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Riemann–Roch for the ring $\mathbb{Z}$

Riemann–Roch pour l'anneau $\mathbb{Z}$

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Abstract. We show that by working over the absolute base $\mathbb{S}$ (the categorical version of the sphere spectrum) instead of $\mathbb{S}[\pm 1]$ improves our previous Riemann–Roch formula for $\overline{\operatorname{Spec}\mathbb{Z}}$. The formula equates the (integer valued) Euler characteristic of an Arakelov divisor with the sum of the degree of the divisor (using logarithms with base 2) and the number 1, thus confirming the understanding of the ring $\mathbb{Z}$ as a ring of polynomials in one variable over the absolute base $\mathbb{S}$, namely $\mathbb{S}[X], 1+1 = X + X^2$.

Résumé. Nous montrons que l'utilisation de la base absolue $\mathbb{S}$ (la version catégorique du spectre en sphère) au lieu de $\mathbb{S}[\pm 1]$, améliore notre formule de Riemann–Roch précédente pour $\overline{\operatorname{Spec}\mathbb{Z}}$. La formule calcule la caractéristique d'Euler (à valeur entière) d'un diviseur d'Arakelov comme la somme du degré du diviseur (en utilisant des logarithmes de base 2) et le nombre 1, confirmant ainsi la compréhension de l'anneau $\mathbb{Z}$ comme un anneau de polynômes en une variable sur la base absolue $\mathbb{S}$,à savoir $\mathbb{S}[X], 1+1 = X + X^2$.

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1. Introduction

In [2] we proved a Riemann–Roch formula for $\overline{\operatorname{Spec}\mathbb{Z}}$ working over the spherical extension $\mathbb{S}[\pm 1] := \mathbb{S}[\mu_{2,+}]$ of the absolute base $\mathbb{S}$. The proof of that result is based on viewing the ring $\mathbb{Z}$ as a ring of polynomials[1] with coefficients in $\mathbb{S}[\pm 1]$ and generator $3 \in \mathbb{Z}$. In the present paper we show that by working over the absolute base $\mathbb{S}$ itself, one obtains the following Riemann–Roch formula.

Theorem 1. *Let D be an Arakelov divisor on $\overline{\operatorname{Spec}\mathbb{Z}}$. Then*[2]

$$\dim_{\mathbb{S}} H^0(D) - \dim_{\mathbb{S}} H^1(D) = \left\lceil \deg_2 D \right\rceil' + 1. \tag{1}$$

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[1] More precisely every integer is uniquely of the form $P(X)$ where P is a polynomial with coefficients in $\{-1,0,1\}$ and $X = 3$, the presentation is given by $1 + 1 = X - 1$.
[2] We use the notation $\deg_2 := \deg / \log 2$.

Here $\lceil x\rceil'$ denotes the right continuous function which agrees with the function ceiling(x) for $x>0$ non-integer, and with –ceiling$(-x)$ for $x<0$ non-integer (see Figure 1).

The proof of (1) follows the same lines as the proof of the Riemann–Roch formula in [2], and views $\mathbb{Z}$ as a ring of polynomials[3] over $\mathbb{S}$ with generator -2. It greatly improves this earlier result as follows:

(1) The term $\mathbf{1}_L$ involving the exceptional set L in the earlier formula is now eliminated.
(2) Formula (1) displays a perfect analogy with the Riemann–Roch formula holding for curves of genus 0.
(3) The canonical divisor $K=-2\{2\}$ has integral degree $\deg_2(K)=-2$.

2. Working over the absolute base $\mathbb{S}$

We let $\Gamma^{\rm op}$ be the opposite of the Segal category (see [3, Chpt. 2] and [1]), it has one object $k_+=\{*,1,\ldots,k\}$ for each integer $k>0$, and the morphisms are morphisms of pointed sets. Covariant functors $\Gamma^{\rm op}\longrightarrow\mathfrak{Sets}_*$ and their natural transformations determine the category $\Gamma\mathfrak{Sets}_*$ of Γ-sets (aka $\mathbb{S}$-modules). When working over the spherical monoidal algebra $\mathbb{S}[\pm1]$ of the (pointed) multiplicative monoid $\{\pm1\}$, the natural $\mathbb{S}[\pm1]$-module associated to a norm on an abelian group A is ($k\in\mathbb{N}$, $\lambda\in\mathbb{R}$)

$$\|HA\|_\lambda(k_+):=\left\{a\in A^k\,\middle|\,\sum|a_j|\le\lambda\right\}. \tag{2}$$

