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New proof and generalization of some results on translated sums over k-almost primes

Nouvelle preuve et généralisation de certains résultats sur des sommes translatées sur k-presque premiers

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Abstract. A sequence $\mathscr{A}$ of strictly positive integers is said to be primitive if none of its terms divides the others, Erdős conjectured that the sum $f(\mathscr{A},0) \le f(\mathbb{N}_1,0)$, where $\mathbb{N}_1$ is the sequence of prime numbers and $f(\mathscr{A},h) = \sum_{a\in\mathscr{A}} \frac{1}{a(\log a+h)}$. In 2019, Laib et al. proved that the analogous conjecture of Erdős $f(\mathscr{A},h) \le f(\mathbb{N}_1,h)$ is false for $h \ge 81$ on a sequence of semiprimes. Recently, Lichtman gave the best lower bound $h = 1.04\cdots$ on semiprimes and he obtained other results for translated sums on k-almost primes with $2 < k \le 20$ and when k sufficiently large. In this note, we propose a new proof of the same result on semiprimes, and we generalize the result on k -almost primes for any $k \ge 2$.

Résumé. Une suite $\mathscr{A}$ d'entiers strictement positifs est dite primitive si aucun de ses termes ne divise les autres, Erdős a conjecturé que la somme $f(\mathscr{A},0) \le f(\mathbb{N}_1,0)$, où $\mathbb{N}_1$ est la suite des nombres premiers et $f(\mathscr{A},h) = \sum_{a\in\mathscr{A}} \frac{1}{a(\log a+h)}$. En 2019, Laib et al. ont prouvé que la conjecture analogue d'Erdős $f(\mathscr{A},h) \le f(\mathbb{N}_1,h)$ est fausse pour $h \ge 81$ sur la suite de nombres semi-premiers. Récemment, Lichtman a donné le meilleur minorant $h = 1.04\cdots$ sur les nombres semi-premiers et il a obtenu d'autres résultats pour des sommes translatées sur k-presque premiers avec $2 < k \le 20$ et lorsque k est suffisamment grand. Dans cette note, nous proposons une nouvelle démonstration du même résultat sur les nombres semi-premiers, et nous généralisons le résultat sur les k-presque premiers pour tout $k \ge 2$.

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1. Introduction

A sequence $\mathscr{A}$ of strictly positive integers is said to be primitive if none of its terms divides the others. It is obvious that the sequence $\mathbb{N}_1$ of prime numbers and the following examples

$$\mathscr{A}_k^d = \{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_d^{\alpha_d} \,|\, \alpha_1,\dots,\alpha_k, d\in\mathbb{N},\ \alpha_1+\cdots+\alpha_d = k\}$$
$$\mathscr{C}_d = \{p_n \in \mathbb{N}_1 \,|\, n > d\},$$
$$\mathscr{B}_k^d = \mathscr{A}_k^d \cup \mathscr{C}_d,$$
$$\mathbb{N}_k = \{n\in\mathbb{N} \,|\, \Omega(n) = k\},$$

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are primitive sequences, where $\Omega(n)$ is the numbers of prime factors of n counted with multiplicity. Note that the sequence $(\mathbb{N}_k)_{k>2}$ is the sequence of k-almost primes and $\mathbb{N}_2$ is the sequence of semiprimes. In [2], Erdős proved that the series $f(\mathscr{A},0)$ converges uniformly for any primitive sequence $\mathscr{A}\neq\{1\}$. In [3], Erdős and Zhang conjectured, for any primitive sequence $\mathscr{A}\neq\{1\}$, that $f(\mathscr{A},0)\leq f(\mathbb{N}_1,0)$. In [1], Cohen compute $f(\mathbb{N}_1,0)$ and he obtained

$$f(\mathbb{N}_1,0)=1.63661632335126086856965800392186367118159707613129\cdots.$$

