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A Comparison of Cohomological Obstructions to the Hasse Principle and to Weak Approximation

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Abstract. We show that certain Tate–Shafarevich groups are unramified which enables us to give an obstruction to the Hasse principle for torsors under tori over p-adic function fields.

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1. Introduction

Let K be the function field of a smooth projective geometrically integral curve X over a p-adic field k. In recent years, Harari–Scheiderer–Szamuely [7] obtained an explicit description of a cohomological obstruction to weak approximation for K-tori and also related this obstruction to an unramified cohomology group (called the reciprocity obstruction). In the meantime, Harari–Szamuely [8] constructed an obstruction to the Hasse principle for K-torsors under tori. By construction of [8, pp. 16-17 and Theorem 4.1], the obstruction to the Hasse principle is essentially coming from the global duality for K-tori, and they showed that this obstruction given by a group determined by "local triviality" is the only one. However, the local triviality condition increases the difficulty of computing the obstruction. This encourages us to find an obstruction which is easier to compute.

On the other hand, the reciprocity obstruction is widely used in the investigation of the Hasse principle and intuitively the obstruction introduced in [8] should be compatible with the reciprocity obstruction. These known results also motivate us to build a connection between the obstruction in [8] and the reciprocity obstruction. Moreover, another benefit of this attempt is that the unramified cohomology group is indeed easier to handle (for example, it vanishes in several known cases). At this stage, we would like to find out the relation between the obstruction in [8] and the unramified cohomology group used in reciprocity obstruction [7, Theorem 4.1]).

Actually the question that whether the canonical image of a certain map is unramified was first raised by Colliot-Thélène (see also [3, Remarque 4.3 (b)]). Later [11, Appendix] obtained a partial answer when the torsor is trivial. Therefore it is an interesting question to describe the cohomological obstruction to the Hasse principle using unramified cohomology groups for general torsors under tori.

Let us make the above statements more precise. Let $X^{(1)}$ be the set of closed points on X. The local ring $\mathscr{O}_{X,v}$ at $v \in X^{(1)}$ is a discrete valuation ring, so we may denote by K_v the completion of K with respect to $v \in X^{(1)}$. Let Y be a K-torsor under a K-torus T. We put $Y_v := Y \times_K K_v$. Let $\mathscr{Y}$ be a smooth integral separated X_0-scheme such that $\mathscr{Y} \times_{X_0} \operatorname{Spec} K \simeq Y$ for some sufficiently small non-empty open subset $X_0 \subset X$. We define the adelic points on Y (which does not depend on the choice of the model $\mathscr{Y}$) by

$$Y(\mathbb{A}_K) := \varinjlim_{U \subset X_0} \left(\prod_{v \notin U} Y(K_v) \times \prod_{v \in U} \mathscr{Y}(\mathscr{O}_v) \right).$$

In [8, page 15-16], Harari and Szamuely constructed a map $\rho_Y : H^3_{\mathrm{lc}}(Y, \mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{Q}/\mathbb{Z}$ where

$$H^3_{\mathrm{lc}}(Y, \mathbb{Q}/\mathbb{Z}(2)) := \operatorname{Ker}\left(\frac{H^3(Y, \mathbb{Q}/\mathbb{Z}(2))}{\operatorname{Im} H^3(K, \mathbb{Q}/\mathbb{Z}(2))} \to \prod_{v \in X^{(1)}} \frac{H^3(Y_v, \mathbb{Q}/\mathbb{Z}(2))}{\operatorname{Im} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))} \right)$$

gives the only obstruction to the Hasse principle as follows.

Theorem (Harari–Szamuely [8, Theorem 5.1]). *Let Y be a K-torsor under a torus T such that $Y(\mathbb{A}_K) \neq \varnothing$. If ρ_Y is identically zero, then $Y(K) \neq \varnothing$.*

Actually, during the proof of the theorem, Harari and Szamuely constructed a finer obstruction than that given by $H^3_{\mathrm{lc}}(Y, \mathbb{Q}/\mathbb{Z}(2))$. Indeed, the Hochschild–Serre spectral sequence $E_2^{p,q} := H^p(K, H^q(\overline{Y}, \mathbb{Q}/\mathbb{Z}(2))) \Rightarrow H^{p+q} := H^{p+q}(Y, \mathbb{Q}/\mathbb{Z}(2))$ yields $H^2(K, H^1(\overline{Y}, \mathbb{Q}/\mathbb{Z}(2))) \to H^3(Y, \mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im} H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ (where we have used implicitly the fact that the cohomological dimension of K is 3). Moreover, [8, Lemma 5.2] allows one to identify $H^2(K, H^1(\overline{Y}, \mathbb{Q}/\mathbb{Z}(2)))$ with $H^2(K, T')$ where T' is the dual torus of T (i.e. T' is the torus such that its module of characters is the module of cocharacters of T). Let $Ш^2(T')$ be the group of everywhere locally trivial elements of $H^2(K, T')$. Restricting to the subgroup $Ш^2(T')$ of $H^2(K, T')$ yields a map

