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Comptes Rendus

Mathématique

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Volume 362 (2024), p. 327-356

Online since: 16 May 2024

https://doi.org/10.5802/crmath.539

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www.centre-mersenne.org — e-ISSN : 1778-3569

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Comptes Rendus. Mathématique
2024, Vol. 362, p. 327-356
https://doi.org/10.5802/crmath.539

Research article / *Article de recherche*
Partial differential equations, Control theory / *Équations aux dérivées partielles, Théorie du contrôle*

Exact controllability for systems describing plate vibrations. A perturbation approach

Contrôle exact pour systèmes décrivant les vibrations des plaques. Une approche perturbative

Marius Tucsnak [*,a], **Megane Bournissou** [a] **and Sylvain Ervedoza** [a]

[a] Institut de Mathématiques de Bordeaux, UMR 5251, Université de Bordeaux, CNRS, Bordeaux INP, F-33400 Talence, France.

E-mails: marius.tucsnak@u-bordeaux.fr, megane.bournissou@u-bordeaux.fr, sylvain.ervedoza@u-bordeaux.fr

Abstract. The aim of this paper is to prove new exact controllability properties of systems described by perturbations of the classical Kirchhoff plate equation. We first consider systems described by an abstract plate equation with a bounded control operator. The generator of these systems is perturbed by bounded operators which are not necessarily compact, thus not falling in the range of application of compactness-uniqueness arguments. Our first main result is abstract and can be informally stated as follows: if the system described by the corresponding unperturbed abstract wave equation, with the same control operator, is exactly controllable (in some time), then the considered perturbed plate system is exactly controllable in arbitrarily small time. The employed methodology is based, in particular, on frequency-dependent Hautus type tests for systems with skew-adjoint operators. When applied to systems described by the classical Kirchhoff equations, our abstract results, combined with some elliptic Carleman-type estimates, yield exact controllability in arbitrarily small time, provided that the system described by the wave equation in the same spatial domain and with the same control operator is exactly controllable. The same abstract results can be used to prove the exact controllability of the system obtained by linearizing the von Kármán plate equation around a real analytic stationary state. This leads, via a fixed-point method, to our second main result: the nonlinear system described by the von Kármán plate equations is locally exactly controllable around any stationary state defined by a real analytic function. We also discuss the possible application of the methods in this paper to systems described by Schrödinger type equations on manifolds or by the related Berger's nonlinear plate equation.

Résumé. L'objectif de ce travail est l'obtention des nouvelles propiétés de contrôlabilité exacte de systèmes décrits par des perturbations de l'équation de plaque de Kirchhoff classique. Nous considérons d'abord les systèmes décrits par une équation des plaques abstraite avec un opérateur de contrôle borné. Le générateur de ces systèmes est perturbé par des opérateurs bornés qui ne sont pas nécessairement compacts, donc hors du domaine d'application des arguments compacité-unicité. Notre premier résultat principal est abstrait et peut être énoncé de manière informelle comme suit : si le système décrit par l'équation d'onde abstraite non perturbée correspondante, avec le même opérateur de contrôle, est exactement contrôlable (dans un certain temps), alors le système de plaques perturbé considéré est exactement contrôlable en un temps arbitrairement petit. La méthodologie employée s'appuie notamment sur des tests de type Hautus dépendant de la fréquence pour des systèmes à opérateurs anti-adjoints. Lorsqu'ils sont appliqués à des systèmes décrits

* Corresponding author.

par les équations classiques de Kirchhoff, nos résultats abstraits, combinés à des estimations elliptiques de type Carleman, donnent une contrôlabilité exacte en un temps arbitrairement petit, à condition que le système décrit par l'équation d'onde dans le même domaine spatial et avec le même l'opérateur de contrôle est exactement contrôlable. Les mêmes résultats abstraits peuvent être utilisés pour prouver la contrôlabilité exacte du système obtenu en linéarisant l'équation de la plaque de von Karman autour d'un état stationnaire analytique réel. Ceci conduit, via une méthode de point fixe, à notre deuxième résultat principal : le système non linéaire décrit par les équations de von Kárman est localement exactement contrôlable autour de tout état stationnaire défini par une fonction analytique réelle. Nous discutons également de l'application possible des méthodes de cet article à des systèmes décrits par des équations de type Schrodinger sur des variétés ou par l'équation de plaque non linéaire de Berger associée.

Funding. This work benefits from the support of ANR project TRECOS, grant ANR-20-CE40-0009.

Manuscript received 21 December 2022, revised 13 July 2023, accepted 14 July 2023.

1. Introduction

The exact controllability for systems described by the linear plate equation, designed as Kirchhoff plate equation in the remaining part of this paper, via a distributed internal control is by now a well-understood subject. The existing type of results asserts that, under appropriate conditions on the domain where the PDE holds and on the support of the control, exact observability holds in arbitrarily small time(see, for instance, Zuazua [36, Appendix 1] for an early result of this type).

A natural question is the robustness of these results when the bilaplacian appearing in the Kirchhoff equation is perturbed by a linear second order differential operator. As far as we know, there is no result in this direction, with the exception of the case when the coefficients of the perturbing operator are small (in an appropriate sense), where this robustness follows from simple functional analytic arguments. For the related problem of exact controllability of systems described by Schrödinger equations with the Laplacian perturbed by multiplication operators the literature contains several results which assume either that a geometric optics condition holds or that the domain in which the Schrödinger equation holds is a disk or a torus, see Anantharaman, Léautaud and Macià [1], Bourgain, Burq and Zworski [7] and references therein.

In this work we assume that the spatial domain occupied by the plate and the control operator are such that the system described by the wave equation in the same domain and with the same control operator is exactly controllable. One of our main results asserts that under this assumption, which is strictly stronger than the exact controllability of the unperturbed Kirchhoff system (see comments in Section 2 below), combined with a unique continuation hypothesis, the perturbed plate system is exactly controllable.

The remaining part of this work is organized as follows. Section 2 is devoted to a detailed description of the general context and to the statement of the main results. In Section 3, we describe, in an abstract setting, a strategy, based on resolvent estimates, to deal with bounded perturbations of the generator of control systems. In Section 4, we give a Hautus-type condition for the exact observability of systems described by abstract Schrödinger and Kirchhoff equations. This test is then used in Section 5 to prove the exact controllability of linear perturbed abstract systems. These abstract results are then used in Section 6 to prove the exact controllability of linear perturbed Kirchhoff equations stated in Theorem 4. In Section 7, using a fixed-point theorem, one can deduce the exact controllability of the nonlinear Von Kármán plate model given in Theorem 6. Finally, in Section 8, we indicate how our methodology can be used to tackle perturbations of the system described by the Schrödinger or plate equations on a class of compact manifolds and of the nonlinear system described by Berger's equation, which can be seen as a simplification of the von Kármán system.

2. Context and statement of the main results

As already mentioned, the exact controllability of the systems we are interested in is strongly related to the similar property for systems described by the Schrödinger and wave equations. In particular, in the case of boundary conditions corresponding (at least for a flat boundary) to the hinged case, these properties can be derived from the corresponding properties of the system described by the Schrödinger equation with homogeneous Dirichlet boundary condition and with the same control operator. Moreover, the exact controllability in any positive time of Schrödinger type systems can be obtained from the exact controllability (in some time) of the corresponding system described by the wave equation as done in Miller [41, Remark 10.3] and Tucsnak and Weiss [47, Section 6.8]. Thus, in the above sense, the exact controllability properties of systems described by the wave equation implies the same property (in arbitrarily small time) for the corresponding systems described by the Schrödinger or Kirchhoff plate equations.

To state the above assertions in a more precise manner, we introduce some notation that will be used in the remaining part of this paper. Let $n \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $\partial\Omega$ of class C^3 or let Ω be a rectangular domain. We first consider the control system, described by a wave equation,

$$\begin{cases} \ddot{v}(t,x) - \Delta v(t,x) = u(t,x)\chi_{\mathscr{O}}(x) & (t \geqslant 0,\ x \in \Omega), \\ v(t,x) = 0 & (t \geqslant 0,\ x \in \partial\Omega), \end{cases} \tag{Σ_{wave}}$$

where $\mathscr{O}$ is an open nonempty subset of Ω, $\chi_{\mathscr{O}} \in L^\infty(\Omega)$ is non negative and positive in $\mathscr{O}$, u is the control function and $\left[\begin{smallmatrix} v \\ \dot{v} \end{smallmatrix}\right]$ is the state trajectory of the system. This system, with input space $L^2(\Omega)$, is clearly well-posed in the state space $H_0^1(\Omega) \times L^2(\Omega)$, where, as usual, for every $m \in \mathbb{N}$ we denote by $H^m(\Omega)$, the space of functions in $L^2(\Omega)$ with distributional derivatives, up to order m in $L^2(\Omega)$ and by $H_0^m(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. Using the same notation as for defining (Σ_{wave}) we define two other systems (Σ_{schrod}) and (Σ_{plate}) which correspond, respectively, to the Schrödinger and Kirchhoff plate equations, by

$$\begin{cases} \dot{z}(t,x) + i\Delta z(t,x) = u(t,x)\chi_{\mathscr{O}}(x) & (t \geqslant 0,\ x \in \Omega), \\ z(t,x) = 0 & (t \geqslant 0,\ x \in \partial\Omega), \end{cases} \tag{Σ_{schrod}}$$

and

$$\begin{cases} \ddot{w}(t,x) + \Delta^2 w(t,x) = u(t,x)\chi_{\mathscr{O}}(x) & (t \geqslant 0,\ x \in \Omega), \\ w(t,x) = 0,\ \Delta w(t,x) = 0 & (t \geqslant 0,\ x \in \partial\Omega). \end{cases} \tag{Σ_{plate}}$$

It is well-known that (Σ_{schrod}) and (Σ_{plate}) are well-posed control systems, both with input space $L^2(\Omega)$ and with state space $L^2(\Omega)$ and $(H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$, respectively. The exact controllability properties of the three above systems are connected by a result that goes back to Lebeau [35] (see also Tucsnak and Weiss [47, Sections 6.7 and 6.8]).

Proposition 1. *Assume that the system* (Σ_{wave}) *is exactly controllable (in some time). Then, the systems* (Σ_{schrod}) *and* (Σ_{plate}) *are exactly controllable in arbitrarily small time.*

Remark 2. It appears that for the exact controllability in some time τ of (Σ_{wave}), a crucial sufficient condition on the control domain is the following:
(BLR): Any light ray, travelling in Ω at unit speed and reflected according to geometric optics laws when it hits the boundary $\partial\Omega$, will hit $\mathscr{O}$ in time $\leqslant \tau$.
This condition was first considered for the wave equation by Rauch and Taylor in [45] for a manifold, by Bardos, Lebeau and Rauch in [2] for bounded open sets Ω with $\partial\Omega$ of class C^∞ (see also [3] in the case of boundary control) and later generalized to domains with $\partial\Omega$ of class C^3 by Burq in [8]. It also has been proved to be sufficient for the Schrödinger equation by Lebeau in [35] (actually, Lebeau deals with boundary control, but the same strategy holds for

internal control). This condition is "almost" necessary in a sense made precise in [3], and we shall refer to it as the *geometric control condition* of Bardos, Lebeau and Rauch or, shortly, the *(BLR)* condition. However, this condition is not necessary for the controllability of systems described by the Schrödinger or the Kirchhoff equations. This can be seen, for instance, in the case when Ω is a rectangular domain, for which it has been shown in Jaffard [28] and Komornik [31] that the exact controllability in arbitrarily small time of the plate equation holds for every open nonempty subset $\mathcal{O}$ of Ω.

The first aim of this paper is to investigate the robustness of the result in Proposition 1 when the Laplacian in (Σ_{schrod}) or the bi-Laplacian in (Σ_{plate}) are perturbed by lower order linear operators. In the case of the Schrödinger equation we have:

Proposition 3. *Assume that the system* (Σ_{wave}) *is exactly controllable (in some time) and let* $a \in L^\infty(\Omega;\mathbb{R})$. *Then the system*

$$\begin{cases} \dot z(t,x) + i\Delta z(t,x) + ia(x)z(t,x) = u(t,x)\chi_{\mathcal{O}}(x) & (t \geqslant 0,\ x\in\Omega), \\ z(t,x) = 0 & (t\geqslant 0,\ x\in\partial\Omega), \end{cases} \tag{1}$$

with state and control space $L^2(\Omega)$, *is exactly controllable in any positive time.*

Although we did not find the above result explicitly stated in the literature, one can say that it makes part of the folklore in the field. For the sake of completeness, we will explain in Section 3 how Proposition 3 follows from known resolvent estimates. Let us also note that when a stronger version of the (BLR) condition holds, such a perturbation can be studied using Carleman estimates which are appropriate to absorb the lower-order terms (possibly depending on time and space), as done, for instance, in Baudouin and Puel [5] or Yuan and Yamamoto [48]. Moreover, it has been shown in Burq and Zworski [9] and Burq, Bourgain and Zworski [7] that if Ω is a rectangular domain then the conclusion of Proposition 3 holds for any open nonempty set $\mathcal{O}\subset\Omega$.

Much less is known for similar perturbations (bounded but not compact in the state space) of the Kirchhoff system (Σ_{plate}). Our main result on linearly perturbed Kirchhoff systems, which is proved in Section 6, is:

Theorem 4. *Let* $(a_{k\ell})_{1\leqslant k,\ell\leqslant n}$ *be functions in* $W^{2,\infty}(\Omega;\mathbb{R})$ *such that*

$$\begin{cases} a_{k\ell} = a_{\ell k} & (1\leqslant k,\ell\leqslant n), \\ \displaystyle\sum_{\ell=1}^n \frac{\partial a_{k\ell}}{\partial x_\ell}(x) = 0 & (k\in\{1,2,\dots,n\},\ x\in\Omega). \end{cases} \tag{2}$$

Let $(b_k)_{1\leqslant k\leqslant n}$ *be functions in* $W^{1,\infty}(\Omega)$ *and let* $c\in L^\infty(\Omega)$. *Moreover, suppose that the system* (Σ_{wave}), *with state space* $H_0^1(\Omega)\times L^2(\Omega)$ *and control space* $L^2(\Omega)$, *is exactly controllable (in some time). Then, the equation*

$$\ddot w(t,x) + \Delta^2 w(t,x) + \sum_{k,\ell=1}^n a_{k\ell}(x)\frac{\partial^2 w}{\partial x_k\partial x_\ell}(t,x) + \sum_{k=1}^n b_k(x)\frac{\partial w}{\partial x_k}(t,x) + c(x)w(t,x) = u(t,x)\chi_{\mathcal{O}}(x) \qquad (t\geqslant 0,\ x\in\Omega), \tag{3}$$

with the boundary conditions

$$w(t,x) = 0,\ \Delta w(t,x) = 0 \qquad (t\geqslant 0,\ x\in\partial\Omega), \tag{4}$$

defines a system, with state space $(H^2(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega)$ *and control space* $L^2(\Omega)$, *which is exactly controllable in any positive time.*

Remark 5. In the particular case when the matrix $(a_{k\ell})_{1\leqslant k,\ell\leqslant n}$ vanishes, the result in Theorem 4 has been proven in Cindea and Tucsnak [15]. Moreover, in the same reference, the exact controllability in some time (not necessarily in an arbitrarily small time) of the system (3)-(4) has been established if $a_{k\ell} = -\alpha\delta_{k\ell}$, where $\alpha \geqslant 0$ and $\delta_{k\ell}$ is the Kronecker symbol.

The second objective of this paper is to prove the local exact controllability around equilibrium states for systems describing the nonlinear vibrations of elastic plates. Our main result in this direction concerns the von Kármán plate model, which is described by the equations

$$\begin{cases} \ddot{w}(t,x) + \Delta^2 w(t,x) + [w,\Phi(w,w)](t,x) = f(x) + u(t,x)\chi_{\mathscr{O}}(x) & (t \geqslant 0,\ x \in \Omega), \\ w(t,x) = \Delta w(t,x) = 0 & (t \geqslant 0,\ x \in \partial\Omega), \\ w(0,x) = w_0(x),\ \dot{w}(0,x) = w_1(x) & (x \in \Omega), \end{cases} \tag{5}$$

where $\Omega \subset \mathbb{R}^2$ is an open, bounded and nonempty set, f is a given force field, the Airy stress function $\Phi(v,w)$ is the solution of the boundary value problem

$$\begin{cases} \Delta^2\Phi(v,w)(t,x) = [v,w](t,x) & (t \geqslant 0,\ x \in \Omega), \\ w(t,x) = \dfrac{\partial w}{\partial \nu}(t,x) = 0 & (t \geqslant 0,\ x \in \partial\Omega), \end{cases} \tag{6}$$

and the bracket $[\cdot,\cdot] : H^2(\Omega) \times H^2(\Omega) \to L^1(\Omega)$ is defined by

$$[\psi,\varphi] = \frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_2^2}\frac{\partial^2\varphi}{\partial x_1^2} - 2\frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial^2\varphi}{\partial x_1\partial x_2} \qquad \left(\psi,\varphi \in H^2(\Omega)\right). \tag{7}$$

In the above system, which is one of the most popular nonlinear models describing the vibrations of elastic plates (see, for instance, Berger and Fife [6], Ciarlet and Rabier [13] for basic facts on this type of model), w stands for the transverse displacement, whereas the in-plane and the rotational inertia are neglected. The control function is $u \in L^2([0,\infty), L^2(\Omega))$, whereas $\mathscr{O}$ is an open nonempty subset of Ω, designing the region where the control acts, and $\chi_{\mathscr{O}} \in L^\infty(\Omega)$ is non negative and positive in $\mathscr{O}$.

