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On direct and inverse problems related to some dilated sumsets

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Abstract. Let A be a nonempty finite set of integers. For a real number m, the set $m \cdot A = \{ma : a \in A\}$ denotes the set of m-dilates of A. In 2008, Bukh initiated an interesting problem of finding a lower bound for the sumset of dilated sets, i.e., a lower bound for $|\lambda_1 \cdot A + \lambda_2 \cdot A + \cdots + \lambda_h \cdot A|$, where $\lambda_1, \lambda_2, \ldots, \lambda_h$ are integers and A be a subset of integers. In particular, for nonempty finite subsets A and B, the problem of dilates of A and B is defined as $A + k \cdot B = \{a + kb : a \in A \text{ and } b \in B\}$. In this article, we obtain the lower bound for the cardinality of $A + k \cdot B$ with $k \geq 3$ and describe sets for which equality holds. We also derive an extended inverse result with some conditions for the sumset $A + 3 \cdot B$.

Keywords. Sum of dilates, direct and inverse problems, additive combinatorics.

2020 Mathematics Subject Classification. 11B13, 11B75.

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1. Introduction

Let A be a finite set of integers and k be any integer. The k-dilation, $k \cdot A$ of A is defined by $k \cdot A = \{ka : a \in A\}$. Classically, there are two types of problems of sumsets in additive number theory, called direct and inverse problems. In *direct problems*, one starts with a set and tries to describe the size of sumsets (of any type) associated with given set, called direct problems. In case of *inverse problems* one starts with the cardinality of a sumsets obtained from direct problem and tries to find the structure of set. More generally, in *extended inverse problems* one tries to find the structure of a sumset by assuming some arbitrary cardinality of a sumset. The main aim is to find the lower bound for the cardinality of sets of type $\lambda_1 \cdot A_1 + \lambda_2 \cdot A_2 + \cdots + \lambda_h \cdot A_h$, where $\lambda_1 \cdot A_1 + \lambda_2 \cdot A_2 + \cdots + \lambda_h \cdot A_h = \{\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_h a_h \mid a_i \in A_i \, and \, \lambda_i \in \mathbb{Z}, \, i = 1, 2, \ldots, h\}$. In 2007, Bukh [3] gave an asymptotically sharp lower bound on the size of sumsets of the form $\lambda_1 \cdot A + \lambda_2 \cdot A + \cdots + \lambda_k \cdot A$, for arbitrary large integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ and integer set A. Bukh derived the lower bound for $\lambda_1 \cdot A + \lambda_2 \cdot A + \cdots + \lambda_k \cdot A$ with some error term $o(|A|)$. He proved that for every vector $\overline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k$ of coprime k-tuple, $|\lambda_1 \cdot A + \lambda_2 \cdot A + \cdots + \lambda_k \cdot A| \geq (|\lambda_1| + |\lambda_2| + \cdots + |\lambda_k|)|A| - o(|A|)$ for a finite set $A \subset \mathbb{Z}$ with the error term $o(|A|)$ depending on $\overline{\lambda}$ only.

Confining ourselves to the sum of only two dilates, it is enough to consider only the sums $m \cdot A + k \cdot B$, where A and B are non empty subsets of integers. When both m and k are equal

