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On Orthogonal Projections of Symplectic Balls

Sur les projections orthogonales de boules symplectiques

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Abstract. We study the orthogonal projections of symplectic balls in $\mathbb{R}^{2n}$ on complex subspaces. In particular we show that these projections are themselves symplectic balls under a certain complexity assumption. Our main result is a refinement of a recent very interesting result of Abbondandolo and Matveyev extending the linear version of Gromov's non-squeezing theorem. We use a conceptually simpler approach where the Schur complement of a matrix plays a central role. An application to the partial traces of density matrices is given.

Résumé. Nous étudions les projections orthogonales de boules symplectiques dans $\mathbb{R}^{2n}$ sur des sous-espaces complexes. En particulier, nous montrons que ces projections sont elles-mêmes des boules symplectiques sous une certaine hypothèse de complexité. Notre résultat principal est une amélioration d'un résultat récent et très intéressant d'Abbondandolo et Matveyev, qui étend la version linéaire du théorème de non-plongement de Gromov. Nous utilisons une approche conceptuellement plus simple où le complément de Schur d'une matrice joue un rôle central. Une application aux traces partielles de matrices densité est donnée.

Keywords. Symplectic ball, orthogonal projection, Gromov's non-squeezing theorem.

Mots-clés. boule symplectique, projection orthogonale, théorème de non-plongement de Gromov.

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1. Introduction

1.1. *What is known*

Let $\sigma = \mathrm{d}p_1 \wedge \mathrm{d}x_1 + \cdots + \mathrm{d}p_n \wedge \mathrm{d}x_n$ be the standard symplectic form on $\mathbb{R}^{2n} \equiv \mathbb{R}^n \times \mathbb{R}^n$; we call *symplectic ball* the image of the ball

$$B^{2n}(z_0, R) = \{z \in \mathbb{R}^{2n} : |z - z_0| \leq R\}$$

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by a symplectic automorphism $S \in \mathrm{Sp}(n)$ (the symplectic group of $(\mathbb{R}^{2n}, \sigma)$). As a consequence of Gromov's non-squeezing theorem [8, 10] the orthogonal projection of a symplectic ball $S(B^{2n}(z_0, R))$ on any two-dimensional symplectic subspace of $(\mathbb{R}^{2n}, \sigma)$ has area at least equal to πR^2. Abbondandolo and Matveyev asked in [3] the question whether this result could be generalized to subspaces with higher dimensions. They showed that the orthogonal projection $\Pi_{\mathbb{V}} S(B^{2n}(z_0, R))$ of $S(B^{2n}(z_0, R))$ onto an arbitrary complex subspace $(\mathbb{V}, \sigma_{|\mathbb{V}})$ of $(\mathbb{R}^{2n}, \sigma)$ such that $\dim \mathbb{V} = 2k$ satisfies

$$\mathrm{Vol}_{\mathbb{V}} \Pi_{\mathbb{V}} S(B^{2n}(z_0, R)) \geq \frac{(\pi R^2)^k}{k!} \tag{1}$$

where $\mathrm{Vol}_{\mathbb{V}}$ is the volume element on $\mathbb{V}$. Notice that $(\pi R^2)^k / k!$ is the volume of the ball $B_{\mathbb{V}}(\Pi_{\mathbb{V}} z_0, R)$ in $\mathbb{V}$:

$$\mathrm{Vol}_{\mathbb{V}}(B_{\mathbb{V}}(\Pi_{\mathbb{V}} z_0, R)) = \frac{(\pi R^2)^k}{k!}. \tag{2}$$

They moreover proved that equality holds in (1) if and only $S^T \mathbb{V}$ is itself a complex subspace of $\mathbb{R}^{2n}$. The inequality (1) implies the linear version of Gromov's theorem when $\dim \mathbb{V} = 2$ and conservation of volume by linear symplectomorphisms when $\mathbb{V} = \mathbb{R}^{2n}$. Abbondandolo and Matveyev proved their results using the classical Wirtinger inequality [4]. Notice that there is a precise equality (see [1, Theorem 7.1]) implying (1).

Results of this type are more subtle and difficult than they might appear at first sight; for instance as Abbondandolo and Matveyev show the inequality (1) does not hold when one replaces S by a nonlinear symplectomorphism f. In fact, one can construct examples where $\mathrm{Vol}_{\mathbb{V}} \Pi_{\mathbb{V}} f(B^{2n}(R))$ can become arbitrarily small. They however make an interesting conjecture, to which we will come back at the end of this paper.

1.2. *What we will do*

We will prove by elementary means a stronger version of (1) and of its extension. We will actually prove (Theorem 3) that the orthogonal projection of a symplectic ball on a symplectic subspace contains a symplectic ball with the same radius in this subspace, and is itself a symplectic ball when the subspace under consideration is complex.

The proof will be done in the particular case where the symplectic space $\mathbb{V}$ is of the type $\mathbb{R}^{2n_A} \oplus 0$ in which case the symplectic orthogonal $\mathbb{V}^\sigma$ is $0 \oplus \mathbb{R}^{2n_B}$; our refinement of (1) says that for every $S \in \mathrm{Sp}(n)$ there exists $S_A \in \mathrm{Sp}(n_A)$ (the symplectic group of $\mathbb{R}^{2n_A} \oplus 0 \equiv \mathbb{R}^{2n_A}$ [1]) such that

$$\Pi_{\mathbb{V}} S(B^{2n}(z_0, R)) \supseteq S_A(B^{2n_A}(z_{0,A}, R)) \tag{3}$$

and $z_{0,A} = \Pi_{\mathbb{V}} z_0$.

