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Loop group schemes and Abhyankar's lemma

Schémas en groupes de lacets et lemme d'Abhyankar

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Abstract. We define the notion of loop reductive group schemes defined over the localization of a regular henselian ring A at a strict normal crossing divisor D. We provide a criterion for the existence of parabolic subgroups of a given type.

Résumé. On définit la notion de schémas en groupes réductifs de lacets au-dessus du localisé d'un anneau hensélien A en un diviseur à croisements normaux stricts D. On établit un critère pour qu'un tel schéma en groupes admette un sous–schéma en groupes paraboliques d'un type donné.

Keywords. Reductive group schemes, normal crossing divisor, parabolic subgroups.

Mots-clés. schémas en groupes réductifs, diviseur à croisements normaux, sous-schéma en groupes paraboliques.

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Soit A un anneau local hensélien régulier muni d'un système de paramètres $f_1,\dots,f_r$. On note k le corps résiduel de A et D le diviseur $D=\mathrm{div}(f_1)+\cdots+\mathrm{div}(f_r)$, il est à croisements normaux stricts. On pose $X=\mathrm{Spec}(A)$ et $U=X\setminus D=\mathrm{Spec}(A_D)$. La théorie d'Abhyankar décrit les revêtements finis étales connexes de U qui sont modérément ramifiés le long de D [8, XIII.2]. Un tel objet est dominé par un revêtement galoisien de la forme

$$B_n=B[T_1^{\pm1},\dots,T_r^{\pm1}]/(T_1^n-f_1,\dots,T_r^n-f_r)$$

où n désigne un entier ≥ 1 premier à la caractéristique de k et B est une A–algèbre galoisienne contenant une racine primitive n–ième de l'unité. Le groupe de Galois $\mathrm{Gal}(B_n/A_D)$ est le produit semi-direct $\mu_n(B)^r\rtimes\mathrm{Gal}(B/A)$ où $\mu_n(B)^r$ agit par multiplication sur les $T_1,\dots,T_r$. Si G est un A-schéma en groupes localement de présentation finie, un 1-cocycle $z:\mathrm{Gal}(B_n/A_D)\to G(B_n)$ est dit de *lacets* (loop en anglais) s'il est à valeurs dans $G(B)\subset G(B_n)$. Cette terminologie est inspirée par l'analogie avec le cas des polynômes de Laurent [6, ch. 3].

On note $\widehat{X}$ l'éclaté de $X=\mathrm{Spec}(A)$ en son point fermé, c'est un schéma régulier [10, §8.1, th. 1.19] et le diviseur exceptionnel $E\subset\widehat{X}$ est un diviseur de Cartier isomorphe à $\mathbb{P}_k^{r-1}$. On note alors $R=\mathcal{O}_{\widehat{X},\eta}$ l'anneau local au point générique η de E. Cet anneau R est de valuation discrète, de corps des fractions K, et de corps résiduel $F=k(E)=k(t_1,\dots,t_{r-1})$ où t_i désigne l'image de

$\frac{f_i}{f_r} \in R$ par l'application de spécialisation $R \to F$. On note alors $v : K^\times \to \mathbb{Z}$ la valuation discrète associée à R et K_v le complété de K. Le résultat principal de cette note est le suivant.

Theorem (extrait du th. 4). *On suppose que G agit sur un A–schéma propre et lisse Z. Soit ϕ un 1–cocycle de lacets pour G. On note ${}_\phi Z/U$ le tordu par ϕ de $Z \times_X U$ Alors les assertions suivantes sont équivalentes :*

(a) $({}_\phi Z)(U) \neq \emptyset$;
(b) $({}_\phi Z)(K_v) \neq \emptyset$.

C'est assez proche d'un résultat sur les polynômes de Laurent [6, §, th. 7.1]. L'application principale concerne le cas d'un schéma en groupes réductifs de lacets. Par définition, un U-schéma en groupes réductifs G est *de lacets* si il est isomorphe à un tordu de sa forme déployée G_0 par un 1–cocycle de lacets à valeurs dans le schéma en groupes des automorphismes $\mathrm{Aut}(G_0)$. On applique alors le résultat ci-dessus à des A–schémas de sous-groupes paraboliques de G_0 d'un type donné (th. 6). On en déduit par exemple que si G est un U-schéma en groupes réductifs *de lacets*, alors G admet un U–schéma en groupes de Borel si et seulement si le K_v–schéma en groupes G_{K_v} est quasi-déployé. Plus généralement l'isotropie de G est contrôlée par l'indice de Tits de G_{K_v}.