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to 1, the sum $A+B$ is called the Minkowski sum of the sets A and B. In a remarkable result by Nathanson [12], it was proven that for non-empty subsets A and B of integers, $|A+B| \geq |A|+|B|-1$, and equality holds if and only if A and B are arithmetic progressions with the same common difference. Also, studying the dilated sumset $m \cdot A + k \cdot B$, when $A = B$ presents an interesting problem. Researchers have dedicated considerable effort to investigate these sumset problems and have made significant advancements in this field of study. In 2010, Cilleruelo et al. [6] proved $|A+3\cdot A| \geq 4|A|-4$ and the equality holds only for $A = \{0,1,3\}$ or $A = \{0,1,4\}$ or $A = 3\{0,1,\ldots n\} \cup (3\{0,1,\ldots,n\}+1)$ and all the affine transforms of these sets. In the same paper, they proposed the conjecture that $|A+k\cdot A| \geq (k+1)|A| - \left\lceil \frac{k^2+2k}{4} \right\rceil$, where A is any set of sufficiently large cardinality. This conjecture has been well studied in the past and is being studied presently. In 2009, Cilleruelo et al. [5] confirmed the conjecture for a prime number k such that $|A| \geq 3(k-1)^2(k-1)!$. In 2014, Du et al. [7] verified the conjecture for k to be prime power and product of two distinct primes such that $|A| \geq (k-1)^2 k!$. Motivated by the work done on the cardinality of $A+k\cdot A$, several authors proved various results on the cardinality of $m\cdot A + k\cdot A$. In 2011, Hamidoune and Rue [9] investigated the scenario, where m is equal to 2 and k is an odd prime number. Consequently, they proved that for an odd prime k and a finite set A of integers with $|A| > 8k^k$, $|2\cdot A + k\cdot A| \geq (k+2)|A| - k^2 - k + 2$. In 2013, Ljujic [11] expanded upon this finding and derived the same limit for cases, where k is a power of an odd prime or a product of two distinct odd primes. In 2013, Balog et al. [1] proved that $|p\cdot A + q\cdot A| \geq (p+q)|A| - (pq)^{(p+q-3)(p+q)+1}$, where $p < q$ are relatively primes and $A \subseteq \mathbb{Z}$. In 2020, Chahal and Pandey [4] handled the case for the cardinality of $3\cdot A + k\cdot A$, under some conditions on A and also generalized this result for $q\cdot A + k\cdot A$, where $q < k$ is an odd prime. In 2017, Freiman et al. [8] proved that if $r \geq 3$, then $|A + r\cdot A| \geq 4|A| - 4$. For $r = 2$, they also obtained an extended inverse result, which states that if $|A+2\cdot A| < 4|A|-4$, then A is a subset of arithmetic progression of length at most $2|A|-3$. In 2019, Bhanja et al. [2] presented an alternative proof of the inequality $|A+r\cdot A| \geq 4|A|-4$ for $r \geq 3$. Additionally, they extended the inverse theorem to the cardinality of the sum of dilates $A+2\cdot B$, where A and B are subsets of the integers.

Let $A = \{a_0, a_1, \ldots, a_{r-1}\}$ be a finite subset of integers such that $a_0 < a_1 < \cdots < a_{r-1}$. Suppose $\ell^*(a_i) = a_i - a_{i-1}$ for all $i = 1,2,\ldots,r-1$. In Section 2, we prove the following direct and inverse problem:

Theorem 1. *Let $k \geq 3$ be a positive integer and let A and B be nonempty finite subsets of integers with the properties such that $|A| \leq |B|$, $\ell^*(a_i) \leq k$, for all $1 \leq i \leq r-1$ and $3 \leq \ell^*(b_j) \leq k$ for all $1 \leq j \leq l-1$. Then $|A+k\cdot B| \geq 3|A|+|B|-4$.*

(Inverse Problem) Furthermore, if $|A+k\cdot B| = 3|A|+|B|-4$, then A and B are arithmetic progressions.

For any set A, we define $c_m(A)$ as the count of distinct classes of A modulo m. In Section 3, we obtain extended inverse problem for $|A+3\cdot B|$. More precisely, we prove the following theorem:

Theorem 2. *Let A, $B \subseteq \mathbb{Z}$ be finite subsets such that $c_3(A) = t$ and $0 \in A$, B with properties*

(1) $d(A) = d(B) = 1$
(2) $\ell(A) \leq \ell(B)$
(3) $h_A \leq h_B$.

If $|A+3\cdot B| = |A| + t(|B|-1) + h \leq 2|A| + t(|B|-2)$ for some integer h, then both A and B are subsets of arithmetic progressions of length at most $|B| + h = |A+3\cdot B| - |A| - (t-1)|B| + t \leq |A|+|B|-3$.

By $d(A)$ we denote the greatest common divisor of $\{a_1 - a_0, a_2 - a_0, \dots, a_{r-1} - a_0\}$. Let $a'_i = (a_i - a_0)/d(A)$ for $i = 1$ to $r-1$ and $\ell(A) = \max(A) - \min(A) = a_{r-1} - a_0$. The set $B = (a'_0, a'_1, \dots, a'_{r-1})$ is called normal form of the set A. Clearly, $a'_0 < a'_1 < \cdots < a'_{r-1}$ and $d(B) = 1$. Define $h_A = \ell(A) + 1 - |A|$ the number of holes in set A.

