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# On the critical behavior for a Sobolev-type inequality with Hardy potential

**Mohamed Jleli** [a] **and Bessem Samet** [a]

[a] Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia
*E-mails:* jleli@ksu.edu.sa, bsamet@ksu.edu.sa

**Abstract.** We investigate the existence and nonexistence of weak solutions to the Sobolev-type inequality $-\partial_t(\Delta u)-\Delta u+\frac{\sigma}{|x|^2}u\geq |x|^\mu|u|^p$ in $(0,\infty)\times B$, under the inhomogeneous Dirichlet-type boundary condition $u(t,x)=f(x)$ on $(0,\infty)\times\partial B$, where $B$ is the unit open ball of $\mathbb{R}^N$, $N\geq 2$, $\sigma>-\left(\frac{N-2}{2}\right)^2$, $\mu\in\mathbb{R}$ and $p>1$. In particular, when $\sigma\neq 0$, we show that the dividing line with respect to existence and nonexistence is given by a critical exponent that depends on $N$, $\sigma$ and $\mu$.

**Keywords.** Sobolev-type inequality; Hardy potential; bounded domain; existence; nonexistence; critical exponent.

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## 1. Introduction

We are concerned with the study of existence and nonexistence of solutions to the Sobolev-type inequality

$$-\partial_t(\Delta u)-\Delta u+\frac{\sigma}{|x|^2}u\geq |x|^\mu|u|^p \quad \text{in } (0,\infty)\times B, \tag{1}$$

where $u=u(t,x)$, $B$ is the unit open ball of $\mathbb{R}^N$, $N\geq 2$, $\sigma>-\left(\frac{N-2}{2}\right)^2$, $\mu\in\mathbb{R}$ and $p>1$. Problem (1) is considered under the Dirichlet-type boundary condition

$$u=f \quad \text{on } (0,\infty)\times\partial B, \tag{2}$$

where $f=f(x)\in L^1(\partial B)$. We mention below some motivations for investigating problems of type (1)–(2).

The corresponding equation to (1) belongs to the class of Sobolev-type equations of the form

$$\partial_t Au+Bu=V(x)F(u), \tag{3}$$

where $A$ and $B$ are linear elliptic operators and $F(u)$ is a nonlinear term with respect to $u$. In our case, we have $Au=-\Delta u$, $Bu=-\Delta u+\frac{\sigma}{|x|^2}u$, $V(x)=|x|^\mu$ and $F(u)=|u|^p$. Equations of type (3) arise in many mathematical models. For instance, the Hoff equation [19] ($Au=-\partial_{xx}u+u$, $Bu=0$, $V=1$, $F(u)=\alpha u+\beta u^3$), the Barenblatt–Zheltov–Kochina equation [10]

($Au = -\Delta u + cu$, $Bu = -\Delta u$, $V = 1$, $F(u) = 0$) that describes nonstationary filtering processes in fissured-porous media, the semiconductor equation [24] ($Au = -\Delta u + u$, $Bu = -\Delta u$, $V = 1$, $Fu = \alpha u^3$) that describes nonstationary processes in crystalline semiconductors, the one-dimensional Boussinesq equation [15] ($Au = -\partial_{xx} u + u$, $Bu = 0$, $V = 1$, $F(u) = \alpha\partial_{xx}\left(|u|^{p-2}u\right)$), and many others.

Sobolev-type equations and inequalities have been studied in various contexts: numerical solutions [6,11,18,31], asymptotic behaviour of solutions [7,8,12,14], inverse problems [17,28–30] and blow-up of solutions [5, 9, 13, 20, 21, 23–25, 27]. In particular, the issue of nonexistence of (weak) solutions to various differential inequalities of Sobolev-type has been investigated in [25]. For instance, the special case of (1) with $\sigma = \mu = 0$ has been studied in the whole space $\mathbb{R}^N$. Namely, it was shown that the Sobolev-type inequality

$$-\partial_t(\Delta u) - \Delta u \geq |u|^p \quad \text{in } (0,\infty)\times\mathbb{R}^N \tag{4}$$

subject to the initial condition

$$u(0,x) = u_0(x) \quad \text{in } \mathbb{R}^N,$$

where $p > 1$ and $u_0 \in L^1(\mathbb{R}^N)$, admits no nontrivial solution, provided that $N \in \{1,2\}$; or

$$N \geq 3, \quad p < \frac{N}{N-2}.$$

An extension of the above result to a time-space-fractional version of (4) has been obtained in [5].

