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Mathématique


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Volume 362 (2024), p. 167-170

Online since: 7 March 2024

<https://doi.org/10.5802/crmath.551>

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www.centre-mersenne.org — e-ISSN : 1778-3569



Research article / *Article de recherche*
Geometry and Topology / *Géométrie et Topologie*

There are no primitive Teichmüller curves in $\text{Prym}(2, 2)$

Il n'y a pas de courbes de Teichmüller primitives dans $\text{Prym}(2, 2)$

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Abstract. We complete the work of Lanneau–Möller [4] to show that there are no primitive Teichmüller curves in $\text{Prym}(2, 2)$.

Résumé. Nous terminons un travail initié par Lanneau et Möller [4] en montrant qu'il n'existe pas de courbes de Teichmüller primitives dans $\text{Prym}(2, 2)$.

Manuscript received 31 January 2023, accepted 25 July 2023.

1. Introduction

Teichmüller curves are closed $\text{GL}(2, \mathbb{R})$ -orbits in the moduli space of translation surfaces $\Omega\mathcal{M}_g$ that descend to isometrically-immersed algebraic curves in the moduli space of genus g Riemann surfaces \mathcal{M}_g . Lanneau–Möller [4] searched for *geometrically primitive* Teichmüller curves, i.e., those not arising from a covering construction, in a certain locus $\text{Prym}(2, 2)$ of genus 3 translation surfaces with two cone-points called *Prym eigenforms*. In the spirit of McMullen's proof that the decagon is the unique primitive Teichmüller curve in $\Omega\mathcal{M}_2(1, 1)$ (see [5, Theorem 6.3]), they reduced their search to considering whether any of 92 candidate Prym eigenforms generate a Teichmüller curve. In this note, we analyze those 92 candidates and show:

Theorem 1. *The Prym locus $\text{Prym}(2, 2) \subset \Omega\mathcal{M}_3$ contains no primitive Teichmüller curves.*

Our main tool is the newly-available SageMath package *Flatsurf* which partially computes the $\text{GL}(2, \mathbb{R})$ -orbit closure of an input translation surface. (See Appendix B of Delecroix–Rüth [2] for an introduction to *Flatsurf*.) Using *Flatsurf*, we construct the 92 candidates Prym eigenforms and find a periodic direction that violates the Veech dichotomy. See Section 2.2 for details.

We thank Erwan Lanneau for his assistance with this project.

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Background

Classifying the translation surfaces that generate Teichmüller curves, called *Veech surfaces*, is a central problem in Teichmüller dynamics. See Hubert–Schmidt [3] for an introduction to Veech surfaces, Möller [8] for a list of known examples of Teichmüller curves, Lanneau–Möller [4] for a history of classification results, and McMullen [7] for a general survey.

While McMullen [6] classified all primitive Teichmüller curves in genus two, we do not have a complete classification of primitive Teichmüller curves in all other genera. In genus three, Bainbridge–Habegger–Möller [1] found a numerical bound on the number of *algebraically primitive* Teichmüller curves, yet the bound is too large for a complete classification. (An *algebraically primitive* Teichmüller curve is one for which the trace field of the generating Veech surface has maximal degree, i.e., degree the genus of the Veech surface. We remark that algebraically primitive Teichmüller curves are automatically geometrically primitive, yet the converse does not hold.) In fact, all but finitely many geometrically primitive Teichmüller curves in genus 3 lie in the Prym loci (see McMullen [7, Theorem 5.5]).

At another level, Lanneau–Möller [4] began searching for Teichmüller curves among translation surfaces having a certain *Prym involution* that generalizes the hyperelliptic involution in genus 2. (See Lanneau–Möller [4] for the definition.) They showed that there are no *geometrically primitive* Teichmüller curves in $\text{Prym}(2, 1, 1)$ and initiated the proof for $\text{Prym}(2, 2)$ that we finish in this note. The remaining case of whether there are primitive Teichmüller curves in the locus $\text{Prym}(1, 1, 1, 1)$ is still open and seems interesting. Outside the Prym loci, Winsor [9] showed that the Veech 14-gon generates the unique algebraically primitive Teichmüller curve in the hyperelliptic component of the stratum $\Omega\mathcal{M}_3(2, 2)$.

2. Proof of Theorem 1

2.1. Candidate surfaces

In this section we review, following Lanneau–Möller [4], how to construct the 92 candidate surfaces that could generate a Teichmüller curve.