The following well known results of Liv [10] and Stanchescu [13] are used frequently for proving our results.

Theorem 3 ([10, 13]). *Let A and B be finite subsets of $\mathbb{N}$ such that $0 \in A \cap B$. Define* $\delta_{A,B} = \begin{cases} 1 & \text{if } \ell(A)=\ell(B) \\ 0 & \text{if } \ell(A)\neq\ell(B) \end{cases}$. *Then the followings hold:*

(1) *If* $\ell(A) = \max(\ell(A), \ell(B)) \geq |A| + |B| - 1 - \delta_{A,B}$ *and* $d(A) = 1$, *then*

$$|A + B| \geq |A| + 2|B| - 2 - \delta_{A,B}$$

(2) *If* $\max(\ell(A), \ell(B)) \leq |A| + |B| - 2 - \delta_{A,B}$, *then*

$$|A + B| \geq \max(\ell(A) + |B|, \ell(B) + |A|).$$

2. Proof of Theorem 1

Direct problem for $|A + k \cdot B|$

Proof. Let $A = \{a_0 < a_1 < \cdots < a_{r-1}\}$ and $B = \{b_0 < b_1 < \cdots < b_{l-1}\}$ be two finite sets of integers satisfying the given conditions. Consider the following sequence of distinct integers in the sumset $A + k \cdot B$,

$$\begin{aligned} a_0 + kb_0 < a_1 + kb_0 < a_0 + kb_1 < a_1 + kb_1 < a_2 + kb_1 \\ < a_1 + kb_2 < a_2 + kb_2 < a_3 + kb_2 < a_2 + kb_3 < a_3 + kb_3 \\ < a_4 + kb_3 < a_3 + kb_4 < \cdots < a_{i-1} + kb_{i-1} < a_i + kb_{i-1} \\ < a_{i-1} + kb_i < a_i + kb_i < a_{i+1} + kb_i \\ < a_i + kb_{i+1} < a_{i+1} + kb_{i+1} < a_{i+2} + kb_{i+1} \\ < a_{i+1} + kb_{i+2} < \cdots < a_{r-2} + kb_{r-1} < a_{r-1} + kb_{r-1} \\ < a_{r-1} + kb_r < a_{r-1} + kb_{r+1} < \cdots < a_{r-1} + kb_{l-1}. \end{aligned} \tag{1}$$

This list contains $3|A| - 2 + l - r = 2|A| + l - 2$ integers. To prove the result it remains to find $|A| - 2$ more integers of $A + k \cdot B$. Take the following list of six consecutive integers of $A + k \cdot B$ from (1) for every $1 \leq i \leq r - 2$,

$$a_{i-1} + kb_{i-1} < a_i + kb_{i-1} < a_{i-1} + kb_i < a_i + kb_i < a_{i+1} + kb_i < a_i + kb_{i+1}. \tag{2}$$

We claim that for each list of integers of type (2), there always exists an integer between $a_{i-1} + kb_{i-1}$ and $a_i + kb_{i+1}$ in the sumset $A + k \cdot B$, which is not in the list (1). Let us verify our claim for every list of type (2). Consider

$$\begin{aligned} & a_i + kb_{i-1} < a_{i+1} + kb_{i-1} < a_{i-1} + kb_i \\ \text{and} \quad & a_{i+1} + kb_i < a_{i-1} + kb_{i+1} < a_i + kb_{i+1}. \end{aligned}$$

Clearly, $a_i + kb_{i-1} < a_{i+1} + kb_{i-1}$ and $a_{i-1} + kb_{i+1} < a_i + kb_{i+1}$ holds for every $i = 1, 2, \dots, r-2$. We need only to prove that $a_{i+1} + kb_{i-1} < a_{i-1} + kb_i$ and $a_{i+1} + kb_i < a_{i-1} + kb_{i+1}$.

On contrary suppose that $a_{i+1} + kb_{i-1} \geq a_{i-1} + kb_i$. It implies $a_{i+1} - a_{i-1} \geq k(b_i - b_{i-1})$, which is a contradiction. As maximum value of $a_{i+1} - a_{i-1}$ can be $2k$, and $3 \leq \ell^*(b_j) \leq k$ for all $1 \leq j \leq l-1$. Hence $a_{i+1} + kb_{i-1} < a_{i-1} + kb_i$ and similarly $a_{i+1} + kb_i < a_{i-1} + kb_{i+1}$.