The issue of existence and nonexistence of solutions to evolution equations and inequalities with Hardy potential in unbounded domains has been considered in several papers. For instance, Hamidi and Laptev [16] invetsigated the higher order evolution inequality

$$\partial_t^k u - \Delta u + \frac{\lambda}{|x|^2} u \geq |u|^p \quad \text{in } (0,\infty)\times\mathbb{R}^N \tag{5}$$

subject to the initial condition

$$\partial_t^{k-1} u(0,x) \geq 0 \quad \text{in } \mathbb{R}^N. \tag{6}$$

where $\partial_t^i u = \frac{\partial^i u}{\partial t^i}$, $N \geq 3$, $k \geq 1$, $p > 1$ and $\lambda \geq -\left(\frac{N-2}{2}\right)^2$. Namely, it was proven that, if one of the following assumptions is satisfied:

$$\lambda \geq 0, \quad 1 < p \leq 1 + \frac{2}{\frac{2}{k} + s^*};$$

or

$$-\left(\frac{N-2}{2}\right)^2 \leq \lambda < 0, \quad 1 < p \leq 1 + \frac{2}{\frac{2}{k} - s_*},$$

where

$$s^* = \frac{N-2}{2} + \sqrt{\lambda + \left(\frac{N-2}{2}\right)^2}, s_* = s^* + 2 - N,$$

then (5)–(6) admits no nontrivial (weak) solution. In the parabolic case, among other problems, Abdellaoui et al. [1] (see also [3]) considered problems of the form

$$\partial_t(u^{p-1}) - \Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q \; (u > 0) \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{7}$$

where $1 < p < N$, $q > 0$ and $0 \leq \lambda < \left(\frac{N-p}{p}\right)^p$. Namely, it was shown that there exist two exponents $q^+(p,\lambda)$ and $F(p,\lambda)$ such that,

(i) if $p-1 < q < F(p,\lambda) < q^+(p,\lambda)$ and $u$ is a solution to (7) satisfying a certain behavior, then $u$ blows-up in a finite time;

(ii) if $F(p,\lambda) < q < q^+(p,\lambda)$, then under suitable condition on $u(0,\cdot)$, (7) admits a global in time positive solution.

We refer also to [22], where (5) with $k = 2$ has been studied in an exterior domain of $\mathbb{R}^N$ under different types of inhomogeneous boundary conditions. Additional results related to parabolic and elliptic equations involving Hardy potential in bounded domains of $\mathbb{R}^N$ can be found in [1–4, 26].

To the best of our knowledge, Sobolev-type equations/inequalities with Hardy potential have not previously been studied.

Before presenting our obtained results, let us give the meaning of solutions to the considered problem. Let

$$\omega = (0,\infty) \times \overline{B}\backslash\{0\}, \quad \Gamma = (0,\infty) \times \partial B.$$

Observe that $\Gamma \subset \omega$. We introduce the set

$$\Phi = \left\{\varphi \in C^3(\omega) : \text{supp}(\varphi) \subset\subset \omega,\, \varphi \geq 0,\, \varphi|_\Gamma = 0\right\}.$$

Here, by $\text{supp}(\varphi) \subset\subset \omega$, we mean that $\text{supp}(\varphi)$ is a compact subset of

$$(0,\infty) \times \left\{x \in \mathbb{R}^N : 0 < |x| \leq 1\right\}.$$

Solutions to (1) under the boundary condition (2) are defined as follows.

**Definition 1.** *Let $N \geq 2$, $\sigma > -\left(\frac{N-2}{2}\right)^2$, $\mu \in \mathbb{R}$, $p > 1$ and $f = f(x) \in L^1(\partial B)$. We say that $u \in L^p_{\text{loc}}(\omega)$ is a weak solution to* (1) *under the boundary condition* (2), *if*

$$\int_\omega |x|^\mu |u|^p \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_\Gamma \left(\partial_\nu \partial_t \varphi - \partial_\nu \varphi\right) f(x)\,\mathrm{d}S_x\,\mathrm{d}t \leq \int_\omega u \left(\Delta\partial_t\varphi - \Delta\varphi + \frac{\sigma}{|x|^2}\varphi\right) \mathrm{d}x\,\mathrm{d}t \tag{8}$$

*for all $\varphi \in \Phi$, where $\nu$ is the outward unit normal to $\partial B$, relative to $B$, and $\partial_\nu$ is the normal derivative on $\partial B$.*

By integration by parts, it can be easily seen that any smooth solution to (1)–(2) is a weak solution in the sense of Definition 1.

For $\sigma > -\left(\frac{N-2}{2}\right)^2$, we introduce the parameter

$$\sigma_N = -\frac{N-2}{2} + \sqrt{\sigma + \left(\frac{N-2}{2}\right)^2}. \tag{9}$$

We introduce also the set

$$L^{1,+}(\partial B) = \left\{ w \in L^1(\partial B) : \int_{\partial B} w(x)\,\mathrm{d}S_x > 0 \right\}.$$

Our main result is sated in the following theorem.