$$\tau : Ш^2(T') \to H^3_{\mathrm{lc}}(Y, \mathbb{Q}/\mathbb{Z}(2)).$$

Let Y be a T-torsor such that $Y(\mathbb{A}_K) \neq \varnothing$. Since $Y(\mathbb{A}_K) \neq \varnothing$ is equivalent to $Y(K_v) \neq \varnothing$ for all $v \in X^{(1)}$, $Y(\mathbb{A}_K) \neq \varnothing$ implies that the class $[Y] \in H^1(K, T)$ actually lies in $Ш^1(T)$. Now we arrive at:

Proposition (Harari–Szamuely, [8, Proposition 5.3]). *Let Y be a T-torsor such that $Y(\mathbb{A}_K) \neq \varnothing$. Then*

$$\rho_Y \circ \tau(\alpha) = \langle [Y], \alpha \rangle$$

holds up to sign for all $\alpha \in Ш^2(T')$, where $\langle -, - \rangle$ is the global duality pairing $Ш^1(T) \times Ш^2(T') \to \mathbb{Q}/\mathbb{Z}$ (see [8, Theorem 4.1]).

The previous theorem follows immediately from the precedent proposition together with the fact that the global duality pairing $Ш^1(T) \times Ш^2(T') \to \mathbb{Q}/\mathbb{Z}$ is perfect. In this way, we obtain a cohomological obstruction to the Hasse principle given by the image of $Ш^2(T')$ in $H^3_{\mathrm{lc}}(Y, \mathbb{Q}/\mathbb{Z}(2))$. Thus the image of $Ш^2(T')$ is the crucial part of the obstruction to the Hasse principle.

As for weak approximation, Harari, Scheiderer and Szamuely announced that the defect to weak approximation for tori can be described by $Ш^2_\omega(T')$, where $Ш^2_\omega(T')$ denotes the subgroup of $H^2(K, T')$ consisting of elements vanishing in $H^2(K_v, T')$ for all but finitely many $v \in X^{(1)}$. To this end, they constructed a pairing (see [7, §4, pp. 18])

$$(-,-)_{\mathrm{WA}} : \prod_{v \in X^{(1)}} Y(K_v) \times H^3_{\mathrm{nr}}(K(Y), \mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{Q}/\mathbb{Z}.$$

Here $H^3_{\mathrm{nr}}(K(Y), \mathbb{Q}/\mathbb{Z}(2))$ denotes the unramified subgroup of $H^3(K(Y), \mathbb{Q}/\mathbb{Z}(2))$ which contains the canonical image of $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ in $H^3(K(Y), \mathbb{Q}/\mathbb{Z}(2))$. See [2, §2 and §4] for general properties of unramified elements and unramified cohomology, respectively.

Theorem (Harari–Scheiderer–Szamuely, [7, Theorem 4.2]). *There is a homomorphism*

$$u : Ш^2_\omega(T') \to H^3_{\mathrm{nr}}(K(T), \mathbb{Q}/\mathbb{Z}(2))$$

such that each family $(t_v) \in T(K_v)$ *annihilated by* $(-, \operatorname{Im} u)_{\mathrm{WA}}$ *lies in the closure* $\overline{T(K)}$ *of* $T(K)$ *in* $\prod T(K_v)$ *with respect to the product topology.*

We are therefore interested in the image of $H^3_{\mathrm{lc}}(Y, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(K(Y), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$. We prove that this map has unramified image[1] and hence the obstruction to the Hasse principle will be easier to handle. The first result is the following:

Theorem (Tian [11, Appendix]). *The image of* $Ш^2_\omega(T') \to H^3(K(T), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ *is unramified, i.e. its image lies in the subquotient* $H^r_{\mathrm{nr}}(K(T), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$.

In particular, the image of $Ш^2(T') \to H^3_{\mathrm{lc}}(T, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(K(T), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ is unramified. This positive answer suggests us to generalize the result to K-torsors under T. Now we arrive at the main result:

Corollary (Corollary 5). *The image of* $Ш^2_\omega(T') \to H^3(Y, \mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im} H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ *is unramified for any* K*-torsor* Y *under* T.