Let η be a stationary solution corresponding to the forcing term f, i.e. satisfying

$$\begin{cases} \Delta^2\eta(x) + \left[\eta,\Phi(\eta,\eta)\right](x) = f(x) & (x \in \Omega), \\ \eta(x) = \Delta\eta(x) = 0 & (x \in \partial\Omega). \end{cases} \tag{8}$$

A natural question is the controllability of the system defined by (5) around the equilibrium η. As far as we know, the first result in this direction has been proved in Lagnese [32], who considered a model including rotational inertia (which simplifies the analysis) and he proved a local controllability result for $\eta = 0$. The proof in [32] can be adapted to the system (5) by using the sharp regularity of the nonlinear term in (5) obtained in Favini et al. [23, 24] (see also Chueshov and Lasiecka [12]). As far as we know, the literature contains no local controllability result for (5) around equilibrium states $\eta \neq 0$, or even for the linearization of the system around such states. Closely related questions are discussed in Eller and Toundykov [22], where the authors consider a plate system with a local nonlinearity containing no derivatives of w and they prove a semiglobal controllability result.

The main novelty we bring in on the system defined by (5) (which involves non-local second order nonlinearities) is that we prove its local exact controllability around any equilibrium η defined by a function which is real analytic on Ω. This analyticity condition could be replaced by a potentially weaker unique continuation assumption, which will be discussed in Remark 28.

The second main result in this paper is:

Theorem 6. *Let $\Omega \subset \mathbb{R}^2$ be a nonempty, open and bounded set, with $\partial\Omega$ of class C^3 or let Ω be a rectangle. Let $f \in L^2(\Omega)$. Assume that $\mathscr{O}$ is an open subset of Ω such that the system* (Σ_{wave}) *is exactly* *controllable (in some time). Moreover, suppose that the function η is in $W^{2,\infty}(\Omega)$, satisfies* (8) *and is analytic in Ω. Then, for every $\tau > 0$, there exists $\varepsilon > 0$ such that for every*

$$w_0 \in H^2(\Omega)\cap H_0^1(\Omega), \qquad w_1 \in L^2(\Omega),$$

with

$$\| w_0 - \eta\|_{H^2(\Omega)} + \| w_1\|_{L^2(\Omega)} \leqslant \varepsilon,$$

there exists $u \in L^2([0,\tau]; L^2(\Omega))$ such that

$$w(\tau,\cdot) = \eta, \qquad \dot{w}(\tau,\cdot) = 0.$$

The proof of this result will be presented in Section 7.

3. Some background on the Hautus test for skew-adjoint systems

The aim of this section is to recall some basic facts on exact controllability and exact observability of systems with skew-adjoint operators, with a focus on the Hautus test and its applications for studying perturbations. For more general background on exact observability and exact controllability, we refer to [47, Ch. 6 and Ch. 11].

Within this section $\mathscr{H}$ (the state space), U (the input space) and Y (the output space) are generic Hilbert spaces. In this work we consider control systems described by

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t) & (t \geqslant 0), \\ z(0) = 0, \end{cases} \tag{9}$$

where $A : \mathscr{D}(A) \subset \mathscr{H} \to \mathscr{H}$ is a skew-adjoint operator generating a unitary C^0-group $\mathbb{T}$ on $\mathscr{H}$ and $B \in \mathscr{L}(U,\mathscr{H})$. We say that the pair (A,B) defines a system, with state space $\mathscr{H}$ and input space U, which is *exactly controllable in time* τ if for every $z_1 \in \mathscr{H}$, there exists $u \in L^2([0,\tau]; U)$ such that the solution of (9) satisfies $z(\tau) = z_1$.

It is well-known that the exact controllability of a well-posed linear system is equivalent to the exact observability of the dual system. This is true, in particular for the three systems (Σ_{wave}), (Σ_{schrod}) and (Σ_{plate}) introduced in Section 1. The duals of these systems can all be written in the form

$$\dot{z} = Az, \qquad y = Cz, \tag{10}$$

where C is a linear bounded operator from $\mathscr{H}$ into Y. Recall that the pair (A,C) is said *exactly observable in time* $\tau > 0$ if there exists $K_\tau > 0$ such that

$$K_\tau^2 \int_0^\tau \|C\mathbb{T}_t z_0\|_Y^2 \, dt \geqslant \|z_0\|_{\mathscr{H}}^2 \qquad (z_0 \in \mathscr{H}).$$

The pair (A,C) is said *exactly observable* if it is exactly observable in some time $\tau > 0$.

Remark 7. In this work we consider only bounded observation operators $C \in \mathscr{L}(\mathscr{H}, Y)$. However, some of our abstract results, in particular those in Section 3 and in Section 4, hold under a weaker assumption on the observation operator, namely that $C \in \mathscr{L}(\mathscr{D}(A), Y)$ is an admissible observation operator for the semigroup $\mathbb{T}$ generated by A, in the sense of [47, Definition 4.3.1]. However, working with admissible operators instead of bounded ones entails some technical issues with respect to the functional setting in which controllability and observability results hold. Thus, due to the main applications we have in mind and for the sake of clarity, the results below are only stated in the case of bounded observation operators.

A widely used necessary and sufficient condition for exact observability of systems with skew-adjoint generator is the following Hautus test, firstly proved in Miller [41, Theorem 5.1]:

Theorem 8. *With the above notation, let A be skew-adjoint on $\mathscr{H}$ and let $C \in \mathscr{L}(\mathscr{H}, Y)$. Then the pair (A, C) is exactly observable if and only if there exist constants M, $m > 0$ such that*

$$M^2 \|(i\omega I - A)z_0\|^2_{\mathscr{H}} + m^2 \|Cz_0\|^2_Y \geqslant \|z_0\|^2_{\mathscr{H}} \qquad (\omega \in \mathbb{R},\ z_0 \in \mathscr{D}(A)). \tag{11}$$

Moreover, if (11) *holds then (A, C) is exactly observable in time τ for any $\tau > M\pi$.*

In the finite-dimensional case, the observability of (A, C) is equivalent to $C\phi \neq 0$ for every eigenvector ϕ of A. The situation is more complicated in the infinite-dimensional case (see [11, 39, 44] for statements without information on the time of observability). A natural analog of the condition $C\phi \neq 0$ for every eigenvector ϕ of A, at least when A is skew-adjoint, is an observability inequality on the wave packets of A. More precisely, we have (see, for instance, [47, Section 6.9]):

Theorem 9. *Assume that A is skew-adjoint on $\mathscr{H}$ and that it has compact resolvents and let $C \in \mathscr{L}(\mathscr{H}, Y)$. We denote by $(\phi_j)_{j \in \mathbb{N}^*}$ an orthonormal basis of eigenvectors of A and by $(i\lambda_j)_{j \in \mathbb{N}^*}$ the corresponding eigenvalues. Moreover, for every $\omega \in \mathbb{R}$ and $r > 0$, we set*

$$J_r(\omega) = \left\{ j \in \mathbb{N}^* \text{ such that } \left|\lambda_j - \omega\right| < r \right\}.$$

The following statements are equivalent.

(1) *: There exist $r, \delta > 0$ such that for all $\omega \in \mathbb{R}$ and for all z of the form $z = \sum_{j \in J_r(\omega)} z_j \phi_j$,*

$$\|Cz\|_Y \geqslant \delta \|z\|_{\mathscr{H}}. \tag{12}$$

(2) *The pair (A, C) is exactly observable.*

An interesting question is to investigate the robustness of exact observability with respect to bounded (but not necessarily small) perturbations $P \in \mathscr{L}(\mathscr{H})$ of the generator. To this purpose, an interesting tool is a reinforced form of the condition (11), in which the constant M is replaced by a positive function tending to zero when $|\omega| \to \infty$, which is a sufficient condition for exact observability in arbitrarily small time, see [42, Corollary 2.14]. Such a frequency-dependent Hautus-type condition allows to deal with bounded skew-adjoint perturbations of the generator. More precisely, we have:

Proposition 10. *Assume that A is skew-adjoint on $\mathscr{H}$ and that it has compact resolvents. Let $C \in \mathscr{L}(\mathscr{H}, Y)$. Suppose that there exist a function $M : \mathbb{R} \to [0, +\infty)$ which tends to zero when $|\omega| \to +\infty$ and a constant $m > 0$ such that*

$$M^2(\omega) \|(i\omega I - A)z_0\|^2_{\mathscr{H}} + m^2 \|Cz_0\|^2_Y \geqslant \|z_0\|^2_{\mathscr{H}}, \qquad (\omega \in \mathbb{R}, z_0 \in \mathscr{D}(A)). \tag{13}$$

Moreover, let $P \in \mathscr{L}(\mathscr{H})$ be a bounded skew-adjoint operator such that $C\phi \neq 0$ for every eigenvector ϕ of $A + P$. Then the pair $(A + P, C)$ is exactly observable in any positive time.

Remark 11. As pointed out in [42, Remark 2.15], Proposition 10 is not necessary to get observability of a system in arbitrarily small time. Indeed, when considering the Schrödinger equation in a 2d torus observed from a strip, there is no function $M : \mathbb{R} \to [0, +\infty)$ which tends to zero when $|\omega| \to +\infty$ so that the resolvent estimate (13) holds, while it is well-known that observability holds in arbitrarily small times (see e.g. [26]).

Proof. First, notice that $A + P$ with $\mathscr{D}(A + P) = \mathscr{D}(A)$ is still skew-adjoint, thus it generates a C^0-group of unitary operators on $\mathscr{H}$.

Using the fact that the pair (A, C) satisfies (13), together with triangular and Young inequalities, one gets that

$$\begin{aligned} 2M^2(\omega) \|(i\omega I - A - P)z_0\|^2_{\mathscr{H}} + 2M^2(\omega) \|P\|^2_{\mathscr{L}(\mathscr{H})} \|z_0\|^2_{\mathscr{H}} \\ + m^2 \|Cz_0\|^2_Y \geqslant \|z_0\|^2_{\mathscr{H}} \qquad (\omega \in \mathbb{R},\ z_0 \in \mathscr{D}(A)). \end{aligned}$$

Since $M(\omega) \to 0$ when $|\omega| \to \infty$, it follows that for every $\gamma > 0$ there exists $c_\gamma > 0$ such that

$$\gamma^2 \|(i\omega I - A - P)z_0\|^2_{\mathscr{H}} + m^2 \|Cz_0\|^2_Y \geqslant \|z_0\|^2_{\mathscr{H}} \qquad \big(|\omega| > c_\gamma,\ z_0 \in \mathscr{D}(A)\big).$$

We have thus shown that (11) holds for "high frequencies". This, combined with the fact that $C\phi \neq 0$ for every eigenvector ϕ of $A+P$ and [47, Proposition 6.6.4], implies the exact observability of $(A+P, C)$ in any time $\tau > \gamma\pi$. □

Proposition 10 allows us to prove, for instance, the robustness of the exact controllability of a system described by the Schrödinger equation with respect to bounded perturbations as stated in Proposition 3.

Proof of Proposition 3. Denote by $A = -i\Delta$ with $\mathscr{D}(A) = H^2(\Omega) \cap H^1_0(\Omega)$ which is skew-adjoint with compact resolvents, $C : \varphi \mapsto \varphi\chi_{\mathscr{O}} \in \mathscr{L}(L^2(\Omega))$ and $P : \varphi \mapsto -ia\varphi \in \mathscr{L}(L^2(\Omega))$ (with $a \in L^\infty(\Omega, \mathbb{R})$) which is also skew-adjoint. By [41, Proof of Theorem 3.4] or [47, Section 6.7], it is known that when (Σ_{wave}) is exactly controllable in some time, the Hautus-type condition (13) holds with $M(\omega) = \frac{M}{\sqrt{\omega}}$ with some constant $M > 0$. Moreover, $C\phi \neq 0$ for every eigenvector ϕ of $A+P$ (see for example [47, Theorem 15.2.1]).

Therefore, Proposition 10 entails that the pair $(A+P, C)$ is exactly observable in any positive time, and thus, by a duality argument, that the Schrödinger equation with a bounded potential (1) is exactly controllable in any positive time. □

As in Theorem 9, the frequency-dependent Hautus-type condition (13) can be equivalently expressed in terms of wave packets, as done in Miller [42, Theorem 2.16]. For the sake of simplicity, we only give below a simplified version of [42, Theorem 2.16], which is sufficient for the present work.

Proposition 12. *Assume that A is skew-adjoint on $\mathscr{H}$ and that it has compact resolvents and let $C \in \mathscr{L}(\mathscr{H}, Y)$. The following statements are equivalent.*

(1) *There exist $\delta > 0$ and $r : \mathbb{R} \to (0, +\infty)$ which tends to infinity when $|\omega| \to +\infty$ such that for all $\omega \in \mathbb{R}$ and z of the form $z = \sum_{j \in J_{r(\omega)}(\omega)} z_j \phi_j$, the inequality* (12) *holds.*
(2) *There exists $m > 0$ and $M : \mathbb{R} \to (0, +\infty)$ which tends to zero when $|\omega| \to +\infty$ such that* (13) *holds.*

Our approach to prove the robustness of the exact observability property for plate equations with respect to bounded perturbations of the generator, as stated in Theorem 4, is to show that the considered system satisfies a frequency-dependent Hautus condition of type (13). To this aim, see Section 4, we first check a frequency-dependent wave packets condition.

However, the situation is more complicated in the case of systems described by the plate equation (3) than in the case of the Schrödinger equation (1), already studied in this section. Indeed, extra difficulties are generated by the fact that, when written in first-order form, the generator of the perturbed system is no longer skew-adjoint. Thus, instead of directly applying Proposition 10, we need a special decomposition in low and high frequency parts of the state space and the application of a simultaneous controllability result.

4. A frequency-dependent Hautus condition for systems describing plate vibrations

In this section, we show that under appropriate assumptions, a class of abstract observation systems, described by plate type equation with distributed observation satisfies a frequency-dependent Hautus-type condition (13). This condition will be essential in the next section where we show that the exact observability property is robust with respect to a class of perturbations of the generator.

Within this section, we continue to denote by $\mathscr{H}$ and Y two Hilbert spaces and we denote by $A_0 : \mathscr{D}(A_0) \to \mathscr{H}$ a positive operator with compact resolvents. If there is no risk of confusion, the inner product and the norm in $\mathscr{H}$ are simply denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. For $\alpha > 0$, we denote by $\mathscr{H}_\alpha$ the space $\mathscr{D}(A_0^\alpha)$ endowed with the graph norm of A_0^α. For $\alpha < 0$ the space $\mathscr{H}_\alpha$ is defined as the dual of $\mathscr{H}_{-\alpha}$ with respect to the pivot space $\mathscr{H}$. Note that for every $\alpha \in \mathbb{R}$ the operator A_0 can be restricted (or extended) to a unitary operator in $\mathscr{L}(\mathscr{H}_\alpha, \mathscr{H}_{\alpha-1})$. Moreover, let $C_0 \in \mathscr{L}(\mathscr{H}, Y)$ be an observation operator.