1. Introduction

In the reference [6], we investigated a theory of loop reductive group schemes over the ring of Laurent polynomials $k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Using Bruhat–Tits' theory, this permitted to relate the study of those group schemes to that of reductive algebraic groups over the field of iterated Laurent series $k((t_1))\dots((t_n))$. The main issue of this note is to start a similar approach for reductive group schemes defined over the localization A_D of a regular henselian ring A at a strict normal crossing divisor D and to relate with algebraic groups defined over a natural field associated to A and D, namely the completion K_v of the fraction field K with respect to the valuation arising from the blow-up of $\mathrm{Spec}(A)$ at its maximal ideal. The example which connects the two viewpoints is $k[\![t_1, \dots, t_n]\!][\frac{1}{t_1}, \dots \frac{1}{t_n}]$ where $K_v \cong k\big(\frac{t_1}{t_n}, \dots, \frac{t_1}{t_{n-1}}\big)((t_n))$.

After defining the notion of loop reductive group schemes in this setting, we show that for this class of group schemes, the existence of parabolic subgroups over the localization A_D is controlled by the parabolic subgroups over K_v (Theorem 6).

2. Tame fundamental group

2.1. *Abhyankar's lemma*

Let $X = \mathrm{Spec}(A)$ be a regular local scheme (not assumed henselian at this stage). Let k be the residue field of A and $p \geq 0$ be its characteristic. We put $\widehat{\mathbb{Z}}' = \prod_{l \neq p} \mathbb{Z}_l$. Let K be the fraction field of A, and let K_s be a separable closure of K. It determines a base point $\xi : \mathrm{Spec}(K) \to X$ so that we can deal with the Grothendieck fundamental group $\Pi_1(X, \xi)$ [8].

Let $(f_1, \dots, f_r)$ be a regular sequence of A and consider the divisor $D = \sum D_i = \sum \mathrm{div}(f_i)$, it has strict normal crossings. We put $U = X \setminus D = \mathrm{Spec}(A_D)$.

We recall that a finite étale cover $V \to U$ is *tamely ramified* with respect to D if the associated étale K–algebra $L = L_1 \times \dots \times L_a$ is tamely ramified at the $D_i's$, that is, for each i, there exists j_i such that for the Galois closure $\widetilde{L}_{j_i}/K$ of L_{j_i}/K, the inertia group associated to v_{D_i} has order prime to p [8, XIII.2.0].

Grothendieck and Murre defined the tame (*modéré* in French) fundamental group $\Pi_1^D(U, \xi)$ with respect to $U \subset X$ as defined in [8, XIII.2.1.3] and [9, §2]. This is a profinite quotient of $\Pi_1(U, \xi)$ whose quotients by open subgroups provides finite Galois tame covers of U.

Let $V \to U$ be a finite étale tame cover. In this case Abhyankar's lemma states that there exists a flat Kummer cover $X' = \mathrm{Spec}(A') \to X$ where

$$A' = A[T_1, \dots, T_r]/(T_1^{n_1} - f_1, \dots, T_r^{n_r} - f_r)$$

and the n_i's are coprime to p such that $V' = V \times_X X' \to X'$ extends uniquely to a finite étale cover $Y' \to X'$ [8, XIII.5.2].

Lemma 1. *Let $V \to U$ be a finite étale cover which is tame. Then* $\mathrm{Pic}(V) = 0$.

Proof. We use the same notation as above. We know that X' is regular [8, XIII.5.1] so a fortiori locally factorial. It follows that the restriction maps $\mathrm{Pic}(X') \to \mathrm{Pic}(V') \to \mathrm{Pic}(V)$ are surjective [7, 21.6.11]. Since A' is finite over the local ring A, it is semilocal so that $\mathrm{Pic}(A') = \mathrm{Pic}(X') = 0$. Thus $\mathrm{Pic}(V) = 0$ as desired. □

From now on we assume that A is henselian. According to [7, 18.5.10], the finite A–ring A' is a finite product of henselian local rings. We observe that $A' \otimes_A k = k[T_1, \dots, T_r]/(T_1^{n_1}, \dots, T_r^{n_r})$ is a local Artinian algebra so that A' is connected. It follows that A' is a henselian local ring. Its maximal ideal is $\mathfrak{m}' = \mathfrak{m} \otimes_A A' + \langle T_1, \dots, T_r \rangle$ so that $A'/\mathfrak{m}' = k$. Since there is an equivalence of categories between finite étale covers of A (resp. A') and étale k–algebras [7, 18.5.15], the base change from A to A' provides an equivalence of categories between the category of finite étale covers of A and that of A'.

