(x^2 + 1)$ and $S(x) = (x^2 - 3x + 1)(x + 1)^2$. Note that if $P(x)$ has roots x_i $(1 \le i \le 4)$, then $S(x)$ has roots x_i^2. Now $S(x)$ is a product of two cyclotomic polynomials and one Salem polynomial of degree 2. Now let L be the $\mathbb{Z}$-module, $\mathbb{Z}[x]/P(x)$, with the inner product with respect to the basis $\{1, x, x^2, x^3\}$ given as follows:

$$\begin{gathered} \langle 1,1\rangle = \langle x^2, x^2\rangle = 2p, \quad \langle 1, x\rangle = \langle x^2, x^3\rangle = q, \quad \langle 1, x^2\rangle = s, \quad \langle 1, x^3\rangle = t, \\ \langle x, x\rangle = \langle x^3, x^3\rangle = 2r, \quad \langle x, x^2\rangle = u, \quad \langle x, x^3\rangle = v, \end{gathered} \tag{19}$$

where $p, q, r, s, t, u, v \in \mathbb{Z}$. Moreover, let g be the multiplication by x on L. The action of g on L is represented by the following matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{20}$$

Lemma 9. *g^2 is an isometry of L if $u = -q$, $v = -2p - 2r - s$, and $t = 2p - q + s$.*

Proof. From (19), the intersection matrix Q_L of L for the basis $\{1, x, x^2, x^3\}$ is the following matrix:

$$\begin{pmatrix} 2p & q & s & t \\ q & 2r & u & v \\ s & u & 2p & q \\ t & v & q & 2r \end{pmatrix}. \tag{21}$$

Note that it has the same intersection numbers corresponding to the ij-th and the $(i+2)(j+2)$-th entries. This is because we want g^2 (multiplication by x^2) to be an isometry of L. Now the Lemma follows from $(g^2)^T Q_L (g^2) = Q_L$ and (3). □

Step 2. By Lemma 4, in order to get a symplectic or anti-symplectic automorphism f on X, we need to check whether $(f \pm Id)Q_L^{-1}$ is integral or not, respectively. We can construct anti-symplectic automorphisms of X by the following simple matrix multiplication.

Lemma 10. *$((g^2)^3 + Id)Q_L^{-1}$ is integral if $2p + s = \pm 1, \pm 2$.*

Now for $s = 2 - 2p$ and $p = q = r = 2$, $L(-1)$ has signature $(1, 3)$. Hence by Theorem 5, there is a K3 surface with Picard lattice $S_X \cong L(-1)$. Moreover, it has no (-2) curves, hence the ample cone is the same with positive cone.

Step 3. g^6 preserves the positive cone, because it has a Salem number as an Eigenvalue. Now by the global Torelli theorem, g^6 induces an anti-symplectic automorphism on X whose Picard lattice is $L(-1)$.

Step 4. g^6 has no fixed curves. If $g^6(C) = C$ for some curve, then the class $[C]$ would be an eigenvector of $g^{6*}|S_X$ with eigenvalue 1. However, the eigenvalues of g^6 are -1 with multiplicity 2 and $9 \pm 4\sqrt{5}$. Moreover, by the topological Lefschetz fixed point formula,

$$n = T(X, g^6) = 2 + \mathrm{trace}(g^{6*}|S_X) + (-1)(22 - \mathrm{rank}\, S_X), \tag{22}$$

where n is the number of fixed points of g^6. Since $\mathrm{trace}(g^{6*}|S_X) = 16$ and g^6 is anti-symplectic, $n = 0$, i.e, g^6 has no fixed points.

4.2.2. *Case 2*

Define $M = U(-2) \oplus \langle -4 \rangle$, where U is the unimodular hyperbolic plane. Then by Theorem 5, there is a projective K3 surface X whose Picard lattice $S_X \cong M$.

One can show that for any $A \in PGL_2(\mathbb{Z})$ with $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, $P = \begin{bmatrix} ad+bc & -ac & bd \\ -2ab & a^2 & -b^2 \\ 2cd & -c^2 & d^2 \end{bmatrix}$ is an isometry of the lattice M. In particular for $A = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right]$, $P = \left[\begin{smallmatrix} 7 & -2 & 6 \\ -4 & 1 & -4 \\ 12 & -4 & 9 \end{smallmatrix}\right]$ is an isometry of the lattice M. Indeed, $P^T Q_M P = Q_M$, where Q_M is the intersection matrix of M. Moreover, since $(P + \mathrm{id})Q_M^{-1}$ is integral, P acts on $A(M)$ as $-\mathrm{id}$ hence it extends to an anti-symplectic automorphism of X by defining $-\mathrm{id}$ on T_X. Moreover P maps an ample divisor $(1,1,0)$ to another ample divisor $(5,-3,8)$ with respect to a basis, $\{u, v, w\}$, where $\{u, v\}$ and $\{w\}$ are the basis of $U(-2)$ and $\langle -4 \rangle$, respectively. By the topological Lefschetz fixed point formula,

$$\begin{aligned} T(X,f) &= 2 + \mathrm{trace}(f^*|S_X) + (-1)(22 - \mathrm{rank}\, S_X) \\ &= 2 + (-1+18) + (-1)(19) \\ &= 0. \end{aligned} \tag{23}$$

Hence it is a free automorphism with positive entropy.