The idea of the proof is simple: we show that the image of $Ш^2_\omega(T')$ in $H^3(Y, \mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im} H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ comes from $H^3(Y^c, \mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im} H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ where Y^c denotes a smooth compactification of Y (see [10, Theorem 3.21 and Corollary 3.22] for the existence of Y^c). Thus the image of $Ш^2_\omega(T')$ in $H^3(K(Y), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ is unramified by [2, Proposition 2.1.8] and the properness of Y^c. To this end, we use purity sequences to relate the cohomology of Y with that of Y^c via a commutative diagram (Lemma 4).

2. Main results

Let T^c be a T-equivariant smooth projective compactification of T over K (see [4, Corollaire 1]). Let $T \subset V_i \subset T^c$ be the open subset of T^c consisting of T-orbits such that $\operatorname{codim}(T^c \setminus V_i, T^c) \geq i$. Let $Y^c = Y \times^T T^c$ and let $U_i = Y \times^T V_i \subset Y^c$. Fix an algebraic closure $\overline{K}$ of K and let $\overline{\mathfrak{X}}$ be the base change of a K-scheme $\mathfrak{X}$ to $\overline{K}$. So $U_0 = Y$ by construction and $\overline{Y^c}$ is cellular by [1, Proposition 2.5 (3)]. For $i \geq 1$, $Z_i := U_i \setminus U_{i-1}$ is smooth of codimension i in U_i.

We begin with the computation of some cohomology groups via purity and then deduce a commutative diagram which tells us $Ш^2_\omega(T')$ is unramified. Throughout we shall simply write $\mathcal{Q}(i) := \mathbb{Q}/\mathbb{Z}(i)$ for $i \in \mathbb{Z}$.

Lemma 1. *Suppose either* (a) $0 \leq r \leq 4$ *and* $i \geq 2$, *or* (b) $0 \leq r \leq 2$ *and* $i \geq 1$. *There are isomorphisms*

$$H^r(U_i, \mathcal{Q}(2)) \simeq H^r\left(Y^c, \mathcal{Q}(2)\right) \quad \textit{and} \quad H^r\left(\overline{U}_i, \mathcal{Q}(2)\right) \simeq H^r\left(\overline{Y^c}, \mathcal{Q}(2)\right).$$

Proof. The purity of $U_{i-1} \subset U_i \supset Z_i$ for $i \geq 1$ yields exact sequences (and similar sequences over $\overline{K}$)

$$H^{r-2i}(Z_i, \mathcal{Q}(2-i)) \to H^r(U_i, \mathcal{Q}(2)) \to H^r(U_{i-1}, \mathcal{Q}(2)) \to H^{r+1-2i}(Z_i, \mathcal{Q}(2-i)) \tag{1}$$

by [5, Corollary 8.5.6]. Thus for $r \leq 4$ and $i \geq 3$, there are isomorphisms

$$H^r(U_i, \mathcal{Q}(2)) \to H^r(U_{i-1}, \mathcal{Q}(2)) \tag{2}$$

by the exact sequence (1). In particular, applying (2) inductively yields isomorphisms $H^r(U_2, \mathcal{Q}(2)) \simeq H^r(Y^c, \mathcal{Q}(2))$ and $H^r(\overline{U}_2, \mathcal{Q}(2)) \simeq H^r(\overline{Y^c}, \mathcal{Q}(2))$ for $r \leq 4$. For case (b), since $\operatorname{codim}(Z_2, U_2) = 2$, the exact sequence (1) for $0 \leq r \leq 2$ and $U_1 \subset U_2 \supset Z_2$ implies $H^r(U_2, \mathcal{Q}(2)) \simeq H^r(U_1, \mathcal{Q}(2))$ and $H^r(\overline{U}_2, \mathcal{Q}(2)) \simeq H^r(\overline{U}_1, \mathcal{Q}(2))$. Therefore we conclude $H^r(U_1, \mathcal{Q}(2)) \simeq H^r(U_2, \mathcal{Q}(2)) \simeq H^r(Y^c, \mathcal{Q}(2))$ and similarly over $\overline{K}$. □

[1] Throughout, the unramified part of $H^3(K(Y), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ means $H^3_{\mathrm{nr}}(K(Y), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$.