With the above notation, the class of systems we consider is:

$$\begin{cases} \ddot{\delta}(t) + A_0^2\delta(t) = 0 & (t \geqslant 0),\\ \delta(0) = \delta_0 \in \mathscr{H},\ \dot{\delta}(0) = \delta_1 \in \mathscr{H}_{-1}, & \\ y(t) = C_0\delta(t) & (t \geqslant 0). \end{cases} \tag{14}$$

The system (14) can be written in a first-order form

$$\begin{cases} \dot{z}(t) = \mathscr{A} z(t) & (t \geqslant 0),\\ z(0) = z_0, & \\ y(t) = Cz(t) & (t \geqslant 0), \end{cases}$$

in the state space $\mathscr{H} \times \mathscr{H}_{-1}$, which is a Hilbert space with the inner product

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_{\mathscr{H}\times\mathscr{H}_{-1}} = \langle f_1, f_2\rangle + \langle A_0^{-1} g_1, A_0^{-1} g_2\rangle,$$

with $\mathscr{A} : \mathscr{D}(\mathscr{A}) \to \mathscr{H} \times \mathscr{H}_{-1}$, $C \in \mathscr{L}(\mathscr{H} \times \mathscr{H}_{-1}, Y)$ and z defined by $\mathscr{D}(\mathscr{A}) = \mathscr{H}_1 \times \mathscr{H}$ and

$$\mathscr{A} = \begin{bmatrix} 0 & I \\ -A_0^2 & 0 \end{bmatrix} \quad \text{i.e.} \quad \mathscr{A}\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0^2 f \end{bmatrix}, \tag{15}$$

$$C = \begin{bmatrix} C_0 & 0 \end{bmatrix}, \tag{16}$$

$$z(t) = \begin{bmatrix} \delta(t) \\ \dot{\delta}(t) \end{bmatrix}, \qquad z_0 = \begin{bmatrix} \delta_0 \\ \delta_1 \end{bmatrix}. \tag{17}$$

Since $A_0^2 > 0$ (see [47, Remark 3.3.7]), according to [47, Proposition 3.7.6], the operator $\mathscr{A}$ is skew-adjoint and $0 \in \rho(\mathscr{A})$. By Stone's theorem, $\mathscr{A}$ generates a unitary group on $\mathscr{H} \times \mathscr{H}_{-1}$.
The main result of this section is:

Theorem 13. *With the above notation and assumptions, suppose that the pair $(\widetilde{A}, \widetilde{C})$ with*

$$\mathscr{D}(\widetilde{A}) = \mathscr{H}_1 \times \mathscr{H}_{\frac{1}{2}}, \qquad \widetilde{A} = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \qquad \widetilde{C} = \begin{bmatrix} 0 & C_0 \end{bmatrix}, \tag{18}$$

defines a system, with state space $\mathscr{H}_{\frac{1}{2}} \times \mathscr{H}$ and output space Y, which is exactly observable (in some time).

Then there exist a function $M_1 : \mathbb{R} \to [0, +\infty)$, which tends to zero when $|\omega| \to \infty$, and a constant $m_1 > 0$ such that

$$M_1^2(\omega)\,\|(i\omega I - \mathscr{A})z_0\|^2_{\mathscr{H}\times\mathscr{H}_{-1}} + m_1^2\|Cz_0\|_Y^2 \geqslant \|z_0\|^2_{\mathscr{H}\times\mathscr{H}_{-1}} \qquad (\omega \in \mathbb{R},\ z_0 \in \mathscr{D}(\mathscr{A})), \tag{19}$$

where $\mathscr{A}$ and C are respectively defined in (15) *and* (16).

Remark 14. It is not difficult to check that the above assumption that the pair $(\widetilde{A}, \widetilde{C})$ defines an exactly observable system with state space $\mathscr{H}_{\frac{1}{2}} \times \mathscr{H}$ and output space Y is equivalent to the fact that $\left(\widetilde{A}, \begin{bmatrix} C_0 & 0 \end{bmatrix}\right)$ defines an exactly observable system with state space $\mathscr{H} \times \mathscr{H}_{-\frac{1}{2}}$ and output space Y. By duality, these conditions are equivalent to the exact controllability of the control system, with state space $\mathscr{H}_{\frac{1}{2}} \times \mathscr{H}$ and input space Y, defined by $(\widetilde{A}, \widetilde{B})$, where $\widetilde{B} = \left[\begin{smallmatrix} 0 \\ C_0^* \end{smallmatrix}\right]$. In the approach that will be presented in Section 6 to prove Theorem 4, this assumption will correspond to the controllability of the system (Σ_{wave}), with state space $H_0^1(\Omega) \times L^2(\Omega)$ and control space $L^2(\Omega)$.

The proof of Theorem 13 relies on the link between a wave packets condition (first introduced in [11]) and resolvent estimates (13) as stated by Miller in [42, Proposition 2.6]. Thus, the proof relies on a wave packet condition for the abstract Kirchhoff system (14) which is deduced from a similar condition for an abstract Schrödinger system. To this end, one needs first to show that (13) holds for a system described by an abstract plate equation.

Proposition 15. *With the notation and assumptions in Theorem 13, there exists a constant* $\gamma_0 > 0$ *such that*

$$\frac{1}{\omega}\left\|\left(\omega^2 I - A_0^2\right)\varphi\right\|_{\mathscr{H}}^2 + \|\omega C_0\varphi\|_Y^2 \geqslant \gamma_0\|\omega\varphi\|_{\mathscr{H}}^2 \qquad \left(\omega > 0,\ \varphi \in \mathscr{D}\left(A_0^2\right)\right). \tag{20}$$

Proof. Since the pair $(\widetilde{A}, \widetilde{C})$ is exactly observable, applying the Hautus type test in [38] (see also [44, Proposition 4.5]), it follows that there exists a constant $\gamma_0 > 0$ such that

$$\left\|(\omega I - A_0)\varphi\right\|_{\mathscr{H}}^2 + \omega\| C_0\varphi\|_Y^2 \geqslant \gamma_0\omega\|\varphi\|_{\mathscr{H}}^2 \qquad \left(\omega > 0,\ \varphi \in \mathscr{D}(A_0)\right). \tag{21}$$

On the other hand, using the fact that $A_0 > 0$ it follows that

$$\left\|\left(\omega^2 I - A_0^2\right)\varphi\right\|_{\mathscr{H}}^2 = \left\|(\omega I + A_0)(\omega I - A_0)\varphi\right\|_{\mathscr{H}}^2 \geqslant \omega^2\left\|(\omega I - A_0)\varphi\right\|_{\mathscr{H}}^2 \qquad \left(\omega > 0,\ \varphi \in \mathscr{D}\left(A_0^2\right)\right).$$

The last estimate and (21) imply the conclusion (20). ☐

As a consequence of the above result we can prove a wave packets condition for the abstract Schrödinger equation.

Proposition 16. *With the notation and assumptions in Theorem 13, let* $(\lambda_n)_{n\in\mathbb{N}^*}$ *be the nondecreasing sequence formed by the eigenvalues of* A_0 *and let* $(\phi_n)_{n\in\mathbb{N}^*}$ *be a corresponding sequence of eigenvectors, forming an orthonormal basis of* $\mathscr{H}$*. Moreover, for every* $\omega,\ r > 0$ *and* $\varepsilon \in (0, \frac{1}{2})$ *we set*

$$I_r(\omega) = \left\{m \in \mathbb{N}^* \text{ such that } |\lambda_m - \omega| < r\right\}, \tag{22}$$

$$r_\varepsilon(\omega) = \begin{cases} \omega^{\frac{1}{2}-\varepsilon} & \left(\omega \geqslant \omega_{0,\varepsilon}\right), \\ \min\left\{\dfrac{\omega}{2}, \dfrac{\rho_\varepsilon}{3}\right\} & \omega \in \left(0, \omega_{0,\varepsilon}\right), \end{cases} \tag{23}$$

where

$$\omega_{0,\varepsilon} = \max\left\{1, \left(\frac{18}{\gamma_0}\right)^{\frac{1}{2\varepsilon}}\right\}$$

(with γ_0 *is the constant in* (20)*) and*

$$\rho_\varepsilon = \inf\left\{\left|\lambda - \mu\right|; \lambda \neq \mu \text{ eigenvalues of } A_0 \text{ in } (0, 2\omega_{0,\varepsilon})\right\}.$$

(Notice that $\rho_\varepsilon > 0$ *because there is only a finite number of eigenvalues of* A_0 *in* $(0, 2\omega_{0,\varepsilon})$*.) Then, for every* $\varepsilon \in (0, \frac{1}{2})$*, there exists* $\gamma_1 > 0$ *such that we have*

$$\|C_0\varphi\|_Y \geqslant \gamma_1\|\varphi\|_{\mathscr{H}} \qquad \left(\omega \geqslant 0,\ \varphi \in \operatorname{span}\{\phi_k\}_{k\in I_{r_\varepsilon(\omega)}(\omega)}\right). \tag{24}$$

Proof. Let $\varepsilon \in (0, \frac{1}{2})$. For the sake of clarity, in this proof, the dependency of $\omega_{0,\varepsilon}$ and ρ_ε with respect to ε is not mentioned. For $\omega \geqslant \omega_0$, we consider φ of the form

$$\varphi = \sum_{m\in I_{r_\varepsilon}(\omega)} c_m\phi_m. \tag{25}$$

Then we clearly have that $\varphi \in \mathscr{D}(A_0^2)$ and

$$\left\|\left(\omega^2 I - A_0^2\right)\varphi\right\|_{\mathscr{H}}^2 = \sum_{m\in I_{r_\varepsilon(\omega)}(\omega)} \left|\omega^2 - \lambda_m^2\right|^2 |c_m|^2 \leqslant \omega^{1-2\varepsilon} \sum_{m\in I_{r_\varepsilon(\omega)}(\omega)} |\omega + \lambda_m|^2|c_m|^2. \tag{26}$$

On the other hand, it is clear that for every $\omega \geqslant 1$ and $m \in I_{r_\varepsilon(\omega)}(\omega)$ we have

$$0 < \omega + \lambda_m < 2\omega + r_\varepsilon(\omega) \leqslant 3\omega.$$

The last inequality and (26) imply that for every φ of the form (25), we have

$$\left\|\left(\omega^2 I - A_0^2\right)\varphi\right\|_{\mathcal{H}}^2 \leqslant 9\omega^{1-2\varepsilon}\|\omega\varphi\|_{\mathcal{H}}^2 \leqslant \frac{\gamma_0\omega}{2}\|\omega\varphi\|_{\mathcal{H}}^2,$$

for $\omega \geqslant \omega_0$. By applying (20) to φ of the form (25), the above estimate leads to (24) for every $\omega \geqslant \omega_0$.

Moreover, by construction of r_ε, for every $\omega \in (0,\omega_0)$, there exists $\lambda = \lambda(\omega)$ such that

$$\operatorname{span}\{\phi_k\}_{k\in I_{r_\varepsilon(\omega)}(\omega)} = \operatorname{Ker}(A_0 - \lambda I).$$

Indeed, if there exist $n, m \in I_{r_\varepsilon(\omega)}(\omega)$ such that $\lambda_n \neq \lambda_m$,

$$\rho \leqslant |\lambda_n - \lambda_m| \leqslant |\lambda_n - \omega| + |\lambda_m - \omega| \leqslant 2r_\varepsilon(\omega) \leqslant \frac{2\rho}{3},$$

which is a contradiction. Therefore, since $C_0\varphi \neq 0$ for every eigenfunction φ of A_0 (because the pair $(\widetilde{A},\widetilde{C})$ is observable), it follows that for every $\omega \in (0,\omega_0)$, there exists $\gamma_1 = \gamma_1(\omega) > 0$ such that

$$\|C_0\varphi\|_Y \geqslant \gamma_1\|\varphi\|_{\mathcal{H}} \qquad \varphi \in \operatorname{span}\left\{\phi_k\right\}_{k\in I_{r_\varepsilon(\omega)}(\omega)}.$$

Finally, using the fact that A_0 has a finite number of eigenvalues in $(0,\omega_0)$, the constant γ_1 can be chosen uniformly with respect to $\omega \in (0,\omega_0)$, giving (24). □

Remark 17. The wave packets condition (24) on the pair (A_0, C_0) allows us to prove the existence of constants $M, m > 0$ such that

$$\frac{M^2}{\omega^{1-2\varepsilon}}\|(\omega I - A_0)z_0\|_{\mathcal{H}}^2 + m^2\|Cz_0\|_Y^2 \geqslant \|z_0\|_{\mathcal{H}}^2 \qquad (\omega \in \mathbb{R},\ z_0 \in \mathcal{D}(A_0)), \tag{27}$$

using the link between wave packets condition and resolvent estimates as stated by Miller in [42, Proposition 2.6] and recalled in Proposition 12. Taking $A_0 = -\Delta$ with $\mathcal{D}(A_0) = H^2(\Omega)\cap H_0^1(\Omega)$ and $C : \varphi \mapsto \varphi\chi_{\mathcal{O}}$, this gives a Hautus type condition (13) for the Schrödinger equation which is weaker than the one already proven in [41, Proof of Theorem 3.4] or [47, Section 6.7] and used in the proof of Proposition 3, in Section 3.

Finally, one can deduce the wave packets condition for the abstract Kirchhoff equation, which leads us to the proof the main result of this section.

Proof of Theorem 13. Let $\mathcal{A}$ be the operator defined in (15), let $(\lambda_n)_{n\in\mathbb{N}^*}$ be the nondecreasing sequence formed by the eigenvalues of A_0 (repeated according to their multiplicity) and let $(\phi_n)_{n\in\mathbb{N}^*}$ be the corresponding eigenvectors of A_0 forming an orthonormal basis of $\mathcal{H}_{-1}$. We set $\phi_{-n} = -\phi_n$ for all $n \in \mathbb{N}^*$. Then (see, for instance, [47, Proposition 3.7.7]) the eigenvalues of $\mathcal{A}$ are $(i\mu_n)_{n\in\mathbb{Z}^*}$ with $\mu_n = \lambda_n$ if $n > 0$ and $\mu_n = -\lambda_{-n}$ if $n < 0$. Moreover, there is in $\mathcal{H}\times\mathcal{H}_{-1}$ an orthonormal basis formed of eigenvectors of $\mathcal{A}$, given by

$$\psi_n = \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{i\mu_n}\phi_n\\ \phi_n\end{bmatrix} \qquad \left(n \in \mathbb{Z}^*\right). \tag{28}$$

With the above notation and introducing, for all $\omega \in \mathbb{R}$ and $r > 0$, the sets

$$J_r(\omega) = \left\{m \in \mathbb{Z}^* \text{ such that } |\mu_m - \omega| < \varepsilon\right\}, \tag{29}$$

we remark that for every $\varepsilon \in (0,\frac{1}{2})$, if the function r_ε defined in (23) is extended by $r_\varepsilon(\omega) = r_\varepsilon(-\omega)$ for $\omega < 0$, we have

$$J_{r_\varepsilon(\omega)}(\omega) = \operatorname{sign}(\omega)\, I_{r_\varepsilon(|\omega|)}(|\omega|) \tag{30}$$

where I_r has been defined in (22). From (28) and (30), it follows that if $\psi = \left[\begin{smallmatrix}\eta\\ \varphi\end{smallmatrix}\right] \in \mathcal{D}(\mathcal{A})$ is in $\operatorname{span}\{\psi_k\}_{k\in J_{r_\varepsilon(\omega)}(\omega)}$, then $\eta \in \operatorname{span}\{\phi_k\}_{k\in I_{r_\varepsilon(|\omega|)}(|\omega|)}$, $\|\psi\|_{\mathcal{H}\times\mathcal{H}_{-1}} = \sqrt{2}\|\eta\|_{\mathcal{H}}$, and $\|C\psi\|_Y = \|C_0\eta\|_Y$. This facts and Proposition 16 imply that

$$\|C\psi\|_Y \geqslant \gamma_1\|\eta\|_{\mathcal{H}} = \frac{\gamma_1}{\sqrt{2}}\|\psi\|_{\mathcal{H}\times\mathcal{H}_{-1}} \qquad \left(\omega \in \mathbb{R},\ \psi = \begin{bmatrix}\eta\\ \varphi\end{bmatrix} \in \operatorname{span}\left\{\psi_k\right\}_{k\in J_{r_\varepsilon(\omega)}(\omega)}\right).$$

The above estimate implies the announced conclusion by applying the frequency-dependent Hautus test given in Miller [42, Theorem 2.16] and recalled in Proposition 12. □

5. Perturbation of abstract Kirchhoff systems

The goal of this section is to use the resolvent estimate (19) to study the robustness of the exact observability property for a system described by an abstract plate equation, with respect to bounded (but not necessarily compact) perturbations of the generator. Notice that the similar result for the perturbed Schrödinger equation given in Proposition 3 has already been dealt with in Section 3.

We continue in this section to use the notation introduced in the previous one. More precisely, $\mathscr{H}$ and Y are Hilbert spaces, $A_0 : \mathscr{D}(A_0) \to \mathscr{H}$ is a positive operator with compact resolvents, and $C_0 \in \mathscr{L}(\mathscr{H}, Y)$. If needed, the spaces $\mathscr{H}$ and Y are identified with their duals. Moreover, if V is another Hilbert space with continuous embedding $V \subset \mathscr{H}$, the dual of V is identified with its dual using the pivot space $\mathscr{H}$. For $\alpha > 0$ we still denote by $\mathscr{H}_\alpha$ the space $\mathscr{D}(A_0^\alpha)$ endowed with the graph norm of A_0^α and we define $\mathscr{H}_{-\alpha}$ as the dual of $\mathscr{H}_\alpha$ with respect to the pivot space $\mathscr{H}$. Moreover, we set $\mathscr{H}_0 := \mathscr{H}$ and $\mathscr{A}$ still is the operator defined in (15). Recall that for every $\alpha \in \mathbb{R}$ we can extend (or restrict) A_0 to a unitary operator from $\mathscr{H}_\alpha$ onto $\mathscr{H}_{\alpha-1}$. With a slight abuse of notation, we shall still denote by A_0 this extension (or restriction).