It follows that $Y' \to X'$ descends uniquely to a finite étale cover $\widetilde{f} : \widetilde{Y} \to X$. From now on, we assume that V is furthermore connected, it implies that

$$H^0(V, \mathcal{O}_V) = B[T_1, \dots, T_r]/(T_1^n - f_1, \dots, T_r^n - f_r)$$

where B is a finite connected étale cover of A. It follows that $V \to U$ is a quotient of a Galois cover of the shape

$$B_n = \big(B[T_1, \dots, T_r]/(T_1^n - f_1, \dots, T_r^n - f_r)\big) \otimes_A A_D \cong B[T_1^{\pm 1}, \dots, T_r^{\pm 1}]/(T_1^n - f_1, \dots, T_r^n - f_r)$$

where B is Galois cover of A containing a primitive n–th root of unity. We notice that B_n is the localization at $T_1 \dots T_r$ of $B_n' = B[T_1, \dots, T_r]/(T_1^n - f_1, \dots, T_r^n - f_r)$. We have

$$\mathrm{Gal}(B_n/A_D) = \left(\prod_{i=1}^r \mu_n(B) \right) \rtimes \mathrm{Gal}(B/A).$$

Passing to the limit we obtain an isomorphism

$$\pi_1^t(U, \xi) \cong \left(\prod_{i=1}^r \widehat{\mathbb{Z}}'(1) \right) \rtimes \pi_1(X, \xi).$$

We denote by $f : U^{sc,t} \to U$ the profinite étale cover associated to the quotient $\pi_1^t(U, \xi)$ of $\pi_1(U, \xi)$. According to [9, Thm. 2.4.2], it is the universal tamely ramified cover of U. It is a localization of the inductive limit $\widetilde{B}'$ of the B_n'. On the other hand we consider the inductive limit $\widetilde{B}$ of the B's and observe that $\widetilde{B}'$ is a $\widetilde{B}$-ring.

2.2. *Blow-up*

We follow a blowing-up construction arising from [7, Lem. 15.1.1.6]. We denote by $\widehat{X}$ the blow-up of $X = \mathrm{Spec}(A)$ at its closed point, this is a regular scheme [10, §8.1, Thm. 1.19] and the exceptional divisor $E \subset \widehat{X}$ is a Cartier divisor isomorphic to $\mathbb{P}_k^{r-1}$. We denote by $R = \mathcal{O}_{\widehat{X},\eta}$ the local ring at the generic point η of E. The ring R is a DVR of fraction field K and of residue field $F = k(E) = k(t_1, \dots, t_{r-1})$ where t_i is the image of $\frac{f_i}{f_r} \in R$ by the specialization map. We denote by $v : K^\times \to \mathbb{Z}$ the discrete valuation associated to R.

We deal now with a Galois extension B_n of A_D as above. Since B is a connected finite étale cover of A, B is regular and local; it is furthermore henselian [7, 18.5.10]. We denote by L the fraction field of B and by L_n that of B_n. We have $[L_n : L] = n^r$. We want to extend the valuation v to L and to L_n.

We denote by $l = B/\mathfrak{m}_B$ the residue field of B, this is a finite Galois field extension of k. Also $(t_1,\dots,t_r)$ is a system of parameters for B. We denote by $w : L^\times \to \mathbb{Z}$ the discrete valuation associated to the exceptional divisor of the blow-up of Spec(B) at its closed point. Then w extends v and L_w/K_v is an unramified extension of degree $[L:K]$ and of residual extension $F_l = l(t_1,\dots,t_{r-1})/k(t_1,\dots,t_{r-1})$.

On the other hand we denote by $w_n : L_n^\times \to \mathbb{Z}$ the discrete valuation associated to the exceptional divisor of the blow-up of Spec(B_n) at its closed point. We put $l_n = B_n/\mathfrak{m}_{B_n}$, we have $l = l_n$. The valuation $\frac{w_n}{n}$ on L_n extends w and its residual extension is $F_{l,n} = l\big(t_1^{1/n},\dots,t_{r-1}^{1/n}\big)/k\big(t_1,\dots,t_{r-1}\big)$ so that $[F_{l,n} : F_l] = n^{r-1}$. Furthermore the ramification index e_n of L_n/L is $\geq n$. Since $n^r \leq e_n\,[F_{l,n} : F_l] \leq [L_n : K] = n^r$ (where the last inequality is [1, §VI.3, Prop. 2]) it follows that $e_n = n$. The same statement shows that the map $L_w \otimes_L L_n \to L_{w_n}$ is an isomorphism. To summarize L_{w_n}/L_w is tamely ramified of ramification index n and of degree n^r. Altogether we have $L_{w_n} = L_w \otimes_K L_n$ so that L_{w_n} is Galois over K_v of group $\prod_i \mu_n(B) \rtimes \mathrm{Gal}(B/A) = \prod_i \mu_n(l) \rtimes \mathrm{Gal}(l/k)$.