Remark 2. Since $\overline{Y^c}$ is a projective cellular variety, $H^i(\overline{Y^c},\mathcal{Q}(2)) = 0$ for $i = 2n-1$ with $n \geq 1$ an integer (see [6, Example 19.1.11] or [1, Théorème 2.6]). Moreover, thanks to $\mathrm{Br}(\overline{Y^c}) = 0$ (since $\overline{Y^c}$ is smooth projective rational) and the Kummer sequence

$$0 \to H^1\left(\overline{Y^c},\mathbb{G}_m\right)/n \to H^2\left(\overline{Y^c},\mu_n\right) \to {}_nH^2\left(\overline{Y^c},\mathbb{G}_m\right) \to 0,$$

we obtain an isomorphism of Galois modules $H^2(\overline{Y^c},\mathcal{Q}(1)) \simeq \mathrm{Pic}(\overline{Y^c}) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ after taking direct limit over all n. Over the algebraically closed field $\overline{K}$, we can (non-canonically) identify $\mathcal{Q}(1)$ with $\mathcal{Q}(2)$, so there is an isomorphism $H^2(\overline{Y^c},\mathcal{Q}(2)) \simeq \mathrm{Pic}(\overline{Y^c}) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ of abelian groups.

Lemma 3.

(1) *We have* $H^1(\overline{U}_1,\mathcal{Q}(2)) = 0$. *The map* $H^2(\overline{U}_1,\mathcal{Q}(2)) \to H^2(\overline{U}_0,\mathcal{Q}(2))$ *induced by* $U_0 \subset U_1$ *is identically zero. In particular, there is an exact sequence of Galois modules:*

$$0 \to H^1\left(\overline{U}_0,\mathcal{Q}(2)\right) \to H^0\left(\overline{Z}_1,\mathcal{Q}(1)\right) \to H^2\left(\overline{U}_1,\mathcal{Q}(2)\right) \to 0. \tag{3}$$

(2) *We have*

$$\mathrm{Im}\left(H^3\left(Y^c,\mathcal{Q}(2)\right) \to H^3\left(U_1,\mathcal{Q}(2)\right)\right) = \mathrm{Ker}\left(H^3\left(U_1,\mathcal{Q}(2)\right) \to H^3\left(\overline{U}_1,\mathcal{Q}(2)\right)\right).$$

Therefore

$$\mathrm{Im}\left(\frac{H^3\left(Y^c,\mathcal{Q}(2)\right)}{H^3(K,\mathcal{Q}(2))} \to \frac{H^3(U_1,\mathcal{Q}(2))}{H^3(K,\mathcal{Q}(2))}\right) = \mathrm{Ker}\left(\frac{H^3(U_1,\mathcal{Q}(2))}{H^3(K,\mathcal{Q}(2))} \to H^3\left(\overline{U}_1,\mathcal{Q}(2)\right)\right). \tag{4}$$

(3) *Consider the diagonal map* $\Delta : H^2(K,H^0(\overline{Z}_1,\mathcal{Q}(1))) \to \prod_v H^2(K_v,H^0(\overline{Z}_1,\mathcal{Q}(1)))$ *and write the image* $(\alpha_v) := \Delta(\alpha)$ *of* $\alpha \in H^2(K,H^0(\overline{Z}_1,\mathcal{Q}(1)))$ *under* Δ *into a family of local elements. Put*

$$Ш^2_\omega\left(H^0\left(\overline{Z}_1,\mathcal{Q}(1)\right)\right) := \left\{\alpha \mid \alpha_v = 0 \text{ for all but finitely many } v \in X^{(1)}\right\}.$$

Then $Ш^2_\omega(H^0(\overline{Z}_1,\mathcal{Q}(1))) = 0$ *and in particular* Δ *is injective.*

Proof.

(1) By Lemma 1 and Remark 2, we have $H^1(\overline{U}_1,\mathcal{Q}(2)) \simeq H^1(\overline{Y^c},\mathcal{Q}(2)) = 0$. Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}\left(\overline{T^c}\right)\otimes\mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathrm{Pic}\left(\overline{T}\right)\otimes\mathbb{Q}/\mathbb{Z} \\ \downarrow & & \downarrow \\ H^2\left(\overline{T^c},\mathcal{Q}(2)\right) & \longrightarrow & H^2\left(\overline{T},\mathcal{Q}(2)\right). \end{array}$$

Here the vertical arrows are induced by the Kummer sequence $0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ together with a fixed identification of $\mathcal{Q}(1)$ and $\mathcal{Q}(2)$ over $\overline{K}$. Note that $\mathrm{Br}(\overline{T^c}) = 0$ and $\mathrm{Pic}(\overline{T}) = 0$. Since the left vertical arrow is an isomorphism by the vanishing of $\mathrm{Br}(\overline{T^c})$, we conclude that $H^2(\overline{Y^c},\mathcal{Q}(2)) \to H^2(\overline{U}_0,\mathcal{Q}(2))$ is identically zero because of the isomorphisms of varieties $\overline{Y^c} \simeq \overline{T^c}$ and $\overline{U}_0 = \overline{Y} \simeq \overline{T}$. Finally, the purity sequence (1) for $U_0 \subset U_1 \supset Z_1$ together with the above vanishing results imply the desired short exact sequence of Galois modules.