The main result of this section is:

Theorem 18. *With the notation and assumptions in Theorem 13, assume that*

$$P_0 \in \mathscr{L}(\mathscr{H}, \mathscr{H}_{-1}) \cap \mathscr{L}(\mathscr{H}_1, \mathscr{H})$$

is a symmetric operator on $\mathscr{H}_{-1}$, with domain $\mathscr{H}$. Let $P := \left[\begin{smallmatrix} 0 & 0 \\ P_0 & 0 \end{smallmatrix}\right] \in \mathscr{L}(\mathscr{H} \times \mathscr{H}_{-1})$ and let $\mathscr{A}_P : \mathscr{D}(\mathscr{A}_P) \to \mathscr{H} \times \mathscr{H}_{-1}$ be the operator defined by

$$\mathscr{D}(\mathscr{A}_P) = \mathscr{D}(\mathscr{A}), \qquad \mathscr{A}_P = \mathscr{A} - P. \tag{31}$$

Moreover, let $C \in \mathscr{L}(\mathscr{H} \times \mathscr{H}_{-1})$ be defined by $C = \begin{bmatrix} C_0 & 0 \end{bmatrix}$ and suppose that

$$\operatorname{Ker}\left(s^2 I + A_0^2 + P_0\right) \cap \operatorname{Ker} C_0 = \{0\} \qquad (s \in \mathbb{C}). \tag{32}$$

Then the system, with state space $\mathscr{H} \times \mathscr{H}_{-1}$ and output space Y, described by the pair $(\mathscr{A}_P, C)$ is exactly observable in any time $\tau > 0$.

The proof of Theorem 18 partially relies on a series of results resenting similarities with those in [47, Section 7.3]. For the sake of completeness, we give the detailed proofs below.

The first result of this series can be seen as a variation of [47, Proposition 7.3.3].

Proposition 19. *With the above notation, $\psi = \left[\begin{smallmatrix} \eta \\ \varphi \end{smallmatrix}\right] \in \mathscr{D}(\mathscr{A}_P)$ is an eigenvector of $\mathscr{A}_P$, associated to the eigenvalue $i\mu$, if and only if η is an eigenvector of $A_0^2 + P_0$, associated to the eigenvalue μ^2, and $\varphi = i\mu\eta$ (note that μ does not have to be real).*

Proof. Suppose that $\mu \in \mathbb{C}$ and $\left[\begin{smallmatrix} \eta \\ \varphi \end{smallmatrix}\right] \in \mathscr{D}(\mathscr{A}_P) \setminus \{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\}$ are such that $\mathscr{A}_P \left[\begin{smallmatrix} \eta \\ \varphi \end{smallmatrix}\right] = i\mu \left[\begin{smallmatrix} \eta \\ \varphi \end{smallmatrix}\right]$. According to the definition of $\mathscr{A}_P$ this is equivalent to

$$\varphi = i\mu\eta \quad \text{and} \quad -A_0^2\eta - P_0\eta = i\mu\varphi = -\mu^2\eta.$$ □

Clearly, $A_0^2 + P_0$, with domain $\mathscr{H}_1$, is self-adjoint on $\mathscr{H}_{-1}$ and it has compact resolvents. According to a classical result (see, for instance, [47, Proposition 3.2.12]) it follows that $A_0^2 + P_0$ is diagonalizable with an orthonormal basis $(\widetilde{\phi}_k)_{k \in \mathbb{N}^*}$ in $\mathscr{H}_{-1}$ formed of eigenvectors of $A_0^2 + P_0$ and with the corresponding family of real eigenvalues $(\widetilde{\lambda}_k)_{k\in\mathbb{N}^*}$ satisfying $\lim_{k\to\infty} |\widetilde{\lambda}_k| = \infty$. Moreover, since for all $z \in \mathcal{H}_1$,

$$\begin{aligned}\left\langle \left(A_0^2 + P_0\right) z, z\right\rangle_{\mathcal{H}_{-1}} &\geqslant \|A_0 z\|^2_{\mathcal{H}_{-1}} - \|P_0\|_{\mathcal{L}(\mathcal{H},\mathcal{H}_{-1})}\|z\|_{\mathcal{H}}\|z\|_{\mathcal{H}_{-1}}\\ &\geqslant \frac{1}{2}\|z\|^2_{\mathcal{H}} - \frac{1}{2}\|P_0\|^2_{\mathcal{L}(\mathcal{H},\mathcal{H}_{-1})}\|z\|^2_{\mathcal{H}_{-1}},\end{aligned}$$

it follows that $\lim_{k\to\infty}\widetilde{\lambda}_k = +\infty$. Hence, without loss of generality, we may assume that the sequence $(\widetilde{\lambda}_k)_{k\in\mathbb{N}^*}$ is nondecreasing. We extend the sequence $(\widetilde{\phi}_k)_{k\in\mathbb{N}^*}$ to a sequence indexed by $\mathbb{Z}^*$ by setting $\widetilde{\phi}_k = -\widetilde{\phi}_{-k}$ for every $k \in \mathbb{Z}_-$. We introduce the real sequence $(\mu_k)_{k\in\mathbb{Z}^*}$ by

$$\mu_k = \sqrt{|\widetilde{\lambda}_k|} \ \text{ if } \ k > 0 \quad \text{and} \quad \mu_k = -\mu_{-k} \ \text{ if } \ k < 0.$$

We denote by

$$W_0 = \operatorname{span}\left\{ \begin{bmatrix} \frac{1}{i\,\mathrm{sign}(k)}\widetilde{\phi}_k \\ \widetilde{\phi}_k \end{bmatrix} \,\middle|\, k \in \mathbb{Z}^*,\ \mu_k = 0 \right\}.$$

If $\mathrm{Ker}(A_0^2 + P_0) = \{0\}$ then of course W_0 is the zero subspace of $\mathcal{H}\times\mathcal{H}_{-1}$. Let $N \in \mathbb{N}^*$ be such that $\widetilde{\lambda}_N > 0$. We denote by

$$W_N = \operatorname{span}\left\{ \begin{bmatrix} \frac{1}{i\mu_k}\widetilde{\phi}_k \\ \widetilde{\phi}_k \end{bmatrix} \,\middle|\, k \in \mathbb{Z}^*,\ |k| < N,\ \mu_k \neq 0 \right\},$$

and define $Y_N = W_0 + W_N$. We also introduce the space

$$V_N = \overline{\operatorname{span}\left\{ \begin{bmatrix} \frac{1}{i\mu_k}\widetilde{\phi}_k \\ \widetilde{\phi}_k \end{bmatrix} \,\middle|\, |k| \geqslant N \right\}}, \tag{33}$$

the closure being taken in $\mathcal{H}\times\mathcal{H}_{-1}$.

Lemma 20. *With the above notation, we have $\mathcal{H}\times\mathcal{H}_{-1} = Y_N \oplus V_N$. Moreover, Y_N and V_N are invariant under the semigroup $\mathbb{T}$ generated by $\mathcal{A}_P$ on $\mathcal{H}\times\mathcal{H}_{-1}$.*

Proof. We adapt below the proof of [47, Lemma 7.3.4].

First, to prove that $\mathcal{H}\times\mathcal{H}_{-1} = Y_N\oplus V_N$, one can show that $Y_N = V_N^\perp$ for a suitable inner product to be defined. To deal with the fact that $A_0^2 + P_0$ is not a positive operator, we introduce a new operator A_1, whose eigenfunctions are the same as the one of $A_0^2 + P_0$, but its eigenvalues are all positive. More precisely, let $A_1 : \mathcal{H}_1 \to \mathcal{H}_{-1}$ be defined by

$$A_1 f = \sum_{\widetilde{\lambda}_k = 0} \langle f, \widetilde{\phi}_k\rangle_{\mathcal{H}}\widetilde{\phi}_k + \sum_{\widetilde{\lambda}_k\neq 0} |\widetilde{\lambda}_k|\langle f, \widetilde{\phi}_k\rangle_{\mathcal{H}}\widetilde{\phi}_k \qquad \left(f \in \mathcal{H}_1\right). \tag{34}$$

Since the family $(\widetilde{\phi}_k)_{k\in\mathbb{N}^*}$ is an orthonormal basis in $\mathcal{H}_{-1}$ and each $\widetilde{\phi}_k$ is an eigenvector of A_1, it follows that A_1 is diagonalizable. Moreover, since the eigenvalues of A_1 are positive, it follows that $A_1 > 0$. Following line by line the proof of [47, Proposition 3.4.9], it can be checked that the inner product on $\mathcal{H}\times\mathcal{H}_{-1}$ defined by

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_1 = \langle f_1, f_2\rangle_{\mathcal{H}} + \left\langle A_1^{-\frac{1}{2}} g_1, A_1^{-\frac{1}{2}} g_2 \right\rangle_{\mathcal{H}} \qquad \left(\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \in \mathcal{H}\times\mathcal{H}_{-1} \right), \tag{35}$$

is equivalent to the original one, meaning that it induces a norm equivalent to the original one.

Let $\mathcal{A}_1$ be the operator on $\mathcal{H}\times\mathcal{H}_{-1}$ defined by

$$\mathcal{D}(\mathcal{A}_1) = \mathcal{H}_1 \times \mathcal{H}, \qquad \mathcal{A}_1 = \begin{bmatrix} 0 & I \\ -A_1 & 0 \end{bmatrix}.$$

According to [47, Proposition 3.7.6], $\mathcal{A}_1$ is skew-adjoint on $\mathcal{X}$ (if endowed with the inner product $\langle\cdot,\cdot\rangle_1$ defined in (35)). Thus, [47, Proposition 3.7.7] entails that $Y_N = V_N^\perp$ (for the inner product $\langle\cdot,\cdot\rangle_1$) giving that $\mathcal{H}\times\mathcal{H}_{-1} = Y_N \oplus V_N$.

We next prove that Y_N and V_N are invariant under the semigroup $\mathbb{T}$ generated by $\mathscr{A}_P$. First, using the fact that $\widetilde{\lambda}_k > 0$ for every $k > N$ and Proposition 19, it follows that V_N is a closed subspace spanned by a set of eigenvectors of $\mathscr{A}_P$, thus is invariant under the action of $\mathbb{T}$. To prove that W_0 is also invariant under the action of $\mathbb{T}$, one can notice that for every k in $\mathbb{Z}^*$ such that $\mu_k = 0$,

$$\mathscr{A}_P \begin{bmatrix} \frac{1}{i\operatorname{sign}(k)}\widetilde{\phi}_k \\ \widetilde{\phi}_k \end{bmatrix} = \begin{bmatrix} \widetilde{\phi}_k \\ 0 \end{bmatrix} = \frac{i\operatorname{sign}(k)}{2}\left(\begin{bmatrix} \frac{1}{i\operatorname{sign}(k)}\widetilde{\phi}_k \\ \widetilde{\phi}_k \end{bmatrix} + \begin{bmatrix} \frac{1}{i\operatorname{sign}(-k)}\widetilde{\phi}_{-k} \\ \widetilde{\phi}_{-k} \end{bmatrix}\right) \in W_0.$$

To prove that W_N is invariant under the action of $\mathbb{T}$, one can first notice that for every k in $\mathbb{Z}^*$, $|k| < N$ such that $\mu_k \neq 0$, $\mu_k^2 = \operatorname{sign}(\widetilde{\lambda}_{|k|})\widetilde{\lambda}_{|k|}$ and $[A_0^2 + P_0]\widetilde{\phi}_k = \widetilde{\lambda}_{|k|}\widetilde{\phi}_k$. Thus, one gets that

$$\mathscr{A}_P \begin{bmatrix} \frac{1}{i\mu_k}\widetilde{\phi}_k \\ \widetilde{\phi}_k \end{bmatrix} = \begin{bmatrix} \widetilde{\phi}_k \\ \frac{\tilde{\lambda}_{|k|}}{i\mu_k}\widetilde{\phi}_k \end{bmatrix} = i\mu_k \begin{bmatrix} \frac{1}{i\mu_k}\widetilde{\phi}_k \\ -\operatorname{sign}(\widetilde{\lambda}_{|k|})\widetilde{\phi}_k \end{bmatrix} \in W_N,$$

because

$$\begin{bmatrix} \frac{1}{i\mu_k}\widetilde{\phi}_k \\ -\widetilde{\phi}_k \end{bmatrix} = \begin{bmatrix} \frac{1}{i\mu_{-k}}\widetilde{\phi}_{-k} \\ \widetilde{\phi}_{-k} \end{bmatrix} \in W_N.$$

Finally, $Y_N = W_0 + W_N$ is invariant under the action of $\mathbb{T}$. □

We are now in a position to prove the main result of this section.

Proof of Theorem 18. We first note that Theorem 13 implies that

$$M_1^2(\omega)\,\|(i\omega I - \mathscr{A}_P - P)\,z_0\|^2_{\mathscr{H}\times\mathscr{H}_{-1}} + m_1^2\|Cz_0\|_Y^2 \geqslant \|z_0\|^2_{\mathscr{H}\times\mathscr{H}_{-1}} \qquad (\omega \in \mathbb{R},\ z_0 \in \mathscr{D}(\mathscr{A}))\,.$$

Using an elementary inequality, we obtain that

$$\begin{aligned} 2M_1^2(\omega)\,\|(i\omega I - \mathscr{A}_P)\,z_0\|^2_{\mathscr{H}\times\mathscr{H}_{-1}} + 2M_1^2(\omega)\|P_0\|^2_{\mathscr{L}(\mathscr{H},\mathscr{H}_{-1})}\|z_0\|^2_{\mathscr{H}\times\mathscr{H}_{-1}} \\ + m_1^2\|Cz_0\|_Y^2 \geqslant \|z_0\|^2_{\mathscr{H}\times\mathscr{H}_{-1}} \qquad (\omega \in \mathbb{R},\ z_0 \in \mathscr{D}(\mathscr{A}))\,. \end{aligned}$$

Since we know from Theorem 13 that $M_1(\omega) \to 0$ when $|\omega| \to \infty$, it follows that for every $\gamma > 0$ there exists $c_\gamma > 0$ such that

$$\gamma^2\,\|(i\omega I - \mathscr{A}_P)z_0\|^2_{\mathscr{H}\times\mathscr{H}_{-1}} + 2m_1^2\|Cz_0\|_Y^2 \geqslant \|z_0\|^2_{\mathscr{H}\times\mathscr{H}_{-1}} \qquad \big(|\omega| \geqslant c_\gamma,\ z_0 \in \mathscr{D}(\mathscr{A})\big). \tag{36}$$

Moreover, using the inner product (35) associated with the operator A_1 defined in (34) (which is equivalent to the original one), (36) implies that for every $\gamma > 0$ there exist $\widetilde{c}_\gamma,\ m_\gamma > 0$ such that

$$\gamma^2\,\|(i\omega I - \mathscr{A}_P)\,z_0\|_1^2 + m_\gamma^2\|Cz_0\|_Y^2 \geqslant \|z_0\|_1^2 \qquad \big(|\omega| \geqslant \widetilde{c}_\gamma,\ z_0 \in \mathscr{D}(\mathscr{A})\big). \tag{37}$$

For $N \in \mathbb{N}^*$ such that $\widetilde{\lambda}_N > 0$, we denote by $\mathscr{A}_{P,N}$ the part of $\mathscr{A}_P$ in V_N, where V_N is the space defined in (33). Since $\mathscr{A}_{P,N}$ coincides with the part of $\mathscr{A}_1$ in V_N, it follows that $\mathscr{A}_{P,N}$ is skew-adjoint on V_N (endowed with the inner product $\langle\cdot,\cdot\rangle_1$). Moreover, using (37), it follows that, for every $\gamma > 0$, there exist $\widetilde{c}_\gamma,\ m_\gamma > 0$ such that the following estimate holds

$$\gamma^2\,\big\|\big(i\omega I - \mathscr{A}_{P,N}\big)\,z_0\big\|_1^2 + m_\gamma^2\|Cz_0\|_Y^2 \geqslant \|z_0\|_1^2 \qquad \big(|\omega| \geqslant \widetilde{c}_\gamma,\ z_0 \in \mathscr{D}(\mathscr{A}) \cap V_N\big). \tag{38}$$

Since $\mathscr{A}_{P,N}$ is skew-adjoint (thus normal) on V_N, it follows that there exists $N_\gamma \in \mathbb{N}^*$ such that

$$\left\|\left(i\omega I - \mathscr{A}_{P,N_\gamma}\right)z_0\right\|_1 \geqslant \gamma^{-1}\|z_0\|_1 \qquad \left(|\omega| < \widetilde{c}_\gamma,\ z_0 \in \mathscr{D}(\mathscr{A}) \cap V_{N_\gamma}\right).$$

The above estimate and (38) imply that for every $\gamma > 0$ there exist $m_\gamma > 0$ and $N_\gamma \in \mathbb{N}^*$ such that

$$\gamma^2\left\|\left(i\omega I - \mathscr{A}_{P,N_\gamma}\right)z_0\right\|_1^2 + m_\gamma^2\|Cz_0\|_Y^2 \geqslant \|z_0\|_1^2 \qquad \left(\omega \in \mathbb{R},\ z_0 \in \mathscr{D}(\mathscr{A}) \cap V_{N_\gamma}\right). \tag{39}$$

The above estimate and the fact that $\mathscr{A}_{P,N_\gamma}$ is skew-adjoint imply, according to the Hautus-type test for systems with skew-adjoint generator proved in Miller [41] (see also [47, Theorem 6.6.1]), that the pair $(\mathscr{A}_{P,N_\gamma}, C_{N_\gamma})$, where C_{N_γ} is the restriction of C to V_{N_γ}, is exactly observable in any time $\tau > \gamma\pi$.