Every surface in $\text{Prym}(2, 2)$ decomposes into horizontal cylinders (after possibly rotating the surface). The combinatorics of the horizontal separatrices matches one of the eight polygonal models in Figure 1. (See also Figure 3 in Lanneau–Möller [4].) In each underlying polygonal model, the Prym involution fixes one cylinder, called C_2 , and permutes the other two, called C_1 and C_3 . The surface is then determined by choices of the widths, heights and twists of the cylinders C_1 and C_2 , as well as of the length of a relative period called the *slit parameter* s . See Proposition 8.1 in Lanneau–Möller [4] for an example of one of the eight possible separatrix diagrams.

In this respect, Lanneau–Möller [4] determine a finite list of possible values for the width and the height of C_2 , the surface being normalized with $w(C_1) = h(C_1) = 1$. More precisely, they first show (see their Theorem 4.5 and Section 6.2) that the only possible trace fields are $\mathbb{Q}[\sqrt{D}]$ for $D \in \{2, 3, 33\}$. They then encode the data of the widths and the heights of the horizontal and vertical cylinders in a *reduced intersection matrix*. In Proposition 5.5 and Sections 6.3–6.5, they compute a finite list of such matrices that could give rise to Teichmüller curves; we recall this list in Table 1 along with the associated width and height of C_2 .

Having established the width and height parameters of the cylinders, it remains to control the length of the slit parameter and the twists of the cylinders. In Section 8.2 of Lanneau–Möller [4]

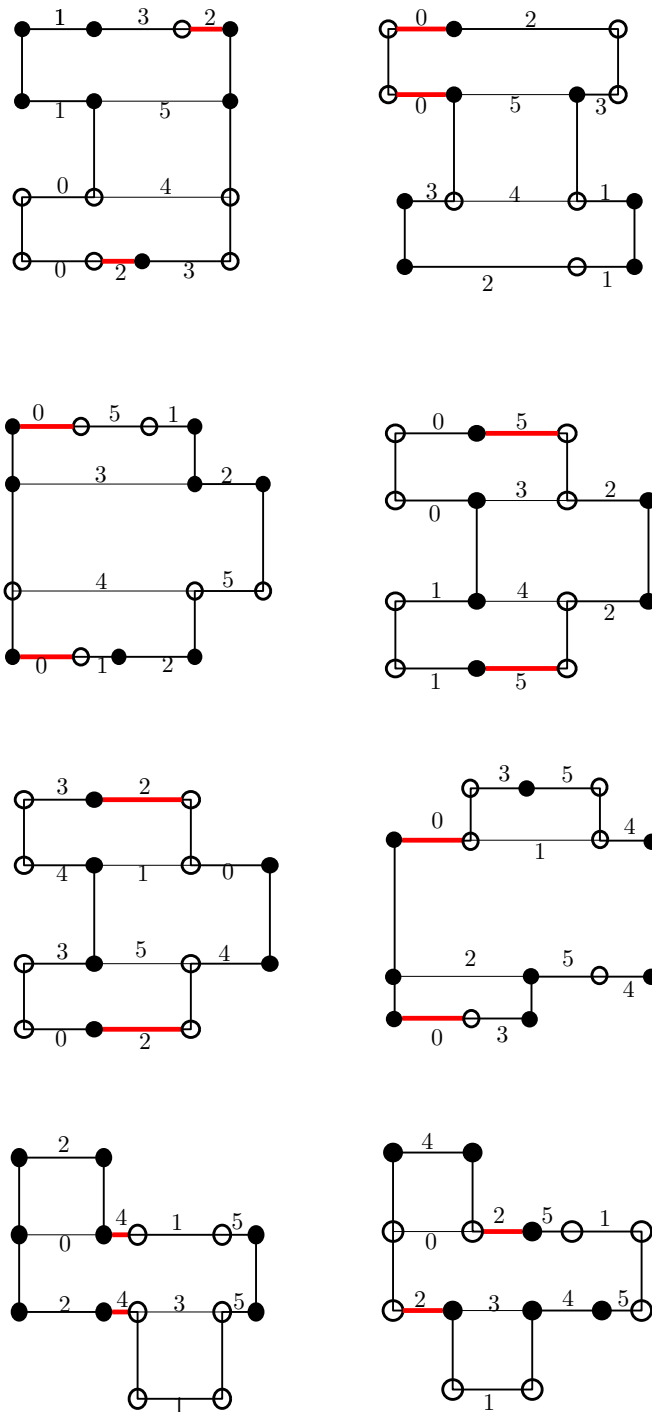


Figure 1. The 8 polygonal models. The slit parameter encodes the length of the bold red relative period.