The above formula is applied at the archimedean place, for subgroups $A\subset\mathbb{R}$ and with $|\cdot|$ denoting the euclidean absolute value. If $\mathbb{S}[\pm1]$ is replaced by the base $\mathbb{S}$, there is a more basic definition of an $\mathbb{S}$-module associated to an arbitrary subset $X\subset A$ containing $0\in A$

Lemma 2. *Let A be an abelian monoid with $0\in A$. Let $X\subset A$ be a subset containing 0. The following condition defines a subfunctor of the $\mathbb{S}$-module HA*

$$(HA)_X(k_+):=\left\{a\in A^k\,\middle|\,\sum_Z a_j\in X,\ \forall\, Z\subset k_+\right\}\subset X^k. \tag{3}$$

Proof. By construction $(HA)_X(k_+)$ is a subset of $HA(k_+)$ containing the base point $a_j=0$, $\forall\ j$. Let $\phi: k_+\to m_+$ be a map preserving the base point $*$, we shall show that $\phi_*((HA)_X(k_+))\subset(HA)_X(m_+)$. Let $a\in(HA)_X(k_+)$. For any $\ell\in m_+$, $\ell\neq *$, one has

$$\phi_*(a)(\ell)=\sum_{\phi^{-1}(\ell)}a_j=\sum_{Z_\ell}a_j,\qquad Z_\ell:=\phi^{-1}(\ell).$$

It follows from (3) that $\phi_*(a)(\ell)\in X$ for all ℓ and that for any pointed subset $Z'\subset m_+$

$$\sum_{\ell\in Z'}\phi_*(a)(\ell)=\sum_Z a_j\in X,\qquad Z=\cup_{\ell\in Z'}Z_\ell.$$

This proves that $\phi_*((HA)_X(k_+))\subset(HA)_X(m_+)$. □

Next proposition shows that for $X=[-\lambda,\lambda]\subset\mathbb{R}$ a symmetric interval, the $\mathbb{S}$-module $(H\mathbb{R})_X$ is a module over the $\mathbb{S}$-algebra $\|H\mathbb{R}\|_1$.

Proposition 3. *Let $\lambda>0$, $X=[-\lambda,\lambda]\subset\mathbb{R}$ a symmetric interval and $(H\mathbb{R})_X$ as in (3). Then*

$$(H\mathbb{R})_X(k_+)=\left\{a\in\mathbb{R}^k\,\middle|\,\sum_{a_j>0}a_j\le\lambda,\ \sum_{a_j<0}\left(-a_j\right)\le\lambda\right\} \tag{4}$$

Moreover, the module action of the $\mathbb{S}$-algebra $H\mathbb{R}$ on itself by multiplication induces an action of the $\mathbb{S}$-algebra $\|H\mathbb{R}\|_1$ on the module $(H\mathbb{R})_X$.

[3]Every integer is uniquely of the form $P(X)$ where P is a polynomial with coefficients in $\{0,1\}$ and $X=-2$, the presentation is $1+1=X+X^2$.

Proof. The condition (4) is fulfilled by all elements of $(H\mathbb{R})_X(k_+)$ since it involves sums on subsets of k_+. Conversely if $a \in \mathbb{R}^k$ fulfills (4) and $Z \subset k_+$ let

$$Z_+ := \{j \in Z \mid a_j > 0\}, \quad Z_- := \{j \in Z \mid a_j < 0\}$$

One has $0 \le \sum_{Z_+} a_j \le \lambda$, $0 \ge \sum_{Z_-} a_j \ge -\lambda$ and thus $-\lambda \le \sum_Z a_j \le \lambda$.
To prove the second statement, let $Y = k_+, Y' = k'_+$ be finite pointed sets and consider the map given by the product

$$m : \|H\mathbb{R}\|_1(Y) \wedge (H\mathbb{R})_X\left(Y'\right) \to (H\mathbb{R})\left(Y \wedge Y'\right)$$