This will be done using the theory of Schur complements and the notion of symplectic spectrum of a positive definite matrix. Since symplectomorphisms are volume-preserving, (3) implies (1). It is however a much stronger statement than (1) because, given two measurable sets Ω and Ω' with the same volume, there does not in general exist a symplectomorphism (let alone a linear one) taking Ω to Ω' as soon as the dimension of the symplectic space exceeds two [10].

We shall then prove (Section 3.2) the same result for the projection on an arbitrary complex symplectic vector subspace $\mathbb{V}$ of $(\mathbb{R}^{2n}, \sigma)$, not necessarily $\mathbb{R}^{2n_A} \oplus 0$.

Note that these results are invariant under phase space translations. We will therefore assume henceforth that $z_0 = 0$.

[1] For the sake of simplicity, we make the identification $\mathbb{R}^{2n_A} \oplus 0 \equiv \mathbb{R}^{2n_A}$. In particular, we write by abuse of language $\Pi_{\mathbb{V}}(z_A, z_B) = z_A$ instead of $(z_A, 0)$.

In Section 4 we will show that, quite interestingly, our main result (and thus the Abbondandolo and Matveyev theorem on the shadows of symplectic balls) can be restated in the language of quantum mechanics as a well-known property about the partial traces of quantum states.

More specifically:

Any partial trace of a Gaussian quantum state is another Gaussian quantum state.

We finally discuss in Section 5 some possible extensions to the non-linear case, pointing out the difficulties.

2. Preliminaries

In what follows M will be a real $2n\times 2n$ positive definite matrix; we will write $M>0$. We will interchangeably use the notaton $Mz\cdot z$ or Mz^2 for the associated quadratic form. We denote by J the standard symplectic matrix $\begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$. We have, $\sigma(z,z')=Jz\cdot z'$ when $z=(x,p)$, $z'=(x',p')$. In this notation the condition $S\in \mathrm{Sp}(n)$ is equivalent to $S^TJS=J$ (or $SJS^T=J$) where S^T is the transpose of S.

2.1. *Williamson's symplectic diagonalization*

By definition the symplectic spectrum of M is the increasing sequence $\lambda_1^\sigma(M)\leq\lambda_2^\sigma(M)\leq\cdots\leq\lambda_n^\sigma(M)$ of numbers $\lambda_j^\sigma(M)>0$ where the $\pm i\lambda_j^\sigma(M)$ are the eigenvalues of JM (which are the same as those of the antisymmetric matrix $M^{1/2}JM^{1/2}$). We will use the following property, known in the literature as "Williamson's symplectic diagonalization theorem" [5, 10]: there exists $S\in\mathrm{Sp}(n)$ such that

$$M=S^TDS,\ D=\begin{pmatrix}\Lambda & 0\\ 0 & \Lambda\end{pmatrix}$$

where Λ is the diagonal matrix whose eigenvalues are the numbers $\lambda_j^\sigma(M)$ (all matrices corresponding here to the standard splitting $z=(x,p)$). The symplectic spectra of M and M^{-1} are inverses of each other in the sense that:

$$\lambda_j^\sigma(M^{-1})=\lambda_{n-j}^\sigma(M)^{-1}\ \text{ for }\ 1\leq j\leq n. \tag{4}$$

We also have the less obvious property ([5, Section 8.3.2])

$$M\leq N\Longrightarrow \lambda_j^\sigma(M)\leq\lambda_j^\sigma(N)\ \text{ for }\ 1\leq j\leq n \tag{5}$$

where $M\leq N$ means $N-M\geq 0$.

The following simple result characterizing positive semi-definiteness in terms of the symplectic spectrum will be very useful for proving Theorem 3:

Lemma 1. *The Hermitian matrix $M+iJ$ is positive semi-definite: $M+iJ\geq 0$ if and only if $\lambda_j^\sigma(M)\geq 1$ for $1\leq j\leq n$.*

Proof. Let $M=S^TDS$ be a Williamson diagonalization of M; since $S^TJS=J$ the condition $M+iJ\geq 0$ is equivalent to $D+iJ\geq 0$. The characteristic polynomial of $D+iJ$ is the product $P(\lambda)=P_1(\lambda)\cdots P_n(\lambda)$ of polynomials

$$P_j(\lambda)=\lambda^2-2\lambda_j^\sigma(M)\lambda+\lambda_j^\sigma(M)^2-1$$

and the eigenvalues of $M+iJ$ are thus the numbers $\lambda_j=\lambda_j^\sigma(M)\pm 1$. The condition $M+iJ\geq 0$ is equivalent to $\lambda_j\geq 0$, that is to $\lambda_j^\sigma(M)\geq 1$ for $j=1,\ldots,n$. □

Notice that if $S \in \mathrm{Sp}(n)$ we have

$$S^T S + iJ \geq 0 \text{ and } SS^T + iJ \geq 0 \tag{6}$$

since $\lambda_j^\sigma(S^T S) = \lambda_j^\sigma(SS^T) = 1$ for all j (because $D = I$ in view of Williamson's diagonalization result).