Denoting by $\widetilde{\mathscr{A}}_{P,N_\gamma}$ the part of $\mathscr{A}_P$ in Y_{N_γ} and by $\widetilde{C}_{N_\gamma}$ the restriction of C to Y_{N_γ}, we obtain that the finite-dimensional system $(\widetilde{\mathscr{A}}_{P,N_\gamma}, \widetilde{C}_{N_\gamma})$ is observable by applying the classical Hautus test thanks to (32). Since $\widetilde{\mathscr{A}}_{P,N_\gamma}$ and $\mathscr{A}_{P,N_\gamma}$ have no common eigenvalues and $(\mathscr{A}_{P,N_\gamma}, C_{N_\gamma})$ is exactly observable in any time larger than $\gamma\pi$, we can apply [47, Theorem 6.4.2] to obtain that $(\mathscr{A}_P, C)$ is exactly observable in any time $\tau > \gamma\pi$. Since $\gamma > 0$ can be arbitrarily small, this implies the conclusion of the theorem. □

As a consequence of Theorem 18, we can obtain a second perturbation result. More precisely, the result below shows that the exact observability property still holds if, besides the perturbation P_0, we add a perturbation Q_0 whose contribution is compact with respect to the topology of the state space.

Corollary 21. *With the notation and assumptions in Theorem 18, let $Q_0 \in \mathscr{L}(\mathscr{H}, \mathscr{H}_{-1})$ be a compact operator and let $Q = \left[\begin{smallmatrix} 0 & 0 \\ Q_0 & 0 \end{smallmatrix}\right] \in \mathscr{L}(\mathscr{H} \times \mathscr{H}_{-1})$. Let $\mathscr{A}_{PQ} : \mathscr{D}(\mathscr{A}_{PQ}) \to \mathscr{H} \times \mathscr{H}_{-1}$ be the operator defined by*

$$\mathscr{D}\left(\mathscr{A}_{PQ}\right) = \mathscr{D}(\mathscr{A}), \qquad \mathscr{A}_{PQ} = \mathscr{A} - P - Q. \tag{40}$$

Then $\mathscr{A}_{PQ}$ generates a C^0-semigroup $\mathbb{T}$ on $\mathscr{H} \times \mathscr{H}_{-1}$. Moreover, assuming that

$$\operatorname{Ker}\left(s^2 I + A_0^2 + P_0 + Q_0\right) \cap \operatorname{Ker} C_0 = \{0\} \qquad (s \in \mathbb{C}), \tag{41}$$

the pair $(\mathscr{A}_{PQ}, C)$ is exactly observable in any time $\tau > 0$.

Proof. The fact that $\mathscr{A}_{PQ}$ generates a C^0-semigroup on $\mathscr{H} \times \mathscr{H}_{-1}$ follows from the obvious property $P + Q \in \mathscr{L}(\mathscr{H} \times \mathscr{H}_{-1})$. Moreover, we can remark that the result in Proposition 19 holds for every $P_0 \in \mathscr{L}(\mathscr{H}, \mathscr{H}_{-1})$ (no symmetry of P_0 is needed), thus, in particular, if we replace P_0 by $P_0 + Q_0$. It follows that $\psi = \left[\begin{smallmatrix} \eta \\ \varphi \end{smallmatrix}\right] \in \mathscr{D}(\mathscr{A}_{PQ})$ is an eigenvector of $\mathscr{A}_{PQ}$, associated to the eigenvalue $i\mu$, if and only if η is an eigenvector of $A_0^2 + P_0 + Q_0$, associated to the eigenvalue μ^2, and $\varphi = i\mu\eta$. This fact and (41) imply that

$$\operatorname{Ker}(sI - \mathscr{A}_{PQ}) \cap \operatorname{Ker} C = \{0\} \qquad (s \in \mathbb{C}).$$

We also note that, under our assumptions, $Q \in \mathscr{L}(\mathscr{H} \times \mathscr{H}_{-1})$ is a compact operator. Moreover, we know from Theorem 18 that the pair $(\mathscr{A}_P, C)$, with $\mathscr{A}_P$ defined in (31), is exactly observable in any time $\tau > 0$. Since $\mathscr{A}_{PQ} = \mathscr{A}_P - Q$, the conclusion follows now using the duality of the exact observability and of exact controllability properties and by applying [20, Theorem 1.2] to deal with the compact perturbation using a compactness-uniqueness method. □

By duality, Corollary 21 yields the following exact controllability result:

Corollary 22. *With the notation and assumptions in Corollary 21, let $R_0 \in \mathscr{L}(\mathscr{H}_1, \mathscr{H})$ be the operator defined by*

$$\langle R_0\varphi, \psi\rangle_{\mathscr{H}} = \langle \varphi, Q_0\psi\rangle_{\mathscr{H}_1, \mathscr{H}_{-1}} \qquad \left(\varphi \in \mathscr{H}_1,\ \psi \in \mathscr{H}\right). \tag{42}$$

Then the equation

$$\ddot{w}(t) + A_0^2 w(t) + (P_0 + R_0) w(t) = C_0^* u(t) \qquad (t \geqslant 0), \tag{43}$$

determines a well-posed control system with state space $\mathscr{H}_1 \times \mathscr{H}$ and input space Y. Moreover, this system is exactly controllable in arbitrarily small time.

Proof. Recall that the Hilbert spaces $\mathscr{H}$ and Y are identified with their duals. Moreover, if V is another Hilbert space, with continuous embedding $V \subset \mathscr{H}$, the dual of V is identified with its dual using the pivot space $\mathscr{H}$.

We next consider, for every $\tau > 0$, the input map $\Phi_\tau \in \mathscr{L}(L^2([0,\tau]; Y), \mathscr{H}_1 \times \mathscr{H})$ defined by

$$\Phi_\tau u = \begin{bmatrix} w(\tau) \\ \dot{w}(\tau) \end{bmatrix} \qquad \left(u \in L^2([0,\tau]; Y)\right), \tag{44}$$

where w is the unique solution of (43) satisfying the initial conditions $w(0) = 0$ and $\dot{w}(0) = 0$. In order to write $\Phi_\tau' \in \mathscr{L}(\mathscr{H}_{-1} \times \mathscr{H}, L^2([0,\tau];Y))$ in a convenient manner we consider the system

$$\begin{cases} \ddot{y}(t) + A_0^2 y(t) + P_0 y(t) + Q_0 y(t) = 0 & (t \in [0,\tau]), \\ y(\tau) = y_1,\ \dot{y}(\tau) = -y_0, \end{cases} \tag{45}$$

where Q_0 is defined by (42). We first assume that $y_1 \in \mathscr{H}_2$ and $y_0 \in \mathscr{H}_1$ so that

$$y \in C([0,\tau];\mathscr{H}_2) \cap C^1([0,\tau];\mathscr{H}_1) \cap C^2([0,\tau];\mathscr{H}).$$

We also (temporarily) assume that $u \in H^1([0,\tau];Y)$ and $u(0) = 0$, so that the solution of (43) with zero initial data satisfies

$$w \in C([0,\tau];\mathscr{H}_2) \cap C^1([0,\tau];\mathscr{H}_1) \cap C^2([0,\tau];\mathscr{H}).$$

The above regularity properties for y and w allow us to take the inner product in $L^2([0,\tau];\mathscr{H})$ of all the terms in (43) by y. In particular, integrating twice by parts with respect to time, we have

$$\int_0^\tau \langle \ddot{w}(t), y(t) \rangle \,\mathrm{d}t = \langle \dot{w}(\tau), y_1 \rangle + \langle w(\tau), y_0 \rangle + \int_0^\tau \langle w(t), \ddot{y}(t) \rangle \,\mathrm{d}t. \tag{46}$$

Moreover, we have

$$\int_0^\tau \langle A_0^2 w(t), y(t) \rangle \,\mathrm{d}t = \int_0^\tau \langle w(t), A_0^2 y(t) \rangle \,\mathrm{d}t. \tag{47}$$

On the other hand, from (5) and (42) it follows that

$$\int_0^\tau \langle P_0 w(t), y(t) \rangle \,\mathrm{d}t = \int_0^\tau \langle w(t), P_0 y(t) \rangle \,\mathrm{d}t, \tag{48}$$

$$\int_0^\tau \langle R_0 w(t), y(t) \rangle \,\mathrm{d}t = \int_0^\tau \langle w(t), Q_0 y(t) \rangle \,\mathrm{d}t. \tag{49}$$

Summing up (46)-(49) and using (45), (43) it follows that for every $y_1 \in \mathscr{H}_2$, $y_0 \in \mathscr{H}_1$ and $u \in H^1([0,\tau];Y)$, with $u(0) = 0$, we have

$$\int_0^\tau \langle u(t), C_0 y(t) \rangle_Y \,\mathrm{d}t = \langle w(\tau), y_0 \rangle + \langle \dot{w}(\tau), y_1 \rangle.$$

By a simple density argument, it follows that

$$\int_0^\tau \langle u(t), C_0 y(t) \rangle_Y \,\mathrm{d}t = \langle A_0 w(\tau), A_0^{-1} y_0 \rangle + \langle \dot{w}(\tau), y_1 \rangle.$$

for every $y_1 \in \mathscr{H}$, $y_0 \in \mathscr{H}_{-1}$ and $u \in L^2([0,\tau];Y)$. By combining the last formula and (44) it follows that

$$\left\langle \Phi_\tau u, \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \right\rangle_{\mathscr{H}_1 \times \mathscr{H}, \mathscr{H}_{-1} \times \mathscr{H}} = \int_0^\tau \langle u(t), C_0 y(t) \rangle_Y \,\mathrm{d}t,$$

for every $y_1 \in \mathscr{H}$, $y_0 \in \mathscr{H}_{-1}$ and $u \in L^2([0,\tau];Y)$. We have thus shown that

$$\left(\Phi_\tau' \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}\right)(t) = C_0 y(t) \qquad \left(y_0 \in \mathscr{H}_{-1}, y_1 \in \mathscr{H}, t \in [0,\tau]\right), \tag{50}$$

where y satisfies (45).

On the other hand, from (45), it is clear that $\left[\begin{smallmatrix} y(t) \\ \dot{y}(t) \end{smallmatrix}\right] = \mathbb{T}_{t-\tau}\left[\begin{smallmatrix} y_1 \\ -y_0 \end{smallmatrix}\right]$, where $\mathbb{T}$ is the C^0-group introduced in Corollary 21. By combining (50) and Corollary 21 it follows that there exists a constant $K_\tau > 0$ such that

$$K_\tau \left\| \Phi_\tau' \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \right\|_{L^2([0,\tau];Y)} \geqslant \left\| \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \right\|_{\mathscr{H}_{-1} \times \mathscr{H}} \qquad \left(y_0 \in \mathscr{H}_{-1}, y_1 \in \mathscr{H}\right).$$

Using a classical result (see, for instance, Barnes [4, Theorem 7]) it follows that Φ_τ is onto from $L^2([0,\tau];Y)$ to $\mathscr{H}_1 \times \mathscr{H}$, which implies the announced exact controllability result. □

6. Proof of the main result on linear systems

The goal of this section is to prove Theorem 4 on the controllability of the perturbations of a plate equation. Within this section, we specify the spaces $\mathscr{H}$, Y and the operators A_0 and C_0 which have been introduced in an abstract context in Sections 4 and 5. More precisely, we set:

- $\mathscr{H} = L^2(\Omega)$, where Ω is an open bounded set of $\mathbb{R}^n$, with $\partial\Omega$ of class C^3 or Ω is a rectangular domain;
- $-A_0$ is the Dirichlet Laplacian on $L^2(\Omega)$. More precisely,
$$\mathscr{D}(A_0) = H^2(\Omega) \cap H_0^1(\Omega), \tag{1}$$
$$A_0\varphi = -\Delta\varphi \qquad \big(\varphi \in \mathscr{D}(A_0)\big); \tag{2}$$
The operator A_0 is positive with compact resolvents;
- $Y = L^2(\Omega)$ and $C_0 \in \mathscr{L}(\mathscr{H}, Y)$ is defined by
$$C_0\varphi = \varphi\chi_{\mathscr{O}} \qquad (\varphi \in \mathscr{H}), \tag{3}$$
where $\mathscr{O}$ is an open subset of Ω and $\chi_{\mathscr{O}} \in L^\infty(\Omega)$ is a nonnegative function which is positive on $\mathscr{O}$.

With $\mathscr{H}$ and A_0 chosen above, it is known (see, for instance, [47, Section 3.6]) that
$$\mathscr{H}_2 = \big\{\varphi \in H^4(\Omega) \cap H_0^1(\Omega) \ \mid \ \Delta\varphi = 0 \ \text{ on } \ \partial\Omega\big\}, \quad \mathscr{H}_1 = H^2(\Omega) \cap H_0^1(\Omega).$$
Moreover, we have
$$\mathscr{H}_{-1} = \big[H^2(\Omega) \cap H_0^1(\Omega)\big]',$$
where $[H^2(\Omega) \cap H_0^1(\Omega)]'$ is the dual of $H^2(\Omega) \cap H_0^1(\Omega)$ with respect to the pivot space $L^2(\Omega)$.

Proof of Theorem 4. The proof consists in applying Corollary 22 with the appropriate choice of spaces and operators. We first remark that, since the system (Σ_{wave}) described by the wave equation is exactly controllable in some time, a standard duality argument implies that the pair $(\widetilde{A}, \widetilde{C})$ defined in (18) is exactly observable in some time. Thus, since the spaces $\mathscr{H}$, Y, the operators A_0, C_0 and the spaces $\mathscr{H}_\alpha$ have been specified in the preamble of this section, it only remains to define the operators P_0 and R_0.

Let $P_0 \in \mathscr{L}(\mathscr{H}_1, \mathscr{H})$ be the operator defined by
$$P_0\varphi = \sum_{k,\ell=1}^{n} a_{k\ell} \frac{\partial^2\varphi}{\partial x_k \partial x_\ell} \qquad \big(\varphi \in \mathscr{H}_1\big). \tag{4}$$
Using (2) and the fact that $(a_{k\ell})_{1 \leqslant k,\ell \leqslant n}$ are real-valued, it is easy to check that P_0 is well-defined and
$$\langle P_0\varphi, \psi\rangle_{\mathscr{H}} = \langle \varphi, P_0\psi\rangle_{\mathscr{H}} \qquad \big(\varphi,\ \psi \in \mathscr{H}_1\big).$$
Moreover, the above formula implies that
$$\big|\langle P_0\varphi, \psi\rangle_{\mathscr{H}}\big| \leqslant \sum_{k,\ell=1}^{n} \|a_{k\ell}\|_{L^\infty(\Omega)} \|\varphi\|_{\mathscr{H}} \|\psi\|_{\mathscr{H}_1} \qquad \big(\varphi,\ \psi \in \mathscr{H}_1\big).$$
It follows that P_0 can be uniquely extended to an unbounded symmetric operator on $\mathscr{H}_{-1}$ (still denoted by P_0), with domain $\mathscr{H}$, by setting
$$\langle P_0\varphi, \psi\rangle_{\mathscr{H}} = \langle \varphi, P_0\psi\rangle_{\mathscr{H}_1, \mathscr{H}_{-1}} \qquad \big(\varphi \in \mathscr{H}_1,\ \psi \in \mathscr{H}\big). \tag{5}$$
Let $R_0 \in \mathscr{L}(\mathscr{H}_1, \mathscr{H})$ be the operator defined by
$$R_0\varphi = \sum_{k=1}^{n} b_k \frac{\partial\varphi}{\partial x_k} + c\varphi \qquad \big(\varphi \in \mathscr{H}_1\big).$$
An integration by parts shows that
$$\langle R_0\varphi, \psi\rangle_{\mathscr{H}} = \langle \varphi, Q_0\psi\rangle_{\mathscr{H}} \qquad \big(\varphi,\ \psi \in \mathscr{H}_1\big),$$

where

$$Q_0\psi = -\operatorname{div}\left(\overline{b}\psi\right) + \overline{c}\psi \qquad \left(\psi \in \mathscr{H}_1\right). \tag{6}$$

From the last two formulas, it follows that Q_0 can be extended uniquely to a compact operator (still denoted by Q_0) in $\mathscr{L}(\mathscr{H}, \mathscr{H}_{-1})$.