It associates to $(\alpha_i) \in \|H\mathbb{R}\|_1(Y)$, $\sum |\alpha_i| \le 1$ and $(a_j) \in (H\mathbb{R})_X(Y')$ the doubly indexed $b := (b_{i,j})$, $b_{i,j} = \alpha_i a_j$ and one needs to show that $b \in (H\mathbb{R})_X(Y \wedge Y')$. Let

$$Y_+ = \{i \in Y \mid \alpha_i > 0\},\ Y_- = \{i \in Y \mid \alpha_i < 0\},\ Y'_+ = \{j \in Y' \mid a_j > 0\},\ Y'_- = \{j \in Y' \mid a_j < 0\}$$

By the rule of signs the pairs (i, j) for which $b_{i,j} > 0$ form the union $Y_+ \times Y'_+ \cup Y_- \times Y'_-$ so that one gets

$$\sum_{b_{i,j}>0} b_{i,j} = \sum_{Y_+ \times Y'_+} \alpha_i a_j + \sum_{Y_- \times Y'_-} (-\alpha_i)\left(-a_j\right) = \sum_{Y_+} \alpha_i \sum_{Y'_+} a_j + \sum_{Y_-} (-\alpha_i) \sum_{Y'_-} \left(-a_j\right) \le \lambda$$

using (4) for the sums over the a_j together with the inequality $\sum_{Y_+} \alpha_i + \sum_{Y_-}(-\alpha_i) \le 1$ (since $\sum |\alpha_i| \le 1$). One treats in a similar way the sum over the negative $b_{i,j}$. $\square$

In general, let $\sigma \in \mathrm{Hom}_\Gamma^{\mathrm{op}}(k_+, 1_+)$ with $\sigma(\ell) = 1\ \forall\ \ell \ne *$ and $\delta(j,k) \in \mathrm{Hom}_\Gamma^{\mathrm{op}}(k_+, 1_+)$, $\delta(j,k)(\ell) := 1$ if $\ell = j$, $\delta(j,k)(\ell) := *$ if $\ell \ne j$.
Given an $\mathbb{S}$-module $\mathscr{F}$ and elements $x, x_j \in \mathscr{F}(1_+)$, $j = 1, \ldots, k$, one writes

$$x = \sum_j x_j \iff \exists\, z \in \mathscr{F}(k_+) \text{ s.t. } \mathscr{F}(\sigma)(z) = x,\ \mathscr{F}\left(\delta(j,k)\right)(z) = x_j,\ \forall\ j. \tag{5}$$

A tolerance relation $\mathscr{R}$ on a set X is a reflexive and symmetric relation on X. Equivalently, $\mathscr{R}$ is a subset $\mathscr{R} \subset X \times X$ which is symmetric and containing the diagonal. We shall denote by $\mathscr{T}$ the category of tolerance relations $(X, \mathscr{R})$. Morphisms in $\mathscr{T}$ are defined by

$$\mathrm{Hom}_{\mathscr{T}}\left((X,\mathscr{R}), (X',\mathscr{R}')\right) := \{\phi : X \to X',\ \phi(\mathscr{R}) \subset \mathscr{R}'\}.$$

We denote $\mathscr{T}_*$ the pointed category under the object $\{*\}$ endowed with the trivial relation. A tolerant $\mathbb{S}$-module is a pointed covariant functor $\Gamma^{\mathrm{op}} \longrightarrow \mathscr{T}_*$ ([2]). We recall below the definition of their dimension.

Definition 4 ([2]). *Let $(E, \mathscr{R})$ be a tolerant $\mathbb{S}$-module. A subset $F \subset E(1_+)$ generates $E(1_+)$ if the following two conditions hold*

(1) *For $x, y \in F$, with $x \ne y \implies (x, y) \notin \mathscr{R}$*
(2) *For every $x \in E(1_+)$ there exists $\alpha_j \in \{0,1\}$, $j \in F$ and $y \in E(1_+)$ such that $y = \sum_F \alpha_j j \in E(1_+)$ in the sense of (5), and $(x, y) \in \mathscr{R}$.*

The dimension $\dim_{\mathbb{S}}(E, \mathscr{R})$ is defined as the minimal cardinality of a generating set F.

