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On the Poisson transform on a homogenous vector bundle over the quaternionic hyperbolic space

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Abstract. Let $K\times_M V$ be the homogenous vector bundle over $K/M = Sp(n)\times Sp(1)/Sp(n-1)\times Sp(1)$ associated to an irreducible representation (δ_ν, V) of Sp(1). We give an image characterization of the Poisson transform $\mathscr{P}_{\lambda,\nu}$ of L^2-section of $K\times_M V$. We also show that $\mathscr{P}_{\lambda,\nu} f$, $f\in L^p(K\times_M V)$ satisfies a Hardy-type estimate.

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1. Introduction and main results

Let G be a connected real semi simple noncompact Lie group with finite center and K be a maximal compact subgroup of G. Then $X = G/K$ is a Riemannian symmetric space of noncompact type. Let $G = KAN$ be an Iwasawa decomposition of G, and let M be the centralizer of A in K. We write $g = \kappa(g)\mathrm{e}^{H(g)}n(g)$, for each $g\in G$ according to $G = KAN$. A central result in harmonic analysis (see [10]) asserts that all joint eigenfunctions F of the algebra $\mathbb{D}(X)$ of invariant differential operators, are Poisson integrals

$$F(g) = \mathscr{P}_\lambda f(g) := \int_K \mathrm{e}^{-(i\lambda+\rho)H(g^{-1}k)} f(k)\,\mathrm{d}k,$$

of a hyperfunction f on K/M, for a generic $\lambda\in\mathfrak{a}_c^*$ (the complexification of $\mathfrak{a}^*$ the real dual of $\mathfrak{a}$).

This result suggests the problem of characterization of the L^p-range of the Poisson transform on the Furstenberg boundary K/M (see [4, 7–9, 11],..), some of these results were generalized to some class of homogenous vector bundles (see [1–3, 5, 6]).

Now we restrict ourselves to the hyperbolic quaternionic space $Sp(n,1)/Sp(n)\times Sp(1)$. We introduce the homogenous vector bundle that we consider in this paper. Let δ_ν be a unitary irreducible representation of $Sp(1)$ realized on a $(\nu+1)$-dimensional Hilbert space $(V, \langle\cdot,\cdot\rangle_\nu)$. We extend δ_ν to a representation τ_ν of K by setting $\tau_\nu\equiv 1$ on $Sp(n)$ (i.e $\tau_\nu = 1\otimes\delta_\nu$). As usual the space of sections of the homogeneous vector bundle $G\times_K V$ associated with τ_ν, will be identified with the space $\Gamma(G/K,\tau_\nu)$ of vector valued functions $F: G\to V$ which are right K-covariant of type τ_ν, i.e.,

$$F(gk) = \tau_\nu(k)^{-1}F(g),\quad \forall\, g\in G,\quad \forall\, k\in K. \tag{1}$$

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We denote by $C^\infty(G/K,\tau_\nu)$ the elements of $\Gamma(G/K,\tau_\nu)$ that are smooth. Let σ_ν denote the restriction of τ_ν to the group $M \simeq Sp(n-1)\times Sp(1)$, and let $K\times_M V$ be the associated homogeneous vector bundle. We identify the section space $L^p(K\times_M V)$ with $L^p(K/M,\sigma_\nu)$, the space of all functions $f: K\to V$ that are L^p with respect to the Haar measure of K and satisfy

$$f(km) = \sigma_\nu\left(m^{-1}\right) f(k)$$

for all $k\in K, m\in M$.

For $\lambda\in\mathbb{C}$ and $f\in L^1(K/M,\sigma_\nu)$, the Poisson transform $\mathscr{P}_{\lambda,\nu} f$ is defined by

$$\mathscr{P}_{\lambda,\nu} f(g) = \int_K \mathrm{e}^{-(i\lambda+\rho)H(g^{-1}k)}\tau_\nu\left(\kappa\left(g^{-1}k\right)\right) f(k)\,\mathrm{d}k$$

where $\rho = 2n+1$ is the half sum of the positive roots with multiplicities (for more detail see section 2).

Let us mention that in [6] for $\lambda\in\mathbb{R}\backslash\{0\}$, we gave an image characterization of the Poisson transform $\mathscr{P}_{\lambda,\nu}$ of L^2-sections, as application we obtain a characterization of the L^2-range of the generalized spectral projections.

The aim of this paper is to give a characterization of the Poisson transform $\mathscr{P}_{\lambda,\nu}$ of L^2-sections and a L^p-Hardy type estimate for $\Re(i\lambda)>0$ and $i\lambda+2n\pm(\nu+1)\notin 2\mathbb{Z}^-$.