To conclude using Corollary 22, we still have to check the unique continuation properties (32) and (41). More precisely, we need to prove that for $\varepsilon \in \{0,1\}$, $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\mu \in \mathbb{C}$ we have

$$\left.\begin{array}{ll} \mu^2\psi + \Delta^2\psi + P_0\psi + \varepsilon Q_0\psi = 0 & \text{in } \Omega, \\ \psi = 0,\ \Delta\psi = 0 & \text{on } \partial\Omega, \\ \psi = 0 & \text{in } \mathscr{O}, \end{array}\right\} \quad \Rightarrow \quad \psi = 0. \tag{7}$$

This unique continuation is a direct consequence of the Carleman estimate given in Theorem 34 of Appendix A. Indeed, denote

$$g = -P_0\psi - \varepsilon Q_0\psi - \mu^2\psi.$$

Applying Theorem 34, there exists a function $\beta \in C^2(\overline{\Omega})$ and a positive constant $C > 0$ such that for all $s \geqslant 1$,

$$\int_\Omega \left(s^2 \left|D^2\psi\right|^2 + s^4|\nabla\psi|^2 + s^6|\psi|^2\right) e^{2s\beta}\,\mathrm{d}x \leqslant C\int_\Omega |g|^2 e^{2s\beta}\,\mathrm{d}x, \tag{8}$$

where $D^2\psi$ designs the Hessian matrix of ψ, $|\cdot|$ stands for the euclidian norm on finite dimensional spaces and we have used the fact that $\psi = 0$ in $\mathscr{O}$. Moreover, using the definition of the operators P_0 and Q_0 given in (4) and (6), one can easily check that, for all $x \in \Omega$,

$$|g(x)| \leqslant \max_{k,\ell=1,\dots,n} \|a_{k\ell}\|_{L^\infty(\Omega)} \left|D^2\psi(x)\right| + \varepsilon \max_{k=1,\dots,n} \|b_k\|_{L^\infty(\Omega)} |\nabla\psi(x)| + \left(\varepsilon\|c\|_{L^\infty(\Omega)} + |\mu|^2\right)|\psi(x)|.$$

Therefore, this estimate combined with (8) implies that for every $s \geqslant 1$,

$$\begin{aligned} \int_\Omega \left(s^2\left|D^2\psi\right|^2 + s^4|\nabla\psi|^2 + s^6|\psi|^2\right)e^{2s\beta} \leqslant C\Big(&\max_{k,\ell=1,\dots,n}\|a_{k\ell}\|^2_{L^\infty(\Omega)} \int_\Omega \left|D^2\psi\right|^2 e^{2s\beta} \\ &+ \varepsilon^2 \max_{k=1,\dots,n}\|b_k\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla\psi|^2 e^{2s\beta} + \left(\varepsilon^2\|c\|^2_{L^\infty(\Omega)} + |\mu|^4\right)\int_\Omega |\psi|^2 e^{2s\beta}\Big). \end{aligned}$$

Taking s large enough in the last inequality, we obtain that $\psi = 0$, which concludes the proof. □

7. Proof of Theorem 6

The main ingredient in the proof of Theorem 6 is an exact controllability result for the system obtained by linearizing (5) around the stationary state $(\eta, 0)$. To write down this system, we insert the formula

$$w(t,x) = \eta(x) + \varepsilon\delta(t,x) \qquad \left(t \geqslant 0,\ x \in \overline{\Omega}\right),$$

in (5) and we develop in a power series with respect to ε. Identifying the terms of order 1, we obtain the system:

$$\begin{cases} \ddot{\delta}(t,x) + \Delta^2\delta(t,x) + [\delta, \Phi(\eta,\eta)] + 2[\eta, \Phi(\eta,\delta)] = u(t,x)\chi_{\mathscr{O}}(x) & (t \geqslant 0,\ x \in \Omega), \\ \delta(t,x) = \Delta\delta(t,x) = 0 & (t \geqslant 0,\ x \in \partial\Omega), \\ \delta(0,x) = \delta_0(x),\ \dot{\delta}(0,x) = \delta_1(x) & (x \in \Omega), \end{cases} \tag{1}$$

where the Airy stress function Φ is the solution of (6). The main result in this section is the following.

Theorem 23. *Assume that Ω, $\mathscr{O}$ and η satisfy the assumptions in Theorem 6. Then* (1) *determines a well-posed control system with state space*

$$\left[H^2(\Omega) \cap H_0^1(\Omega)\right] \times L^2(\Omega),$$

and control space $L^2(\Omega)$. Moreover, this system is exactly controllable in any time $\tau > 0$.

To write (1) as a well-posed control system, we have to introduce some spaces and operators. To this aim, we first recall some properties of the Airy stress function Φ defined in (6) given in [12, Corollary 1.4.4].

Proposition 24. *For every $p \in [1,\infty]$, the Airy stress function Φ defines a bounded bilinear operator from $H^2(\Omega)\times H^2(\Omega)$ to $W^{2+\frac{2}{p},p}(\Omega)\cap H_0^2(\Omega)$. In particular, there exists a positive constant K_Ω such that*

$$\|\Phi(v,w)\|_{W^{2,\infty}(\Omega)} \leqslant K_\Omega \|v\|_{H^2(\Omega)}\|w\|_{H^2(\Omega)} \qquad \big(v,\ w\in H^2(\Omega)\big). \tag{2}$$

Then, we also recall the following property of the von Kármán bracket defined in (7), given in [12, Proposition 1.4.5].

Proposition 25. *For any $u \in H^2(\Omega)$ and $v, w \in H^2\cap H_0^1(\Omega)$, the following relation holds*

$$\int_\Omega [u,v]\,w\mathrm{d}x = \int_\Omega [u,w]\,v\mathrm{d}x.$$

Corollary 26. *For every $\eta\in H^2(\Omega)$, the mapping $P_{0,1}:\delta\mapsto[\eta,\Phi(\eta,\delta)]$ defines a linear bounded operator from $H^2\cap H_0^1(\Omega)$ to $L^2(\Omega)$. Moreover, $P_{0,1}$ can be extended to an operator in $\mathscr{L}(L^2(\Omega),[H^2\cap H_0^1(\Omega)]')$ and is symmetric on $[H^2\cap H_0^1(\Omega)]'$.*

Proof. The fact that for every $\eta\in H^2(\Omega)$, we have $P_{0,1}\in\mathscr{L}(H^2(\Omega),L^2(\Omega))$ is a direct consequence of (2). Moreover, for every $\delta,\psi\in H^2\cap H_0^1(\Omega)$,

$$\langle P_{0,1}\delta,\psi\rangle_{L^2(\Omega)} = \langle \delta, P_{0,1}\psi\rangle_{L^2(\Omega)}. \tag{3}$$

To prove this relation, introduce the operator A_D defined by

$$\begin{cases}\mathscr{D}(A_D)=H^4(\Omega)\cap H_0^2(\Omega),\\ A_D\varphi=\Delta^2\varphi \qquad \big(\varphi\in\mathscr{D}(A_D)\big).\end{cases}$$

This operator is known to be positive on $L^2(\Omega)$ and the definition of $P_{0,1}$ can be rewritten as

$$P_{0,1}\delta=\left[\eta,A_D^{-1}[\eta,\delta]\right] \qquad \big(\delta\in H^2(\Omega)\cap H_0^1(\Omega)\big).$$

Using Proposition 25 and the self-adjointness of A_D^{-1}, it follows that, for every $\delta,\psi\in H^2\cap H_0^1(\Omega)$,

$$\begin{aligned}\langle P_{0,1}\delta,\psi\rangle_{L^2(\Omega)} &= \langle \left[\eta,A_D^{-1}[\eta,\delta]\right],\psi\rangle_{L^2(\Omega)} = \langle A_D^{-1}[\eta,\delta],[\eta,\psi]\rangle_{L^2(\Omega)}\\ &= \langle [\eta,\delta],A_D^{-1}[\eta,\psi]\rangle_{L^2(\Omega)} = \langle \delta,\left[\eta,A_D^{-1}[\eta,\psi]\right]\rangle_{L^2(\Omega)} = \langle\delta,P_{0,1}\psi\rangle_{L^2(\Omega)}.\end{aligned}$$

Then, the relation (3) and the continuity of $P_{0,1}$ from $H^2\cap H_0^1(\Omega)$ to $L^2(\Omega)$ imply that there exists $C>0$ such that for all $\delta,\psi\in H^2\cap H_0^1(\Omega)$,

$$\left|\langle P_{0,1}\delta,\psi\rangle_{L^2(\Omega)}\right| \leqslant C\|\delta\|_{L^2(\Omega)}\|\psi\|_{H^2\cap H_0^1(\Omega)}.$$

Therefore, $P_{0,1}$ can be extended uniquely to an operator in $\mathscr{L}(L^2(\Omega),[H^2\cap H_0^1(\Omega)]')$ (still denoted by $P_{0,1}$) with

$$\langle P_0\varphi,\psi\rangle_{L^2(\Omega)} = \langle\varphi,P_0\psi\rangle_{H^2\cap H_0^1(\Omega),[H^2\cap H_0^1(\Omega)]'} \qquad \big(\varphi\in H^2\cap H_0^1(\Omega),\ \psi\in L^2(\Omega)\big).$$

□

Proposition 27. *For all $\eta\in W^{2,\infty}(\Omega)$, we define the operator $P_{0,2}\in\mathscr{L}(H^2\cap H_0^1(\Omega),L^2(\Omega))$ by*

$$P_{0,2}\delta=\left[\delta,\Phi(\eta,\eta)\right] \qquad \big(\delta\in H^2\cap H_0^1(\Omega)\big).$$

Then, there exist functions $(a_{k\ell})_{1\leqslant k,\ell\leqslant 2}$ in $H^2(\Omega)$ such that

$$P_{0,2}\delta=\sum_{k,\ell=1}^{2} a_{k\ell}\frac{\partial^2\delta}{\partial x_k\partial x_\ell} \qquad \big(\delta\in H^2\cap H_0^1(\Omega)\big), \tag{4}$$

with

$$\begin{cases} a_{k\ell} = a_{\ell k} \in H^2(\Omega) & (k,\ \ell \in \{1,2\}), \\ \sum_{\ell=1}^{2} \frac{\partial a_{k\ell}}{\partial x_\ell} = 0 & (k \in \{1,2\}). \end{cases} \tag{5}$$

Moreover, $P_{0,2}$ can be extended to an operator in $\mathscr{L}(L^2(\Omega), [H^2 \cap H_0^1(\Omega)]')$ which is symmetric on $[H^2 \cap H_0^1(\Omega)]'$.

Finally, if η is analytic, then the functions $(a_{k\ell})_{1\leqslant k,\ell\leqslant 2}$ in (4) *are also analytic.*

Proof. The definition of $P_{0,2}$ implies that (4) holds with

$$a_{11} = \frac{\partial^2}{\partial x_2^2}\Phi(\eta,\eta), \qquad a_{12} = a_{21} = -\frac{\partial^2}{\partial x_1 \partial x_2}\Phi(\eta,\eta), \qquad a_{22} = \frac{\partial^2}{\partial x_1^2}\Phi(\eta,\eta). \tag{6}$$

The fact that $(a_{k\ell})_{1\leqslant k,\ell\leqslant 2}$ satisfies (5) is a direct consequence of (6). The regularity of $(a_{k\ell})_{1\leqslant k,\ell\leqslant 2}$ follows from the elliptic regularity: As η is in $W^{2,\infty}(\Omega)$, $[\eta,\eta]$ is in $L^2(\Omega)$ and thus, $\Phi(\eta,\eta)$ is in $H^4(\Omega)$.

The fact that $P_{0,2}$ is in $\mathscr{L}(\mathscr{H},\mathscr{H}_{-1}) \cap \mathscr{L}(\mathscr{H}_1,\mathscr{H})$ and it is symmetric on $\mathscr{H}_{-1}$, can be checked as in the proof of Theorem 4, in Section 6. ☐

We are now in a position to prove the main result in this section.

Proof of Theorem 23. To prove Theorem 23, we apply Corollary 22

with the spaces $\mathscr{H}, Y$, the operators A_0 and C_0 given at the beginning of Section 6 and $P_0 = P_{0,1} + P_{0,2}$, with $P_{0,1}$ and $P_{0,2}$ defined in Corollary 26 and Proposition 27, respectively.

Moreover, since the system (Σ_{wave}) is supposed to be exactly controllable in some time, we can use a standard duality argument (see Remark 14), to deduce that the pair $(\widetilde{A}, \widetilde{C})$ defined in (18) is exactly observable in some time.

Therefore, to apply Corollary 22, it remains to prove the unique continuation (32): if φ is the solution of

$$\begin{aligned} &s^2\varphi + \Delta^2\varphi + P_{0,1}\varphi + P_{0,2}\varphi = 0 && \text{in } \Omega, \\ &\varphi = 0,\ \Delta\varphi = 0 && \text{on } \partial\Omega, \\ &\varphi = 0, && \text{in } \mathscr{O}, \end{aligned}$$

for some $s \in \mathbb{C}$, then $\varphi = 0$. Using (4), the above property is equivalent to proving that if φ and Γ satisfy for some $s \in \mathbb{C}$

$$\begin{cases} s^2\varphi + \Delta^2\varphi + \sum_{k,\ell=1}^2 a_{k\ell}\frac{\partial^2\varphi}{\partial x_k \partial x_\ell} + [\eta,\Gamma] = 0 & \text{in } \Omega, \\ \varphi = 0,\ \Delta\varphi = 0 & \text{on } \partial\Omega, \\ \varphi = 0, & \text{in } \mathscr{O}, \end{cases} \tag{7}$$

and

$$\begin{cases} \Delta^2\Gamma = [\eta,\varphi] & \text{in } \Omega, \\ \Gamma = 0,\ \frac{\partial\Gamma}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

then $\varphi = 0$. This follows from the fact that φ is analytic on Ω, which in turn is a consequence of the analyticity of η and of the coefficients $(a_{k\ell})_{1\leqslant k,\ell\leqslant 2}$ in Ω (see Proposition 27) and of the classical results in [46, Section 4.1.4] or [30, Ch.7]. ☐

Remark 28. The analyticity of η, assumed in Theorem 6, is used only to ensure the following unique continuation property: if φ and Γ satisfy (7)–(8) for some $s \in \mathbb{C}$, then φ vanishes everywhere (and thus Γ too). This unique continuation property may hold with different assumptions on η. One could, for instance, use the algebraic resolubility method of Gromov to give sufficient algebraic conditions on the derivatives of η guaranteeing that the unique continuation holds for (7)–(8). More precisely, these conditions would require that a large determinant involving derivatives of η to be non zero, similarly to Condition (1.8) in Duprez and Lissy [18, Theorem 1.2] (see also [17, 19] or [37, Ex. 1, Section 1.3, p. 18–19]). Nevertheless, we have no reason to think that this unique continuation property holds for any η smooth enough. However, due to the the above considerations, we conjecture that this property generically holds for smooth η.

We next consider the nonlinear controlled system

$$\ddot{\delta}(t,x)+\Delta^2\delta(t,x)+[\delta,\Phi(\eta,\eta)](t,x)+2[\eta,\Phi(\delta,\eta)](t,x)+[\eta,\Phi(\delta,\delta)](t,x) \\ +2[\delta,\Phi(\eta,\delta)](t,x)+[\delta,\Phi(\delta,\delta)](t,x)=u(t,x)\chi_{\mathscr{O}}(x) \qquad (t\geqslant 0,\ x\in\Omega), \quad (9)$$

with the boundary conditions and initial conditions

$$\delta(t,x)=\Delta\delta(t,x)=0 \qquad (t\geqslant 0,\ x\in\partial\Omega), \quad (10)$$

$$\delta(0,x)=\delta_0(x), \quad \dot{\delta}(0,x)=\delta_1(x) \qquad (x\in\Omega). \quad (11)$$

It is easily seen that Theorem 6 (with $w=\delta+\eta$) directly follows from the result below.