3. Dimension of H^0 over $\mathbb{S}$

Let $m \in \mathbb{N}$, and $I_m = [-m, m] \cap \mathbb{Z}$. Next lemma follows from (5) and Definition 4.

Lemma 5. *The dimension $\dim_{\mathbb{S}}((H\mathbb{Z})_{I_m})$ is the smallest cardinality of a subset $G \subset I_m$ such that for any $j \in I_m$ there exists a subset $Z \subset G$ with $\sum_Z i = j$ and $\sum_{Z'} i \in I_m$ for any $Z' \subset Z$.*

The number of elements of I_m is $2m+1$ and the number of subsets of G is $2^{\#G}$, thus one has the basic inequalities

$$\#G \geq \log_2(2m+1) > \log_2(2m), \qquad \dim_{\mathbb{S}}((H\mathbb{Z})_{I_m} \geq \lceil \log_2(m) \rceil + 1. \tag{6}$$

Here $x \mapsto \lceil x \rceil$ denotes the ceiling function which associates to x the smallest integer $> x$. For $m = 1$ one needs the two elements $\{-1, 1\}$ to generate, while for $m = 2$ one selects the three elements $\{-2, 1, 2\}$. For $m = 3$ one takes the three elements $\{-3, 1, 2\}$ while for $m = 4$ one takes the 4 elements $\{-3, -1, 1, 3\}$.

In general, one uses the following result.

Lemma 6. *Let $n \in \mathbb{N}$ and $I := [-a, a] \subset \mathbb{Z}$, where $2^{n-1} \leq a < 2^n$.*

(i) *If $n > 4$ there exist n distinct elements $\alpha_j \in (0, a)$ such that $\sum \alpha_j = a$ and that any element $z \in [0, a]$ can be written as a partial sum $z = \sum_Z \alpha_j$.*

(ii) *The minimal number of $\mathbb{S}$-generators of $(H\mathbb{Z})_I$ is $n+1$.*

Proof. (i) We have $\sum_0^{n-1} 2^j = 2^n - 1 \geq a$ and $\sigma := \sum_0^{n-2} 2^j = 2^{n-1} - 1 < a$. The idea is to adjoin to the set $T := \{2^j \mid 0 \leq j \leq n-2\}$, whose cardinality is $n-1$ and whose sum is $\sigma < a$, another element $a - \sigma$ so that the full sum is a. The first try is by taking $F = T \cup \{a - \sigma\}$. Assume first that $a - \sigma \notin T$. The partial sums obtained from F are the union of the interval $[0, \sigma]$ with the interval $[a - \sigma, a]$ and these two intervals cover $[0, a]$, since $a - \sigma + \sigma = a$ while $a - \sigma \leq \sigma + 1$. If $a - \sigma \in T$ one has for some $k \geq 0$ that $a = \sigma + 2^k$. To avoid the repetition we adopt the following rules for $2^{n-1} \leq a < 2^n$

(Case 1) If $a = 2^{n-1}$ we let $F := \{2^j \mid 0 \leq j \leq n-3\} \cup \{2^{n-2} - 2\} \cup \{3\}$

(Case 2) If $a \neq 2^{n-1}$ and $a - \sigma \in T$, let $F := \{2^j \mid 0 \leq j \leq n-3\} \cup \{2^{n-2} - 1\} \cup \{a - \sigma + 1\}$

(Case 3) If $a \neq 2^{n-1}$ and $a - \sigma \notin T$, let $F := T \cup \{a - \sigma\}$

Since by hypothesis $n > 4$ one has $2^{n-2} - 2 > 2^{n-3}$, so in Case 1. one gets $\#F = n$ and the sum of elements of F is $a = 2^{n-1}$. The partial sums of elements of $\{2^j \mid 0 \leq j \leq n-3\}$ cover the interval $J = [0, 2^{n-2} - 1]$. By adding $2^{n-2} - 2$ to elements of J one obtains the interval $J + 2^{n-2} - 2 = [2^{n-2} - 2, 2^{n-1} - 3]$ whose union with J is $[0, 2^{n-1} - 3]$, then by imputing the element $3 \in F$ one sees that the partial sums cover $[0, a]$.