**Theorem 2.** *Let $N \geq 2$, $\sigma > -\left(\frac{N-2}{2}\right)^2$, $\mu \in \mathbb{R}$ and $p > 1$.*

(I) *Let $f \in L^{1,+}(\partial B)$. If*

$$\sigma_N p < \sigma_N - \mu - 2, \tag{10}$$

*then problem* (1) *under the boundary condition* (2) *admits no weak solution.*

(II) *If*

$$\sigma_N p > \sigma_N - \mu - 2, \tag{11}$$

*then problem* (1) *under the boundary condition* (2) *admits stationary solutions for some $f \in L^{1,+}(\partial B)$.*

The proof of part (I) of Theorem 2 is based on nonlinear capacity estimates specifically adapted to the operator $-\Delta\cdot + \frac{\sigma}{|x|^2}\cdot$, the domain $(0,\infty)\times B$, and the boundary condition (2). Part (II) of Theorem 2 is proved by the construction of explicit solutions.

Now, let us consider the case $\sigma = 0$ and $N \geq 3$. In this case, by (9), one has $\sigma_N = 0$. Hence, (10) reduces to $\mu < -2$, and (11) reduces to $\mu > -2$. Thus, from Theorem 2, we deduce the following result.

**Corollary 3.** *Let $N \geq 3$, $\sigma = 0$, $\mu \in \mathbb{R}$ and $p > 1$.*

(I) *Let $f \in L^{1,+}(\partial B)$. If $\mu < -2$, then problem* (1) *under the boundary condition* (2) *admits no weak solution.*

(II) *If $\mu > -2$, then problem* (1) *under the boundary condition* (2) *admits stationary solutions for some $f \in L^{1,+}(\partial B)$.*

**Remark 4.** From Corollary 3, we deduce that in the case $\sigma = 0$ and $N \geq 3$, $\mu^* = -2$ is a critical parameter for problem (1) under the boundary condition (2).

Next, let us consider the case $-\left(\frac{N-2}{2}\right)^2 < \sigma < 0$ and $N \geq 3$. In this case, by (9), one has $\sigma_N < 0$. Hence, (10) reduces to $p > 1 - \frac{\mu+2}{\sigma_N}$, and (11) reduces to $p < 1 - \frac{\mu+2}{\sigma_N}$. Then, by Theorem 2, we obtain the following result.

**Corollary 5.** *Let $N \geq 3$, $-\left(\frac{N-2}{2}\right)^2 < \sigma < 0$ and $\mu \in \mathbb{R}$.*

(I) *Let $f \in L^{1,+}(\partial B)$.*

(a) *If $\mu \leq -2$, then for all $p > 1$, problem* (1) *under the boundary condition* (2) *admits no weak solution.*

(b) *If $\mu > -2$, then for all*

$$p > 1 - \frac{\mu+2}{\sigma_N},$$

*problem* (1) *under the boundary condition* (2) *admits no weak solution.*

(II) *If $\mu > -2$, then for all*

$$1 < p < 1 - \frac{\mu+2}{\sigma_N},$$

*problem* (1) *under the boundary condition* (2) *admits stationary solutions for some $f \in L^{1,+}(\partial B)$.*

**Remark 6.** From Corollary 5, we deduce that in the case $-\left(\frac{N-2}{2}\right)^2 < \sigma < 0$ and $N \geq 3$, the dividing line with respect to existence and nonexistence is given by the critical exponent

$$p^* = p^*(N,\sigma,\mu) = \begin{cases} 1 & \text{if } \mu \leq -2, \\ 1 - \frac{\mu+2}{\sigma_N} & \text{if } \mu > -2. \end{cases}$$

Namely,

(i) if $f \in L^{1,+}(\partial B)$ and $p > p^*$, then problem (1) under the boundary condition (2) admits no weak solution;

(ii) if $1 < p < p^*$, then problem (1) under the boundary condition (2) admits solutions for some $f \in L^{1,+}(\partial B)$.

Finally, let us consider the case $\sigma > 0$ and $N \geq 2$. In this case, by (9), one has $\sigma_N > 0$. Hence, (10) reduces to $p < 1 - \frac{\mu+2}{\sigma_N}$, and (11) reduces to $p > 1 - \frac{\mu+2}{\sigma_N}$. Thus, we deduce from Theorem 2 the following result.

**Corollary 7.** *Let $N \geq 2$, $\sigma > 0$ and $\mu \in \mathbb{R}$.*

(I) *Let $f \in L^{1,+}(\partial B)$. If $\mu < -2$, then for all*

$$1 < p < 1 - \frac{\mu+2}{\sigma_N},$$

*problem* (1) *under the boundary condition* (2) *admits no weak solution.*

(II) *If $\mu \geq -2$, then for all $p > 1$, problem* (1) *under the boundary condition* (2) *admits stationary solutions for some $f \in L^{1,+}(\partial B)$.*

(III) *If* $\mu < -2$, *then for all*

$$p > 1 - \frac{\mu+2}{\sigma_N},$$

*problem* (1) *under the boundary condition* (2) *admits stationary solutions for some* $f \in L^{1,+}(\partial B)$.