Theorem 29. *Under the assumptions in Theorem 6, for every $\tau>0$ there exists $\alpha>0$ such that for every*

$$\delta_0\in H^2(\Omega)\cap H_0^1(\Omega), \qquad \delta_1\in L^2(\Omega),$$

with

$$\|\delta_0\|_{H^2(\Omega)}+\|\delta_1\|_{L^2(\Omega)}\leqslant\alpha,$$

there exists $u\in L^2([0,\tau];L^2(\Omega))$ such that the solution of (9)-(11) *satisfies*

$$\delta(\tau,\cdot)=0, \qquad \dot{\delta}(\tau,\cdot)=0.$$

Proof. Let $\tau>0$. In this proof, for convenience, the dependency of the objects in this proof with respect to τ is not mentioned. First, from the exact controllability in time τ of the linearized equation (1), stated in Theorem 23, it follows that there exists a continuous linear operator

$$\mathscr{L}: \quad \left(H^2\cap H_0^1(\Omega)\right)\times L^2(\Omega)\times L^2\left([0,\tau];L^2(\Omega)\right) \quad\to\quad L^2\left([0,\tau];L^2(\Omega)\right)$$

such that for every $\delta_0\in H^2\cap H_0^1(\Omega)$, $\delta_1\in L^2(\Omega)$ and $g\in L^2([0,\tau];L^2(\Omega))$, the solution δ_g of

$$\begin{cases}\ddot{\delta}_g+\Delta^2\delta_g+\left[\delta_g,\Phi(\eta,\eta)\right]+2\left[\eta,\Phi(\eta,\delta_g)\right]=g+u_g\chi_{\mathscr{O}} & (t\geqslant 0,\ x\in\Omega),\\ \delta_g(t,x)=\Delta\delta_g(t,x)=0 & (t\geqslant 0,\ x\in\partial\Omega),\\ \delta_g(0,x)=\delta_0(x),\ \dot{\delta}_g(0,x)=\delta_1(x) & (x\in\Omega),\end{cases} \quad (12)$$

with $u_g=\mathscr{L}(\delta_0,\delta_1,g)$, satisfies

$$\delta_g(\tau,\cdot)=0 \quad\text{and}\quad \dot{\delta}_g(\tau,\cdot)=0. \quad (13)$$

Indeed, the solution δ_g of (12) can be written as $\delta_g=\delta_{g,\text{lin}}+\delta_{g,\text{cont}}$, where $\delta_{g,\text{lin}}$ is the solution of

$$\begin{cases}\ddot{\delta}_{g,\text{lin}}+\Delta^2\delta_{g,\text{lin}}+\left[\delta_{g,\text{lin}},\Phi(\eta,\eta)\right]+2\left[\eta,\Phi(\eta,\delta_{g,\text{lin}})\right]=g & (t\geqslant 0,\ x\in\Omega),\\ \delta_{g,\text{lin}}(t,x)=\Delta\delta_{g,\text{lin}}(t,x)=0 & (t\geqslant 0,\ x\in\partial\Omega),\\ \delta_{g,\text{lin}}(0,x)=\delta_0(x),\ \dot{\delta}_{g,\text{lin}}(0,x)=\delta_1(x) & (x\in\Omega),\end{cases}$$

and u_g is the control given by Theorem 23 such that the solution $\delta_{g,\text{cont}}$ of

$$\begin{cases}\ddot{\delta}_{g,\text{cont}}+\Delta^2\delta_{g,\text{cont}}+\left[\delta_{g,\text{cont}},\Phi(\eta,\eta)\right]+2\left[\eta,\Phi(\eta,\delta_{g,\text{cont}})\right]=u_g\chi_{\mathscr{O}} & (t\geqslant 0,\ x\in\Omega),\\ \delta_{g,\text{cont}}(t,x)=\Delta\delta_{g,\text{cont}}(t,x)=0 & (t\geqslant 0,\ x\in\partial\Omega),\\ \delta_{g,\text{cont}}(T,x)=0,\ \dot{\delta}_{g,\text{cont}}(T,x)=0 & (x\in\Omega),\end{cases}$$

satisfies

$$\delta_{g,\text{cont}}(0,x)=-\delta_{g,\text{lin}}(T,x),\ \dot{\delta}_{g,\text{cont}}(0,x)=-\dot{\delta}_{g,\text{lin}}(T,x) \qquad (x\in\Omega).$$

(Note that, since (1) determines a well-posed control system with state space $[H^2(\Omega)\cap H^1_0(\Omega)]\times L^2(\Omega)$ and control space $L^2(\Omega)$, the control can be chosen such that it depends continuously (in $L^2([0,\tau];L^2(\Omega))$) and linearly on the data to be controlled (in $[H^2(\Omega)\cap H^1_0(\Omega)]\times L^2(\Omega)$).)

Our goal is to prove the local exact controllability of the nonlinear system (9) via a fixed-point argument. To this aim, let $\delta_0 \in H^2\cap H^1_0(\Omega)$ and $\delta_1\in L^2(\Omega)$. We construct a map $\mathscr{G}: L^2([0,\tau];L^2(\Omega))\to L^2([0,\tau];L^2(\Omega))$ by setting, for $g\in L^2([0,\tau];L^2(\Omega))$,

$$\mathscr{G}(g)=\left[\eta,\Phi\left(\delta_g,\delta_g\right)\right]+2\left[\delta_g,\Phi\left(\eta,\delta_g\right)\right]+\left[\delta_g,\Phi\left(\delta_g,\delta_g\right)\right] \tag{14}$$

where δ_g is the solution of (12) with the source term g and the control

$$u_g=\mathscr{L}(\delta_0,\delta_1,g). \tag{15}$$

To conclude the proof of the theorem, it clearly suffices to check the existence of a fixed-point of $\mathscr{G}$.

Step 1: The map $\mathscr{G}$ is well-defined. First, using the property of the Airy function given in Proposition 24 and the definition (7) of the bracket $[\cdot,\cdot]$, there exists $C>0$ such that, for every $\delta_0\in H^2\cap H^1_0(\Omega)$, $\delta_1\in L^2(\Omega)$, $g\in L^2([0,\tau];L^2(\Omega))$,

$$\left\|\mathscr{G}(g)\right\|_{L^2([0,\tau];L^2(\Omega))}\leqslant C\sum_{i=2}^{3}\left(\|\delta_g\|_{C([0,\tau];H^2(\Omega))}+\|\dot{\delta}_g\|_{C([0,\tau];L^2(\Omega))}\right)^i.$$

Moreover, using the continuity of $\mathscr{L}$ (see (15)), it follows that there exists $C>0$ such that, for every $\delta_0\in H^2\cap H^1_0(\Omega)$, $\delta_1\in L^2(\Omega)$, $g\in L^2([0,\tau];L^2(\Omega))$, the solution δ_g of (12) satisfies

$$\|\delta_g\|_{C([0,\tau];H^2(\Omega))}+\|\dot{\delta}_g\|_{C([0,\tau];L^2(\Omega))}\leqslant C\left(\|\delta_0\|_{H^2(\Omega)}+\|\delta_1\|_{L^2(\Omega)}+\|g\|_{L^2([0,\tau];L^2(\Omega))}\right). \tag{16}$$

Combining the two previous estimates, one gets the existence of $C>0$ such that

$$\|\mathscr{G}(g)\|_{L^2([0,\tau];L^2(\Omega))}\leqslant C\sum_{i=2}^{3}\left(\|\delta_0\|^i_{H^2(\Omega)}+\|\delta_1\|^i_{L^2(\Omega)}+\|g\|^i_{L^2([0,\tau];L^2(\Omega))}\right). \tag{17}$$

Step 2: The map $\mathscr{G}$ maps B_r to itself. Let $C>0$ be the constant in (17). Let $r>0$ such that

$$C\left(r+r^2\right)<\frac{1}{2}, \tag{18}$$

and define the associated ball of $L^2([0,\tau];L^2(\Omega))$ by

$$B_r=\{g\in L^2\left([0,\tau];L^2(\Omega)\right);\ \|g\|_{L^2([0,\tau];L^2(\Omega))}\leqslant r\}.$$

Let $\alpha>0$ be such that

$$2C\left(\alpha^2+\alpha^3\right)<\frac{r}{2}, \tag{19}$$

and let $\delta_0\in H^2\cap H^1_0(\Omega)$ and $\delta_1\in L^2(\Omega)$ satisfy

$$\|\delta_0\|_{H^2(\Omega)}+\|\delta_1\|_{L^2(\Omega)}\leqslant\alpha. \tag{20}$$

Using (17), (18) and (19), it follows that for every $g\in B_r$ we have

$$\|\mathscr{G}(g)\|_{L^2([0,\tau];L^2(\Omega))}\leqslant 2C\left(\alpha^2+\alpha^3\right)+C\left(r^2+r^3\right)\leqslant r.$$

Consequently, the ball B_r in invariant under the action of $\mathscr{G}$.

Step 3: The map $\mathscr{G}$ is a contraction on B_r. Let r and α satisfy the conditions (18) and (19) introduced at Step 2.

We first remark that there exists $C>0$ such that for every $g_1,g_2\in B_r$ we have

$$\|\mathscr{G}(g_1)-\mathscr{G}(g_2)\|_{L^2([0,\tau];L^2(\Omega))}\leqslant Cr\left[\|\delta_{g_1}-\delta_{g_2}\|_{C([0,\tau];H^2(\Omega))}+\|\dot{\delta}_{g_1}-\dot{\delta}_{g_2}\|_{C([0,\tau];L^2(\Omega))}\right], \tag{21}$$

where δ_{g_i} is the solution of (12) with the source term g_i and the control u_{g_i}. To avoid repetitions, we detail the above estimate only for the first term in the definition (14) of $\mathscr{G}$. To this aim, we use Proposition 24, to obtain the existence of $C>0$ with

$$\begin{aligned}\left\|[\eta,\Phi(\delta_{g_1},\delta_{g_1})]-[\eta,\Phi(\delta_{g_2},\delta_{g_2})]\right\|_{L^2([0,\tau];L^2(\Omega))}\\ \leqslant C\Big[\left\|\delta_{g_1}-\delta_{g_2}\right\|_{C([0,\tau];H^2(\Omega))}+\left\|\dot\delta_{g_1}-\dot\delta_{g_2}\right\|_{C([0,\tau];L^2(\Omega))}\Big]\\ \times\Big[\left\|\delta_{g_1}+\delta_{g_2}\right\|_{C([0,\tau];H^2(\Omega))}+\left\|\dot\delta_{g_1}+\dot\delta_{g_2}\right\|_{C([0,\tau];L^2(\Omega))}\Big].\end{aligned}$$

Moreover, using (16) and (19) one gets that for every $g_1,g_2\in B_r$ we have

$$\left\|\delta_{g_1}+\delta_{g_2}\right\|_{C([0,\tau];H^2(\Omega))}+\left\|\dot\delta_{g_1}+\dot\delta_{g_2}\right\|_{C([0,\tau];L^2(\Omega))}\leqslant Cr.$$

The other nonlinear terms in $\mathscr{G}$ can be tackled in similar manner, leading to (21). Then, as before, using (16), one can deduce from (21) that, for all $g_1,g_2\in B_r$,

$$\left\|\mathscr{G}(g_1)-\mathscr{G}(g_2)\right\|_{L^2([0,\tau];L^2(\Omega))}\leqslant Cr\|g_1-g_2\|_{L^2([0,\tau];L^2(\Omega))}.$$

Hence, reducing r if needed, one gets that $\mathscr{G}$ is a strict contraction on B_r.

Conclusion. Thus, by the Banach fixed-point theorem, the map $\mathscr{G}$ has a fixed point, which concludes the proof as explained before. □

8. Comments and related questions

8.1. *Perturbed Schrödinger and plate equations on surfaces of variable curvature*

The aim of this subsection is to show that the general results from Section 5 can be applied to systems governed by the Schrödinger and plate type equations in other geometrical situations, namely without assuming that the system described by the corresponding abstract wave equation is exactly observable. To this purpose, let (M,g) be a compact smooth Riemannian manifold whose geodesic flow has the Anosov property. We refer to Dyatlov, Jin and Nonnemacher [21, Section 2.1] for the precise definition of this concept, recalling here just the fact that it includes the case of surfaces with negative Gauss curvature. The result below can be seen as a counterpart in a different geometrical setting of the result given in Proposition 3 in an Euclidian context. The remarkable fact is that, unlike in Proposition 3, the control region can be an arbitrary open set.

Proposition 30. *With the above notation, let Δ_g be the Laplace-Beltrami operator on M, let $\mathscr{O}$ be an open nonempty subset of M, let $\chi_{\mathscr{O}}$ be the indicator function of $\mathscr{O}$ and let $a\in L^\infty(M;\mathbb{R})$. Then the system described by*

$$\dot z(t,x)+i\Delta_g z(t,x)+ia(x)z(t,x)=u(t,x)\chi_{\mathscr{O}}(x)\qquad (t\geqslant 0,\ x\in M),\tag{22}$$

with state and control space $L^2(M)$, is exactly controllable in any positive time.

Proof. Let $-A_0$ be the Laplace-Beltrami operator on $\mathscr{H}=L^2(M)$. It is known that A_0 is a densely defined operator and positive on $\mathscr{H}$. Let $C\in\mathscr{L}(\mathscr{H})$ be the multiplication operator by $\chi_{\mathscr{O}}$. The major ingredient of this proof is the result in [21, Theorem 2], which implies the existence of $K=K_{M,\mathscr{O}}>0$, $m>0$ and $\omega_0>0$ such that

$$K^2\frac{\log(\omega)}{\omega}\|(\omega I-A_0)z_0\|^2_{\mathscr{H}}+m^2\|Cz_0\|^2_{\mathscr{H}}\geqslant\|z_0\|^2_{\mathscr{H}},\qquad (\omega\geqslant\omega_0,z_0\in\mathscr{D}(A_0)).\tag{23}$$

On the other hand, $C\phi\neq 0$ for every eigenvector ϕ of $-iA_0-P$, where P is the multiplication operator by ia (this is the unique continuation property for eigenvectors of the Laplace operator with potential, which is classical, see for instance [34, Chap. 21]). Thus, we can apply Proposition 10 to conclude that $(-iA_0-P,C)$ is exactly observable in any positive time. By duality, it follows that

$(iA_0 + P, C)$ with state and input space $\mathscr{H}$, is exactly controllable in any positive time, which is the announced conclusion. □

Remark 31. When $a = 0$, the exact controllability in any time of the Schrödinger equation (22) has already been proved in [21, Theorem 5] when M has the Anosov property and in [29, Theorem 1.3] when M is a compact hyperbolic surface.

To state a similar result for systems described by the plate equation we use the gradient and divergence operators on Riemannian manifolds, denoted by ∇_g and div_g, respectively. We refer, for instance, to Călin and Chang [10, Ch. 2] for the definition of these operators.

Proposition 32. *With the assumptions and notation in Proposition 30, let $b \in C^\infty(M;\mathbb{R})$ and $c \in L^\infty(M)$. Then the system described by*

$$\ddot{w}(t,x) + \Delta_g^2 w(t,x) + \operatorname{div}_g\left(b(x)\left(\nabla_g w\right)(t,x)\right) + c(x)w(t,x) = u(t,x)\chi_{\mathscr{O}}(x) \qquad (t \geqslant 0,\ x \in M), \quad (24)$$

with state space $H^2(M) \times L^2(M)$ and control space $L^2(M)$, is exactly controllable in any positive time.

Proof. Let $\mathscr{H} := L^2(M)$ and denote by $-A_0$ the Laplace-Beltrami operator on $\mathscr{H}$. Then A_0 is a positive operator on $\mathscr{H}$, with domain $\mathscr{H}_1 = H^2(M)$. Consider the linear operators

$$\begin{cases} C_0\varphi = \varphi\chi_{\mathscr{O}} & (\varphi \in \mathscr{H}), \\ P_0\varphi = \operatorname{div}_g\left(b\nabla_g\varphi\right) & (\varphi \in \mathscr{H}_1), \\ Q_0\varphi = \overline{c}\varphi & (\varphi \in \mathscr{H}). \end{cases}$$

It can be easily checked that, under our assumptions, we have that $P_0 \in \mathscr{L}(\mathscr{H}_1, \mathscr{H})$ is symmetric on $\mathscr{H}$ and that $Q_0 \in \mathscr{L}(\mathscr{H})$. Moreover, by applying [34, Chap. 21], it follows that

$$\operatorname{Ker}\left(s^2 I + A_0^2 + P_0\right) \cap \operatorname{Ker} C_0 = \operatorname{Ker}\left(s^2 I + A_0^2 + P_0 + Q_0\right) \cap \operatorname{Ker} C_0 = \{0\} \qquad (s \in \mathbb{C}),$$

To conclude, it would be sufficient to apply a result similar to Corollary 22 in order to obtain that the equation

$$\ddot{w}(t) + A_0^2 w(t) + (P_0 + R_0)w(t) = C_0^* u(t) \qquad (t \geqslant 0),$$

with $R_0 = Q_0^*$, determines a system, with state space $\mathscr{H}_1 \times \mathscr{H}$ and control space $\mathscr{H}$, which is exactly controllable in arbitrarily small time. This would imply, using the obvious facts that R_0 is just the multiplication operator by c and $C_0^* = C_0$, the announced conclusion.