In Case 2. one obtains similarly $\#F = n$ since $a - \sigma + 1 \notin T$ and the sum of elements of F is $\sigma + a - \sigma = a$. The partial sums of elements of $\{2^j \mid 0 \leq j \leq n-3\}$ cover the interval $J = [0, 2^{n-2} - 1]$ and using $2^{n-2} - 1$ added to elements of J one obtains the interval $J + 2^{n-2} - 1 = [2^{n-2} - 1, 2^{n-1} - 2]$ whose union with J is $J' = [0, 2^{n-1} - 2] = [0, \sigma - 1]$. Adding $a - \sigma + 1$ to J' one obtains the interval $J" = [a - \sigma + 1, a]$. Since $a - \sigma \in T$ one has $a - \sigma \leq 2^{n-2}$, hence $a - \sigma + 1 \leq \sigma - 1$, so that the lowest element of $J"$ belongs to J' and $J' \cup J" = [0, a]$.

In Case 3. the partial sums of elements of F cover $[0, a]$ as explained above.

(ii) Let k be the minimal number of $\mathbb{S}$-generators of $(H\mathbb{Z})_I$. By (6) one has $k \geq n+1$. It remains to show that there exists a generating set of cardinality $n+1$. We assume first that $n > 4$ and thus, by (i), let $\alpha_j \in (0, a)$ be n distinct elements fulfilling (i). Let $F = \{-a\} \cup \{\alpha_j\} \subset [-a, a]$. By construction $\#F = n+1$. To show that F is an $\mathbb{S}$-generating set of $(H\mathbb{Z})_I$ one needs to check the conditions of Lemma 5. By construction the sum of positive elements of F is a and the sum of its negative elements is $-a$ thus any partial sum of elements of F belongs to $I = [-a, a]$. Moreover the partial sums of positive elements of F cover the interval $[0, a]$ by (i), and using the element $-a$ one covers $I = [-a, a]$.

For $n \leq 4$ one has $a \leq 15$ and one can list generating sets of cardinality $n+1$ as follows

$$\{-1,1\}, \{-3,1,2\}, \{-6,1,2,3\}, \{-7,1,2,4\}, \{-10,1,2,3,4\}, \{-11,1,2,3,5\}$$

$$\{-12,1,2,3,6\}, \{-13,1,2,3,7\}, \{-14,1,2,4,7\}, \{-15,1,2,4,8\}$$

These sets are of the same type as those constructed for $n > 4$; for the other values one has

$$\{-3,-1,1,3\}, \{-4,-1,2,3\}, \{-7,-1,1,2,5\}, \{-8,-1,1,3,5\}.$$

The value $a = 2$ requires 3 generators $\{-2,1,2\}$ and it is the only one for which the set F of generators cannot be chosen in such a way that the sum of its positive elements is a and the sum of its negative elements is $-a$. One nevertheless checks that all elements are obtained as an admissible sum. □

Theorem 7. *Let D be an Arakelov divisor on $\overline{\mathrm{Spec}\,\mathbb{Z}}$. If $\deg(D) \geq 0$ one has*

$$\dim_{\mathbb{S}} H^0(D) = \left\lceil \deg_2 D \right\rceil + 1. \tag{7}$$

Proof. One may assume that $D = \delta\{\infty\}$ where $\delta = \deg(D) \geq 0$. One has $H^0(D) = (H\mathbb{Z})_I$ where $I = [-e^{\delta}, e^{\delta}]$, using the classical relation between the degree of the divisor and the associated compact subset in adeles[4]. Let $n \in \mathbb{N}$, $n \geq 1$, such that $2^{n-1} \leq e^{\delta} < 2^n$. The integer part a of e^{δ} fulfills $2^{n-1} \leq a < 2^n$ and one has $H^0(D) = (H\mathbb{Z})_{[-a,a]}$. Thus, by Lemma 6 one gets $\dim_{\mathbb{S}} H^0(D) = n+1$. By definition $\deg_2 D := \deg D/\log 2$. The conditions $2^{n-1} \leq e^{\delta} < 2^n$ mean that $n-1 \leq \deg_2 D < n$ and show that the least integer $> \deg_2 D$ is equal to n which proves (7). □

4. Dimension of H^1 over $\mathbb{S}$

We define the following sequence of integers:

$$j(n) := \frac{1}{3}(-2)^n - \frac{1}{2}(-1)^n + \frac{1}{6} \qquad n \in \mathbb{N}. \tag{8}$$