**Remark 8.** Let $N \geq 2$ and $\sigma > 0$. From Corollary 7, we deduce that, if $\mu < -2$, the dividing line with respect to existence and nonexistence is given by the critical exponent

$$p_* = p_*(N,\sigma,\mu) = 1 - \frac{\mu+2}{\sigma_N}.$$

Namely,

(i) if $f \in L^{1,+}(\partial B)$ and $1 < p < p_*$, then problem (1) under the boundary condition (2) admits no weak solution;

(ii) if $p > p^*$, then problem (1) under the boundary condition (2) admits solutions for some $f \in L^{1,+}(\partial B)$.

However, if $\mu \geq -2$, the problem admits no critical behavior, that is, for all $p > 1$, stationary solutions exists for some $f \in L^{1,+}(\partial B)$.

The rest of the paper is organized as follows. In Section 2, we establish some useful preliminary estimates. In Section 3, we prove Theorem 2.

Throughout the paper, the symbols $C$ or $C_i$ denote always generic positive constants, which are independent of the scaling parameters $T$ and $R$, and the solution $u$. Their values could be changed from one line to another. We will use frequently the notation $T, R, \ell \gg 1$, to indicate that the above parameters are sufficiently large.

## 2. Preliminaries

Let $N \geq 2$, $\sigma > -\left(\frac{N-2}{2}\right)^2$, $\mu \in \mathbb{R}$ and $p > 1$. We introduce the function

$$K(x) = |x|^{2-N-\sigma_N}\left(1 - |x|^{2\sigma_N+N-2}\right), \quad x \in \overline{B}\backslash\{0\},$$

where $\sigma_N$ is given by (9). It can be easily seen that the function $K$ satisfies the following properties.

**Lemma 9.** *The following properties hold:*

(i) $K \geq 0$;

(ii) $-\Delta K + \frac{\sigma}{|x|^2} K = 0$ *in* $B\backslash\{0\}$;

(iii) $K|_{\partial B} = 0$;

(iv) $\partial_\nu K = 2 - N - 2\sigma_N$.

Next, we introduce two cut-off functions $\alpha$ and $\beta$ satisfying:

$$\alpha \in C^\infty([0,\infty)), \quad \alpha \geq 0, \quad \text{supp}(\alpha) \subset\subset (0,1) \tag{12}$$

and

$$\beta \in C^\infty([0,\infty)), \quad 0 \leq \beta \leq 1, \quad \beta(s) = 0 \text{ if } 0 \leq s \leq \frac{1}{2}, \quad \beta(s) = 1 \text{ if } s \geq 1. \tag{13}$$

For $T, R, \ell \gg 1$, let

$$\alpha_T(t) = \alpha^\ell\left(\frac{t}{T}\right), \quad t \geq 0 \tag{14}$$

and

$$\beta_R(x) = K(x)\beta^\ell(R|x|), \quad x \in B\backslash\{0\}. \tag{15}$$

We consider the family of functions $\{\varphi_{T,R,\ell}\}_{T,R,\ell \gg 1}$, where

$$\varphi_{T,R,\ell}(t,x) = \varphi(t,x) = \alpha_T(t)\beta_R(x), \quad (t,x) \in \omega. \tag{16}$$

**Lemma 10.** *For $T, R, \ell \gg 1$, the function $\varphi$ defined by* (16) *belongs to* $\Phi$. *Moreover, we have*

$$\partial_\nu \varphi(t,x) = (2 - N - 2\sigma_N)\alpha_T(t), \quad (t,x) \in \Gamma \tag{17}$$

*and*

$$\partial_\nu \partial_t \varphi(t,x) = (2 - N - 2\sigma_N)\alpha'_T(t), \quad (t,x) \in \Gamma. \tag{18}$$

**Proof.** By the definition of $\varphi$, it can be easily seen that for $T, R, \ell \gg 1$, we have

$$\varphi \in C^3(\omega), \quad \mathrm{supp}(\varphi) \subset\subset \omega.$$

Furthermore, by (12), (13), (14), (15), (16) and Lemma 9 (i), (iii), we have

$$\varphi \geq 0, \quad \varphi|_\Gamma = 0.$$

Consequently, we have $\varphi \in \Phi$. On the other hand, by (13) and (15), we have

$$\beta_R(x) = K(x), \quad \frac{1}{R} \leq |x| < 1. \tag{19}$$

Then (17) and (18) follow from (16), (19) and Lemma 9 (iv). $\square$

For $T, R, \ell \gg 1$, let $\varphi$ be the function defined by (16).