2.2. *Block-matrix partitions and Schur complements*

Let $\mathbb{R}^{2n_A} \equiv \mathbb{R}^{2n_A} \oplus 0$ and $\mathbb{R}^{2n_B} \equiv 0 \oplus \mathbb{R}^{2n_B}$ be two symplectic subspaces of $\mathbb{R}^{2n}$. We split $(\mathbb{R}^{2n}, \sigma)$ as a direct sum $(\mathbb{R}^{2n_A} \oplus \mathbb{R}^{2n_B}, \sigma_A \oplus \sigma_B)$ where σ_A and σ_B are, respectively, the restrictions of σ to $\mathbb{R}^{2n_A}$ and $\mathbb{R}^{2n_B}$. We write $z \in \mathbb{R}^{2n}$ as $z = (z_A, z_B) = z_A \oplus z_B$ with $z_A = (x_A, p_A) \in \mathbb{R}^{2n_A}$ and $z_B = (x_B, p_B) \in \mathbb{R}^{2n_B}$.

We denote by Π_A (resp. Π_B) the orthogonal projection $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n_A}$ (resp. $\mathbb{R}^{2n_B}$). We choose symplectic bases $\mathscr{B}_A$, $\mathscr{B}_B$ in $\mathbb{R}^{2n_A}$ and $\mathbb{R}^{2n_B}$ and identify linear mappings $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ with their matrices in the symplectic basis $\mathscr{B} = \mathscr{B}_A \oplus \mathscr{B}_B$ of $\mathbb{R}^{2n}$. Such a matrix will be written as

$$M = \begin{pmatrix} M_{AA} & M_{AB} \\ M_{BA} & M_{BB} \end{pmatrix} \tag{7}$$

the blocks M_{AA}, M_{AB}, M_{BA}, M_{BB} having dimensions $2n_A \times 2n_A$, $2n_A \times 2n_B$, $2n_B \times 2n_A$, $2n_B \times 2n_B$, respectively. Similarly, the standard symplectic matrix J will be split as

$$J = J_A \oplus J_B \equiv \begin{pmatrix} J_A & 0 \\ 0 & J_B \end{pmatrix}$$

where J_A (resp. J_B) is the standard symplectic matrix in $(\mathbb{R}^{2n_A}, \sigma_A)$ (resp. $(\mathbb{R}^{2n_B}, \sigma_B)$).

Since M is positive definite and symmetric the upper-left and lower-right blocks in (7) are themselves positive-definite and symmetric: $M_{AA} > 0$ and $M_{BB} > 0$. In particular the Schur complements

$$M/M_{BB} = M_{AA} - M_{AB} M_{BB}^{-1} M_{BA} \tag{8}$$

$$M/M_{AA} = M_{BB} - M_{BA} M_{AA}^{-1} M_{AB} \tag{9}$$

are well defined and invertible [15], and the inverse of the matrix M is given by the formula

$$M^{-1} = \begin{pmatrix} (M/M_{BB})^{-1} & -(M/M_{BB})^{-1} M_{AB} M_{BB}^{-1} \\ -M_{BB}^{-1} M_{BA} (M/M_{BB})^{-1} & (M/M_{AA})^{-1} \end{pmatrix}. \tag{10}$$

2.3. *Orthogonal projections of ellipsoids in* $\mathbb{R}^{2n}$

We will also need the following general characterization of the orthogonal projection of an ellipsoid on a subspace:

Lemma 2. *Let* Π_A *be the orthogonal projection* $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n_A}$ *and*

$$\Omega = \{z \in \mathbb{R}^{2n} : Mz^2 \leq R^2\}.$$

We have

$$\Pi_A \Omega = \{z_A \in \mathbb{R}^{2n_A} : (M/M_{BB}) z_A^2 \leq R^2\}. \tag{11}$$

Proof. Let us set $Q(z) = Mz^2 - R^2$; the boundary $\partial\Omega$ of the hypersurface $Q(z) = 0$ is defined by

$$M_{AA} z_A^2 + 2M_{BA} z_A \cdot z_B + M_{BB} z_B^2 = R^2. \tag{12}$$

A point z_A belongs to $\partial\Pi_A\Omega$ if and only if the normal vector to $\partial\Omega$ at the point $z = (z_A, z_B)$ is parallel to $\mathbb{R}^{2n_A}$ hence the constraint

$$\partial_z Q(z) = 2Mz \in \mathbb{R}^{2n_A} \oplus 0 .$$

This is equivalent to the condition $M_{BA}z_A + M_{BB}z_B = 0$, that is to $z_B = -M_{BB}^{-1}M_{BA}z_A$. Inserting z_B in (12) shows that the boundary $\partial\Pi_A\Omega$ is the set of all $z_A \in \mathbb{R}^{2n_A}$ such that $(M/M_{BB})z_A^2 = R^2$ hence formula (11). □

Interchanging A and B the orthogonal projection of Ω on $\mathbb{R}^{2n_B}$ is similarly given by

$$\Pi_B\Omega = \{z_B \in \mathbb{R}^{2n_B} : (M/M_{AA})z_B^2 \leq R^2\}. \tag{13}$$

3. Orthogonal Projections of Symplectic Balls

3.1. *The main result: statement and proof*

Let us now prove the main result. We assume again the matrix M is written in block-form (7). To simplify notation we also assume that all balls $B^{2n}(z_0,R)$ are centered at the origin and set $B^{2n}(0,R) = B^{2n}(R)$.The case of a general ball $B^{2n}(z_0,R)$ trivially follows using the translation $z \longmapsto z + z_0$.