Let Ω denote the Casimir element of the Lie algebra $\mathfrak{g}$ of G, viewed as a differential operator acting on $C^\infty(G/K,\tau_\nu)$. $\mathscr{P}_{\lambda,\nu}$ map $L^p(K/M,\sigma_\nu)$ into $\mathscr{E}_\lambda(G/K,\tau_\nu)$ the space of eigensections of the Casimir operator the space of all $F\in C^\infty(G/K,\tau_\nu)$ satisfying

$$\Omega F = -\left(\lambda^2+\rho^2-\nu(\nu+2)\right)F. \tag{2}$$

We define the Hardy type norm $\|\cdot\|_{\lambda,p}$ for $F\in C^\infty(G/K,\tau_\nu)$, by

$$\| F\|_{\lambda,p} = \sup_{t>0} e^{(\rho-\Re(i\lambda))t}\left(\int_K \|F(ka_t)\|_{\tau_\nu}^p\,\mathrm{d}k\right)^{\frac{1}{p}},$$

where $\|\cdot\|_{\tau_\nu}$ is the norm on V.

We introduce the subspace $\mathscr{E}_\lambda^p(G/K,\tau_\nu)$ of $\mathscr{E}_\lambda(G/K,\tau_\nu)$, defined by

$$\mathscr{E}_\lambda^p(G/K,\tau_\nu) = \left\{F\in\mathscr{E}_\lambda(G/K,\tau_\nu);\quad \|F\|_{\lambda,p}<\infty\right\}.$$

The first main result in this paper can be stated as follows.

Theorem 1. *Let $\lambda\in\mathbb{C}$ such that $\Re(i\lambda)>0$ and $i\lambda+2n\pm(\nu+1)\notin 2\mathbb{Z}^-$.*

The map $\mathscr{P}_{\lambda,\nu}$ is a topological isomorphism of the space $L^2(K/M,\sigma_\nu)$ onto the space $\mathscr{E}_\lambda^2(G/K,\tau_\nu)$.

Moreover, there exists a positive constant γ_λ such that for every $f\in L^2(K/M,\sigma_\nu)$, we have

$$|\mathbf{c}_\nu(\lambda)|\,\|f\|_2 \le \|\mathscr{P}_{\lambda,\nu} f\|_{\lambda,2} \le \gamma_\lambda \|f\|_2, \tag{3}$$

where $\mathbf{c}_\nu(\lambda)$ is the Harish–Chandra $\mathbf{c}$-function associated with τ_ν given by

$$\mathbf{c}_\nu(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\rho-1)\Gamma(i\lambda)}{\Gamma\left(\frac{i\lambda+\nu+\rho}{2}\right)\Gamma\left(\frac{i\lambda+\rho-\nu-2}{2}\right)}.$$

Remark 2. We should notice that for λ real the asymptotic behaviour of Poisson transform $\mathscr{P}_{\lambda,\nu} f$ is different from the case $\lambda\in\mathbb{C}\backslash\mathbb{R}$, the above Theorem 1 complete our study on the L^2-range of $\mathscr{P}_{\lambda,\nu}$ initiated in [6].

The second main result in this paper is a Hardy type estimate of the Poisson transform $\mathscr{P}_{\lambda,\nu} f$, $f\in L^p(K/M,\sigma_\nu)$), stated as follows.

Theorem 3. *Let $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$ and $i\lambda + 2n \pm (\nu+1) \notin 2\mathbb{Z}^-$. There exists a positive constant γ_λ such that for every $f \in L^p(K/M, \sigma_\nu)$ with $1 < p < +\infty$, we have*

$$|\mathbf{c}_\nu(\lambda)| \|f\|_p \leq \left\|\mathscr{P}_{\lambda,\nu} f\right\|_{\lambda,p} \leq \gamma_\lambda \|f\|_p, \tag{4}$$

Remark 4. Since $\mathscr{P}_{\lambda,\nu}(L^p(K/M,\sigma_\nu))$ is a subspace of $\mathscr{E}_\lambda(G/K,\tau_\nu)$, then from the right side of (4) we deduce that $\mathscr{P}_{\lambda,\nu}$ is a continuous map from $L^p(K/M,\sigma_\nu)$ into $\mathscr{E}^p_\lambda(G/K,\tau_\nu)$.

We think that by similar reasoning as in [2], $\mathscr{P}_{\lambda,\nu}$ is an isomorphism from $L^p(K/M,\sigma_\nu)$ onto $\mathscr{E}^p_\lambda(G/K,\tau_\nu)$.