**Lemma 11.** *The following estimate holds:*

$$\int_{\mathrm{supp}(\Delta\partial_t\varphi)} |x|^{\frac{-\mu}{p-1}} \varphi^{\frac{-1}{p-1}} \left|\Delta\partial_t\varphi\right|^{\frac{p}{p-1}} \,\mathrm{d}x\,\mathrm{d}t \leq C T^{1-\frac{p}{p-1}} \left(\ln R + R^{\sigma_N - 2 + \frac{\mu+2p}{p-1}}\right). \tag{20}$$

**Proof.** By the definition of $\varphi$, we obtain

$$\begin{aligned}\int_{\mathrm{supp}(\Delta\partial_t\varphi)} & |x|^{\frac{-\mu}{p-1}} \varphi^{\frac{-1}{p-1}} \left|\Delta\partial_t\varphi\right|^{\frac{p}{p-1}} \,\mathrm{d}x\,\mathrm{d}t \\ &= \left(\int_0^T \alpha_T^{\frac{-1}{p-1}}(t)|\alpha'_T(t)|^{\frac{p}{p-1}}\,\mathrm{d}t\right)\left(\int_{\mathrm{supp}(\Delta\beta_R)} |x|^{\frac{-\mu}{p-1}} \beta_R^{\frac{-1}{p-1}}(x)|\Delta\beta_R(x)|^{\frac{p}{p-1}}\,\mathrm{d}x\right).\end{aligned} \tag{21}$$

On the other hand, by (14), we have

$$\alpha'_T(t) = \ell T^{-1} \alpha^{\ell-1}\left(\frac{t}{T}\right)\alpha'\left(\frac{t}{T}\right),$$

which implies by (12) and (14) that

$$\alpha_T^{\frac{-1}{p-1}}(t)|\alpha'_T(t)|^{\frac{p}{p-1}} \leq C T^{-\frac{p}{p-1}} \alpha^{\ell - \frac{p}{p-1}}\left(\frac{t}{T}\right), \quad 0 < t < T.$$

Integrating, we get

$$\int_0^T \alpha_T^{\frac{-1}{p-1}}(t)|\alpha'_T(t)|^{\frac{p}{p-1}}\,\mathrm{d}t \leq C T^{-\frac{p}{p-1}} \int_0^T \alpha^{\ell-\frac{p}{p-1}}\left(\frac{t}{T}\right)\mathrm{d}t = C T^{1-\frac{p}{p-1}} \int_0^1 \alpha^{\ell-\frac{p}{p-1}}(s)\,\mathrm{d}s,$$

that is,

$$\int_0^T \alpha_T^{\frac{-1}{p-1}}(t)|\alpha'_T(t)|^{\frac{p}{p-1}}\,\mathrm{d}t \leq C T^{1-\frac{p}{p-1}}. \tag{22}$$

Furthermore, by (15), we have

$$\begin{aligned}\Delta(\beta_R(x)) &= \Delta\left(K(x)\beta^\ell(R|x|)\right) \\ &= \beta^\ell(R|x|)\Delta K(x) + K(x)\Delta\beta^\ell(R|x|) + 2\nabla K(x)\cdot\nabla\beta^\ell(R|x|),\end{aligned}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^N$, which implies by Lemma 9 (ii) that

$$\Delta(\beta_R(x)) = \sigma|x|^{-2}K(x)\beta^\ell(R|x|) + K(x)\Delta\beta^\ell(R|x|) + 2\nabla K(x)\cdot\nabla\beta^\ell(R|x|).$$

Hence, from (13), we deduce that

$$\int_{\mathrm{supp}(\Delta\beta_R)} |x|^{\frac{-\mu}{p-1}} \beta_R^{\frac{-1}{p-1}}(x)|\Delta\beta_R(x)|^{\frac{p}{p-1}}\,\mathrm{d}x \leq C(I_1 + I_2 + I_3), \tag{23}$$

where

$$I_1=\int_{\frac{1}{2R}<|x|<1}|x|^{-\frac{\mu+2p}{p-1}}K(x)\beta^{\ell}(R|x|)\,\mathrm{d}x,$$

$$I_2=\int_{\frac{1}{2R}<|x|<\frac{1}{R}}|x|^{\frac{-\mu}{p-1}}K(x)\left|\Delta\beta^{\ell}(R|x|)\right|^{\frac{p}{p-1}}\beta^{\frac{-\ell}{p-1}}(R|x|)\,\mathrm{d}x,$$

$$I_3=\int_{\frac{1}{2R}<|x|<\frac{1}{R}}|x|^{\frac{-\mu}{p-1}}K^{\frac{-1}{p-1}}(x)|\nabla K(x)|^{\frac{p}{p-1}}\beta^{\frac{-\ell}{p-1}}(R|x|)\left|\nabla\beta^{\ell}(R|x|)\right|^{\frac{p}{p-1}}\,\mathrm{d}x.$$