The reason for which Corollary 22 cannot be directly applied is that it relies upon several preliminary results, namely those of Section 4 and Section 5. Nevertheless, the conclusion of this corollary holds here. Indeed, the only ingredient of the proof of Corollary 22 which cannot be adapted in an obvious manner to the context of the present proof is Theorem 13. However, we can easily show that the conclusion of Theorem 13 is still true in the context of the current proof. Indeed, the exact controllability of the corresponding wave equation, assumed in Theorem 13 and not necessarily holding here, is used in the proof of Theorem 13 only to obtain the resolvent estimate (21). This resolvent estimate can be replaced by (23), which holds in our context and which is clearly sufficient to obtain the conclusion of Theorem 13. We have thus shown that indeed the conclusion of Corollary 22 holds here, which ends our proof. □

8.2. *Small time controllability for the Berger plate equation*

In this subsection, we consider a system that can be seen as an asymptotic limit of the Von Kármán equations (see Perla Menzala and Zuazua [40], Nayfeh and Mook [43]) and we show that, using Corollary 22, we can easily improve the known results on the controllability of an associated system. More precisely, Berger's model for an elastic plate filling the domain $\Omega \subset \mathbb{R}^2$ and hinged on the boundary $\partial\Omega$ is:

$$\begin{cases} \ddot{w}(t,x) + \Delta^2 w(t,x) - \left(a + b\int_\Omega |\nabla w|^2 \,\mathrm{d}x\right)\Delta w(t,x) = f(x) + u\chi_{\mathscr{O}} & (t \geqslant 0,\ x \in \Omega), \\ w(t,x) = \Delta w(t,x) = 0 & (t \geqslant 0,\ x \in \partial\Omega), \\ w(0,x) = 0,\ \dot{w}(0,x) = 0 & (x \in \Omega). \end{cases} \tag{25}$$

In the above system, f is a given force field and a is a real constant. The constant b is supposed to be positive. Let η be a stationary solution corresponding to the forcing term f, i.e. satisfying

$$\begin{cases} \Delta^2\eta(x) - \left(a + b\int_\Omega |\nabla\eta|^2 \,\mathrm{d}x\right)\Delta\eta(x) = f(x) & (x \in \Omega), \\ \eta(x) = \Delta\eta(x) = 0 & (x \in \partial\Omega). \end{cases}$$

The main result in this subsection is:

Proposition 33. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with C^2 boundary and let $\mathscr{O} \subset \Omega$ be an open and nonempty subset of Ω such that* (Σ_{wave}) *(introduced in Section 1) is exactly controllable in some time. Then the nonlinear system* (25) *is locally exactly controllable in any positive time $\tau > 0$, i.e., for every $\tau > 0$ there exists $M > 0$ such that for every $\left[\begin{smallmatrix} w_0 \\ w_1 \end{smallmatrix}\right] \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$, with $\|w_0 - \eta\|^2_{H^2(\Omega)} + \|w_1\|^2_{L^2(\Omega)} \leqslant M^2$, there exists $u \in L^2([0,\tau]; L^2(\mathscr{O}))$ such that the solution w of* (25) *satisfies*

$$w(\tau,\cdot) = \eta, \quad \dot{w}(\tau,\cdot) = 0.$$

A weaker version of the above result, which was yielding the exact controllability in some (not necessarily small) time has been proved in [14], in the case $\eta = 0$. The fact that the arbitrarily small controllability time was not obtained in [14] is due to the following limitation: in the above-referred work, the exact controllability of the linearized system was proved only for large enough time (see Remark 5 above).

Sketch of the proof of Proposition 33. With the results of the present paper at hand, we can apply Theorem 4, with $a_{k\ell} = -\alpha\delta_{k\ell}$, where

$$\alpha = a + b\|\nabla\eta\|^2_{L^2(\Omega)},$$

$\delta_{k\ell}$ is the Kronecker symbol, $b_k = 0$ and $c = 0$, to obtain that the system obtained by linearizing (25) around η is exactly controllable in any positive time. Using this fact, the approach proposed in [14] can be easily adapted to obtain the local exact controllability of (33) in arbitrarily small time and thus proving Proposition 33. □

8.3. *Conclusions and open questions*

In this work we have developed a perturbation approach for abstract control systems described by Kirchhoff and Schrödinger type equations. More precisely, after writing the equations as a first-order system, the perturbations we have considered are bounded, but not necessarily compact, with respect to the natural topology of the state space. Consequently, a compactness-uniqueness based methodology cannot be directly applied. Nevertheless, we show that, under the assumption that the system described by the wave equation on the same spatial domain and with the same control operator is exactly controllable, we can apply a frequency domain perturbation argument. This methodology yields robustness of the exact controllability property with respect to this type of perturbation, with some extra assumptions (similar to those appearing in compactness-uniqueness approaches) on the unique continuation of the eigenvectors of the perturbed systems. The new results obtained on the systems obtained by perturbing the classical

Kirchhoff plate equation by terms involving up to second-order derivatives with respect to the space variable provide, in particular, enough estimates to tackle the local controllability for some nonlinear plate systems. More precisely, they allow to obtain the local, *around a sufficiently smooth equilibrium state, exact controllability in arbitrarily small time*, for systems described by the nonlinear von Kármán or Berger plate equations.

The main limitation of our approach is the systematic use of the assumption that the system described by the wave equation on the same spatial domain and with the same control operator is exactly controllable. In the case of systems described by the Schrödinger equation with homogeneous Dirichlet conditions in rectangular domains, this assumption has been removed in [9] and [7], provided that the perturbation preserves the skew-adjoint nature of the generator. We can conjecture that similar results hold for systems described by the plate equation in rectangular domains, opening the way to the local exact controllability of rectangular von Kármán plates with controls localized in an arbitrary control region. However, since in this case the generator of the corresponding first-order system is not skew-adjoint, adapting the arguments (which are not perturbation theory-based ones) from [9] and [7] to the case of systems described by the plate equation does not seem an obvious task.

Acknowledgements

The authors are grateful to Nicolas Burq for helpful suggestions and discussions. We are also indebted to the referee for her / his suggestions and comments on the first version of this work.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

Appendix A. A Carleman estimate for the bi-Laplacian

The goal of this section is to prove the global Carleman estimate for the bi-Laplacian which has been used in the proof of Theorem 4.

To give the precise statement of this result, we introduce some notation, which will be used in all the remaining part of this section. Firstly, given $n \in \mathbb{N}$, the euclidian norms on $\mathbb{C}^n$ and $\mathscr{M}_n(\mathbb{C})$ are denoted by $|\cdot|$. We denote by Ω a nonempty bounded open set of $\mathbb{R}^n$ with a C^2 boundary or a rectangular domain and by $\mathscr{O}$ an open and nonempty subset of Ω. Moreover, for $g \in H^2(\Omega)$, we write $(D^2 g)(x)$ and $\nabla g(x)$ for the Hessian matrix and the gradient of g at $x \in \Omega$, respectively.

For the remaining part of this section, let α be a $C^2(\overline{\Omega})$ function satisfying

$$\forall\, x \in \partial\Omega,\ \alpha(x) = 0, \quad \forall\, x \in \Omega,\ \alpha(x) > 0, \quad \text{and} \quad \inf_{\Omega \setminus \mathscr{O}} |\nabla \alpha| > 0. \tag{26}$$

The existence of a function α with the above properties has been proved in Fursikov and Imanuvilov [25] (see also [47, Chapter 14] or [16, Lemma 2.68]).

We are now in a position to state the main result in this section.

Theorem 34. *With the above notation and assumptions, there exist a constant $C > 0$ and $\widehat{\lambda} > 0$ such that for every $s \geqslant 1$ and every $\psi \in H^4(\Omega)$ satisfying $\psi = \Delta\psi = 0$ on $\partial\Omega$ we have*

$$\int_\Omega \left(s^2 \left|D^2\psi\right|^2 + s^4 \left|\nabla\psi\right|^2 + s^6 |\psi|^2 \right) e^{2s\beta}\, \mathrm{d}x \\ \leqslant C \left(\int_\Omega \left|\Delta^2\psi\right|^2 e^{2s\beta}\, \mathrm{d}x + \int_{\mathscr{O}} \left[s^3 |\Delta\psi|^2 + s^6 |\psi|^2 \right] e^{2s\beta}\, \mathrm{d}x \right), \tag{27}$$

where β is given by $\beta = e^{\hat{\lambda}\alpha}$, with $\alpha \in C^2(\overline{\Omega})$ as in (26).

We refer to [33, Section 1.4, Proposition 1.1] for the proof of the optimality of the powers of the parameter s in the Carleman estimate (27).

Remark 35. The following global Carleman estimate for the bi-Laplacian is known from [15, Proposition 1]: for every $a > 0$, there exist $\widehat{s} \geqslant 1$, $\widehat{\lambda} > 0$ and a constant $C > 0$ such that for every $s \geqslant \widehat{s}$, for every $\psi \in H^4(\Omega)$ satisfying $\psi = \Delta\psi = 0$ on $\partial\Omega$,

$$\int_\Omega \left(s^4|\nabla\psi|^2 + s^6|\psi|^2\right)e^{2s\beta}\,\mathrm{d}x \leqslant C\left(\int_\Omega \left|\Delta^2\psi - a\Delta\psi\right|^2 e^{2s\beta}\,\mathrm{d}x + \int_{\mathscr{O}}\left[s\left|\nabla(\Delta\psi)\right|^2 + s^3|\Delta\psi|^2 + s^4|\nabla\psi|^2 + s^6|\psi|^2\right]e^{2s\beta}\,\mathrm{d}x\right), \tag{28}$$

where $\beta = e^{\hat{\lambda}\alpha}$ with $\alpha \in C^2(\overline{\Omega})$ as in (26). However, (28) is not sufficient to prove the unique continuation (7), since one needs to consider the case $a = 0$ and to estimate all the second order derivatives of ψ from the right hand side of the Carleman estimate (27) or (28).

The strategy used to prove Theorem 34 is the same as the one in [15, Proposition 1]. More precisely, we apply twice a global Carleman estimate for the Laplacian, which is deduced from Imanuvilov [27, Lemma 2.7]:

Theorem 36. *With the notation and assumptions in Theorem 34, there exist $\widehat{\lambda} > 0$ and a constant $C > 0$ such that for every $s \geqslant 1$, for every $y \in H^2(\Omega) \cap H^1_0(\Omega)$ we have*

$$\int_\Omega \left(\frac{1}{s}\left|D^2y\right|^2 + s|\nabla y|^2 + s^3|y|^2\right)e^{2s\beta}\,\mathrm{d}x \leqslant C\left(\int_\Omega \left|\Delta y\right|^2 e^{2s\beta}\,\mathrm{d}x + s^3\int_{\mathscr{O}}|y|^2 e^{2s\beta}\,\mathrm{d}x\right), \tag{29}$$

where $\beta = e^{\hat{\lambda}\alpha}$ with $\alpha \in C^2(\overline{\Omega})$ as in (26).

Proof. Let $T > 0$. From [27, Lemma 2.7], one gets the existence of $\widehat{\lambda} > 0$ such that for every $\lambda \geqslant \widehat{\lambda}$ there exist $\widehat{s} \geqslant 1$ and $C > 0$ such that for every $s \geqslant \widehat{s}$, $t \in (0, T)$ and $y \in H^2(\Omega) \cap H^1_0(\Omega)$,

$$\int_\Omega \left(\frac{1}{s\beta_{T,\lambda}(t)}\left|D^2y\right|^2 + s\lambda^2\beta_{T,\lambda}(t)|\nabla y|^2 + s^3\lambda^4\left(\beta_{T,\lambda}(t)\right)^3|y|^2\right)e^{2s\delta_{T,\lambda}(t)}\,\mathrm{d}x \leqslant C\left(\int_\Omega \left|\Delta y\right|^2 e^{2s\delta_{T,\lambda}(t)}\,\mathrm{d}x + s^3\lambda^4\int_{\mathscr{O}}\left(\beta_{T,\lambda}(t)\right)^3|y|^2 e^{2s\delta_{T,\lambda}(t)}\,\mathrm{d}x\right),$$

where the weight functions $\beta_{T,\lambda}(t)$ and $\delta_{T,\lambda}(t)$ are defined by

$$\beta_{T,\lambda}(t,x) = \frac{e^{\lambda\alpha(x)}}{[t(T-t)]^2} \quad \text{and} \quad \delta_{T,\lambda}(t,x) = \frac{e^{\lambda\alpha(x)} - e^{\lambda^2\|\alpha\|_\infty}}{[t(T-t)]^2},$$

with $\alpha \in C^2(\overline{\Omega})$ satisfying (26).

Applying the above estimate at time $t = T/2$, taking $\lambda = \widehat{\lambda}$ and replacing s by $16s/T^4$, we get the existence of $\widehat{s} \geqslant 1$ such that for all $s \geqslant \widehat{s}$ we have

$$\int_\Omega \left(\frac{1}{s\beta}\left|D^2y\right|^2 + s\lambda^2\beta|\nabla y|^2 + s^3\lambda^4\beta^3|y|^2\right)e^{2s\left(\beta - e^{\hat{\lambda}^2\|\alpha\|_\infty}\right)}\,\mathrm{d}x \leqslant C\left(\int_\Omega \left|\Delta y\right|^2 e^{2s\left(\beta - e^{\hat{\lambda}^2\|\alpha\|_\infty}\right)}\,\mathrm{d}x + s^3\lambda^4\int_{\mathscr{O}}\beta^3|y|^2 e^{2s\left(\beta - e^{\hat{\lambda}^2\|\alpha\|_\infty}\right)}\,\mathrm{d}x\right),$$

with $\beta = e^{\hat{\lambda}\alpha}$. Multiplying by $e^{2se^{\hat{\lambda}^2\|\alpha\|_\infty}}$ on both sides, and bounding β from below and from above (which can be done independently of s), we get (29) for $s \geqslant \widehat{s}$. If $\widehat{s} > 1$, applying (29) for the value $\widehat{s}$ of the Carleman parameter, straightforward bounds on $e^{2s\beta}$ for $s \in [1, \widehat{s}]$ yields (29) for all $s \geqslant 1$. This concludes the proof. □

Using twice the Carleman estimate (29) for the Laplacian, one can deduce the Carleman estimate (27) for the bi-Laplacian and prove Theorem 34.

Proof of Theorem 34. Let $g \in L^2(\Omega)$ and $\psi \in H^4(\Omega)$ satisfying $\psi = \Delta\psi = 0$ on $\partial\Omega$. Then $y = \Delta\psi$ satisfies

$$\begin{cases} \Delta y = \Delta^2 \psi & \text{in } \Omega, \\ y = 0 & \text{in } \partial\Omega. \end{cases}$$

Then, by applying the Carleman estimate (29) (neglecting the terms involving derivatives of order one and two in y, i.e. of order three and four in ψ), one gets the existence of $\widehat{\lambda} > 0$ and $C > 0$ such that for every $s \geqslant 1$,

$$s^3 \int_\Omega |\Delta\psi|^2 e^{2s\beta} \, \mathrm{d}x \leqslant C \left(\int_\Omega \left|\Delta^2 \psi\right|^2 e^{2s\beta} \, \mathrm{d}x + s^3 \int_{\mathscr{O}} |\Delta\psi|^2 e^{2s\beta} \, \mathrm{d}x \right). \tag{30}$$

One the other hand, applying the Carleman estimate (29) to ψ, we have
for all $s \geqslant 1$,

$$\int_\Omega \left(\frac{1}{s} \left|D^2\psi\right|^2 + s|\nabla\psi|^2 + s^3|\psi|^2 \right) e^{2s\beta} \, \mathrm{d}x \leqslant C \left(\int_\Omega |\Delta\psi|^2 e^{2s\beta} \, \mathrm{d}x + s^3 \int_{\mathscr{O}} |\psi|^2 e^{2s\beta} \, \mathrm{d}x \right). \tag{31}$$

Combining (30) and (31), we deduced that for every $s \geqslant 1$, estimate (27) holds. □

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