Theorem 3. *Let $S \in \mathrm{Sp}(n)$.*

(i) *There exists $S_A \in \mathrm{Sp}(n_A)$ such that*

$$\Pi_A(S(B^{2n}(R)) \supseteq S_A(B^{2n_A}(R)); \tag{14}$$

(ii) *We have*

$$\Pi_A(S(B^{2n}(R)) = S_A(B^{2n_A}(R)) \tag{15}$$

if and only if $S = (S_A \oplus S_B)U$ for some $S_B \in \mathrm{Sp}(n_B)$, and some unitary automorphism $U \in U(n) = \mathrm{Sp}(n) \cap O(2n)$ in which case we also have

$$\Pi_B(S(B^{2n}(R)) = S_B(B^{2n_B}(R)) \tag{16}$$

Proof. (i). The symplectic ball $S(B^{2n}(R))$ consists of all $z \in \mathbb{R}^{2n}$ such that $Mz^2 \leq R^2$ where $M = (SS^T)^{-1}$. It follows from Lemma 2 that $\Pi_A S(B^{2n}(R))$ is determined by the inequality $(M/M_{BB})z_A^2 \leq R^2$. We are going to show that the symplectic eigenvalues $\lambda_j^{\sigma_A}(M/M_{BB})$ are all ≤ 1. The inclusion (14) will then follow since we have, in view of Williamson's diagonalization result,

$$M/M_{BB} = (S_A^{-1})^T D_A S_A^{-1} \tag{17}$$

for some $S_A \in \mathrm{Sp}(n_A)$ and

$$D_A = \begin{pmatrix} \Lambda_A & 0 \\ 0 & \Lambda_A \end{pmatrix} \ , \ \Lambda_A = \mathrm{diag}(\lambda_1^{\sigma_A}(M/M_{BB}),\ldots,\lambda_{n_A}^{\sigma_A}(M/M_{BB})). \tag{18}$$

It follows that:

$$\begin{aligned}(M/M_{BB})\,z_A^2 &= \left((S_A^{-1})^T D_A S_A^{-1}\right) z_A^2 \\ &= D_A\left(S_A^{-1}z_A\right)^2 \leq |S_A^{-1}z_A|^2\end{aligned} \tag{19}$$

The condition $z \in S_A(B^{2n_A}(R))$ being equivalent to $|S_A^{-1}z_A|^2 \leq R^2$ thus implies $(M/M_{BB})z_A^2 \leq R^2$ and hence $S_A(B^{2n_A}(R)) \subseteq \Pi_A S(B^{2n}(R))$.

To prove that we indeed have

$$\lambda_j^{\sigma_A}((M/M_{BB})) \leq 1 \ \text{ for } 1 \leq j \leq n_A \tag{20}$$

we begin by noting that the symplectic eigenvalues $\lambda_j^{\sigma}(M)$ of $M = (SS^T)^{-1}$ are all trivially equal to one, and hence also those of its inverse M^{-1}: $\lambda_j^{\sigma}(M^{-1}) = 1$ for $1 \leq j \leq n$. In view of Lemma 1 the Hermitian matrix $M^{-1} + iJ$ is positive semidefinite:

$$M^{-1} + iJ \geq 0 \tag{21}$$

which implies that (cf. (10))

$$(M/M_{BB})^{-1} + iJ_A \geq 0 \tag{22}$$

(recall that $J = J_A \oplus J_B$). Applying now Lemma 1 to $(M/M_{BB})^{-1}$ this implies that the inequalities (20) must hold (cf. (4)).

(ii). In view of (11) in Lemma 2 the equality (15) will hold if and only if $M/M_{BB} = (S_A S_A^T)^{-1}$. If $S = (S_A \oplus S_B)U$, then $M = (SS^T)^{-1} = \left[\left(S_A S_A^T\right) \oplus \left(S_B S_B^T\right)\right]^{-1}$ implies $M/M_{BB} = (S_A S_A^T)^{-1}$ and we have the equality (15).

Conversely, suppose the equality (15) holds. From the inversion formula (10) we have

$$M^{-1} = \begin{pmatrix} (M/M_{BB})^{-1} & X \\ X^T & (M/M_{AA})^{-1} \end{pmatrix}. \tag{23}$$

where $X = -(M/M_{BB})^{-1} M_{AB} M_{BB}^{-1}$; since $M^{-1} \in \mathrm{Sp}(n)$ is symmetric it satisfies the symplecticity condition

$$M^{-1}(J_A \oplus J_B)M^{-1} = J_A \oplus J_B$$

which is in turn equivalent to the set of conditions

$$(M/M_{BB})^{-1} J_A (M/M_{BB})^{-1} + XJ_B X^T = J_A \tag{24}$$

$$(M/M_{BB})^{-1} J_A X + XJ_B (M/M_{AA})^{-1} = 0 \tag{25}$$

$$X^T J_A X + (M/M_{AA})^{-1} J_B (M/M_{AA})^{-1} = J_B. \tag{26}$$

Since

$$M/M_{BB} = (S_A S_A^T)^{-1} \in \mathrm{Sp}(n_A), \tag{27}$$

it follows from (24) that

$$XJ_B X^T = 0. \tag{28}$$

Multiplying the identity (25) on the right by $(M/M_{AA})X^T$ and using (28), we obtain

$$(M/M_{BB})^{-1} J_A X (M/M_{AA}) X^T = 0$$

that is

$$X(M/M_{AA})X^T = 0. \tag{29}$$

Since $(M/M_{AA}) > 0$, this is possible if and only if $X = 0$. Finally, from (26) we conclude that $(M/M_{AA})^{-1} \in \mathrm{Sp}(n_B)$. Moreover, since $(M/M_{AA})^{-1}$ is symmetric and positive definite, there exists $S_B \in \mathrm{Sp}(n_B)$, such that $(M/M_{AA})^{-1} = S_B S_B^T$.