Due to some technical difficulties we are blocked, we hope to return to the problem in the near future.

We now describe the plan of the paper.

In Section 2, we recall some basic known results of harmonic analysis on the quaternionc hyperbolic spaces, In Section 3, we define the Poisson transform $\mathscr{P}_{\lambda,\nu}$. In Section 4, we prove Theorem 3. Section 5, is devoted to prove Theorem 1.

2. Preliminaries

In this section we recall some background of harmonic analysis on the quaternionic hyperbolic space.

Let $G = Sp(n,1)$ be the group of all linear transformations of the right $\mathbb{H}$-vector space $\mathbb{H}^{n+1}$ which preserve the sesquilinear form

$$[u,v] = \sum_{j=1}^{n} \overline{u}_j\, v_j - \overline{u}_{n+1}\, v_{n+1}, \quad u = (u_j), v = (v_j) \in \mathbb{H}^{n+1},$$

where $q \to \overline{q}$ is the standard involution of $\mathbb{H}$.

By $Sp(q)$ we denote the group of all $q \times q$ matrices over $\mathbb{H}$ keeping the inner product on $\mathbb{H}^n$ $\langle u,v\rangle = \sum_{j=1}^{q} \overline{u}_j\, v_j$ invariant. In particular $Sp(1)$ is identified with the group of quaternions of norm equal to one. Let

$$K = \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}; u \in Sp(n), v \in Sp(1) \right\} \simeq Sp(n) \times Sp(1).$$

K is a maximal compact subgroup of G. The quaternionic hyperbolic space is the rank one symmetric space G/K of the noncompact type. It can be realized as the unit ball $\mathbb{B}(\mathbb{H}^n) = \{x \in \mathbb{H}^n; |\, x \,| < 1\}$. The group G acts on $\mathbb{B}(\mathbb{H}^n)$ by the fractional transformations

$$x \mapsto g.x = (ax+b)(cx+d)^{-1},$$

for $g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$. Denote by $\mathfrak{g}$ the Lie algebra of G; $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is the vector space of matrices of the form

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}, x \in \mathbb{H}^n \right\}, \quad \text{and} \quad \mathfrak{k} = \left\{ \begin{pmatrix} X & 0 \\ 0 & q \end{pmatrix}, X^* + X = 0, q + \overline{q} = 0 \right\},$$

where X^* is the conjugate transpose of the matrix X and $q \in \mathbb{H}$.

Let $H = (\begin{smallmatrix} 0_n & e_1 \\ {}^t e_1 & 0 \end{smallmatrix}) \in \mathfrak{p}$ with ${}^t e_1 = (1,0,\cdots,0)$. Then $\mathfrak{a} = \mathbb{R}H$ is a Cartan subspace in $\mathfrak{p}$, and $A = \{a_t = \exp tH; t \in \mathbb{R}\}$ is the corresponding analytic subgroup where

$$a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}.$$

The group M is equal to

$$M = \left\{ g = \begin{pmatrix} q & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & q \end{pmatrix}, m \in Sp(n-1), |\, q \,| = 1 \right\} \simeq Sp(n-1) \times Sp(1).$$

Let $\alpha \in \mathfrak{a}^*$ be defined by $\alpha(H) = 1$. Then a system Σ of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ is $\Sigma = \{\pm\alpha, \pm 2\alpha\}$ if $n \geq 2$ and $\Sigma = \{\pm 2\alpha\}$ if $n = 1$, with Weyl group $W \simeq \{\pm Id\}$. A positive subsystem of roots corresponding to the positive Weyl chamber $\mathfrak{a}^+ \simeq \mathbb{R}^+$ in $\mathfrak{a}$ is $\Sigma^+ = \{\alpha, 2\alpha\}$ if $n \geq 2$ and $\Sigma^+ = \{2\alpha\}$ if $n = 1$.

Let $\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ be the direct sum of the positive root subspaces, with $\dim \mathfrak{g}_\alpha = 4(n-1)$ and $\dim \mathfrak{g}_{2\alpha} = 3$ and N the corresponding subgroup of G. Then the half sum of the positive restricted roots with multiplicities counted ρ is equal to $(2n+1)\alpha$, and shall be viewed as a real number $\rho = 2n+1$ by the identification $\mathfrak{a}_c^* \simeq \mathbb{C}$ via $\lambda\alpha \leftrightarrow \lambda$.