(2) The purity for $U_1 \subset U_2 \supset Z_2$ induces a commutative diagram with exact rows

$$\begin{array}{ccccc} H^3\left(U_2,\mathcal{Q}(2)\right) & \longrightarrow & H^3\left(U_1,\mathcal{Q}(2)\right) & \longrightarrow & H^0\left(Z_2,\mathcal{Q}(0)\right) \\ \downarrow & & \downarrow & & \downarrow \\ H^3\left(\overline{U}_2,\mathcal{Q}(2)\right) & \longrightarrow & H^3\left(\overline{U}_1,\mathcal{Q}(2)\right) & \longrightarrow & H^0\left(\overline{Z}_2,\mathcal{Q}(0)\right). \end{array}$$

Note that the map $H^0(Z_2, \mathcal{Q}(1)) \to H^0(\overline{Z}_2, \mathcal{Q}(1))$ is injective. According to Lemma 1 and Remark 2, we have

$$H^3\left(\overline{U}_2, \mathcal{Q}(2)\right) \simeq H^3\left(\overline{Y^c}, \mathcal{Q}(2)\right) = 0.$$

Thus a diagram chasing yields

$$\operatorname{Im}\left(H^3(U_2, \mathcal{Q}(2)) \to H^3(U_1, \mathcal{Q}(2))\right) = \operatorname{Ker}\left(H^3(U_1, \mathcal{Q}(2)) \to H^3\left(\overline{U}_1, \mathcal{Q}(2)\right)\right).$$

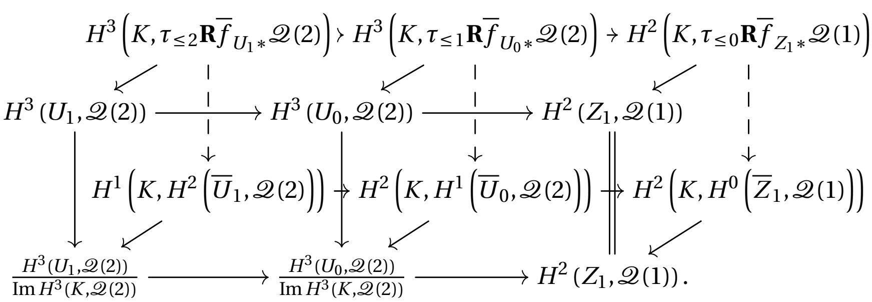
Recall that $H^3(U_2, \mathcal{Q}(2)) \simeq H^3(Y^c, \mathcal{Q}(2))$ by Lemma 1, we are done.

(3) Note first that $H^0(\overline{Z}_1, \mathcal{Q}(1))$ is isomorphic to a direct sum of copies of $\mathcal{Q}(1)$ as Galois modules and hence $Ш^2_\omega(H^0(\overline{Z}_1, \mathcal{Q}(1)))$ is a direct sum of copies of $Ш^2_\omega(\mathcal{Q}(1))$. For a field L of characteristic zero, we have a short exact sequence $0 \to \mathcal{Q}(1) \to \overline{L}^\times \to Q \to 0$ of Galois modules, where the quotient Q is uniquely divisible. Taking cohomology yields an isomorphism $H^2(L, \mathcal{Q}(1)) \simeq \operatorname{Br} L$. Finally, applying $H^2(L, \mathcal{Q}(1)) \simeq \operatorname{Br} L$ to $L = K, K_v$ and taking $Ш^2_\omega(\mathbb{G}_m) = 0$ ([7, Lemma 3.2 (a)]) into account yield $Ш^2_\omega(\mathcal{Q}(1)) \simeq Ш^2_\omega(\mathbb{G}_m) = 0$. □

In the diagram below, we denote by HU/HL, VF/VB/VL/VM/VR for the horizontal upper/lower, vertical front/back/left/middle/right face, respectively. To avoid confusion, in VB we draw dashed vertical arrows. So all the other faces are uniquely determined.

In the proof of Corollary 5 we only need the commutativity of the left cube and the exactness of the upper row of HL in Lemma 4, but we still construct and show the commutativity of the two cubes because all the involved arrows arise naturally.