Altogether

$$M^{-1} = SS^T = \begin{pmatrix} S_A S_A^T & 0 \\ 0 & S_B S_B^T \end{pmatrix} \tag{30}$$

Let us write $P = S_A \oplus S_B$. By the polar decomposition of symplectic matrices (see [5, Proposition 2.19]), there exist matrices $R_S, R_P \in \mathrm{Sp}(n)$, symmetric and positive-definite and matrices $U_S, U_P \in U(n)$ such that:

$$S = R_S U_S\,, \qquad P = R_P U_P\,. \tag{31}$$

Notice that $R_S = (SS^T)^{1/2}$ and $R_P = (PP^T)^{1/2}$. From (30,31), we have that:

$$SS^T = PP^T \Longleftrightarrow R_S^2 = R_P^2 \Longleftrightarrow R_S = R_P\,.$$

If we define $U = U_P^{-1} U_S \in U(n)$ and multiply the last identity by U_S on the right, we obtain:

$$R_S U_S = R_P U_P U_P^{-1} U_S \Longleftrightarrow S = PU\,,$$

and the result follows. □

We remark that the proof above actually provides the means to calculate explicitly the symplectic automorphisms S_A in (14). Recapitulating, it is constructed as follows: given $S \in \mathrm{Sp}(n)$ calculate

$$M = (SS^T)^{-1} = \begin{pmatrix} M_{AA} & M_{AB} \\ M_{BA} & M_{BB} \end{pmatrix}$$

and then obtain the Schur complement (8)

$$M/M_{BB} = M_{AA} - M_{AB}M_{BB}^{-1}M_{BA}.$$

The matrix S_A is then obtained from (27) (observe that S_A is only defined up to a symplectic rotation, but this ambiguity is irrelevant since $B^{2n_A}(R)$ is rotationally invariant).

3.2. *Discussion and extension*

In the previous sections we used the abbreviated notation $\Pi_A(z_A \oplus z_B) = z_A$, etc. In this subsection, we shall restore the notation $\Pi_A(z_A \oplus z_B) = z_A \oplus 0$ whenever necessary for the sake of clarity.

Consider a general complex symplectic subspace $\mathbb{V}$ of $(\mathbb{R}^{2n}, \sigma)$ (that is such that $J\mathbb{V} = \mathbb{V}$) and let $\mathbb{V}^\sigma$ be its symplectic orthocomplement.

Choose symplectic bases $\mathscr{B}_\mathbb{V}$ of $\mathbb{V}$ and $\mathscr{B}_{\mathbb{V}^\sigma}$ of $\mathbb{V}^\sigma$ such that their union $\mathscr{B}_\mathbb{V} \cup \mathscr{B}_{\mathbb{V}^\sigma}$ is a symplectic orthonormal basis of $\mathbb{R}^{2n}$ (this is easily done using the symplectic version of the Gram–Schmidt construction for orthogonal bases; see [5]). Set $\dim \mathbb{V} = 2n_A$ and $\dim \mathbb{V}^\sigma = 2n_B$ (hence $n = n_A + n_B$) and let $\mathscr{B}_A$ and $\mathscr{B}_B$ be symplectic bases of $(\mathbb{R}^{2n_A}, \sigma_A)$ and $(\mathbb{R}^{2n_B}, \sigma_B)$, respectively, such that $\mathscr{B}_A \cup \mathscr{B}_B$ is a symplectic orthonormal basis of $(\mathbb{R}^{2n}, \sigma)$. Let U be the linear mapping $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ defined by $\mathscr{B}_\mathbb{V} = U(\mathscr{B}_A)$ and $\mathscr{B}_{\mathbb{V}^\sigma} = U(\mathscr{B}_B)$; clearly U is symplectic and orthogonal. The mapping $\Pi_\mathbb{V} : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ defined by $\Pi_\mathbb{V} = U\Pi_A U^{-1}$ is then the projection onto $\mathbb{V}$ along $\mathbb{V}^\sigma$. In the basis $\mathscr{B}_\mathbb{V} \cup \mathscr{B}_{\mathbb{V}^\sigma}$ it has the representation

$$\Pi_\mathbb{V} = \begin{pmatrix} I_\mathbb{V} & 0 \\ 0 & 0 \end{pmatrix}, \tag{32}$$

where $I_\mathbb{V}$ is the $2n_A \times 2n_A$ identity matrix. An element of the symplectic group $\mathrm{Sp}_\mathbb{V}(n_A)$ is represented in this basis as:

$$M_\mathbb{V} = \begin{pmatrix} S_\mathbb{V} & 0 \\ 0 & I_{\mathbb{V}^\sigma} \end{pmatrix}, \tag{33}$$

where $I_{\mathbb{V}^\sigma}$ is the $2n_B \times 2n_B$ identity matrix and $S_\mathbb{V} \in \mathrm{Sp}(n_A)$. We thus have:

$$M_\mathbb{V} = UM_AU^{-1}, \tag{34}$$

with

$$M_A = \begin{pmatrix} S_A & 0 \\ 0 & I_B \end{pmatrix}, \tag{35}$$

for some $S_A \in \mathrm{Sp}(n_A)$ and I_B the $2n_B \times 2n_B$ identity matrix.

We denote by $B_\mathbb{V}^{2n_A}(R)$ the ball of radius $R > 0$ in $\mathbb{V}$:

$$B_\mathbb{V}^{2n_A}(R) \oplus 0 = U\left(B^{2n_A}(R) \oplus 0\right). \tag{36}$$

Similar definitions apply to $\mathbb{V}^\sigma$.