3. Poisson transform

In this section we define the Poisson transform on the vector bundle $G \times_K V$ over $Sp(n,1)/Sp(n) \times Sp(1)$ associated with τ_ν and derive some results referring to [12, 13] and [14] for more details on the subject. Let $P = MAN$ the minimal standard parabolic subgroup of G. For $\lambda \in \mathbb{C}$ we consider the representation $\sigma_{\lambda,\nu}$ of P on V defined by $\sigma_{\lambda,\nu}(man) = a^{\rho - i\lambda}\sigma_\nu(m)$. Then $\sigma_{\lambda,\nu}$ defines a principal series representations of G on the Hilbert space

$$H^{\lambda,\nu} := \left\{ f : G \to V \,\middle|\, f(gman) = \sigma_{\lambda,\nu}^{-1}(man) f(g), \quad \forall\, man \in MAN, f_{|K} \in L^2 \right\},$$

where G acts by the left regular representation. We shall denote by $C^{-\omega}(G, \sigma_{\lambda,\nu})$ the space of its hyperfunctions vectors. By the Iwasawa decomposition, the restriction map from G to K gives an isomorphism from $H^{\lambda,\nu}$ onto the space $L^2(K, \sigma_\nu)$. This yields the compact picture of $H^{\lambda,\nu}$, with the group action given by

$$\pi_{\lambda,\nu}(g) f(k) = \mathrm{e}^{(i\lambda - \rho)H(g^{-1}k)} f\left(\kappa\left(g^{-1}k\right)\right).$$

By $C^{-\omega}(K/M, \sigma_\nu)$ we denote the space of its hyperfunctions vectors. The Poisson transform $\mathscr{P}_{\lambda,\nu}$ is the continuous, linear, G-equivariant map from $C^{-\omega}(G/P, \sigma_{\lambda,\nu})$ to $C^\infty(G/K, \tau_\nu)$ defined by

$$\mathscr{P}_{\lambda,\nu} f(g) = \int_K \tau_\nu(k) f(gk)\, \mathrm{d}k,$$

The integral is a formal notation with the meaning that the hyperfunction $\tau_\nu(.) f(g.)$ on K has to be applied to the constant function 1. In the compact picture the Poisson transform is given by

$$\mathscr{P}_{\lambda,\nu} f(g) = \int_K \mathrm{e}^{-(i\lambda + \rho)H(g^{-1}k)} \tau_\nu\left(\kappa\left(g^{-1}k\right)\right) f(k)\, \mathrm{d}k.$$

Let $\mathbb{D}(G/K, \tau_\nu)$ denote the algebra of left invariant differential operators on $C^\infty(G/K, \tau_\nu)$.

Proposition 5 ([6]).

(i) *$\mathbb{D}(G/K, \tau_\nu)$ is the algebra generated by the Casimir operator Ω of $\mathfrak{g}$.*

(ii) *For $\lambda \in \mathbb{C}, \nu \in \mathbb{N}$, the Poisson transform $\mathscr{P}_{\lambda,\nu}$ maps $C^{-\omega}(G/P, \sigma_{\lambda,\nu})$ to $\mathscr{E}_\lambda(G/K, \tau_\nu)$.*

We end this section by recalling a result of Olbrich [12] on the range of the Poisson transform on vector bundles which reads in our case as follows

Theorem 6 ([12]). *Let $\nu \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that*

(i) $-2i\lambda \notin \mathbb{N}$

(ii) $i\lambda + \rho \notin 2\mathbb{Z}^- - \nu \cup 2\mathbb{Z}^- + \nu + 2$.

Then the Poisson transform $\mathscr{P}_{\lambda,\nu}$ is a K-isomorphism from $C^{-\omega}(K/M, \sigma_\nu)$ onto $\mathscr{E}_\lambda(G/K, \tau_\nu)$.

4. Fatou-type Theorem

The proof of the right part of the estimate (4) follows from the following result.

Proposition 7. *Let $1 < p < +\infty$, and $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$. There exists a positive constant γ_λ such that for every $f \in L^p(K/M, \sigma_\nu)$, we have*

$$\sup_{t>0} e^{(\rho-\Re(i\lambda))t} \left(\int_K \|\mathscr{P}_{\lambda,\nu} f(ka_t)\|_{\tau_\nu}^p \, dk \right)^{\frac{1}{p}} \leq \gamma_\lambda \|f\|_p. \tag{5}$$