Next our aim is to show that for any two consecutive lists of six integers of the form (2), we always have two distinct integers of $A+k\cdot B$, that are not included in (1). Let us consider two lists of six integers

$$a_{i-1}+kb_{i-1} < a_i+kb_{i-1} < a_{i-1}+kb_i < a_i+kb_i < a_{i+1}+kb_i < a_i+kb_{i+1} \quad (3)$$

$$\text{and} \quad a_i+kb_i < a_{i+1}+kb_i < a_i+kb_{i+1} < a_{i+1}+kb_{i+1} < a_{i+2}+kb_{i+1} < a_{i+1}+kb_{i+2}. \quad (4)$$

Observe that, x and y in $A+k\cdot B$ such that $a_{i-1}+kb_{i-1} < x < a_i+kb_{i+1}$ and $a_i+kb_i < y < a_{i+1}+kb_{i+2}$, where x, y not in lists (3), (4). Our purpose is to show that either $x \neq y$ or there exists integer $z \neq x(=y)$ such that $z \in A+k\cdot B$ and lies between $a_{i-1}+kb_{i-1}$ and $a_{i+1}+kb_{i+2}$.

We claim that there exist two integers $x = a_{i-1}+kb_{i+1}$ and $y = a_{i+2}+kb_i$ satisfying $a_{i+1}+kb_i < a_{i-1}+kb_{i+1}$ and $a_{i+2}+kb_i < a_{i-1}+kb_{i+1}$.

For first identity, assume that $a_{i+1}+kb_i \geq a_{i-1}+kb_{i+1}$, which contradicts as $3 \leq \ell^*(b_j) \leq k$ for all $1 \leq j \leq l-1$. Similarly, if possible, let $a_{i+2}+kb_i \geq a_{i-1}+kb_{i+1}$, again a contradiction. Now, if $a_{i-1}+kb_{i+1} \neq a_{i+2}+kb_i$, then we get two distinct integers $x = a_{i-1}+kb_{i+1}$ and $y = a_{i+2}+kb_i$, which are not in (3) and (4). If $x = a_{i-1}+kb_{i+1} = a_{i+2}+kb_i = y$, we prove that in this case there also exist a new integer $z = a_i+kb_{i+2}$ in (4), which is different from $x = y$. Clearly, $z > y = x$. We have to check that $a_i+kb_{i+2} > a_{i+2}+kb_{i+1}$. If possible, let

$$a_i+kb_{i+2} \leq a_{i+2}+kb_{i+1}$$

$$a_{i+2}-a_i \geq k(b_{i+2}-b_{i+1}).$$

Since the maximum value of $a_{i+2}-a_i$ is $2k$. Therefore, our assumption is incorrect, leading to the confirmation and proof of our claim.

Thus in each case, we get two distinct elements of $A+k\cdot B$, which are not in (3) and (4). Hence, we get $|A|-2$ extra integers of $A+k\cdot B$, which are not included in (1). Consequently, $|A+k\cdot B| \geq 3|A|+|B|-4$.

Inverse Problem for $|A+k\cdot B|$

Let us begin with the case $|A| = |B| = r$ and assume that $A = \{a_0 < a_1 < \cdots < a_{r-1}\}$ and $B = \{b_0 < b_1 < \cdots < b_{r-1}\}$. The sumset $A+k\cdot B$ contains the following strictly increasing sequence of $3|A|-2$ integers

$$\begin{aligned} a_0+kb_0 &< a_1+kb_0 < a_0+kb_1 < a_1+kb_1 < a_2+kb_1 \\ &< a_1+kb_2 < a_2+kb_2 < a_3+kb_2 < a_2+kb_3 < a_3+kb_3 \\ &< a_4+kb_3 < a_3+kb_4 < \cdots < a_i+kb_i < a_{i+1}+kb_i \\ &< a_i+kb_{i+1} < a_{i+1}+kb_{i+1} < a_{i+2}+kb_{i+1} \\ &< a_{i+1}+kb_{i+2} < \cdots < a_{r-2}+kb_{r-1} < a_{r-1}+kb_{r-1}. \end{aligned} \quad (5)$$

Observe that the above sequence contains $|A|-2$ extra integers from the cardinality of $|A+k\cdot B| = 4|A|-4$.