Equipped with these definitions we can now prove the following generalization of Theorem 3:

Theorem 4. *Let $S \in \mathrm{Sp}(n)$.*

(i) *There exists $M_\mathbb{V} \in \mathrm{Sp}_\mathbb{V}(n_A)$ of the form* (34), (35) *in the basis $\mathscr{B}_\mathbb{V} \cup \mathscr{B}_{\mathbb{V}^\sigma}$, such that*

$$\Pi_\mathbb{V}(S(B^{2n}(R)) \supseteq S_\mathbb{V}(B_\mathbb{V}^{2n_A}(R)); \tag{37}$$

(ii) *We have*

$$\Pi_{\mathbb{V}}(S(B^{2n}(R)) = S_{\mathbb{V}}(B^{2n_A}_{\mathbb{V}}(R)) \tag{38}$$

if and only if $SV^{-1} = (S_{\mathbb{V}} \oplus S_{\mathbb{V}^\sigma})$ *in the basis* $\mathscr{B}_{\mathbb{V}} \cup \mathscr{B}_{\mathbb{V}^\sigma}$*, for some* $S_{\mathbb{V}^\sigma} \in \mathrm{Sp}(n_B)$*, and some unitary automorphism* $V \in U(n) = \mathrm{Sp}(n) \cap O(2n)$ *in which case we also have*

$$\Pi_{\mathbb{V}^\sigma}(S(B^{2n}(R)) = S_{\mathbb{V}^\sigma} B^{2n_B}_{\mathbb{V}^\sigma}(R)) \tag{39}$$

Proof. (i). We have

$$\Pi_{\mathbb{V}}(S(B^{2n}(R))) = U\Pi_A U^{-1}(S(B^{2n}(R))). \tag{40}$$

Theorem 3 implies that there exists

$$M_A = \begin{pmatrix} S_A & 0 \\ 0 & I_B \end{pmatrix}, \tag{41}$$

with $S_A \in \mathrm{Sp}(n_A)$ such that

$$\Pi_A(U^{-1}(S(B^{2n}(R)))) \supseteq M_A(B^{2n_A}(R) \oplus 0) . \tag{42}$$

Consequently, from (40), (42):

$$\Pi_{\mathbb{V}}(S(B^{2n}(R))) \supseteq UM_A(B^{2n_A}(R) \oplus 0) . \tag{43}$$

Setting $M_{\mathbb{V}} = UM_AU^{-1}$ (and obtain $M_{\mathbb{V}}$ of the form (34), (35)) and using (36), we finally obtain:

$$\Pi_{\mathbb{V}}(S(B^{2n}(R))) \supseteq M_{\mathbb{V}}\left(B^{2n_A}_{\mathbb{V}}(R) \oplus 0\right) . \tag{44}$$

By abbuse of notation we write again $M_{\mathbb{V}}\big(B^{2n_A}_{\mathbb{V}}(R) \oplus 0\big) = S_{\mathbb{V}}\big(B^{2n_A}_{\mathbb{V}}(R)\big)$, and the result follows.

(ii). Suppose that

$$\Pi_{\mathbb{V}}(S(B^{2n}(R))) = M_{\mathbb{V}}\left(B^{2n_A}_{\mathbb{V}}(R) \oplus 0\right) . \tag{45}$$

With the previous arguments in reverse order, we conclude that:

$$\Pi_A U^{-1} S\left(B^{2n}(R)\right) = M_A\left(B^{2n_A} \oplus 0\right) , \tag{46}$$

for $M_A = UM_{\mathbb{V}}U^{-1}$.

From Theorem 3 (ii), there exist $S_B \in \mathrm{Sp}(n_B)$ and $V' \in U(n)$, such that:

$$U^{-1}S = (S_A \oplus S_B)\, V' \Longleftrightarrow S = U\,(S_A \oplus S_B)\, U^{-1}UV' . \tag{47}$$

Setting $V = UV' \in U(n)$ and $U\,(S_A \oplus S_B)\, U^{-1} = S_{\mathbb{V}} \oplus S_{\mathbb{V}^\sigma}$, the result follows. □

Theorem 4 implies *de facto* the Abbondandolo and Matveyev result (1): since $S_{\mathbb{V}} \in \mathrm{Sp}(n_A)$ is volume-preserving formula (37) implies that

$$\mathrm{Vol}_{2n_A} \Pi_{\mathbb{V}}(S(B^{2n}(R)) \geq \frac{(\pi R^2)^{n_A}}{n_A!} . \tag{48}$$

Similarly, the equality (38) implies

$$\mathrm{Vol}_{2n_A} \Pi_{\mathbb{V}}(S(B^{2n}(R)) = \frac{(\pi R^2)^{n_A}}{n_A!}. \tag{49}$$

According to Abbondandolo and Matveyev, the previous equality holds if and only if the subspace $S^T\mathbb{V}$ is complex, that is $JS^T\mathbb{V} = S^T\mathbb{V}$. Let us verify from our results that this is indeed the case. From Theorem 4 (i), we have the inclusion (37), which is equivalent to (cf. (42))

$$\Pi_A(U^{-1}(S(B^{2n}(R)))) \supseteq M_A(B^{2n_A}(R) \oplus 0) , \tag{50}$$

wheras the equality (38) is equivalent to:

$$\Pi_A(U^{-1}(S(B^{2n}(R)))) = M_A(B^{2n_A}(R) \oplus 0) . \tag{51}$$

If the inclusion (50) is strict, then (49) cannot hold. Therefore, (51) must be valid.