Proof. We recall that $\mathscr{P}_{\lambda,\nu} f(ka_t) = \int_K e^{-(i\lambda+\rho)H(a_{-t}k^{-1}h)} f(h) dh$, then

$$\left\|\mathscr{P}_{\lambda,\nu} f(ka_t)\right\|_{\tau_\nu} \leq \int_K e^{-(\Re(i\lambda)+\rho)H(a_{-t}k^{-1}h)} \left\|f(h)\right\|_{\tau_\nu} dh$$

the right term of the above inequality can be written as a convolution over the compact group K

$$\int_K e^{-(\Re(i\lambda)+\rho)H(a_{-t}k^{-1}h)} \left\|f(h)\right\|_{\tau_\nu} dh = \left(\left\|f(\cdot)\right\|_{\tau_\nu} * e_{\lambda,t} \right)(k)$$

where $e_{\lambda,t}(h) = e^{-(\Re(i\lambda)+\rho)H(a_{-t}h^{-1})}$, hence by using Hausdorff–Young inequality we get

$$\left(\int_K \left\|\mathscr{P}_{\lambda,\nu} f(ka_t)\right\|_{\tau_\nu}^p dk \right)^{\frac{1}{p}} \leq \|e_{\lambda,t}\|_1 \|f\|_p$$

then the desired result follows from the following estimate

$$\left\|e_{\lambda,t}\right\|_1 = \int_K e_{\lambda,t}(k) = \varphi_{-i\Re(i\lambda)}^{(\rho-\frac{1}{2},-\frac{1}{2})}(t) \leq \gamma_\lambda e^{(\Re(i\lambda)-\rho)t},$$

where γ_λ is a positive constant, $\varphi_\lambda^{(\alpha,\beta)}$ is the Jacobi function and the last inequality above follows from the fact that for $\Re(i\lambda) > 0$ the Jacobi function $\varphi_\lambda^{(\alpha,\beta)}$ verify as $t \to +\infty$ the following identities

$$\varphi_\lambda^{(\alpha,\beta)}(t) = e^{(i\lambda-\alpha-\beta-1)t} \left(\frac{2^{-i\lambda+\alpha+\beta+1}\Gamma(1+\alpha)\Gamma(i\lambda)}{\Gamma\left(\frac{i\lambda+\alpha+\beta+1}{2}\right)\Gamma\left(\frac{i\lambda+\alpha-\beta+1}{2}\right)} + \mathbf{o}(1) \right)$$

□

Theorem 8 (Fatou type Theorem). *Let $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$ and $i\lambda + 2n \pm (\nu+1) \notin 2\mathbb{Z}^-$, then we have*

$$\lim_{t \to +\infty} e^{(\rho-\Re(i\lambda))t} \mathscr{P}_{\lambda,\nu} f(ka_t) = \mathbf{c}_\nu(\lambda) f(k) \tag{6}$$

(i) *uniformly for $f \in C^\infty(K/M, \sigma_\nu)$*

(ii) *in the $L^p(K, V)$ for $f \in L^p(K/M, \sigma_\nu)$ with $1 < p < +\infty$, where $\mathbf{c}_\nu(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\rho-1)\Gamma(i\lambda)}{\Gamma(\frac{i\lambda+\nu+\rho}{2})\Gamma(\frac{i\lambda+\rho-\nu-2}{2})}$.*

Proof. For (i) see [13] and [14], it remains to show (ii).

Let $\epsilon > 0$, and $f \in L^p(K/M, \sigma_\nu)$. Since $C^\infty(K/M, \sigma_\nu)$ is dense in $L^p(K/M, \sigma_\nu)$ there exists $\varphi \in C^\infty(K/M, \sigma_\nu)$ such that

$$\|\varphi - f\|_p \leq \varepsilon.$$

For every $t > 0$ and $k \in K$ we put $\mathscr{P}_{\lambda,\nu}^t f(k) := \mathscr{P}_{\lambda,\nu} f(ka_t)$, then we have

$$\begin{aligned} \left\|e^{(\rho-\Re(i\lambda))t} \mathscr{P}_{\lambda,\nu}^t f(k) - \mathbf{c}_\nu(\lambda) f(k)\right\|_p &\leq \left\|e^{(\rho-\Re(i\lambda))t} \mathscr{P}_{\lambda,\nu}^t (f-\varphi)(k)\right\|_p \\ &+ \left\|e^{(\rho-\Re(i\lambda))t} \mathscr{P}_{\lambda,\nu}^t \varphi(k) - \mathbf{c}_\nu(\lambda)\varphi(k)\right\|_p + |\mathbf{c}_\nu(\lambda)| \left\|f - \varphi\right\|_p \end{aligned}$$

from the first part (i) and Lebesgue Theorem we get

$$\lim_{t\to+\infty}\left\| e^{(\rho-\Re(i\lambda))t}\mathscr{P}^t_{\lambda,\nu}\varphi(k)-\mathbf{c}_\nu(\lambda)\varphi(k)\right\|_p=0.$$