Since $a_{i-1}+kb_i < a_i+kb_i < a_i+kb_{i+1}$, $a_{i-1}+kb_i < a_{i-1}+kb_{i+1} < a_i+kb_{i+1}$ and also the cardinality of $|A+k\cdot B| = 4|A|-4$, it implies $a_i+kb_i = a_{i-1}+kb_{i+1}$, which gives $a_i-a_{i-1} = k(b_{i+1}-b_i)$ for $i = 1,2,\ldots,r-2$. Similarly, from the inequalities $a_{i-1}+kb_{i-1} < a_{i-1}+kb_i < a_i+kb_i$ and $a_{i-1}+kb_{i-1} < a_i+kb_{i-1} < a_i+kb_i$, we have $a_{i-1}+kb_i = a_i+kb_{i-1}$. Thus $a_i-a_{i-1} = k(b_i-b_{i-1})$ for $i = 1,2,\ldots,r-2$. This completes the proof for the case $|A| = |B|$.

Further assume that $|A| < |B|$ and let $A = \{a_0 < a_1 < \cdots < a_{r-1}\}$ and $B = \{b_0 < b_1 < \cdots < b_{l-1}\}$. Suppose $0 \leq m \leq l-r$. Let $B = B_0^{(m)} \cup B_1^{(m)} \cup B_2^{(m)}$, where $B_0^{(m)} = \{b_0, b_1, \ldots, b_{m-1}\}, B_1^{(m)} = \{b_m, b_{m+1}, \ldots, b_{m+r-1}\}$, $B_2^{(m)} = \{b_{m+r}, b_{m+r+1}, \ldots, b_{l-1}\}$.

Therefore, $A+k\cdot B \supseteq (a_0+k\cdot B_0^{(m)})\cup(A+k\cdot B_1^{(m)})\cup(a_{r-1}+k\cdot B_2^{(m)})$. It implies that $|a_0+k\cdot B_0^{(m)}| = m$, $|A+k\cdot B_1^{(m)}| \geq 4r-4$, $|a_{r-1}+k\cdot B_2^{(m)}| = l-m-r$. Thus

$$\begin{aligned} 3r+l-4 &= |A+k\cdot B| \\ &\geq |a_0+k\cdot B_0^{(m)}| + |A+k\cdot B_1^{(m)}| + |a_{r-1}+k\cdot B_2^{(m)}| \\ &\geq m+4r-4+l-m-r \\ &= 3r+l-4. \end{aligned}$$

Hence the proof of the result. □

3. Extended Inverse Problem for $|A+3\cdot B|$

Proof. Let $A=\{a_0,a_1,\ldots,a_{r-1}\}$ and $B=\{b_0,b_1,\ldots,b_{l-1}\}$, where $a_0<a_1<\cdots<a_{r-1}$ and $b_0<b_1<\cdots<b_{l-1}$. Without loss of generality, we can assume that $a_0=0$ and $b_0=0$. Let A_0, A_1 and A_2 be three distinct congruence classes of A, such that $A_0\subseteq 3\mathbb{Z}$, $A_1\subseteq 3\mathbb{Z}+1$ and $A_2\subseteq 3\mathbb{Z}+2$. We further assume that $|A_0|=m\geq 1$, $|A_1|=n\geq 0$, $|A_2|=p\geq 0$, and thus we have $r=m+n+p$.

Case 1. $|A_0|=m\geq 1$**,** $|A_1|=n\geq 1$**,** $|A_2|=p=0$ **i.e.** $c_3(A)=2$**.** Assume that

$$\begin{aligned} A_0 &= \{0=3x_0<3x_1<\cdots<3x_{m-1}\}, \\ A_0^* &= \frac{1}{3}\cdot A_0 = \{0=x_0<x_1<\cdots<x_{m-1}\}, \\ A_1 &= \{3y_0+1<3y_1+1<\cdots<3y_{n-1}+1\}, \\ A_1^* &= \frac{1}{3}\cdot(A_1-1)-y_0 = \{0<y_1-y_0<y_2-y_0<\cdots<y_{n-1}-y_0\}, \end{aligned}$$