Lemma 4. *There is an exact commutative diagram with surjective vertical arrows*



Proof. Let $j : \overline{U}_0 \to \overline{U}_1$ and $i : \overline{Z}_1 \to \overline{U}_1$ be open and closed immersions, respectively. We consider the exact sequence $0 \to i_* i^! \mathcal{Q}(2) \to \mathcal{Q}(2) \to j_* j^* \mathcal{Q}(2) \to i_* R^1 i^! \mathcal{Q}(2) \to 0$ of étale sheaves over U_1 (for example, see the proof of [5, Corollary 8.5.6]). The isomorphism $R^q j_* \mathcal{Q}(2) \simeq i_* R^{q+1} i^! \mathcal{Q}(2)$ for $q \ge 1$ yields a distinguished triangle

$$i_* R i^! \mathcal{Q}(2) \to \mathcal{Q}(2) \to R j_* j^* \mathcal{Q}(2) \to i_* R i^! \mathcal{Q}(2)[1]. \tag{5}$$

According to [5, Theorem 8.5.2] (or see *loc. cit.* pp. 467 bottom for a more explicit version), we have $R i^! \mathcal{Q}(2) \simeq i^* \mathcal{Q}(1)[-2]$. Let $\overline{f}_{\mathfrak{X}}$ denotes the structural morphism over $\overline{K}$ of a K-scheme $\mathfrak{X}$. Then we obtain $\overline{f}_{U_1} \circ i = \overline{f}_{Z_1}$ and hence $\mathbf{R}\overline{f}_{U_1} i_* R i^! \mathcal{Q}(2) = \mathbf{R}\overline{f}_{Z_1} i^* \mathcal{Q}(1)[-2] = \mathbf{R}\overline{f}_{Z_1} \mathcal{Q}(1)[-2]$. Now applying the functor $\mathbf{R}\overline{f}_{U_1 *}$ to (5) yields a distinguished triangle

$$\mathbf{R}\overline{f}_{Z_1 *}\mathcal{Q}(1)[-2] \to \mathbf{R}\overline{f}_{U_1 *}\mathcal{Q}(2) \to \mathbf{R}\overline{f}_{U_0 *}\mathcal{Q}(2) \to \mathbf{R}\overline{f}_{Z_1 *}\mathcal{Q}(1)[-1]. \tag{6}$$

- The lower row of VB is obtained by taking Galois cohomology of (3).
- We construct the upper row of VB. The left upper horizontal arrow is constructed as follows. Let $\mathcal{F}$ be an étale sheaf over $\overline{\mathfrak{X}}$ for some K-scheme $\mathfrak{X}$. Then we have $H^2(\mathbf{R}\overline{f}_{\mathfrak{X}*}\mathcal{F}) =$

$R^2\overline{f}_{\mathfrak{X}*}\mathscr{F} \simeq H^2(\overline{\mathfrak{X}},\mathscr{F})$. Consider the commutative diagram of short exact sequences in the derived category of étale sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\tau}_{\leq 1}\mathbf{R}\overline{f}_{U_1*}\mathscr{Q}(2) & \longrightarrow & \tau_{\leq 2}\mathbf{R}\overline{f}_{U_1*}\mathscr{Q}(2) & \longrightarrow & H^2\left(\overline{U}_1,\mathscr{Q}(2)\right)[-2] & \longrightarrow & 0 \\ & & \downarrow & \swarrow & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{\tau}_{\leq 1}\mathbf{R}\overline{f}_{U_0*}\mathscr{Q}(2) & \longrightarrow & \tau_{\leq 2}\mathbf{R}\overline{f}_{U_0*}\mathscr{Q}(2) & \longrightarrow & H^2\left(\overline{U}_0,\mathscr{Q}(2)\right)[-2] & \longrightarrow & 0 \end{array} \tag{7}$$

where the vertical arrows are induced by the middle arrow of (6), and the rows come from respective short exact sequences in the category of complexes of étale sheaves. Since the right vertical map is zero by Lemma 3 (1), the arrow $\tau_{\leq 2}\mathbf{R}\overline{f}_{U_1*}\mathscr{Q}(2) \to \tau_{\leq 2}\mathbf{R}\overline{f}_{U_0*}\mathscr{Q}(2)$ factors uniquely through $\widetilde{\tau}_{\leq 1}\mathbf{R}\overline{f}_{U_0*}\mathscr{Q}(2)$ by the universal property of kernels. But the quasi-isomorphism $\tau_{\leq 1}\mathbf{R}\overline{f}_{U_0*}\mathscr{Q}(2) \to \widetilde{\tau}_{\leq 1}\mathbf{R}\overline{f}_{U_0*}\mathscr{Q}(2)$ becomes an isomorphism in the derived category, so we obtain a map $\tau_{\leq 2}\mathbf{R}\overline{f}_{U_1*}\mathscr{Q}(2) \to \tau_{\leq 1}\mathbf{R}\overline{f}_{U_0*}\mathscr{Q}(2)$.