On the other hand by using (5) we find that

$$\left\| e^{(\rho-\Re(i\lambda))t}\mathscr{P}^t_{\lambda,\nu}(f-\varphi)(k)\right\|_p\le\gamma_\lambda\left\|f-\varphi\right\|_p\le\epsilon\gamma_\lambda$$

therefore $\forall\,\epsilon>0$, we have

$$\lim_{t\to+\infty}\left\| e^{(\rho-\Re(i\lambda))t}\mathscr{P}^t_{\lambda,\nu}f(k)-\mathbf{c}_\nu(\lambda)f(k)\right\|_p\le\epsilon\left(\gamma_\lambda+|\mathbf{c}_\nu(\lambda)|\right)$$

hence the result follows. □

Now we are able to prove the Theorem 3.

Proof. The right side of the estimation (4) is just a reformulation of Proposition 7. For the left side, from (ii) in Theorem 8 we deduce that there exists a sequence $t_j\to+\infty$ as j $\to+\infty$ such that

$$\lim_{j\to+\infty}e^{(\rho-\Re(i\lambda))t_j}\mathscr{P}_{\lambda,\nu}f\left(ka_{t_j}\right)=\mathbf{c}_\nu(\lambda)f(k)$$

almost everywhere, hence

$$\lim_{j\to+\infty}e^{(\rho-\Re(i\lambda))pt_j}\left\|\mathscr{P}_{\lambda,\nu}f\left(ka_{t_j}\right)\right\|^p_{\tau_\nu}=|\mathbf{c}_\nu(\lambda)|^p\left\|f(k)\right\|^p_{\tau_\nu}$$

almost everywhere, therefore by using the classical Fatou's lemma we get

$$\begin{aligned}|\mathbf{c}_\nu(\lambda)|^p\left\|f\right\|^p_p&=\int_K\lim_{j\to+\infty}e^{p(\rho-\Re(i\lambda))t_j}\left\|\mathscr{P}_{\lambda,\nu}f\left(ka_{t_j}\right)\right\|^p_{\tau_\nu}dk\\&\le\lim_{j\to+\infty}e^{p(\rho-\Re(i\lambda))t_j}\int_K\left\|\mathscr{P}_{\lambda,\nu}f\left(ka_{t_j}\right)\right\|^p_{\tau_\nu}dk\\&\le\left\|\mathscr{P}_{\lambda,\nu}f\right\|^p_{\lambda,p}.\end{aligned}$$

and the desired result follows. □

5. The L^2-range of the Poisson transform

We first recall some results of harmonic analysis on the homogeneous vector bundle $K\times_M V$ associated to the representation σ_ν of M.
We recall from the introduction that the space $L^2(K\times_M V)$ is identified with the space $L^2(K/M,\sigma_\nu)$ of V-valued functions f on K which satisfy $f(km)=\sigma_\nu(m^{-1})f(k)$ $(k\in K, m\in M)$ and $\|f\|\in L^2(K)$.

Let $\widehat{K}$ be the set of unitary equivalence classes of irreducible representations of K. For $\delta\in\widehat{K}$ let V_δ denote the representation space of δ with $d_\delta=\dim V_\delta$. We denote by $\widehat{K}(\sigma_\nu)$ the set of $\delta\in\widehat{K}$ such that σ_ν occurs in $\delta\,|_M$ with multiplicity $m_\delta>0$.

The decomposition of $L^2(K/M,\sigma_\nu)$ under K (the group K acts by left translations on this space) is given by the Frobenius reciprocity law

$$L^2\left(K/M,\sigma_\nu\right)=\bigoplus_{\delta\in\widehat{K}(\sigma_\nu)}V_\delta\otimes Hom_M(V,V_\delta),$$

where $v\otimes L$, for $v\in V_\delta, L\in Hom_M(V,V_\delta)$ is identified with the function $(v\otimes L)(k)=L^*(\delta(k^{-1})v)$, where L^* denotes the adjoint of L.
For each $\delta\in\widehat{K}(\sigma_\nu)$ let $(L_j)_{j=1}^{m_\delta}$ be an orthonormal basis of $Hom_M(V,V_\delta)$ with respect to the inner product $<L_1,L_2>=\frac{1}{\nu+1}Tr(L_1L_2^*)$. Let $\{v_1,\cdots,v_{d_\delta}\}$ be an orhonormal basis of V_δ. Then

$$f^\delta_{ij}:k\to\sqrt{\frac{d_\delta}{\nu+1}}L^*_i\left(\delta\left(k^{-1}\right)v_j\right),1\le i\le m_\delta,\quad 1\le j\le d_\delta,\quad \delta\in\widehat{K}(\sigma)$$

form an orthonormal basis of $L^2(K/M,\sigma_\nu)$.