In [3], Erdős and Zhang proved that $f(\mathscr{A},0)\leq 1.84$ and in [9], Lichtman and Pomerance proved that $f(\mathscr{A},0)<1.781\cdots$. In [10], Zhang proved that $f(\mathbb{N}_k,0)<f(\mathbb{N}_1,0)$ for all $k>1$. Recently, in [5], recently, Laib et al. proved that $f(\mathscr{A},h)>f(\mathbb{N}_1,h)$ for $h\geq 81$ and the author improved it later in [4] for $h\geq 60$. Also, in [6], Laib and Rezzoug proved that for all $k>1$ and

$$h\geq\frac{kk!e^{k+1}}{(k+1)^{k-1}-k!},\quad f(\mathbb{N}_k,h)>f\left(\mathscr{B}_k^{d_k},h\right)>f(\mathbb{N}_1,h).$$

For the purposes of this note, we define h_k as follows

Definition 1. *For any integer $k>1$, the value h_k is defined as the greatest lower bound of the set of all positive real numbers, where for any $h\geq h_k$, we have $f(\mathbb{N}_k,h)\geq f(\mathbb{N}_1,h)$.*

Note that for $k\leq 10$, we have $h_2=1.04466$, $h_3=0.98214$, $h_4=0.93018$, $h_5=0.89038$, $h_6=0.86146$, $h_7=0.84126$, $h_8=0.82759$, $h_9=0.81859$ and $h_{10}=0.81280$.
In [8], Lichtman gave the best lower bound $h_2=1.04\cdots$ for semiprimes, and other best result h_k for k-almost primes when $2<k\leq 20$.
Motivated by the above works, we show in this paper by different methods the general case $k>1$. Our main results are given by the following two theorems.

Theorem 2. *For all $k>1$ and any $h>h_k$, we have $f(\mathbb{N}_k,h)>f(\mathbb{N}_1,h)$, where h_k is the common limit of the decreasing sequence $(x_n^{(k)})_{n\geq 0}$, and the increasing sequence $(y_n^{(k)})_{n\geq 0}$ defined as*

$$\begin{cases} x_0^{(k)}=\frac{kk!e^{k+1}}{(k+1)^{k-1}-k!},\\ f\left(\mathbb{N}_k,x_n^{(k)}\right)=f\left(\mathbb{N}_1,x_{n+1}^{(k)}\right)\end{cases}\quad\text{and}\quad\begin{cases} y_0^{(k)}=0,\\ f\left(\mathbb{N}_k,y_n^{(k)}\right)=f\left(\mathbb{N}_1,y_{n+1}^{(k)}\right).\end{cases}$$

Moreover $f(\mathbb{N}_k,h)>f(\mathbb{N}_1,h_\infty)$, for $h>h_\infty$, where $h_\infty=0.8035236546387282\cdots$ is the unique real solution of the equation $f(\mathbb{N}_1,h)=1$, for all k sufficiently large.

The case when $k=2$ becomes a special case of the previous theorem. A computer calculation gives $h_2=1.0446\cdots$, so we show the next result as the general case of [8, Theorem 2], and which confirms that all $h_k\leq h_2$.

Theorem 3. *For $h>h_2=1.0446\cdots$ and for all $k>1$, we have*

$$f(\mathbb{N}_k,h)>f(\mathbb{N}_1,h).$$

The proof strategy used by Lichtman in [8] is based on the introduction of the zeta function for k-almost primes as

$$P_k(s)=\sum_{n\in\mathbb{N}_k}\frac{1}{n^s},$$

where $P_1(s):=P(s)$ denotes the prime zeta function. He proved the decreasing of the function $s\mapsto P_k(s)-P(s)$ over the interval $(1,s_k')$ with the condition $t_k<s_k'$, for $2\leq k\leq 20$ where t_k,s_k' are, respectively, the solutions of the equations

$$P_k(s)/\left(2^{-s}+3^{-s}\right)=1\quad\text{and}\quad P_{k-1}(s)=1.$$

We present improvements and other proofs of [8, Theorems 1 and 2]. Our proofs are based on the following lemmas.