The right upper horizontal arrow is obtained by $\mathbf{R}\overline{f}_{U_0*}\mathscr{Q}(2) \to \mathbf{R}\overline{f}_{Z_1*}\mathscr{Q}(1)[-1]$ from (6). Indeed, we have a map

$$\tau_{\leq 1}\mathbf{R}\overline{f}_{U_0*}\mathscr{Q}(2) \to \tau_{\leq 1}\left(\mathbf{R}\overline{f}_{Z_1*}\mathscr{Q}(1)[-1]\right) = \left(\tau_{\leq 0}\mathbf{R}\overline{f}_{Z_1*}\mathscr{Q}(1)\right)[-1].$$

- The vertical arrows of VB are induced by the distinguished triangle (see [9, pp. 303, (12.3.2) and (12.3.3)])

  $$\tau_{\leq j-1}\mathbf{R}\overline{\pi}_*\mathscr{Q}(2) \to \tau_{\leq j}\mathbf{R}\overline{\pi}_*\mathscr{Q}(2) \to H^j\left(\overline{\mathfrak{X}},\mathscr{Q}(2)\right)[-j] \to \tau_{\leq j-1}\mathbf{R}\overline{\pi}_*\mathscr{Q}(2)[1]$$

  with $\overline{\pi} : \overline{\mathfrak{X}} \to \overline{K}$ the structural morphism. The map

  $$H^3\left(K,\tau_{\leq 2}\mathbf{R}\overline{f}_{U_1*}\mathscr{Q}(2)\right) \to H^1\left(K,H^2\left(\overline{U}_1,\mathscr{Q}(2)\right)\right)$$

  is *surjective* since $H^4(K,\mathbb{Q}/\mathbb{Z}(2)) = 0$. Similarly, the other vertical arrows are also surjective.
- Taking Galois cohomology of the triangle (6) yields an exact sequence

  $$H^3\left(K,\mathbf{R}f_{U_1*}\mathscr{Q}(2)\right) \to H^3\left(K,\mathbf{R}f_{U_0*}\mathscr{Q}(2)\right) \to H^2\left(K,\mathbf{R}f_{Z_1*}\mathscr{Q}(1)\right),$$

  i.e. $H^3(U_1,\mathscr{Q}(2)) \to H^3(U_0,\mathscr{Q}(2)) \to H^2(Z_1,\mathscr{Q}(1))$ is exact. The horizontal rows of VF are constructed.
- All the vertical arrows of VF are canonical projections.
- Vertical arrows of HU (those arrows of the form $H^i(K,\tau_{\leq j}\mathbf{R}\overline{f}_{\mathfrak{X}*}\mathscr{F}) \to H^i(\mathfrak{X},\mathscr{F})$) are induced by the canonical map $\tau_{\leq i}A^* \to A^*$ of complexes, and that of HL are obtained by the Hochschild–Serre spectral sequence $H^p(K,H^q(\overline{\mathfrak{X}},\mathscr{Q}(i))) \Rightarrow H^{p+q}(\mathfrak{X},\mathscr{Q}(i))$, where $\mathfrak{X}$ is a variety defined over K.

Now we check the commutativity of the diagram.

- VL, VM and VR commute by construction of the Hochschild–Serre spectral sequence (see [8, pp. 17]).
- VF commutes by construction. For VB, we have a diagram in the derived category of étale sheaves

  $$\begin{array}{ccccc} \tau_{\leq 2}\mathbf{R}\overline{f}_{U_1*}\mathscr{Q}(2) & \longrightarrow & \tau_{\leq 1}\mathbf{R}\overline{f}_{U_0*}\mathscr{Q}(2) & \longrightarrow & \tau_{\leq 0}\mathbf{R}\overline{f}_{Z_1*}\mathscr{Q}(1)[-1] \\ \downarrow & & \downarrow & & \downarrow \\ H^2\left(\overline{U}_1,\mathscr{Q}(2)\right)[-2] & \longrightarrow & H^1\left(\overline{U}_0,\mathscr{Q}(2)\right)[-1] & \longrightarrow & H^0\left(\overline{Z}_1,\mathscr{Q}(1)\right)[-1] \end{array}$$

  where the vertical arrows come from truncations. This diagram commutes because the rows are both obtained from the triangle (6). Hence applying the function $H^3(K,-)$ yields the commutativity of VB.