We denote by Δ the diagonal embedding $\mu_n(l) \subset \prod_i \mu_n(l)$. We put $L_{w_n}^\Delta = L_n^{\Delta(\mu_n(B))}$. Since t_r is an uniformizing parameter of K_v and since $\Delta(\zeta)\,.\,t_r = \zeta . t_r$ for each $\zeta \in \mu_n(B)$, it follows that $(L_{w_n})^\Delta$ is the maximal unramified extension of L_{w_n}/K_v.

2.3. *Loop cocycles and loop torsors*

Let G be an X–group scheme locally of finite presentation. A loop cocycle is an element of $Z^1\big(\pi_1^t(U), G(\widetilde{B})\big)$ and it defines a Galois cocycle in $Z^1(\pi_1^t(U), G(U^{sc,t}))$. We denote by $Z^1_{loop}(\pi_1^t(U), G(U^{sc,t}))$ the image of the map $Z^1\big(\pi_1^t(U), G(\widetilde{B})\big) \to Z^1(\pi_1^t(U), G(U^{sc,t}))$ and by $H^1_{loop}(U,G)$ the image of the map

$$Z^1\big(\pi_1^t(U), G(\widetilde{B})\big) \longrightarrow H^1(\pi_1^t(U), G(U^{sc,t})) \longrightarrow H^1(U,G).$$

We say that a G-torsor E over U (resp. an fppf sheaf G-torsor) is a loop torsor if its class belongs to $H^1_{loop}(U,G) \subset H^1(U,G)$.

A given class $\gamma \in H^1_{loop}(U,G)$ is represented by a 1–cocycle $\phi : \mathrm{Gal}(B_n/A_D) \to G(B)$ for some cover B_n/A as above. Its restriction $\phi^{ar} : \mathrm{Gal}(B/A) \to G(B)$ to the subgroup $\mathrm{Gal}(B/A)$ of $\mathrm{Gal}(B_n/A_D)$ is called the "arithmetic part" and the other restriction $\phi^{geo} : \prod_i \mu_n(B) \to \mathfrak{G}(B)$ is called the geometric part. We observe that ϕ^{geo} is a B-group homomorphism.

Furthermore for $\sigma \in \mathrm{Gal}(B/A)$ and $\tau \in \prod_i \mu_n(B)$ the computation of [6, p. 16] shows that $\phi^{geo}(\sigma\tau\sigma^{-1}) = \phi^{ar}(\sigma)\,{}^\sigma\phi(\tau)\,\phi^{ar}(\sigma)^{-1}$ so that ϕ^{geo} descends to a homomorphism of A-group schemes $\phi^{geo} : \mu_n^r \to {}_{\phi^{ar}}G$. This provides a parameterization of loop cocycles.

Lemma 2.

(1) *For B_n/A_D as above, the map $\phi \mapsto (\phi^{ar}, \phi^{geo})$ provides a bijection between $Z^1_{loop}\big(\mathrm{Gal}(B_n/A_D), G(B)\big)$ and the couples (z,η) where $z \in Z^1\big(\mathrm{Gal}(B/A), G(B)\big)$ and $\eta : \prod_i \mu_n \to {}_zG$ is an A–group homomorphism.*

(2) *The map $\phi \mapsto (\phi^{ar}, \phi^{geo})$ provides a bijection between $Z^1_{loop}\big(\pi^1(U,\xi)^t, G(\widetilde{B})\big)$ and the couples (z,η) where $z \in Z^1\big(\pi^1(X,\xi), G(\widetilde{B})\big)$ and $\eta : \prod_{i=1}^r \widehat{\mathbb{Z}}' \to {}_zG$ is an A–group homomorphism.*

Proof. This is similar with [6, Lem. 3.7]. □

We examine more closely the case of a finite étale X–group scheme $\mathfrak{F}$ of constant degree d.

Lemma 3.

(1) $\mathfrak{F}(\widetilde{B}) = \mathfrak{F}(X^{sc}) = \mathfrak{F}(U^{sc,t})$.

(2) *We assume that* d *is prime to* p. *We have* $H^1_{loop}(U,\mathfrak{F}) = H^1(U,\mathfrak{F})$.