For $f\in L^2(K/M,\sigma_\nu)$ we have the Fourier series expansion $f(k)=\sum_{\delta\in\widehat{K}(\sigma)}\sum_{i=1}^{m_\delta}\sum_{j=1}^{d_\delta}a_{ij}^\delta f_{ij}^\delta(k)$ with

$$\| f\|_2^2=\sum_{\delta\in\widehat{K}(\sigma)}\sum_{i=1}^{m_\delta}\sum_{j=1}^{d_\delta}\left|a_{ij}^\delta\right|^2.$$

We define for $\delta\in\widehat{K}(\sigma)$ and $\lambda\in\mathbb{C}$, the Eisenstein integral

$$\Phi_{\lambda,\delta}^L(g)=\int_K \mathrm{e}^{-(i\lambda+\rho)H(g^{-1}k)}\tau_\nu\left(\kappa\left(g^{-1}k\right)\right)L^*\delta\left(k^{-1}\right)\mathrm{d}k,\quad L\in Hom_M(V,V_\delta). \tag{7}$$

It is easy to see that $\Phi_{\lambda,\delta}^L$ satisfies the following identity

$$\Phi_{\lambda,\delta}^L(k_1gk_2)=\tau_\nu\left(k_2^{-1}\right)\Phi_{\lambda,\delta}^L(g)\delta\left(k_1^{-1}\right),\quad k_1,k_2\in K,\, g\in G.$$

and for every $g\in G$ and $v\in V$ we have

$$\mathscr{P}_{\lambda,\nu}\left(L^*\delta^{-1}(\cdot)v\right)(ka_t)=\Phi_{\lambda,\delta}^L(ka_t)v \tag{8}$$

In order to prove Theorem 1 we need the following Lemma.

Lemma 9. *Let $\lambda\in\mathbb{C}$ such that $\Re(i\lambda)>0$ and $i\lambda+2n\pm(\nu+1)\notin 2\mathbb{Z}^-$, $\delta\in\widehat{K}(\sigma)$. Then for $L,S\in Hom_M(V,V_\delta)$ we have*

$$\lim_{t\to+\infty}e^{2(\rho-\Re(i\lambda))t}Tr\left[\Phi_{\lambda,\delta}^L(a_t)^*\Phi_{\lambda,\delta}^S(a_t)\right]=|\mathbf{c}_\nu(\lambda)|^2\,Tr\left[LS^*\right] \tag{9}$$

Proof. Since $\mathscr{P}_{\lambda,\nu}(L^*\delta^{-1}(\cdot)v)(ka_t)=\Phi_{\lambda,\delta}^L(ka_t)v$, for every L $\in Hom_M(V,V_\delta)$ and $v\in V_\delta$, then by using (6) in particular for k = e (the identity element), we find that

$$\lim_{t\to+\infty}e^{(\rho-\Re(i\lambda))t}\Phi_{\lambda,\delta}^L(a_t)v=\mathbf{c}_\nu(\lambda)L^*(v). \tag{10}$$

We have

$$\begin{aligned}\lim_{t\to+\infty}e^{2(\rho-\Re(i\lambda))t}Tr\left[\Phi_{\lambda,\delta}^L(a_t)^*\Phi_{\lambda,\delta}^S(a_t)\right]&=\lim_{t\to+\infty}e^{2(\rho-\Re(i\lambda))t}\sum_{i=1}^{d_\delta}\left\langle\Phi_{\lambda,\delta}^S(a_t)v_i,\Phi_{\lambda,\delta}^L(a_t)v_i\right\rangle\\&=\sum_{i=1}^{d_\delta}\lim_{t\to+\infty}\left\langle e^{(\rho-\Re(i\lambda))t}\Phi_{\lambda,\delta}^S(a_t)v_i,e^{(\rho-\Re(i\lambda))t}\Phi_{\lambda,\delta}^L(a_t)v_i\right\rangle\end{aligned}$$

where $\{v_1,\cdots,v_{d_\delta}\}$ is an orhonormal basis of V_δ. Therefore by (10) we get

$$\begin{aligned}\lim_{t\to+\infty}e^{2(\rho-\Re(i\lambda))t}Tr\left[\Phi_{\lambda,\delta}^L(a_t)^*\Phi_{\lambda,\delta}^S(a_t)\right]&=|\mathbf{c}_\nu(\lambda)|^2\sum_{i=1}^{d_\delta}\left\langle S^*v_i,L^*(v_i)\right\rangle\\&=|\mathbf{c}_\nu(\lambda)|^2\,Tr\left[LS^*\right].\end{aligned}$$

□

Now we are able to prove Theorem 1.