- HU commutes by construction of truncation and by functoriality of Galois cohomology.

Thus the horizontal lower face HL commutes by diagram chasing. □

Corollary 5. *The image of* $Ш^2_\omega(T')$ *in* $H^3(Y,\mathscr{Q}(2))/\operatorname{Im} H^3(K,\mathscr{Q}(2))$ *is unramified.*

Proof. By Lemma 4 and functoriality, the following diagram commutes:

$$\begin{array}{ccccc} & & Ш^2_\omega(T') & \longrightarrow & Ш^2_\omega\left(H^0\left(\overline{Z}_1,\mathscr{Q}(1)\right)\right) \\ & & \iota_T\downarrow & & \iota_Z\downarrow \\ H^1\left(K,H^2\left(\overline{U}_1,\mathscr{Q}(2)\right)\right) & \xrightarrow{\Phi} & H^2\left(K,H^1\left(\overline{U}_0,\mathscr{Q}(2)\right)\right) & \longrightarrow & H^2\left(K,H^0\left(\overline{Z}_1,\mathscr{Q}(1)\right)\right) \\ \mathrm{HS}_1\downarrow & & \mathrm{HS}_0\downarrow & & \downarrow \\ \dfrac{H^3(U_1,\mathscr{Q}(2))}{\operatorname{Im} H^3(K,\mathscr{Q}(2))} & \xrightarrow[\Psi]{} & \dfrac{H^3(Y,\mathscr{Q}(2))}{\operatorname{Im} H^3(K,\mathscr{Q}(2))} & \longrightarrow & H^2(Z_1,\mathscr{Q}(1)). \end{array}$$

Here ι_T and ι_Z are respective inclusions. According to Lemma 3, the second row is exact and we have $Ш^2_\omega(H^0(\overline{Z}_1,\mathscr{Q}(1)))=0$. Subsequently a diagram chasing shows that $\operatorname{Im}(\iota_T)\subset\operatorname{Im}\Phi$. By commutativity of the left lower square, $\operatorname{Im}(\mathrm{HS}_0\circ\iota_T)\subset\operatorname{Im}(\mathrm{HS}_0\circ\Phi)=\operatorname{Im}(\Psi\circ\mathrm{HS}_1)$. The composite map

$$H^1\left(K,H^2\left(\overline{U}_1,\mathscr{Q}(2)\right)\right)\to\frac{H^3(U_1,\mathscr{Q}(2))}{\operatorname{Im} H^3(K,\mathscr{Q}(2))}\to H^3\left(\overline{U}_1,\mathscr{Q}(2)\right)$$

factors through $H^1(\overline{K},H^2(\overline{U}_1,\mathscr{Q}(2)))=0$ by the functoriality of the Hochschild–Serre spectral sequence. Thus we obtain from Lemma 3 (2) that

$$\operatorname{Im}\mathrm{HS}_1\subset\operatorname{Ker}\left(\frac{H^3(U_1,\mathscr{Q}(2))}{\operatorname{Im} H^3(K,\mathscr{Q}(2))}\to H^3\left(\overline{U}_1,\mathscr{Q}(2)\right)\right)=\operatorname{Im}\left(\frac{H^3(Y^c,\mathscr{Q}(2))}{H^3(K,\mathscr{Q}(2))}\to\frac{H^3(U_1,\mathscr{Q}(2))}{H^3(K,\mathscr{Q}(2))}\right).$$

Thus the image of $Ш^2_\omega(T')$ in $H^3(Y,\mathscr{Q}(2))/\operatorname{Im} H^3(K,\mathscr{Q}(2))$ comes from $H^3(Y^c,\mathscr{Q}(2))$, i.e. $Ш^2_\omega(T')$ is unramified. In particular, the image of $Ш^2(T')$ in $H^3_{\mathrm{lc}}(Y,\mathscr{Q}(2))$ is unramified. □

If we restrict to the subgroup $Ш^2(T')$ of $Ш^2_\omega(T')$, then its image lies in the subgroup $H^3_{\mathrm{lc}}(Y,\mathbb{Q}/\mathbb{Z}(2))$ of $H^3(Y,\mathbb{Q}/\mathbb{Z}(2))/\operatorname{Im} H^3(K,\mathbb{Q}/\mathbb{Z}(2))$. Now the image of $Ш^2(T')$ in $H^3(K(Y),\mathbb{Q}/\mathbb{Z}(2))/H^3(K,\mathbb{Q}/\mathbb{Z}(2))$ is unramified by Corollary 5.

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The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than his research organizations.

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