(3) *We assume that* d *is prime to* p. *Let* $f : \mathfrak{F} \to \mathfrak{H}$ *be a homomorphism of* A*–group schemes (locally of finite type). Then* $f_*\Big(H^1(U,\mathfrak{F})\Big) \subset H^1_{loop}(U,\mathfrak{H})$.

Proof. (1). We are given a cover B_n/A_D as above such that $\mathfrak{F}_{B_n} \cong \Gamma_{B_n}$ is finite constant. as above. Since B and B_n are connected, the map $\mathfrak{F}(B) \to \mathfrak{F}(B_n)$ reads as the identity $\Gamma \cong \mathfrak{F}(B) \to \mathfrak{F}(B_n) \cong \Gamma$ so is bijective. By passing to the limit we get $\mathfrak{F}(\widetilde{B}) = \mathfrak{F}(U^{sc,t})$.

(2). Let $\mathfrak{E}$ be a $\mathfrak{F}$–torsor over U. This is a finite étale U–scheme. Since U is noetherian and connected, we have a decomposition $\mathfrak{E} = V_1 \times_U \cdots \times_U V_l$ where each V_i is a connected finite étale U–scheme of constant degree d_i. We have $d_1 + \cdots + d_l = d$ so that we can assume that d_1 is prime to p. We have then $\mathfrak{E}(V_1) \neq \emptyset$.

It follows that $f_1 : V_1 \to U$ is a finite étale cover so that there exists a factorization $U^{sc,t} \to V_1 \xrightarrow{h} U$ of f so that $\mathfrak{E}(U^{sc,t}) \neq \emptyset$. Therefore $[\mathfrak{E}]$ arises from $H^1(\pi_1^t(U,\xi),\mathfrak{F}(U^{sc,t})) \subset H^1(U,\mathfrak{F})$. It follows that $H^1(\pi_1^t(U,\xi),\mathfrak{F}(U^{sc,t})) \xrightarrow{\sim} H^1(U,\mathfrak{F})$. We use now (1) and obtain the desired bijection $H^1(\pi_1^t(U,\xi),\mathfrak{F}(B)) \xrightarrow{\sim} H^1(U,\mathfrak{F})$.

(3). This follows readily from (2). □

2.4. *Twisting by loop torsors*

We assume that the A–group scheme G acts on an A–scheme Z. Let $\phi : (\prod_i^r \mu_n)(B) \rtimes \mathrm{Gal}(B/A) \to G(B)$ be a loop cocycle. It gives rise to an A–action of μ_n^r on ${}_{\phi^{ar}}Z$. We denote by $({}_{\phi^{ar}}Z)^{\phi^{geo}}$ the fixed point locus for this action, it is representable by a closed A–subscheme of ${}_{\phi^{ar}}Z$ [3, A.8.10.(1)]. We have a closed embedding $({}_{\phi^{ar}}Z)^{\phi^{geo}} \times_X U \subset {}_{\phi}Z$ of U-schemes.

3. Fixed points method

Theorem 4. *Let* $X = \mathrm{Spec}(A)$ *be a henselian regular local scheme and* $U = X \setminus D$ *as above. We denote by* $v : K^\times \to \mathbb{Z}$ *the discrete valuation associated to the exceptional divisor* E *of the blow-up of* X *at its closed point.*

Let G *be an affine* A*-group scheme of finite presentation acting on a proper smooth* A*–scheme* Z*. Let* ϕ *be a loop cocycle for* G*. Then* $Y = \big({}_{\phi^{ar}}Z\big)^{\phi^{geo}}$ *is a smooth proper* A*–scheme and the following are equivalent:*

(i) $({}_{\phi}Z)(K_v) \neq \emptyset$;

(ii) $Y(k) \neq \emptyset$;

(iii) $Y(U) \neq \emptyset$;

(iv) $({}_{\phi}Z)(U) \neq \emptyset$.

This is quite similar with the fixed point theorem [6, §, Thm. 7.1]. The following example makes the connection.