From Theorem 3 (ii) there exist $S_B \in \mathrm{Sp}(n_B)$ and $V \in U(n)$, such that:

$$U^{-1}S = (S_A \oplus S_B)\,V\,. \tag{52}$$

Since $\left(S_A^T \oplus S_B^T\right)\left(\mathbb{R}^{2n_A} \oplus 0\right) = \left(\mathbb{R}^{2n_A} \oplus 0\right) = J\left(\mathbb{R}^{2n_A} \oplus 0\right)$, $U, V \in U(n)$, and $U\left(\mathbb{R}^{2n_A} \oplus 0\right) = \mathbb{V}$, we conclude that:

$$\begin{aligned} &V^T J\left(\mathbb{R}^{2n_A} \oplus 0\right) = V^T\left(\mathbb{R}^{2n_A} \oplus 0\right) \\ &\Longleftrightarrow JV^T\left(\mathbb{R}^{2n_A} \oplus 0\right) = V^T\left(\mathbb{R}^{2n_A} \oplus 0\right) \\ &\Longleftrightarrow JV^T\left(S_A^T \oplus S_B^T\right)\left(\mathbb{R}^{2n_A} \oplus 0\right) = V^T\left(S_A^T \oplus S_B^T\right)\left(\mathbb{R}^{2n_A} \oplus 0\right) \\ &\Longleftrightarrow JS^T U\left(\mathbb{R}^{2n_A} \oplus 0\right) = S^T U\left(\mathbb{R}^{2n_A} \oplus 0\right) \Longleftrightarrow JS^T\mathbb{V} = S^T\mathbb{V}\,, \end{aligned} \tag{53}$$

which shows that $S^T\mathbb{V}$ is indeed complex.

4. Application to the Partial Trace of a Density Operator

Let $\widehat{\rho}$ be a density operator on $L^2(\mathbb{R}^n)$ with Wigner distribution ρ. Thus $\widehat{\rho}$ is of trace class with trace $\mathrm{Tr}(\widehat{\rho}) = 1$ and is positive semidefinite: $\widehat{\rho} \geq 0$. Using the spectral theorem there exists a sequence $(\psi_j)_j$ of orthonormal functions in $L^2(\mathbb{R}^n)$ and a sequence of nonnegative numbers $(\lambda_j)_j$ summing up to one such that

$$\widehat{\rho} = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j| \;;$$

the Wigner distribution of $\widehat{\rho}$ is then the convex sum $\rho = \sum_j \lambda_j W\psi_j$ where $W\psi_j$ is the usual Wigner transform of ψ_j. Assume now that ρ is a Gaussian function

$$\rho(z) = \frac{1}{(2\pi)^n\sqrt{\det\Sigma}}e^{-\frac{1}{2}\Sigma^{-1}z\cdot z} \tag{54}$$

where the covariance matrix Σ is a symmetric positive definite real $2n\times 2n$ matrix. Such a function is the Wigner distribution of a density operator if and only if the quantum condition

$$\Sigma + \frac{i\hbar}{2}J \geq 0 \tag{55}$$

holds [5, 7, 13]; this condition ensures us that $\widehat{\rho} \geq 0$. Let

$$\Omega = \left\{z \in \mathbb{R}^{2n} : \frac{1}{2}\Sigma^{-1}z\cdot z \leq 1\right\} \tag{56}$$

be the covariance ellipsoid of $\widehat{\rho}$. Condition (55) is equivalent to the geometric condition [5]

$$\text{There exists } S \in \mathrm{Sp}(n) \text{ such that } S(B^{2n}(\sqrt{\hbar})) \subset \Omega\,. \tag{57}$$

Let us now define the partial trace

$$\rho_A(z_A) = \int_{\mathbb{R}^{n_B}} \rho(z_A, z_B)\mathrm{d}z_B \tag{58}$$

and denote by $\widehat{\rho}_A$ the Weyl operator $(2\pi\hbar)^{n_A}\mathrm{Op_W}(\rho_A)$ with symbol ρ_A. While it is clear that $\widehat{\rho}_A$ satisfies $\mathrm{Tr}(\widehat{\rho}_A) = 1$ it is not immediately obvious that the positivity condition $\widehat{\rho}_A \geq 0$ holds, i.e. that $\widehat{\rho}_A$ is a density operator. This actually immediately follows from Theorem 3: setting $M = \frac{\hbar}{2}\Sigma^{-1}$, we have

$$\rho(z) = (\pi\hbar)^{-n}(\det M)^{1/2}e^{-\frac{1}{\hbar}Mz^2} \tag{59}$$

and a straightforward calculation of Gaussian integrals shows that

$$\rho_A(z_A) = (\pi\hbar)^{-n_A}(\det M/M_{BB})^{1/2}e^{-\frac{1}{\hbar}(M/M_{BB})z_A^2}\,. \tag{60}$$

Moreover, the covariance ellipsoid

$$\Omega_A = \{z_A : (M/M_{BB})z_A^2 \leq \hbar\} \tag{61}$$

is the orthogonal projection $\Pi_A\Omega$ on $\mathbb{R}^{2n_A}$ of the covariance ellipsoid Ω of $\widehat{\rho}$ (cf. Lemma 2). In view of (57) Ω contains a ball $S(B^{2n}(\sqrt{\hbar}))$, and by Theorem 3 Ω_A also contains a ball $S_A(B^{2n_A}(\sqrt{\hbar}))$. Hence, by (57) $\Sigma_A = \frac{\hbar}{2}(M/M_{BB})^{-1}$ satisfies the quantum condition

$$\Sigma_A + \frac{i\hbar}{2} J_A \geq 0 \tag{62}$$

and we conclude that $\widehat{\rho}_A$ is a quantum state.