2. Lemmas

Lemma 4. *Let $f, g \in C^1[a,b]$, be two strictly decreasing functions on the interval $[a,b]$, such that $g(b) > f(b)$ and $g(a) < f(a)$ and let $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ be two sequences defined as*

$$\begin{cases} x_0 = b, \ \ g(x_i) = f(x_{i+1}), \\ y_0 = a, \ \ g(y_i) = f(y_{i+1}), \ \ i \geq 0. \end{cases}$$

Then, the sequence $(x_n)_{n\geq 0}$ is strictly decreasing and the sequence $(y_n)_{n\geq 0}$ is strictly increasing. Furthermore, if there exist $i, j \in \mathbb{N}$ such that the function $x \mapsto g(x) - f(x)$ is monotonic on the interval $[y_j, x_i]$, then these sequences are adjacent i.e. share a common limit.

Proof. Let $(h_n(x))_{n\geq 0}$ and $(k_n(x))_{n\geq 0}$ be two sequences of continuous and strictly increasing functions defined on the interval $[y_n, x_n]$ by $h_n(x) = g(x_n) - f(x)$ and $k_n(x) = g(y_n) - f(x)$. To prove this lemma we proceed by induction on $n \geq 0$. Indeed, for $n = 0$, since h_0 is strictly increasing with $h_0(b) > 0 > h_0(a)$, by the intermediate value theorem, there exists a unique $x_1 \in (a,b)$ such that $h_0(x_1) = 0$. That is, $g(b) = f(x_1)$. Similarly, there exists a unique $y_1 \in (a,b)$ such that $g(a) = f(y_1)$. And since g is strictly decreasing, $0 < g(a) - g(b) = f(y_1) - f(x_1)$ so f strictly decreasing gives $x_1 > y_1$. Assume that $b > x_1 > \cdots > x_n > y_n > \cdots > y_1 > a$, $g(x_n) > f(x_n)$ and $g(y_n) < f(y_n)$, since h_n is strictly increasing with $h_n(x_n) > 0 > h_n(y_n)$, by the intermediate value theorem, there exists a unique $x_{n+1} \in (y_n, x_n)$ such that $h_n(x_{n+1}) = 0$. That is, $g(x_n) = f(x_{n+1})$. Similarly, there exists a unique $y_{n+1} \in (y_n, x_n)$ such that $g(y_n) = f(y_{n+1})$. And since g is strictly decreasing, $0 < g(y_n) - g(x_n) = f(y_{n+1}) - f(x_{n+1})$ so f strictly decreasing gives $x_{n+1} > y_{n+1}$.

For the second part of lemma, we have the two sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ are bounded and monotonic. So, they converge, respectively, to some values l_1 and l_2 with $x_n > l_1 \geq l_2 > y_n$ for any $n \geq 0$, where $g(l_1) = f(l_1)$ and $g(l_2) = f(l_2)$. By the fact that the function $x \mapsto g(x) - f(x)$ is monotonic on the interval $[y_j, x_i]$ for some indices i and j, it follows that $g(x) = f(x)$ has a unique root $l \in [y_j, x_i]$. So necessarily $l = l_1 = l_2$ which completes the proof. □

Lemma 5 (Lichtman, 2022). *For $k \geq 1$, we have*

$$P_k(s) = \sum_{j=1}^{k} \frac{P_{k-j}(s) P(js)}{k} \quad \text{and} \quad f(\mathbb{N}_k, h) = \int_1^{\infty} P_k(s) e^{(1-s)h} ds.$$

Lemma 6. *For $k > 13$ and $h \geq 0$ the series*

$$D_k(h) := \int_1^{\infty} \left(\frac{(P(s))^k}{k!} - P(s) \right) e^{(1-s)h} ds$$

has a unique real zero $h = l_k$ where $l_k < 1.9$.