Let us estimate the terms $I_i$, $i=1,2,3$. Since $0\le\beta\le 1$, by the definition of $K$, we obtain

$$\begin{aligned}I_1&\le\int_{\frac{1}{2R}<|x|<1}|x|^{-\frac{\mu+2p}{p-1}}K(x)\,\mathrm{d}x\\&\le\int_{\frac{1}{2R}<|x|<1}|x|^{-\frac{\mu+2p}{p-1}+2-N-\sigma_N}\,\mathrm{d}x\\&=C\int_{r=\frac{1}{2R}}^{1}r^{-\frac{\mu+2p}{p-1}+1-\sigma_N}\,\mathrm{d}r\\&=C\begin{cases}1 & \text{if } -\frac{\mu+2p}{p-1}+2-\sigma_N>0,\\ R^{\sigma_N-2+\frac{\mu+2p}{p-1}} & \text{if } -\frac{\mu+2p}{p-1}+2-\sigma_N<0,\\ \ln R & \text{if } -\frac{\mu+2p}{p-1}+2-\sigma_N=0,\end{cases}\end{aligned}$$

which yields

$$I_1\le C\left(\ln R+R^{\sigma_N-2+\frac{\mu+2p}{p-1}}\right).\tag{24}$$

On the other hand, by (9), (15) and the definition of $K$, for $\frac{1}{2R}<|x|<\frac{1}{R}$, we have

$$\left|\Delta\beta^{\ell}(R|x|)\right|\le CR^2\beta^{\ell-2}(R|x|),\quad \left|\nabla\beta^{\ell}(R|x|)\right|\le CR\beta^{\ell-2}(R|x|)\tag{25}$$

and

$$C_1R^{\sigma_N+N-2}\le K(x)\le C_2R^{\sigma_N+N-2},\quad |\nabla K(x)|\le CR^{\sigma_N+N-1}.\tag{26}$$

Thus, due to $0\le\beta\le1$, and using (25) and (26), we obtain

$$I_2\le CR^{\sigma_N-2+\frac{\mu+2p}{p-1}}\tag{27}$$

and

$$I_3\le CR^{\sigma_N-2+\frac{\mu+2p}{p-1}}.\tag{28}$$

Finally, in view of (21), (22), (23), (24), (27) and (28), we obtain (20). □

**Lemma 12.** *The following estimate holds:*

$$\int_{\mathrm{supp}(\varphi)}|x|^{\frac{\mu}{p-1}}\varphi^{\frac{-1}{p-1}}\left|-\Delta\varphi+\frac{\sigma}{|x|^2}\varphi\right|^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t\le CTR^{\sigma_N-2+\frac{\mu+2p}{p-1}}.\tag{29}$$

**Proof.** By the definition of the function $\varphi$, we obtain

$$\begin{aligned}&\int_{\mathrm{supp}(\varphi)}|x|^{\frac{\mu}{p-1}}\varphi^{\frac{-1}{p-1}}\left|-\Delta\varphi+\frac{\sigma}{|x|^2}\varphi\right|^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t\\&\qquad=\left(\int_0^T\alpha_T(t)\,\mathrm{d}t\right)\left(\int_{\mathrm{supp}(\beta_R)}|x|^{\frac{\mu}{p-1}}\beta_R^{\frac{-1}{p-1}}(x)\left|-\Delta\beta_R(x)+\frac{\sigma}{|x|^2}\beta_R(x)\right|^{\frac{p}{p-1}}\,\mathrm{d}x\right).\end{aligned}\tag{30}$$

On the other hand, by (14), we have

$$\begin{aligned}\int_0^T \alpha_T(t)\,\mathrm{d}t &= \int_0^T \alpha^\ell\left(\frac{t}{T}\right)\mathrm{d}t\\ &= T\int_0^1 \alpha^\ell(s)\,\mathrm{d}s,\end{aligned}$$

that is,

$$\int_0^T \alpha_T(t)\,\mathrm{d}t = CT. \tag{31}$$

moreover, using similar calculations as that done in the proof of Lemma 11, we obtain

$$\int_{\mathrm{supp}(\beta_R)} |x|^{\frac{\mu}{p-1}} \beta_R^{\frac{-1}{p-1}}(x) \left|-\Delta\beta_R(x) + \frac{\sigma}{|x|^2}\beta_R(x)\right|^{\frac{p}{p-1}} \mathrm{d}x \le CR^{\sigma_N - 2 + \frac{\mu+2p}{p-1}}. \tag{32}$$