Remark 11. Similarly we can do this for $U(-2) \oplus \langle -4 \rangle^{\oplus n}$ for $n \le 8$ to get similar free automorphisms of positive entropy by considering the isometry $h = (f, -\mathrm{id}, \ldots, -\mathrm{id})$, where f is given as above.

4.2.3. *Case 3*

As in Section 3, we define a lattice L of signature $(1,3)$ for the Salem polynomial $x^4 - x^3 - x^2 - x + 1$ and let g be the isometry of multiplication by x. By the construction, this is the characteristic polynomial of g and the characteristic polynomial of g^5 is $x^4 - 16x^3 + 14x^2 - 16x + 1$. Now by Theorem 5, there is a projective K3 surface X whose Picard lattice $S_X \cong L(2)$. Since $(g^5 + \mathrm{id})L(2)^{-1}$ is integral, we can extend g^5 to an isometry of $H^2(X,\mathbb{Z})$ by defining $-\mathrm{id}$ on the transcendental lattice. Since $L(2)$ has no (-2) curves and g^5 preserves the ample cone(or equivalently, the positive cone), g^5 is an anti-symplectic automorphism of X by Torelli theorem. Now by the topological Lefschetz fixed point formula as above, g^5 has no fixed points. Hence g^5 is an automorphism with positive entropy and the characteristic polynomial is the Salem polynomial of degree 4: $x^4 - 16x^3 + 14x^2 - 16x + 1$.

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References

[1] E. Bayer-Fluckiger, "Isometries of lattices and automorphisms of K3 surfaces", 2021, https://arxiv.org/abs/2107.07583.

[2] S. Brandhorst, "On the stable dynamical spectrum of complex surfaces", *Math. Ann.* **377** (2020), no. 1-2, p. 421-434.

[3] S. Cantat, "Dynamics of automorphisms of compact complex surfaces", in *Frontiers in complex dynamics*, Princeton Mathematical Series, vol. 51, Princeton University Press, 2014, p. 463-514.

[4] M. Gromov, "On the entropy of holomorphic maps", *Enseign. Math.* **49** (2003), no. 3-4, p. 217-235.

[5] K. Hashimoto, J. Keum, K. Lee, "*K*3 surfaces with Picard number 2, Salem polynomials and Pell equation", *J. Pure Appl. Algebra* **224** (2020), no. 1, p. 432-443.

[6] C. T. McMullen, "Dynamics on $K3$ surfaces: Salem numbers and Siegel disks", *J. Reine Angew. Math.* **545** (2002), p. 201-233.
[7] ———, "$K3$ surfaces, entropy and glue", *J. Reine Angew. Math.* **658** (2011), p. 1-25.
[8] ———, "Automorphisms of projective $K3$ surfaces with minimum entropy", *Invent. Math.* **203** (2016), no. 1, p. 179-215.
[9] V. V. Nikulin, "Integral Symmetric bilinear forms and some of their applications", *Math. USSR, Izv.* **14** (1980), p. 103-167.
[10] K. Oguiso, "Free Automorphisms of positive entropy on smooth Kähler surfaces", in *Algebraic geometry in East Asia – Taipei 2011*, Advanced Studies in Pure Mathematics, vol. 65, Mathematical Society of Japan, 2015, p. 187-199.
[11] I. I. Pjateckiĭ-Šapiro, I. R. Šhafarevič, "Torelli's theorem for algebraic surfaces of type K3", *Izv. Akad. Nauk SSSR, Ser. Mat.* **35** (1971), p. 530-572.
[12] P. Reschke, "Salem numbers and automorphisms of complex surfaces", *Math. Res. Lett.* **19** (2012), no. 2, p. 475-482.
[13] ———, "Salem numbers and automorphisms of abelian surfaces", *Osaka J. Math.* **54** (2017), no. 1, p. 1-15.
[14] R. Salem, *Algebraic numbers and Fourier analysis*, The Wadsworth Mathematics Series, Wadsworth International Group, 1983.
[15] Y. Yomdin, "Volume growth and entropy", *Isr. J. Math.* **57** (1987), p. 285-300.
[16] S. Zhao, "Automorphismes loxodromiques de surfaces abèliennes rèelles", *Ann. Fac. Sci. Toulouse, Math.* **28** (2019), no. 1, p. 109-127.