Example 5. We assume that $A = k[\![t_1,\dots,t_r]\!]$ for a field k and $k[U] = k[\![t_1,\dots,t_n]\!]\big[\frac{1}{t_1},\dots,\frac{1}{t_r}\big]$. We consider an affine algebraic k–group G acting on a smooth proper k–scheme Z. In this case $K = k((t_1,\dots,t_r))$ and A embeds in $k\big(\frac{t_1}{t_r},\dots,\frac{t_{r-1}}{t_r}\big)[\![t_r]\!]$ so that K embeds in $k\big(\frac{t_1}{t_n},\dots,\frac{t_{r-1}}{t_r}\big)((t_r))$ which is nothing but the complete field K_v. If Q is a loop G-torsor over U, the statement is then that

${}^{Q}Z(U) \neq \emptyset$ if and only if ${}^{Q}Z(K_v) \neq \emptyset$. Taking a cocycle $\phi \in Z^1(\pi_1(U)^t, G(k_s))$ for E, this rephrases by the equivalence between $({}_\phi Z)(U) \neq \emptyset$ and $({}_\phi Z)(K_v) \neq \emptyset$.

What we have from [6, Thm. 7.1] (in characteristic zero but this extends to this tame setting) is the equivalence between $({}_\phi Z)(k[t_1^{\pm 1}, \dots, t_r^{\pm 1}]) \neq \emptyset$ and $({}_\phi Z)\big(k((t_1))\dots((t_r))\big) \neq \emptyset$. Since $({}_\phi Z)(k[t_1^{\pm 1}, \dots, t_r^{\pm 1}]) \subset ({}_\phi Z)(U)$ and $({}_\phi Z)\big(K_v\big) \subset ({}_\phi Z)\big(k((t_1))\dots((t_r))\big)$, it follows that this special case of Theorem 4 is a consequence of the fixed point result of [6].

We proceed to the proof of Theorem 4.

Proof. According to [3, A.8.10.(1)], $Y = \big({}_{\phi^{ar}}(Z^{\phi^{geo}})\big)$ is a closed A–scheme of ${}_{\phi^{ar}}Z$ so it is proper. It is smooth over X according to point (2) of the same reference. Let $\phi : \mathrm{Gal}(B_n/A_D) \to G(B)$ be the loop 1-cocycle for some Galois cover B_n/A_D as above for some n prime to p. Up to replacing G by ${}_{\phi^{ar}G}$ and G by ${}_{\phi^{ar}Z}$, we can assume that $\phi^{ar} = 1$ without lost of generality.

(ii)⇒(iii). Since Y_k is the special fiber of the smooth X–scheme Y, Hensel's lemma shows that $Y(A) \to Y(k)$ is onto. Since $Y(k)$ is not empty, it follows that $Y(A)$ is not empty and so is $Y(U)$.

(iii)⇒(iv). Since $Y(U) \subset {}_\phi Z(U)$, $Y(U) \neq \emptyset$ implies that ${}_\phi Z(U) \neq \emptyset$.

(iv)⇒(i). This is obvious.

(i)⇒(ii). We assume that $({}_\phi Z)(K_v) \neq \emptyset$. By definition we have

$$({}_\phi Z)(K_v) = \big\{ z \in Z(L_{w_n}) \,\big|\, \phi(\sigma).\sigma(z) = z \quad \forall\, \sigma \in \mathrm{Gal}(L_n/K) \big\}$$

and our assumption is that this set is non-empty. Let O_{w_n} be the valuation ring of L_{w_n}. Since Z is proper over X, we have the specialization map $Z(L_{w_n}) \to Z_k(F_{l,n})$. We get that the set

$$\big\{ z \in Z_k(F_{l,n}) \,\big|\, \phi(\sigma).\sigma(z) = z, \ \ \forall\, \sigma \in \mathrm{Gal}(L_{w_n}/K_v) \big\}$$

is not empty. Since we have an embedding

$$F_{l,n} = l\big(t_1^{1/n}, \dots, t_{r-1}\big) \hookrightarrow l\big(\big(t_1^{1/n}\big)\big)\dots\big(\big(t_{r-1}^{1/n}\big)\big)$$

in a higher field of Laurent series, successive specializations along the coordinates $t_1^{1/n}, \dots, t_{r-1}^{1/n}$ show similarly that the set

$$\Big\{ z \in (Z_k)\big(l\big) \,\Big|\, \phi(\sigma).\sigma(z) = z, \ \ \forall\, \sigma \in \mathrm{Gal}(L_{w_n}/K_v) \Big\} \tag{1}$$

is not empty. Since $\eta^{ar} = 1$, this set is $(Z_k)^{\eta^{geo}}(k)$. Thus $Y(k) = (Z_k)^{\eta^{geo}}(k)$ is non empty. □