Hence, (20) follows from (30), (31) and (32). □

## 3. Proof of the main result

**Proof of Theorem 2.**

**(I).** Suppose that $u \in L^p_{\mathrm{loc}}(\omega)$ is a weak solution to (1) under the boundary condition (2). Then, by (8), for all $\varphi \in \Phi$, there holds

$$\begin{aligned}\int_\omega |x|^\mu |u|^p \varphi\,\mathrm{d}x\,\mathrm{d}t &+ \int_\Gamma \left(\partial_\nu\partial_t\varphi - \partial_\nu\varphi\right) f(x)\,\mathrm{d}S_x\,\mathrm{d}t\\ &\le \int_\omega |u| \left|\Delta\partial_t\varphi\right| \mathrm{d}x\,\mathrm{d}t + \int_\omega |u| \left|-\Delta\varphi + \frac{\sigma}{|x|^2}\varphi\right| \mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{33}$$

On the other hand, by means of Young's inequality, we have

$$\begin{aligned}\int_\omega |u| \left|\Delta\partial_t\varphi\right| \mathrm{d}x\,\mathrm{d}t &= \int_\omega |x|^{\frac{\mu}{p}} |u| \varphi^{\frac{1}{p}} |x|^{\frac{-\mu}{p}} \varphi^{\frac{-1}{p}} \left|\Delta\partial_t\varphi\right| \mathrm{d}x\,\mathrm{d}t\\ &\le \frac{1}{2}\int_\omega |x|^\mu |u|^p \varphi\,\mathrm{d}x\,\mathrm{d}t + C\int_{\mathrm{supp}(\Delta\partial_t\varphi)} |x|^{\frac{-\mu}{p-1}} \varphi^{\frac{-1}{p-1}} \left|\Delta\partial_t\varphi\right|^{\frac{p}{p-1}} \mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{34}$$

and

$$\begin{aligned}\int_\omega |u| &\left|-\Delta\varphi + \frac{\sigma}{|x|^2}\varphi\right| \mathrm{d}x\,\mathrm{d}t\\ &\le \frac{1}{2}\int_\omega |x|^\mu |u|^p \varphi\,\mathrm{d}x\,\mathrm{d}t + C\int_{\mathrm{supp}(\varphi)} |x|^{\frac{\mu}{p-1}} \varphi^{\frac{-1}{p-1}} \left|-\Delta\varphi + \frac{\sigma}{|x|^2}\varphi\right|^{\frac{p}{p-1}} \mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{35}$$

provided that

$$\begin{aligned}&\int_{\mathrm{supp}(\Delta\partial_t\varphi)} |x|^{\frac{-\mu}{p-1}} \varphi^{\frac{-1}{p-1}} \left|\Delta\partial_t\varphi\right|^{\frac{p}{p-1}} \mathrm{d}x\,\mathrm{d}t < \infty,\\ &\int_{\mathrm{supp}(\varphi)} |x|^{\frac{\mu}{p-1}} \varphi^{\frac{-1}{p-1}} \left|-\Delta\varphi + \frac{\sigma}{|x|^2}\varphi\right|^{\frac{p}{p-1}} \mathrm{d}x\,\mathrm{d}t < \infty.\end{aligned} \tag{36}$$

In view of (33), (34) and (35), we obtain

$$\begin{aligned}&\int_\Gamma \left(\partial_\nu\partial_t\varphi - \partial_\nu\varphi\right) f(x)\,\mathrm{d}S_x\,\mathrm{d}t\\ &\le C\left(\int_{\mathrm{supp}(\Delta\partial_t\varphi)} |x|^{\frac{-\mu}{p-1}} \varphi^{\frac{-1}{p-1}} \left|\Delta\partial_t\varphi\right|^{\frac{p}{p-1}} \mathrm{d}x\,\mathrm{d}t + \int_{\mathrm{supp}(\varphi)} |x|^{\frac{\mu}{p-1}} \varphi^{\frac{-1}{p-1}} \left|-\Delta\varphi + \frac{\sigma}{|x|^2}\varphi\right|^{\frac{p}{p-1}} \mathrm{d}x\,\mathrm{d}t\right).\end{aligned} \tag{37}$$

Next, for $T, R, \ell \gg 1$, we consider the function $\varphi$ defined by (16). By Lemma 10, we know that $\varphi \in \Phi$. Moreover, by Lemmas 11 and 12, (36) holds. Consequently, for $T, R, \ell \gg 1$, (37) holds for the function $\varphi$ defined by (16). On the other hand, thanks to (17) and (18), we have

$$\int_\Gamma \left(\partial_\nu \partial_t \varphi - \partial_\nu \varphi\right) f(x)\, \mathrm{d}S_x\, \mathrm{d}t = (2-N-2\sigma_N) \int_0^\infty \int_{\partial B} \left(\alpha_T'(t) - \alpha_T(t)\right) f(x)\, \mathrm{d}S_x\, \mathrm{d}t.$$