Table 1. The list of trace fields and reduced matrices that could give rise to Teichmüller curves¹, and the corresponding width and height of the cylinder C_2 .

Trace field	Reduced intersection matrix	$w(C_2)$	$h(C_2)$
$\mathbb{Q}[\sqrt{2}]$	$\begin{pmatrix} 72 & 48 \\ 24 & 18 \end{pmatrix}$	$\frac{\sqrt{2}}{2}$	$2\sqrt{2}$
$\mathbb{Q}[\sqrt{3}]$	$\begin{pmatrix} 72 & 24 \\ 12 & 6 \end{pmatrix}$	$\frac{-1+\sqrt{3}}{2}$	$-2 + 2\sqrt{3}$
$\mathbb{Q}[\sqrt{3}]$	$\begin{pmatrix} 72 & 24 \\ 48 & 18 \end{pmatrix}$	$\frac{1+\sqrt{3}}{2}$	$2 + 2\sqrt{3}$
$\mathbb{Q}[\sqrt{3}]$	$\begin{pmatrix} 36 & 12 \\ 30 & 12 \end{pmatrix}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$
$\mathbb{Q}[\sqrt{33}]$	$\begin{pmatrix} 6 & 24 \\ 12 & 54 \end{pmatrix}$	$\frac{3+\sqrt{33}}{2}$	$\frac{3+\sqrt{33}}{6}$

and the corresponding code, they explicitly enumerate the finite list of possible values for such parameters that would lead to one of each admissible reduced intersection matrix. This amounts to a finite list of candidate surfaces (see Table 2 of Lanneau–Möller [4]). In the code joint to this note, we list all those candidate surfaces and construct them using *Flatsurf*.

2.2. Cylinder decompositions

Given a translation surface M , *Flatsurf* can efficiently generate many saddle connections on M . For each such saddle connection, it can compute the decomposition of the straight-line flow on M in that direction into cylinders and minimal components. Observe that if *Flatsurf* finds a direction for which M decomposes completely into cylinders, yet those cylinders have incommensurable moduli, then this direction witnesses that M does not satisfy the Veech dichotomy (See Hubert–Schmidt [3, Theorem 1]). Our code analyzes each of the 92 candidates and in each case finds a direction that is not completely parabolic. We conclude that none of the candidates is Veech, proving Theorem 1.

References

- [1] M. Bainbridge, P. Habegger, M. Möller, “Teichmüller curves in genus three and just likely intersections in $G_m^n \times G_a^n$ ”, *Publ. Math., Inst. Hautes Étud. Sci.* **124** (2014), p. 1-98.
- [2] V. Delecroix, J. Rüth, “A new orbit closure in genus 8?”, 2021, <https://arxiv.org/abs/2110.05407>.
- [3] P. Hubert, T. Schmidt, “An Introduction to Veech Surfaces”, in *Handbook of dynamical systems. Volume 1B*, Elsevier, 2006, p. 501-526.
- [4] E. Lanneau, M. Moeller, “Non-Existence and Finiteness Results for Teichmüller Curves in Prym Loci”, *Exp. Math.* **31** (2019), p. 621-636.
- [5] C. T. McMullen, “Teichmüller curves in genus two: Torsion divisors and ratios of sines”, *Invent. Math.* **165** (2006), no. 3, p. 651-672.
- [6] ———, “Dynamics of $SL_2(\mathbb{R})$ over moduli space in genus two”, *Ann. Math.* **165** (2007), no. 2, p. 397-456.
- [7] ———, “Billiards and Teichmüller curves”, <https://people.math.harvard.edu/~ctm/papers/home/text/papers/tsurv/tsurv.pdf>, 2021.
- [8] M. Möller, “Geometry of Teichmüller curves”, in *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures*, World Scientific, 2018, p. 2017-2034.
- [9] K. Winsor, “Uniqueness of the Veech 14-gon”, 2022, <https://arxiv.org/abs/2210.09702>.

¹The last two reduced matrices of Lanneau–Möller [4] do not generate Teichmüller curves according to Table 2 of [4], so we do not consider them.