Conversely, if Theorem 3 does not hold, $\Omega_A = \Pi_A\Omega$ does not contain a symplectic ball, and by (57) Σ_A does not satisfy the quantum condition (55). Hence, $\widehat{\rho}_A$ will not be a quantum state.

We conclude that, in view of the relation between (55) and (57), our main result (and thus the Abbondandolo and Matveyev theorem) is equivalent to the statment:

Any partial trace of a Gaussian quantum state is another Gaussian quantum state.

the proof of which usually requires the use of the rather complicated KLM (Kastler–Loupias–Miracle–Sole) conditions [6].

5. Perspectives and Remarks

A first natural question that arises is whether Theorem 3 can be extended in some way to nonlinear symplectic mappings, that is to general symplectomorphisms of $(\mathbb{R}^{2n},\sigma)$. The first answer is that there are formidable roadblocks to the passage from the linear to the nonlinear case, as shortly mentioned in the Introduction. For instance, Abbondandolo and Matveyev [3] show, elaborating on ideas of Guth [9], that for every $\varepsilon > 0$ one can find a symplectomorphism f of $(\mathbb{R}^{2n},\sigma)$ defined near $B^{2n}(0,1)$ such that

$$\operatorname{Vol}(\Pi_{\mathbb{V}} f(B^{2n}(0,1)) < \varepsilon.$$

They however speculate in [3] that their projection result might still hold true when the linear symplectic automorphism $S \in \operatorname{Sp}(n)$ is replaced with a symplectomorphism f of $(\mathbb{R}^{2n},\sigma)$ close to a linear one.

The conjecture that (1) holds true when S is replaced by a nonlinear symplectomorphism which is sufficiently close to a linear one has been recently proved in [2] in the case $\dim\mathbb{V} = 4$ and, in general in Corollary 3 of [1].

It would be interesting to apply our methods to tackle this difficult problem. In particular, it would be interesting to know whether the main result of our paper also generalizes to nonlinear symplectomorphism: Is it true that if the symplectomorphism f is close enough to a linear one, then the set $\Pi_A(f(B(R)))$ contains the image of the ball of radius R in $\mathbb{R}^{2n_A}$ by a (nonlinear) symplectomorphism of R^{2n_A}?

Also, Theorem 3 could be used to shed some light on packing problems which form a notoriously difficult area of symplectic topology (see the review [12] by Schlenk).

Given the partitioning $\mathbb{R}^{2n} = \mathbb{R}^{2n_A} \oplus \mathbb{R}^{2n_B}$ it seems natural to expect some connection between orthogonal projections of symplectic balls and the separability/entanglement problem in quantum mechanics [7, 11, 14]. We intend to address this problem in a future work.

References

[1] A. Abbondandolo, G. Benedetti, “On the local systolic optimality of Zoll contact forms”, *Geom. Funct. Anal.* **33** (2023), p. 299-363.

[2] A. Abbondandolo, B. Bramham, U. L. Hryniewicz, P. A. S. Salomão, “Sharp systolic inequalities for Reeb flows on the three-sphere”, *Invent. Math.* **211** (2018), no. 2, p. 687-778.

[3] A. Abbondandolo, R. Matveyev, “How large is the shadow of a symplectic ball?”, *J. Topol. Anal.* **5** (2013), no. 1, p. 87-119.

[4] H. Federer, *Geometric measure theory*, Grundlehren der Mathematischen Wissenschaften, vol. 153, Springer, 1969.

[5] M. de Gosson, *Symplectic geometry and quantum mechanics*, Operator Theory: Advances and Applications, vol. 166, Springer, 2006.

[6] ———, *Symplectic methods in harmonic analysis and in mathematical physics*, Pseudo-Differential Operators. Theory and Applications, vol. 7, Springer, 2011.

[7] ———, "On Density Operators with Gaussian Weyl Symbols", in *Proceedings to the Conference MLTFA18 – Microlocal and Time-Frequency Analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser, 2018, p. 191-206.

[8] M. Gromov, "Pseudo holomorphic curves in symplectic manifolds", *Invent. Math.* **82** (1985), p. 307-347.

[9] L. Guth, "Symplectic embeddings of polydisks", *Invent. Math.* **172** (2008), no. 3, p. 477-489.

[10] H. Hofer, E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser Advanced Texts, Birkhäuser, 1994.

[11] L. Lami, A. Serafini, G. Adesso, "Gaussian entanglement revisited", *New J. Phys.* **20** (2018), article no. 023030.

[12] F. Schlenk, "Symplectic embedding problems, old and new", *Bull. Am. Math. Soc.* **55** (2018), no. 2, p. 139-182.

[13] R. Simon, E. C. G. Sudarshan, N. Mukunda, "Gaussian-Wigner distributions in quantum mechanics and optics", *Phys. Rev. A* **36** (1987), no. 8, p. 3868-3880.

[14] R. F. Werner, M. M. Wolf, "Bound Entangled Gaussian States", *Phys. Rev. Lett.* **86** (2001), no. 16, p. 3658-3661.

[15] F. Zhang (ed.), *The Schur Complement and its Applications*, Numerical Methods and Algorithms, vol. 4, Springer, 2005.