Then $\ell(A_0^*)=x_{m-1}<a_{r-1}=\ell(A)$ and $\ell(A_1^*)=y_{n-1}-y_0<a_{r-1}=\ell(A)$. Now

$$\begin{aligned} |A+3\cdot B| &= |(A_0\cup A_1)+3\cdot B| \\ &= |A_0+3\cdot B|+|A_1+3\cdot B| \\ &= |3\cdot A_0^*+3\cdot B|+|3\cdot(A_1^*+y_0)+1+3\cdot B| \\ &= |A_0^*+B|+|A_1^*+B|. \end{aligned}$$

Further, We prove two inequalities in Claim 4 and Claim 5.

Claim 4. $\ell(B)\leq l+\max(m,n)-2\leq l+r-3$.

Proof of Claim 4. Since $\ell(B)\geq\ell(A)>\ell(A_0^*)$ and $\ell(B)\geq\ell(A)>\ell(A_1^*)$, therefore $\delta_{B,A_0^*}=\delta_{B,A_1^*}=0$.

Let's consider the case where $m\leq n$. Assuming Claim 4 is false, then $\ell(B)\geq l+n-1=|B+|A_1^*|-1\geq l+m-1=|B|+|A_0^*|-1$ and $d(B)=1$. Thus by Theorem 3, $|A_0^*+B|\geq l+2|A_0^*|-2=l+2m-2$ and $|A_1^*+B|\geq l+2|A_1^*|-2=l+2n-2$

Hence $|A+3\cdot B|\geq 2l+2r-4$, which contradicts to our hypothesis.

In the case where $n\leq m$, we can obtain the result by following the same approach as described earlier.

Thus $\ell(B)\leq l+\max(m,n)-2$. Since $r=m+n$ and $\max(m,n)\leq r-1$, therefore $\ell(B)\leq l+\max(m,n)-2\leq l+r-3$. This completes the proof of Claim 4. □

Claim 5. $|A+3\cdot B|\geq|A|+2(|B|-1)+h_B$.

Proof of Claim 5. Assume the case $m \le n$. According to Claim 4, it is evident that $\ell(B) \le l+n-2$. Additionally, referring to Theorem 3, we have $|A_1^* + B| \ge (n+l-1)+h_B$. Consequently,

$$\begin{aligned}|A+3\cdot B| &= |A_0^*+B| + |A_1^*+B| \\ &\ge (|A_0^*|+|B|-1)+(n+l-1)+h_B \\ &\ge (m+l-1)+(n+l-1)+h_B \\ &= 2l+r-2+h_B.\end{aligned}$$

Similarly for the remaining case $(n \le m)$, $|A+3\cdot B| \ge 2l+r-2+h_B$. Thus, we obtain that h_B satisfies $0 \le h_B \le |A+3\cdot B|-(2l+r-2) = h \le r-3$. Therefore, $B \subseteq \{b_0, b_0+1, b_0+2, \ldots, b_{l-1}\} \subseteq \{0,1,\ldots,b_{l-1}\}$ and B is an arithmetic progression of length at most $b_{l-1}+1 = l+h_B \le l+h \le r+l-3$. As $\ell(A) \le \ell(B)$, the set A is also contained in A.P. of length at most $r+l-3$. The result can be easily verified for the case $|A_0| = m \ge 1$, $|A_1| = n = 0$ and $|A_2| = p \ge 1$. □

Case 2. $|A_0| = m \ge 1$**,** $|A_1| = n \ge 1$**,** $|A_2| = p \ge 1$ **i.e.** $c_3(A) = 3$**.** Assume that

$$\begin{aligned}A_0 &= \{0 = 3x_0 < 3x_1 < \cdots < 3x_{m-1}\}, \\ A_0^* &= \frac{1}{3}\cdot A_0 = \{0 = x_0 < x_1 < \cdots < x_{m-1}\}, \\ A_1 &= \{3y_0+1 < 3y_1+1 < \cdots < 3y_{n-1}+1\}, \\ A_1^* &= \frac{1}{3}\cdot(A_1-1)-y_0 = \{0 < y_1-y_0 < y_2-y_0 < \cdots < y_{n-1}-y_0\}, \\ A_2 &= \{3z_0+2 < 3z_1+2 < \cdots < 3z_{p-1}+2\}, \\ A_2^* &= \frac{1}{3}\cdot(A_2-2)-z_0 = \{0 < z_1-z_0 < z_2-z_0 < \cdots < z_{p-1}-z_0\}.\end{aligned}$$