Proof. Let $k > 13$. According to Lemma 5 and from Theorem of Zhang [10], we get

$$\int_1^{\infty} \frac{(P(s))^k}{k!} e^{(1-s)0} ds < \int_1^{\infty} P_k(s) e^{(1-s)0} ds < \int_1^{\infty} P(s) e^{(1-s)0} ds. \tag{1}$$

Then, by [7, equation (5.10)], we have

$$0 < P(s) - \log\left(\frac{\alpha}{s-1}\right) < 1.4(s-1), \quad s \in [1,2],$$

where $\alpha = \exp(-\sum_{m\geq 2} P(m)/m) = 0.72926\cdots$.
Thus, for every $k > 13$, since $\log(\alpha/0.0134) < 4$, we get

$$\begin{aligned} \int_1^{1.0134} (P(s))^k ds &> \int_0^{0.0134} \left(\log\left(\frac{\alpha}{s}\right)\right)^k ds \\ &= \alpha \int_{\log(\alpha/0.0134)}^{\infty} u^k e^{-u} du \\ &> \alpha \Gamma(k+1, 4) > 0.72926 k!, \end{aligned}$$

where $\Gamma(k+1,4)$ is the incomplete Gamma function. It follows that

$$\begin{aligned}\int_1^{\infty} \frac{(P(s))^k}{k!} e^{(1-s)1.9} ds &\geq \int_1^{1.0134} \frac{(P(s))^k}{k!} e^{(1-s)1.9} ds \\ &> e^{(-0.0134)1.9} \int_1^{1.0134} \frac{(P(s))^k}{k!} ds \\ &> 0.72926 e^{(-0.0134)\times 1.9} \\ &\geq 0.71 > 0.702 \\ &\geq \int_1^{\infty} P(s)\, e^{(1-s)\times 1.9} ds.\end{aligned}$$

The series $\frac{(P(s))^k}{k!} - P(s)$ changes sign from positive to negative around its unique real root $\sigma_k > 1$, where $P(\sigma_k) = (k!)^{\frac{1}{k-1}}$. So we have

$$D_k(h)\, e^{(\sigma_k - 1)h} = \int_1^{\sigma_k} \left(\frac{(P(s))^k}{k!} - P(s)\right) e^{(\sigma_k - s)h} ds - \int_{\sigma_k}^{\infty} \left(P(s) - \frac{(P(s))^k}{k!}\right) e^{(\sigma_k - s)h} ds.$$

The expression given represents the difference between two integrals, both of which have positive integrands. The first integral is monotonically increasing, whereas the second is monotonically decreasing with respect to the variable $h \geq 0$. Consequently, the difference between the two integrals is monotonically increasing with respect to $h \geq 0$. Moreover, since $D_k(1.9) > 0$, it follows that for all $h \geq 1.9$, we have $\int_1^{\infty} \frac{(P(s))^k}{k!} e^{(1-s)h} ds > \int_1^{\infty} P(s) e^{(1-s)h} ds$. Combining this with equation (1), we can conclude that the real zero $h = l_k$ of the series $D_k(h)$ is unique and lies in the interval $[0, 1.9]$. This completes the proof. □

3. Proofs of Theorems 2 and 3

Proof of Theorem 2. For $2 \leq k \leq 13$, the two series of functions $h \mapsto f(\mathbb{N}_k, h)$ and $h \mapsto f(\mathbb{N}_1, h)$, satisfy the conditions of Lemma 4 on the interval $[y_0^{(k)}, x_0^{(k)}]$. So by Lemma 4 the sequence $(x_n^{(k)})_{n\geq 0}$ is strictly decreasing and the sequence $(y_n^{(k)})_{n\geq 0}$ is strictly increasing, then they converge. The series of functions $h \mapsto f(\mathbb{N}_k, h)$ converges uniformly according to Erdős [2] and the series of derivatives functions

$$h \mapsto \frac{d}{dh} f(\mathbb{N}_k, h) := f'(\mathbb{N}_k, h) = -\sum_{a \in \mathbb{N}_k} \frac{1}{a\left(\log(a) + h\right)^2}$$

converges uniformly on the interval $[y_0^{(k)}, x_0^{(k)}]$. Also, for any $h \in [y_1^{(k)}, x_5^{(k)}]$, we have

$$\begin{aligned}f'(\mathbb{N}_k, h) - f'(\mathbb{N}_1, h) &> f'\left(\mathbb{N}_k, y_1^{(k)}\right) - f'\left(\mathbb{N}_1, x_5^{(k)}\right) \\ &= \Delta_k \\ &\geq f'\left(\mathbb{N}_2, y_1^{(2)}\right) - f'\left(\mathbb{N}_1, x_5^{(2)}\right) \\ &= f'(\mathbb{N}_2, 0.5183) - f'(\mathbb{N}_1, 1.56114) \\ &= -0.2173\cdots + 0.2209\cdots \\ &\geq 0.0036 > 0.\end{aligned}$$