Notice that by (9), we have $2-N-2\sigma_N < 0$. Hence, by (12) and (14), we get

$$\begin{aligned} \int_\Gamma \left(\partial_\nu \partial_t \varphi - \partial_\nu \varphi\right) f(x)\, \mathrm{d}S_x\, \mathrm{d}t &= C\left(\int_0^T \left(\alpha^\ell\left(\frac{t}{T}\right) - \ell T^{-1} \alpha^{\ell-1}\left(\frac{t}{T}\right)\alpha'\left(\frac{t}{T}\right)\right) \mathrm{d}t\right)\left(\int_{\partial B} f(x)\, \mathrm{d}S_x\right) \\ &= CT\left(\int_0^1 \left(\alpha^\ell(s) - \ell T^{-1}\alpha^{\ell-1}(s)\alpha'(s)\right) \mathrm{d}s\right)\left(\int_{\partial B} f(x)\, \mathrm{d}S_x\right). \end{aligned} \tag{38}$$

Furthermore, by the dominated convergence theorem, we have

$$\lim_{T\to\infty} \int_0^1 \left(\alpha^\ell(s) - \ell T^{-1}\alpha^{\ell-1}(s)\alpha'(s)\right) \mathrm{d}s = \int_0^1 \alpha^\ell(s)\, \mathrm{d}s > 0.$$

Thus, for $T \gg 1$, one has

$$\int_0^1 \left(\alpha^\ell(s) - \ell T^{-1}\alpha^{\ell-1}(s)\alpha'(s)\right) \mathrm{d}s \ge C.$$

Since $f \in L^{1,+}(\partial B)$, we deduce from (38) that

$$\int_\Gamma \left(\partial_\nu \partial_t \varphi - \partial_\nu \varphi\right) f(x)\, \mathrm{d}S_x\, \mathrm{d}t \ge CT \int_{\partial B} f(x)\, \mathrm{d}S_x. \tag{39}$$

Then, using (37), (39), Lemmas 11 and 12, we obtain

$$T\int_{\partial B} f(x)\, \mathrm{d}S_x \le C\left[T^{1-\frac{p}{p-1}}\left(\ln R + R^{\sigma_N - 2 + \frac{\mu+2p}{p-1}}\right) + TR^{\sigma_N - 2 + \frac{\mu+2p}{p-1}}\right],$$

that is,

$$\int_{\partial B} f(x)\, \mathrm{d}S_x \le C\left(T^{\frac{-p}{p-1}} \ln R + T^{\frac{-p}{p-1}} R^\iota + R^\iota\right), \tag{40}$$

where

$$\iota = \sigma_N - 2 + \frac{\mu+2p}{p-1}.$$

Notice that due to (10), we have $\iota < 0$. Hence, taking $T = R$ in (40), and passing to the limit as $R \to \infty$, we obtain $\int_{\partial B} f(x)\, \mathrm{d}S_x \le 0$, which is a contradiction with $f \in L^{1,+}(\partial B)$. This completes the proof of part (I) of Theorem 2.

**(II).** Let

$$\max\left\{2-N-\sigma_N, \frac{-(\mu+2)}{p-1}\right\} < \delta < \sigma_N \tag{41}$$

and

$$0 < \varepsilon < \left(-\delta^2 + (2-N)\delta + \sigma\right)^{\frac{1}{p-1}}. \tag{42}$$

Notice that by (9), we have $2-N-\sigma_N < \sigma_N$. Moreover, due to (11), there holds $\frac{-(\mu+2)}{p-1} < \sigma_N$. Hence, the set of $\delta$ satisfying (41) is nonempty. Notice also that $2-N-\sigma_N$ and $\sigma_N$ are the roots of the polynomial function

$$F(\delta) = -\delta^2 + (2-N)\delta + \sigma.$$

Hence, for all $\delta$ satisfying (41), one has $F(\delta) > 0$. Thus, the set of $\varepsilon$ satisfying (42) is nonempty. We consider functions of the form

$$u_{\delta,\varepsilon}(x) = \varepsilon|x|^\delta, \quad x \in B\backslash\{0\}.$$

Elementary calculations show that

$$-\Delta u_{\delta,\varepsilon} + \frac{\sigma}{|x|^2} u_{\delta,\varepsilon} = \varepsilon F(\delta)|x|^{\delta-2}. \tag{43}$$

Then, thanks to (41), (42) and (43), we obtain

$$-\Delta u_{\delta,\varepsilon}+\frac{\sigma}{|x|^2}u_{\delta,\varepsilon}\geq \varepsilon^p|x|^{\delta p+\mu}=|x|^\mu u_{\delta,\varepsilon}^p.$$

Thus, $u_{\delta,\varepsilon}$ is a stationary solution to (1)–(2) with $f\equiv\varepsilon$. This completes the proof of part (II) of Theorem 2. □

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