4. Parabolic subgroups of loop reductive group schemes

4.1. *Chevalley groups*

Let G_0 be Chevalley group defined over $\mathbb{Z}$. Let T_0 be a maximal split $\mathbb{Z}$-subtorus of G_0 together with a Borel subgroup B_0 containing it. We denote by Δ_0 the Dynkin diagram of (G_0, B_0, T_0). We denote by $G_{0,ad}$ the adjoint quotient of G_0 and by G_0^{sc} the simply connected covering of DG_0. We have a map $\mathrm{Aut}(G_0) \to \mathrm{Aut}(G_0^{sc}) \xrightarrow{\sim} \mathrm{Aut}(G_{0,ad})$ and a fundamental exact sequence

$$1 \longrightarrow G_{0,ad} \longrightarrow \mathrm{Aut}(G_{0,ad}) \longrightarrow \mathrm{Out}(G_{0,ad}) \longrightarrow 1$$

where $\mathrm{Out}(G_{0,ad}) \xrightarrow{\sim} \mathrm{Aut}(\Delta_0)$. We recall that there is a bijection $I \mapsto P_{0,I}$ between the finite subsets of Δ_0 and the parabolic subgroups of G_0 containing B_0 [4, XXVI.3.8]; it is increasing for the inclusion order, in particular $B_0 = P_{0,\emptyset}$ and $G_0 = P_{0,\Delta_0}$. We consider the total scheme Par_{G_0} of parabolic subgroups of G_0, it is a projective smooth $\mathbb{Z}$–scheme equipped with a type map $\mathbf{t} : \mathrm{Par}_{G_0} \to \mathrm{Of}(\Delta_0)$ where $\mathrm{Of}(\Delta_0)$ stands for the finite constant scheme attached to the set of subsets of Δ_0 [4, XXVI.3]. The fiber at I is denoted by $\mathrm{Par}_{G_0,I}$, it has connected fibers and is the scheme of parabolic subgroups of G_0 of type I. We have a natural action of $\mathrm{Aut}(G_0)$ on Par_{G_0}. As in [5, §5.1], we denote by $\mathrm{Aut}_I(G_0)$ the stabilizer of I for this action. By construction $\mathrm{Aut}_I(G_0)$ acts on $\mathrm{Par}_{G_0,I}$.

4.2. *Definition*

Let G be a reductive U-group scheme in the sense of Demazure–Grothendieck [4, XIX]. Since U is connected and G is locally splittable [4, XXII.2.2] for the étale topology, G is an étale form of a Chevalley group G_0 as above defined over $\mathbb{Z}$.

We say that G is a *loop group scheme* if the $\mathrm{Aut}(G_0)$-torsor $Q = \mathrm{Isom}(G_0, G)$ (defined in [4, XXIV.1.9]) is a loop $\mathrm{Aut}(G_0)$-torsor. We denote by $G_{0,ad}$ the adjoint quotient of G_0 and by G_0^{sc} the simply connected covering of DG_0. We have a map $\mathrm{Aut}(G_0) \to \mathrm{Aut}(G_0^{sc}) \xrightarrow{\sim} \mathrm{Aut}(G_{0,ad})$ which permits to see G_{ad} (resp. G^{sc}) as twisted forms of $G_{0,ad}$ (resp. G_0^{sc}) so that G_{ad} and G^{sc} are also loop reductive group schemes. We consider the map $\mathrm{Aut}(G_0) \to \mathrm{Aut}(G_{0,ad}) \to \mathrm{Out}(G_{0,ad}) \xrightarrow{\sim} \mathrm{Aut}(\Delta_0)$.

If $\phi : \mathrm{Gal}(B_n/A_D) \to \mathrm{Aut}(G_0)(B)$ is a loop cocycle, we get an action of $\mathrm{Gal}(B_n/A_D)$ on Δ_0 called the star action. If I is stable under the star action, we can twist $\mathrm{Par}_{G_0,I}$ by ϕ and deal with the scheme ${}_\phi\mathrm{Par}_{G_0,I}$ which is the scheme of parabolic subgroup schemes of G of type I.

4.3. *Parabolics*

Theorem 6. *Assume that G is a loop reductive U-group scheme and let $\phi : \mathrm{Gal}(B_n/A_D) \to \mathrm{Aut}(G_0)(B)$ be a loop cocycle such that $G \cong {}_\phi G_0$. Let $I \subset \Delta_0$ be a subset stable under the star action defined by ϕ. Then the following are equivalent:*

(i) *G admits a U–parabolic subgroup of type I;*
(ii) *the k–morphism $\eta_k^{geo} : \mu_n^r \to \mathrm{Aut}({}_{\eta^{ar}}G_0)_k = \big({}_{\eta^{ar}}\mathrm{Aut}(G_0)\big)_k$ normalizes a parabolic k–subgroup of ${}_{\eta^{ar}}G_{0,k}$ of type I;*
(iii) *G_{K_v} admits a parabolic subgroup of type I.*

Proof. Without loss of generality we can assume that G is adjoint. Our assumption on the star action is rephrased by saying that ϕ takes values in $\mathrm{Aut}_I(G_0)$. We apply Theorem 4 to the action of $\mathrm{Aut}_I(G_0)$ on the proper A-scheme $\mathrm{Par}_{G_0,I}$. We consider the A-scheme $Y = ({}_{\phi^{ar}}\mathrm{Par}_{G_0,I})^{\phi^{geo}}$. Theorem 4 shows that the following statements are equivalent.