So the two sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ converge to the common limit h_k which is the unique solution of $f(\mathbb{N}_k, h) = f(\mathbb{N}_1, h)$ and $f(\mathbb{N}_k, h) > f(\mathbb{N}_1, h)$ for any $h > h_k$.
For $k > 13$, we have $P_k(s) > \frac{(P(s))^k}{k!}$, so from Lemma 5 and Lemma 6, we get

$$f(\mathbb{N}_k, h) > \int_1^{\infty} \frac{(P(s))^k}{k!} e^{(1-s)h} ds > f(\mathbb{N}_1, h), \text{for any } h \geq 1.9.$$

As $f(\mathbb{N}_k, 0) < f(\mathbb{N}_1, 0)$, so there exists $h_k \in [0, 1.9]$ such that $f(\mathbb{N}_k, h_k) = f(\mathbb{N}_1, h_k)$ and $f(\mathbb{N}_k, h) > f(\mathbb{N}_1, h)$ for $h > h_k$. However, for any $h \in [0, 1.9]$, we have

$$\begin{aligned} f^{'}(\mathbb{N}_k, h) - f^{'}(\mathbb{N}_1, h) &> f^{'}(\mathbb{N}_k, 0) - f^{'}(\mathbb{N}_1, 1.9) \\ &> -\frac{f(\mathbb{N}_1, 0)}{k\log(2)} - f^{'}(\mathbb{N}_1, 1.9) \\ &\geq -\frac{1.63\cdots}{14\log(2)} + 0.1764\cdots \\ &\geq 0.00778 > 0, \end{aligned}$$

this implies that the solution h_k is unique, which is the common limit of the two sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$. For the second part of Theorem 1, we use the same method used in [8], which gives us the unique root of $f(\mathbb{N}_1, h) - 1$ is $h_\infty = 0.8035236546387282\cdots$. This completes the proof. □

Proof of Theorem 3. The proof of the theorem comes directly from the inequality $f(\mathbb{N}_k, h_2) > f(\mathbb{N}_1, h_2)$ for all $k > 1$ of [8] and the inequality $f(\mathbb{N}_k, 0) < f(\mathbb{N}_1, 0)$ for all $k > 1$ of Zhang [10], so the unique solution h_k which is in the interval $[0, 1.9]$, becomes in the interval $[0, h_2]$. Then for $h < h_k$, we have $f(\mathbb{N}_k, h) - f(\mathbb{N}_1, h) < 0$ and for $h > h_k$ we have $f(\mathbb{N}_k, h) - f(\mathbb{N}_1, h) > 0$, and since $h_2 > h_k$, this completes the proof. □

4. Calculations table

For $2 \leq k \leq 13$, by using Mathematica[1] we compute $(x_i^{(k)})_{1\leq i\leq 5}$ and $(y_1^{(k)})$ as the unique roots of the equations $f(\mathbb{N}_k, x_n^{(k)}) = f(\mathbb{N}_1, x_{n+1}^{(k)})$ and $f(\mathbb{N}_k, y_n^{(k)}) = f(\mathbb{N}_1, y_{n+1}^{(k)})$, respectively, where $x_0^{(k)} = \frac{kk!e^{k+1}}{(k+1)^{k-1} - k!}$ and $y_0^{(k)} = 0$. We remark that, for $k \geq 5$ the sequences $(x_n^{(k)})_{n\geq 0}$ and $(y_n^{(k)})_{n\geq 0}$ converge rapidly to h_k which is the unique solution of $f(\mathbb{N}_k, h) = f(\mathbb{N}_1, h)$, and for $k \geq 11$ the two digits after the point of h_k are fixed with that of h_∞, then for $k \geq 16$ the three digits after the point of h_k are fixed with that of h_∞, ...etc.
So we can ask the question:

Open question: The sequence $(h_k)_{k\geq 2}$, decrease and converges rapidly to

$$h_\infty = 0.8035236546387282\cdots.$$

For $13 < k \leq 25$, we compute the unique root h_k of the equation $f(\mathbb{N}_k, h) = f(\mathbb{N}_1, h)$ by using Mathematica. [2]

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[1] $P[k - \texttt{integer}, s-] := \texttt{If}[k == 1, \texttt{PrimeZetaP}[s], \texttt{Expand}[(\texttt{Sum}[P[1, j*s]*P[k-j, s], j, 1, k-1] + P[1, k*s])/k]]$
$\texttt{FindRoot}[\texttt{NIntegrate}[P[k,s]/e^{[(s-1)x_n^{(k)}]}, s, 1, \infty] == \texttt{NIntegrate}[\texttt{PrimeZetaP}[s]/e^{[(s-1)x_{n+1}^{(k)}]}, s, 1, \infty], x_{n+1}^{(k)}, 0.1]$,
$\texttt{FindRoot}[\texttt{NIntegrate}[P[k,s]/e^{[(s-1)y_n^{(k)}]}, s, 1, \infty] == \texttt{NIntegrate}[\texttt{PrimeZetaP}[s]/e^{[(s-1)y_{n+1}^{(k)}]}, s, 1, \infty] y_{n+1}^{(k)}, 0.1]$,
$\Delta_k = \texttt{NIntegrate}[((1-s)P[k,s])/e^{[(s-1)y_1^{(k)}]}, s, 1, \infty] - \texttt{NIntegrate}[((1-s)\texttt{PrimeZetaP}[s])/e^{[(s-1)x_5^{(k)}]}, s, 1, \infty]$

[2] $\texttt{FindRoot}[\texttt{NIntegrate}[P[k,s]/e^{[(s-1)X]}, s, 1, \infty] == \texttt{NIntegrate}[\texttt{PrimeZetaP}[s]/e^{[(s-1)X]}, s, 1, \infty], X, 0.1]$

Table 1. The values of $(x_i^{(k)})_{1\le i\le 5}$, $y_1^{(k)}$, h_k and Δ_k, for $2 \le k \le 13$.

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_4^{(k)}$	$x_5^{(k)}$	$y_1^{(k)}$	h_k	Δ_k
2	23.4028	8.47027	3.86317	2.21842	1.56114	0.5183	1.04466	0.00361
3	11.0425	2.65532	1.33427	1.06189	1.00053	0.73419	0.98214	0.25936
4	6.84739	1.4812	0.99408	0.93783	0.93110	0.80975	0.93018	0.32833
5	4.70836	1.09508	0.90330	0.89120	0.89043	0.82987	0.89038	0.36537
6	3.43161	0.94071	0.86419	0.86155	0.86146	0.83018	0.86146	0.38850
7	2.61004	0.87232	0.84184	0.84127	0.84126	0.82474	0.84126	0.40332
8	2.05938	0.83979	0.84167	0.82772	0.82759	0.81875	0.82759	0.41277
9	1.68123	0.82336	0.81862	0.81860	0.81860	0.81381	0.81859	0.41872
10	1.41757	0.81465	0.81280	0.81280	0.81280	0.81021	0.81280	0.42242
11	1.23208	0.80986	0.80915	0.80915	0.80915	0.80775	0.80915	0.42469
12	1.10107	0.80716	0.80689	0.80689	0.80689	0.80613	0.80689	0.42605
13	1.00855	0.80562	0.80551	0.80551	0.80551	0.80551	0.80551	0.42687

Table 2. The values of h_k, for $13 < k \le 25$.

k	h_k	k	h_k	k	h_k	k	h_k
14	0.80469	17	0.80374	20	0.80356	23	0.80353
15	0.80420	18	0.80365	21	0.80355	24	0.80352
16	0.80391	19	0.80359	22	0.80355	25	0.80352

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