The first values of $j(n)$ are then: $0, 1, -2, 5, -10, 21, -42, 85, -170, 341, -682, 1365, -2730, \ldots$

Lemma 8. *Let $G(n) = \{(-2)^j \mid 0 \leq j < n\}$. The map σ from the set of subsets of $G(n)$ to $\mathbb{Z}$ defined by $\sigma(Z) := \sum_Z j$ is a bijection with the interval $\Delta(n) := [j(k), j(k) + 2^n - 1]$ where $k = k(n) := 2E(n/2) + 1$, ($E(x)$ = integral part of x).*

Proof. The map σ is injective and covers an interval $[a, b]$. The lower bound a is the sum of powers $a = \sum_{0 \leq \ell < \frac{n-1}{2}} (-2)^{2\ell+1}$ and the upper bound is the sum of powers $b = \sum_{0 \leq \ell < \frac{n}{2}} (-2)^{2\ell}$. We list the first intervals as follows

$$\Delta(1) = [0,1],\ \Delta(2) = [-2,1],\ \Delta(3) = [-2,5],\ \Delta(4) = [-10,5],\ \Delta(4) = [-10,21], \ldots$$ □

We refer to [2, Appendix A, B], for the interpretation of $H^1(D)$ in terms of the tolerant $\mathbb{S}$-module $(U(1), d)_{\lambda}$, $\lambda = e^{\deg D}$. At level 1 the tolerance relation on the abelian group $\mathbb{R}/\mathbb{Z}$ is given by the condition $d(x, y) \leq \lambda$.

Proposition 9. *Let $U(1)$ be the abelian group $\mathbb{R}/\mathbb{Z}$ endowed with the canonical metric d of length 1. Let $\lambda \in \mathbb{R}_{>0}$, $U(1)_{\lambda}$ the tolerant $\mathbb{S}$-module $(U(1), d)_{\lambda}$. Then*

$$\dim_{\mathbb{S}} U(1)_{\lambda} = \begin{cases} m & \text{if } 2^{-m-1} \leq \lambda < 2^{-m}, \\ 0 & \text{if } \lambda \geq \frac{1}{2}. \end{cases} \tag{9}$$

Proof. For $\lambda \geq \frac{1}{2}$, any element of $U(1)_{\lambda} = (\mathbb{R}/\mathbb{Z}, d)_{\lambda}$ is at distance $\leq \lambda$ from 0, thus one can take $F = \emptyset$ as generating set since, by convention, $\sum_{\emptyset} = 0$. Thus $\dim_{\mathbb{S}} U(1)_{\lambda} = 0$. Next, we assume $\lambda < \frac{1}{2}$. Let $F \subset U(1)$ be a generating set and let $k = \#F$. One easily sees that there are at most 2^k elements of the form $\sum_F \alpha_j j$, $\alpha_j \in \{0,1\}$. The subsets $\{x \in U(1) \mid d(x, \sum_F \alpha_j j) \leq \lambda\}$ cover $U(1)$, and since each of them has measure 2λ one gets the inequality $2\lambda \cdot 2^k \geq 1$. Thus $k \geq \frac{-\log\lambda - \log 2}{\log 2}$. When $\frac{-\log\lambda - \log 2}{\log 2} = m$ is an integer, one has $\lambda = 2^{-m-1}$. Let $F(m) = \{(-2)^{-j} \mid 1 \leq j \leq m\}$. The minimal distance between two elements of $F(m)$ is the distance between 2^{-m+1} and -2^{-m} which

[4] Note that $e^{\deg D} = 2^{\deg_2(D)}$

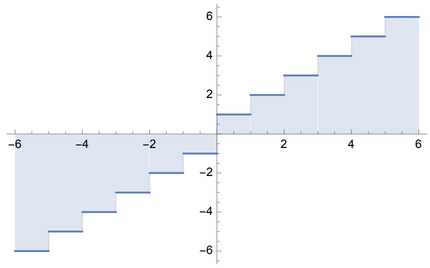

Figure 1. Graph of $\dim_{\mathbb{S}} H^0(D) - \dim_{\mathbb{S}} H^1(D) - 1$ as a function of $\deg_2 D$.