(i′) $({}_\phi\mathrm{Par}_{G_0,I})(U) \neq \emptyset$;
(ii′) $Y(k) \neq \emptyset$.
(iii′) $({}_\phi\mathrm{Par}_{G_0,I})(K_v) \neq \emptyset$.

Clearly (i′) is equivalent to condition (i) of the Theorem and similarly we have (iii′) $\Longleftrightarrow$ (iii). It remains to establish the equivalence between (ii) and (ii′).

Assume that $({}_{\phi^{ar}}\mathrm{Par}_{G_0,I})^{\phi^{geo}}(k)$ is not empty and pick a k–point z. Then the stabilizer $({}_{\phi^{ar}}G_0)_z$ is a k–parabolic subgroup of ${}_{\phi^{ar}}G_0$ of type I which is stabilized by the action ϕ_k^{geo}. In other words, ϕ_k^{geo} normalizes $({}_{\phi^{ar}}G)_z$. Conversely we assume that ${}_{\phi^{ar}}G$ admits a k–parabolic subgroup of type I normalized by ϕ^{geo}. It defines then a point $z \in ({}_{\phi^{ar}}\mathrm{Par}_{G_0,I})(k)$ which is fixed by ϕ^{geo}. □

4.4. *An example*

Assume that the residue field k is not of characteristic 2 and consider the diagonal quadratic form of dimension 2^r

$$q = \sum_{I \subset \{1,\dots,r\}} u_I\, t^I (x_I)^2$$

where $t_I = \prod_{i\in I} t_i$ and $u_I \in A^\times$. Then $\mathrm{SO}(q)$ is a loop reductive group scheme over U. Since the projective quadric $\{q = 0\}$ is a scheme of parabolic subgroups of $\mathrm{SO}(q)$, Theorem 6 shows that q is isotropic over A_D if and only if q is isotropic over K_v. The 2-dimensional case is related with [2, proof of Theorem 3.1].

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References

[1] N. Bourbaki, *Algèbre commutative, Ch. 1 à 10*, Springer, 1961-1998.

[2] J.-L. Colliot-Thélène, R. Parimala, V. Suresh, "Patching and local-global principles for homogeneous spaces over function fields of p-adic curves", *Comment. Math. Helv.* **87** (2012), no. 4, p. 1011-1033.

[3] B. Conrad, G. Prasad, *Pseudo-reductive groups*, Annals of Mathematics Studies, vol. 191, Cambridge University Press, 2016, vii+245 pages.

[4] M. Demazure, A. Grothendieck (eds.), *Séminaire de Géométrie algébrique de l'I. H. E. S., 1963-1964, schémas en groupes, dirigé par M. Demazure et A. Grothendieck*, Lecture Notes in Mathematics, vol. 151-153, Springer, 1970.

[5] P. Gille, "Sur la classification des schémas en groupes semi-simples", in *Autour des schémas en groupes, III*, Panoramas et Synthèses, vol. 47, Société Mathématique de France, 2015, p. 39-110.

[6] P. Gille, A. Pianzola, *Torsors, reductive group schemes and extended affine Lie algebras*, Memoirs of the American Mathematical Society, vol. 1063, American Mathematical Society, 2013, v+112 pages.

[7] A. Grothendieck, *Eléments de Géométrie Algébrique IV*, vol. 20, 24, 28, 32, Springer, 1964-1967, rédigé avec la colloboration de J. Dieudonné.

[8] ——— (ed.), *Séminaire de Géométrie algébrique de l'I.H.E.S., Revêtements étales et groupe fondamental, dirigé par A. Grothendieck*, Documents Mathématiques, vol. 3, Société Mathématique de France, 2003, xviii+325 pages.

[9] A. Grothendieck, J. P. Murre, *The tame fundamental group of a formal neighbourhood of a divisor with normal crossings on a scheme*, Lecture Notes in Mathematics, vol. 208, Springer, 1971, viii+133 pages.

[10] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, 2002, xv+577 pages.