Proof. From Proposition 7 and Theorem 6 it follows that $\mathscr{P}_{\lambda,\nu}$ is a continuous map from $L^2(K/M,\sigma_\nu)$ into $\mathscr{E}_\lambda^2(G/K,\tau_\nu)$.

We now prove the sufficiently condition, let $F\in\mathscr{E}_\lambda^2(G/K,\tau_\nu)$. Then by Theorem 6 there exists a functional f such that $F=\mathscr{P}_{\lambda,\nu}f$. By using the Fourier series expansion

$$f(k)=\sum_{\delta\in\widehat{K}(\sigma)}\sum_{i=1}^{m_\delta}\sum_{j=1}^{d_\delta}a_{ij}^\delta f_{ij}^\delta(k)$$

$F=\mathscr{P}_{\lambda,\nu}f$ can be written as

$$F(g)=\sum_{\delta\in\widehat{K}(\sigma)}\sqrt{\frac{d_\delta}{\nu+1}}\sum_{j=1}^{d_\delta}\sum_{i=1}^{m_\delta}a_{ij}^\delta\Phi_{\lambda,\delta}^{L_i}(g)v_j\quad\text{in}\quad C^\infty(G,\tau_\nu).$$

Moreover by using Schur relations and a straightforward computation we find that

$$\int_K \|F(ka_t)\|_\tau^2 \,\mathrm{d}k = \frac{1}{\nu+1}\sum_{\delta\in\widehat{K}(\sigma_\nu)}\sum_{j=1}^{d_\delta}\sum_{i,i'=1}^{m_\delta} a_{ij}^\delta\overline{a_{i'j}^\delta}\,Tr\left[\Phi_{\lambda,\delta}^{L_{i'}}(a_t)\Phi_{\lambda,\delta}^{L_i}(a_t)\right].$$

Let Λ be a finite subset in $\widehat{K}(\sigma)$, since $\| F \|_{\lambda,2} < \infty$ it follows that

$$\frac{1}{\nu+1}\sum_{\delta\in\Lambda}\sum_{j=1}^{d_\delta} e^{2(\rho-\Re(i\lambda))t}\sum_{i,i'=1}^{m_\delta} a_{ij}^\delta\overline{a_{i'j}^\delta}\,Tr\left[\Phi_{\lambda,\delta}^{L_{i'}}(a_t)\Phi_{\lambda,\delta}^{L_i}(a_t)\right] \le \| F \|_{\lambda,2}^2 < \infty. \tag{11}$$

By using (9) we find that

$$\begin{aligned}\lim_{t\to+\infty} e^{2(\rho-\Re(i\lambda))t}\sum_{i,i'=1}^{m_\delta} a_{ij}^\delta\overline{a_{i'j}^\delta}\,Tr\left[\Phi_{\lambda,\delta}^{L_{i'}}(a_t)\Phi_{\lambda,\delta}^{L_i}(a_t)\right] &= |\mathbf{c}_\nu(\lambda)|^2\sum_{i,i'=1}^{m_\delta} a_{ij}^\delta\overline{a_{i'j}^\delta}\,Tr\left[L_iL_{i'}^*\right]\\ &= (\nu+1)\,|\mathbf{c}_\nu(\lambda)|^2\sum_{i=1}^{\delta}\left|a_{ij}^\delta\right|^2.\end{aligned} \tag{12}$$

By going to the limit in (11) when $t\to+\infty$, using (12) we get

$$|\mathbf{c}_\nu(\lambda)|^2\sum_{\delta\in\Lambda}\sum_{j=1}^{d_\delta}\sum_{i=1}^{m_\delta}\left|a_{ij}^\delta\right|^2 \le \| F \|_{\lambda,2}^2 < \infty.$$

Since the subset Λ is arbitrary in $\widehat{K}(\sigma)$, then

$$\|f\|_2^2 = \sum_{\delta\in\widehat{K}(\sigma)}\sum_{j=1}^{d_\delta}\sum_{i=1}^{m_\delta}\left|a_{ij}^\delta\right|^2 \le |\mathbf{c}_\nu(\lambda)|^{-2}\,\| F \|_{\lambda,2}^2 < \infty.$$

Thus $f\in L^2(K/M,\sigma_\nu)$ and the proof is finished. □

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Declaration of interests

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