is $3\cdot 2^{-m} = 6\lambda$. Let us show that $F(m)$ is a generating set. By Lemma 8 any integer q in the interval $\Delta(m)$ can be written as $q = \sum_{i=0}^{m-1} \alpha_i(-2)^i$, with $\alpha_i \in \{0,1\}$. One then gets

$$q\cdot(-2)^{-m} = \sum_{i=0}^{m-1} \alpha_i(-2)^{i-m} = \sum_{j=1}^{m} \alpha_{m-j}(-2)^{-j}.$$

Let $y \in \mathbb{R}/\mathbb{Z}$, lift y to an element x of the interval $(-2)^{-m}[j(k(m)), j(k(m))+2^m)$ which is connected of length 1 and is a fundamental domain for the action of $\mathbb{Z}$ by translation. Then there exists an integer $q \in \Delta(m)$ such that $|(-2)^m x - q| \le \frac{1}{2}$. Hence $d(x, q\cdot(-2)^{-m}) \le 2^{-m-1} = \lambda$. This proves that $F(m)$ is a generating set (see Definition 4) and one derives $\dim_{\mathbb{S}} U(1)_\lambda = m$. Assume now that $\frac{-\log\lambda - \log 2}{\log 2} \in (m, m+1)$, where m is an integer, i.e. that $\lambda \in (2^{-m-2}, 2^{-m-1})$. For any generating set F of cardinality k one has $k \ge \frac{-\log\lambda-\log 2}{\log 2} > m$ so that $k \ge m+1$. The subset $F(m+1) = \{(-2)^{-j} \mid 1 \le j \le m+1\}$ fulfills the first condition of Definition 4 since the minimal distance between two elements of $F(m+1)$ is $3\cdot 2^{-m-1}$ which is larger than $\lambda < 2^{-m-1}$. As shown above, the subset $F(m+1)$ is generating for $\lambda = 2^{-m-2}$ and a fortiori for $\lambda > 2^{-m-2}$. Thus one obtains $\dim_{\mathbb{S}} U(1)_\lambda = m+1$ and (9) is proven. □

5. Riemann–Roch formula

We can now formulate the main result of our paper

Theorem 10. *Let D be an Arakelov divisor on $\overline{\operatorname{Spec}\mathbb{Z}}$. Then*

$$\dim_{\mathbb{S}} H^0(D) - \dim_{\mathbb{S}} H^1(D) = \left\lceil \deg_2 D \right\rceil' + 1 \tag{10}$$

where $\lceil x\rceil'$ is the right continuous function which agrees with ceiling(x) for $x > 0$ non-integer and with -ceiling$(-x)$ for $x < 0$ non-integer (see Figure 1).

Proof. For $\deg_2 D \ge 0$ one has $\lambda = e^{\deg D} \ge 1$ and hence by (9) one gets $\dim_{\mathbb{S}} H^1(D) = 0$, so (10) follows from Theorem 7. For $\deg_2 D < 0$ one has $\dim_{\mathbb{S}} H^0(D) = 0$ since the empty set is a generating set. For $\deg_2 D \in [-m-1, -m)$ where $m \in \mathbb{N}$ one has, by (9), $\dim_{\mathbb{S}} H^1(D) = m$. Thus the left hand side of (10) is $-m$ while the right hand side is

$$\left\lceil \deg_2 D \right\rceil' + 1 = -m$$

by definition of the function $\lceil x\rceil'$ as the right continuous function which agrees with ceiling(x) for $x > 0$ non-integer and with -ceiling$(-x)$ for $x < 0$ non-integer. □

Declaration of interests

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References

[1] A. Connes, C. Consani, "Absolute algebra and Segal's γ-rings: au dessous de $\overline{\mathrm{Spec}(\mathbb{Z})}$", *J. Number Theory* **162** (2016), p. 518-551.

[2] ———, "Riemann-Roch for $\overline{\mathrm{Spec}\,\mathbb{Z}}$", to appear in *Bulletin des Sciences Mathématiques*, 2023, https://arxiv.org/abs/2205.01391.

[3] B. I. Dundas, T. G. Goodwillie, R. McCarthy, *The local structure of algebraic K-theory*, Algebra and Applications, vol. 18, Springer, 2013.