Then $\ell(A_0^*) = x_{m-1} < a_{r-1} = \ell(A)$, $\ell(A_1^*) = y_{n-1}-y_0 < a_{r-1} = \ell(A)$ and $\ell(A_2^*) = z_{p-1}-z_0 < a_{r-1} = \ell(A)$. Now

$$\begin{aligned}|A+3\cdot B| &= |(A_0\cup A_1\cup A_2)+3\cdot B| \\ &= |A_0+3\cdot B| + |A_1+3\cdot B| + |A_2+3\cdot B| \\ &= |3\cdot A_0^*+3\cdot B| + |3\cdot(A_1^*+y_0)+1+3\cdot B| + |3\cdot(A_2^*+z_0)+2+3\cdot B| \\ &= |A_0^*+B| + |A_1^*+B| + |A_2^*+B|.\end{aligned}$$

Furthermore, we establish two inequalities in claim 1 and claim 2.

Claim 6. $\ell(B) \le l + \max(m,n,p) - 2 \le l+r-3$.

Proof of Claim 6. Since $\ell(B) \ge \ell(A) > \ell(A_0^*)$, $\ell(B) \ge \ell(A) > \ell(A_1^*)$ and $\ell(B) \ge \ell(A) > \ell(A_2^*)$, therefore $\delta_{B,A_0^*} = \delta_{B,A_1^*} = \delta_{B,A_2^*} = 0$.

Let's start by considering the case where $m \le n \le p$. Assuming that Claim 6 is false, we can deduce that $\ell(B) \ge l+p-1 = |B|+|A_2|-1 \ge l+n-1 = |B|+|A_1|-1 \ge l+m-1 = |B|+|A_0^*|-1$, while $d(B) = 1$. Thus by Theorem 3

$$|A_0^*+B| \ge l+2|A_0^*|-2 = l+2m-2, \quad |A_1^*+B| \ge l+2|A_1^*|-2 = l+2n-2$$
$$\text{and} \quad |A_2^*+B| \ge l+2|A_2^*|-2 = l+2p-2. \tag{6}$$

Hence $|A+3\cdot B| \ge 3l+2r-6$, which contradicts our hypothesis.

For all the remaining cases we obtain the result by proceeding like above. Thus $\ell(B) \le l+\max(m,n,p)-2$. Since $r = m+n+p$ and $\max(m,n,p) \le r-1$, therefore, $\ell(B) \le l+\max(m,n,p)-2 \le l+r-3$. This completes the proof of Claim 6. □

Claim 7. $|A+3\cdot B| \ge |A|+3(|B|-1)+h_B$.

Proof of Claim 7. Assume the case $m \le n \le p$. By Claim 6, observe that $\ell(B) \le l + p - 2$.

Also the By Theorem 1.3, $|A_2^* + B| \ge (p + l - 1) + h_B$ and thus

$$\begin{aligned}|A + 3\cdot B| &= |A_0^* + B| + |A_1^* + B| + |A_2^* + B|\\ &\ge (|A_0^*| + |B| - 1) + (|A_1^*| + |B| - 1) + |A_2^* + B|\\ &\ge (m + l - 1) + (n + l - 1) + (p + l - 1) + h_B\\ &= 3l + r - 3 + h_B.\end{aligned}$$

Similarly for all the remaining cases $|A + 3\cdot B| \ge 3l + r - 3 + h_B$. Therefore, we can deduce that h_B satisfies the inequality $0 \le h_B \le |A + 3\cdot B| - (3l + r - 3) = h \le r - 3$. Consequently, it follows that $B \subseteq \{b_0, b_0 + 1, b_0 + 2, \dots, b_{l-1}\} \subseteq \{0, 1, \dots, b_{l-1}\}$ and B forms an arithmetic progression with a length of at most $b_{l-1} + 1 = l + h_B \le l + h \le r + l - 3$. Since $\ell(A) \le \ell(B)$, the set A is also contained within an arithmetic progression with a length of at most $r + l - 3$. This establishes the desired result. By combining both cases, we obtain the overall result. □

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